

A primer for unstable motivic homotopy theory

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Abstract

In this expository article, we give the foundations, basic facts, and first examples of unstable motivic homotopy theory with a view towards the approach of Asok-Fasel to the classification of vector bundles on smooth complex affine varieties. Our focus is on making these techniques more accessible to algebraic geometers.

Key Words. Vector bundles, projective modules, motivic homotopy theory, Postnikov systems, algebraic K -theory.

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1 Introduction

This primer is intended to serve as an introduction to the basic facts about Morel and Voevodsky’s motivic, or \mathbb{A}^1 , homotopy theory [MV99], [Voe98], with a focus on the unstable part of the theory. It was written following a week-long summer school session on this topic led by Antieau at the University of Utah in July 2015. The choice of topics reflects what we think might be useful for algebraic geometers interested in learning the subject.

In our view, the starting point of the development of unstable motivic homotopy theory is the resolution of Serre’s conjecture by Quillen [Qui76] and Suslin [Sus76]. Serre asked in [Ser55] whether every finitely generated projective module over $k[x_1, \dots, x_n]$ is free when k is a field. Put another way, the question is whether

$$\mathrm{Vect}_r(\mathrm{Spec} k) \rightarrow \mathrm{Vect}_r(\mathbb{A}_k^n)$$

is a bijection for all $n \geq 1$, where $\mathrm{Vect}_r(X)$ denotes the set of isomorphism classes of rank r vector bundles on X . Quillen and Suslin showed that this is true and in fact proved the analogous statement when k is replaced by a Dedekind domain. This suggested the following conjecture.

Conjecture 1.1 (Bass-Quillen). *Let X be a regular noetherian affine scheme of finite Krull dimension. Then, the pullback map $\mathrm{Vect}_r(X) \rightarrow \mathrm{Vect}_r(X \times \mathbb{A}^n)$ is a bijection for all $r \geq 1$ and all $n \geq 1$.*

The Bass-Quillen conjecture has been proved in many cases, but not yet in full generality. Lindel [Lin81] prove the conjecture when X is essentially of finite type over a field, and Popescu proved it when X is the spectrum of an unramified regular local ring (see [Swa98, Theorem 2.2]). Piecing these results together one can, for example, allow X to be the spectrum of a ring with the property that all its localizations at maximal ideals are smooth over a Dedekind ring with a perfect residue field see [AHW15a, Theorem 5.2.1]. For a survey of other results in this direction, see [Lam06, Section VIII.6].

If X is a reasonable topological space, such as a manifold, simplicial complex, or CW complex, then there are also bijections $\mathrm{Vect}_r^{\mathrm{top}}(X) \rightarrow \mathrm{Vect}_r^{\mathrm{top}}(X \times I^1)$, where I^1 is the unit interval and $\mathrm{Vect}_r^{\mathrm{top}}$ denotes the set of isomorphism classes of rank r topological complex vector bundles on X . Thus, the Quillen-Suslin theorem and the Bass-Quillen conjecture suggest that there might be a homotopy theory for schemes in which the affine line \mathbb{A}^1 plays the role of the contractible unit interval.

Additional evidence for this hypothesis is provided by the fact that many important cohomology theories for smooth schemes are \mathbb{A}^1 -invariant. For example, the pullback maps in Chow groups

$$\mathrm{CH}^*(X) \rightarrow \mathrm{CH}^*(X \times_k \mathbb{A}^1),$$

Grothendieck groups

$$K_0(X) \rightarrow K_0(X \times_k \mathbb{A}^1),$$

and étale cohomology groups

$$H_{\text{ét}}^*(X, \mu_\ell) \rightarrow H_{\text{ét}}^*(X \times_k \mathbb{A}^1, \mu_\ell)$$

are isomorphisms when X is smooth over a field k and ℓ is invertible in k .

Now, we should note immediately, that the functor $\text{Vect}_r : \text{Sm}_k^{\text{op}} \rightarrow \text{Sets}$ is *not* itself \mathbb{A}^1 -invariant. Indeed, there are vector bundles on $\mathbb{P}^1 \times_k \mathbb{A}^1$ that are not pulled back from \mathbb{P}^1 . The reader can construct a vector bundle on the product such that the restriction to $\mathbb{P}^1 \times \{1\}$ is a non-trivial extension E of $\mathcal{O}(1)$ by $\mathcal{O}(-1)$ while the restriction to $\mathbb{P}^1 \times \{0\}$ is $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. Surprisingly, for affine schemes, this proves not to be a problem: forcing Vect_r to be \mathbb{A}^1 -invariant produces an object which still has the correct values on smooth affine schemes.

The construction of (unstable) motivic homotopy theory over a quasi-compact and quasi-separated scheme S takes three steps. The first stage is a homotopical version of the process of passing from a category of schemes to the topos of presheaves on the category. Specifically, one enlarges the class of spaces from Sm_S , the category of smooth schemes over S , to the category of presheaves of simplicial sets $\text{sPre}(\text{Sm}_S)$ on Sm_S . An object X of $\text{sPre}(\text{Sm}_S)$ is a functor $X : \text{Sm}_S^{\text{op}} \rightarrow \text{sSets}$, where sSets is the category of simplicial sets, one model for the homotopy theory of CW complexes. Presheaves of sets give examples of simplicial presheaves by viewing a set as a discrete space. There is a Yoneda embedding $\text{Sm}_S \rightarrow \text{sPre}(\text{Sm}_S)$ as usual. In the next stage, one imposes a descent condition, namely focusing on those presheaves that satisfy the appropriate homotopical version of Nisnevich descent. We note that one can construct motivic homotopy theory with other topologies, as we will do later in Section 5. The choice of Nisnevich topology is motivated by the fact that it is the coarsest topology where we can prove the purity theorem (Section 7) and the finest where we can prove representability of K -theory (Section 6.1). The result is a homotopy theory enlarging the category of smooth schemes over S but which does not carry any information about the special role \mathbb{A}^1 is to play. In the third and final stage, the projection maps $X \times_S \mathbb{A}^1 \rightarrow X$ are formally inverted.

In practice, care must be taken in each stage; the technical framework we use in this paper is that of model categories, although one could equally use ∞ -categories instead, as has been done recently by Robalo [Rob15]. Model categories, Quillen functors (the homotopical version of adjoint pairs of functors), homotopy limits and colimits, and Bousfield's theory of localization are all explained in the lead up to the construction of the motivic homotopy category.

When S is regular and noetherian, algebraic K -theory turns out to be representable in $\text{Spc}_S^{\mathbb{A}^1}$, as are many of its variants. A pleasant surprise however is that despite the fact that Vect_r is not \mathbb{A}^1 -invariant on all of Sm_S , its \mathbb{A}^1 -localization still has the correct values on smooth affine schemes over k . This is a crucial result of Morel [Mor12, Chapter 8], which was simplified in the Zariski topology by Schlichting [Sch15], and simplified and generalized by Asok, Hoyois, and Wendt [AHW15a]. This fact is at the heart of applications of motivic homotopy theory to the classification of vector bundles on smooth affine complex varieties by Asok and Fasel.

We describe now the contents of the paper. As mentioned above, the document below reflects topics the authors decided should belong in a first introduction to \mathbb{A}^1 -homotopy theory, especially for people coming from algebraic geometry. Other surveys in the field which focus on different aspects of the theory include [Dun07], [Lev08], [Lev16], [Mor06]; a textbook reference for the ideas covered in this survey is [Mor12]. Voevodsky's ICM address pays special attention to the topological motivation for the theory in [Voe98].

Some of these topics were the focus of Antieau's summer school course at the AMS Summer Institute in Algebraic Geometry at the University of Utah in July 2015. This includes the material in Section 2 on topological vector bundles. This section is meant to

explain the power of the Postnikov tower approach to classification problems and entice the reader to dream of the possibilities were this possible in algebraic geometry.

The construction of the motivic homotopy category is given in Section 3 after an extensive introduction to model categories, simplicial presheaves, and the Nisnevich topology. Other topics are meant to fill gaps in the literature, while simultaneously illustrating the techniques common to the field. Section 4 establishes the basic properties of motivic homotopy theory over S . It is meant to be a kind of cookbook and contains many examples, exercises, and computations. In Section 5 we define and give examples of classifying spaces BG for linear algebraic groups G , and perform some calculations of their homotopy sheaves. The answers will involve algebraic K -theory which is discussed in Section 6. Following [MV99], with some modifications, we give a self-contained proof that algebraic K -theory is representable in the \mathbb{A}^1 -homotopy category, and we identify its representing object as the \mathbb{A}^1 -homotopy type of a classifying space BGL_∞ . In Section 7, we prove the critical purity theorem which is the source of Gysin sequences. A brief vista at the end of the paper, in Section 8, illustrates how all of this comes together to classify vector bundles on smooth affine schemes. Finally, in Section 9, we gather some miscellaneous additional exercises.

Many things are not in this paper. We view the biggest omission as the exclusion of a presentation of the first non-zero homotopy sheaves of punctured affine spaces. Morel proved that

$$\pi_n^{\mathbb{A}^1}(\mathbb{A}^{n+1} - \{0\}) \cong \mathbf{K}_{n+1}^{\text{MW}},$$

the $(n+1)$ st unramified Milnor-Witt K -theory sheaf where $n \geq 1$. A proof may be found in [Mor12, Chapter 6].

Other topics we would include granted unlimited time include stable motivic homotopy theory, and in particular the motivic spectral sequence, the stable connectivity theorem of Morel, the theory of algebraic cobordism due to Levine-Morel [LM07], motivic cohomology and the work of Voevodsky and Rost on the Bloch-Kato conjecture, and the work [DI10] of Dugger and Isaksen on the motivic Adams spectral sequence.

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2 Classification of topological vector bundles

We introduce the language of Postnikov towers and illustrate their use through several examples involving the classification of topological vector bundles. The point is to tempt the reader to imagine the power these tools would possess in algebraic geometry if they existed.

General references for the material here include Hatcher and Husemoller’s books [Hat02, Hus75].

2.1 Postnikov towers and Eilenberg-MacLane spaces

Let S^i denote the i -sphere, embedded in \mathbb{R}^{i+1} as the unit sphere, and let $s = (1, 0, \dots, 0)$ be the basepoint. Recall that if (X, x) is a pointed space, then

$$\pi_i(X, x) = [(S^i, s), (X, x)]_*$$

the set of homotopy classes of pointed maps from the i -sphere to X . The set of path-components $\pi_0(X, x)$ is simply a set pointed by the component containing x . The fundamental group is $\pi_1(X, x)$, a not-necessarily-abelian group. The groups $\pi_i(X, x)$ are abelian for $i \geq 2$. For a path-connected space, $\pi_i(X, x)$ does not depend on x , so we will often omit x from our notation and write $\pi_i(X)$ or $\pi_i X$.

Definition 2.1. A map of spaces $f : X \rightarrow Y$ is an n -**equivalence** if $\pi_0(f) : \pi_0(X) \rightarrow \pi_0(Y)$ is a bijection and if for each choice of a basepoint $x \in X$, the induced map

$$\pi_i(f) : \pi_i(X, x) \rightarrow \pi_i(Y, f(x))$$

is an isomorphism for each $i < n$ and a surjection for $i = n$. The map f is a **weak homotopy equivalence** if it is an ∞ -equivalence.

Typically we are interested in working with spaces up to weak homotopy equivalence. The correct notion of a fibration in this setting is a Serre fibration. Let D^n denote the n -disk and I^1 the unit interval. A map $p : E \rightarrow B$ is a **Serre fibration** (or simply a **fibration** as we will not use any other notion of fibration for maps of topological spaces) if for every diagram

$$\begin{array}{ccc} D^n \times \{0\} & \longrightarrow & E \\ \downarrow i & \nearrow \text{dotted} & \downarrow p \\ D^n \times I^1 & \longrightarrow & B \end{array}$$

of solid arrows, there exists a dotted lift making both triangles commute. In other words, p has the **right lifting property** with respect to the maps $D^n \times \{0\} \rightarrow D^n \times I^1$. This property is equivalent to having the right lifting property with respect to all maps $A \times I^1 \cup X \times \{0\} \rightarrow X$ for all CW pairs (X, A) .

There is a functorial way of replacing an arbitrary map $f : X \rightarrow Y$ by a Serre fibration. Let P_f be the space consisting of pairs (x, ω) where $x \in X$ and $\omega : I^1 \rightarrow Y$ such that $\omega(0) = f(x)$. There is a natural inclusion $X \rightarrow P_f$ sending x to (x, c_x) , where c_x is the constant path at $f(x)$, and there is a natural map $P_f \rightarrow Y$ sending (x, ω) to $\omega(1)$.

Exercise 2.2. Show that the map $X \rightarrow P_f$ is a homotopy equivalence and that $P_f \rightarrow Y$ is a fibration.

Given a fibration $p : E \rightarrow B$ and a basepoint $e \in E$, the subspace $F = p^{-1}(p(e))$ is the **fiber** of p at $p(e)$. The point e is inside F . The crucial fact about Serre fibrations is that the sequence

$$(F, e) \rightarrow (E, e) \rightarrow (B, p(e))$$

gives rise to a long exact sequence

$$\cdots \rightarrow \pi_n(F, e) \rightarrow \pi_n(E, e) \rightarrow \pi_n(B, p(e)) \rightarrow \pi_{n-1}(F, e) \rightarrow \cdots \rightarrow \pi_0(F, e) \rightarrow \pi_0(E, e) \rightarrow \pi_0(B, p(e))$$

of homotopy groups. Some explanation of ‘exactness’ is required in low-degrees as they are only groups or pointed sets. We refer to [BK72, Section IX.4].

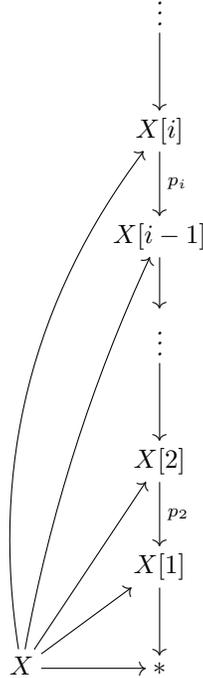
Definition 2.3. The **homotopy fiber** $F_f(y)$ of a map $f : X \rightarrow Y$ over a point $y \in Y$ is the fiber of $P_f \rightarrow Y$ over y . A sequence $(F, x) \rightarrow (X, x) \xrightarrow{f} (Y, y)$ of pointed spaces is a **homotopy fiber sequence** if

1. Y is path-connected,
2. $f(F) = y$, and
3. the natural map $F \rightarrow F_f(y)$ is a weak homotopy equivalence.

The homotopy fiber sequences are those which behave just as well as fiber sequences from the point of view of their homotopy groups.

Exercise 2.4. Given a space X and a point $x \in X$, the based loop space $\Omega_x X$ is the homotopy fiber of $x \rightarrow X$. When X is path-connected or the basepoint is implicit, we will write ΩX for $\Omega_x X$.

Theorem 2.5. *Let X be a path-connected space, so that $\pi_0(X, x) = *$. There exists a commutative diagram of pointed spaces*



such that

1.
$$\pi_j(X[i]) \cong \begin{cases} \pi_j X & j \leq i, \\ 0 & j > i; \end{cases}$$
2. $X \rightarrow X[i]$ is an $(i + 1)$ -equivalence;
3. each map $X[i + 1] \rightarrow X[i]$ is a Serre fibration;
4. the natural map $X \rightarrow \lim_i X[i]$ is a weak homotopy equivalence.

The space $X[i]$ is the **i th Postnikov section** of X , and the diagram is called the **Postnikov tower** of X .

Proof. See Hatcher [Hat02, Chapter 4]. The basic idea is that one builds $X[i]$ from X by first attaching cells to X to kill π_{i+1} . Then, attaching cells to the result to kill π_{i+2} , and so on. \square

Definition 2.6. Let $i > 0$ and let G be a group, abelian if $i > 1$. A $K(G, i)$ -**space** is a connected space Y such that $\pi_i(Y) \cong G$ and $\pi_j(Y) = 0$ for $j \neq i$. As a class, these are referred to as **Eilenberg-MacLane spaces**.

Exercise 2.7. The homotopy fiber of $X[i] \rightarrow X[i-1]$ is a $K(\pi_i X, i)$ -space.

Suppose that Y is another space, and we want to construct a map $Y \rightarrow X$. We can hope to start with a map $Y \rightarrow X[1]$, lift it to a map $Y \rightarrow X[2]$, and on up the Postnikov tower. Using the fact that X is the limit of the tower, we would have constructed a map $Y \rightarrow X$.

What we need is a way of knowing when a map $Y \rightarrow X[i]$ lifts to a map $Y \rightarrow X[i+1]$. We need an **obstruction theory** for such lifts. Before getting into the details in the topological setting, we consider an example from algebra. Let

$$0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$$

be an exact sequence of abelian groups. Let $g : H \rightarrow G$ be a homomorphism. When can we lift g to a map $f : H \rightarrow F$? The extension is classified by a class $p \in \text{Ext}^1(G, E)$. This can be viewed as a map $G \rightarrow E[1]$ in the derived category $D(\mathbb{Z})$. Composing with f , we get the pulled back extension $f^*p \in \text{Ext}^1(H, E)$, viewed either as the composition $H \rightarrow G \rightarrow E[1]$ in $D(\mathbb{Z})$, or as an induced extension

$$0 \rightarrow E \rightarrow F' \rightarrow H \rightarrow 0.$$

Now, we know that g lifts if and only if the extension F' splits if and only if $f^*p = 0 \in \text{Ext}^1(H, E)$. The theory we explain now is a **nonabelian** version of this example.

Definition 2.8. A homotopy fiber sequence $F \rightarrow X \xrightarrow{p} Y$ is **principal** if there is a delooping B of F (so that $\Omega B \simeq F$) and a map $k : Y \rightarrow B$ such that p is homotopy equivalent to the homotopy fiber of k . We will call k the **classifying map** of the principal fiber sequence.

Example 2.9. The reduced cohomology $\tilde{H}^{i+1}(X, A)$ of a pointed space (X, x) with coefficients in an abelian group A is the kernel of the restriction map $H^{i+1}(X, A) \rightarrow H^{i+1}(\{x\}, A)$. If $i+1 > 0$, then the reduced cohomology is isomorphic to the unreduced cohomology. Recall that the reduced cohomology of X can be represented as $\tilde{H}^{i+1}(X, A) = [(X, x), K(A, i+1)]_*$. That is, Eilenberg-MacLane spaces represent cohomology classes. Given a cohomology class $k \in \tilde{H}^{i+1}(X, A)$ viewed as a map $X \rightarrow K(A, i+1)$, the homotopy fiber of k is a space Y with $Y \rightarrow X$ having homotopy fiber $K(A, i)$.

Suppose that $k \in \tilde{H}^2(X, \mathbb{Z})$. The homotopy fiber sequence one gets is $K(\mathbb{Z}, 1) \rightarrow Y \rightarrow X$. Since $S^1 \simeq \mathbb{C}^* \simeq K(\mathbb{Z}, 1)$, we see that k corresponds to a topological complex line bundle (up to homotopy) $Y \rightarrow X$, as expected.

Lemma 2.10. Let $F \rightarrow X \rightarrow Y$ be a principal fibration classified by $k : Y \rightarrow B$ (so that $F \simeq \Omega B$). Let Z be a CW complex. Then, a map $Z \rightarrow Y$ lifts to $Z \rightarrow X$ if and only if the composition $Z \rightarrow Y \rightarrow B$ is nullhomotopic.

Proof. This follows from the definition of a fibration. Indeed, we can assume that $Y \rightarrow B$ is a fibration (by replacing the map by a fibration using P_k if necessary) and that X is the fiber over the basepoint. Applying the right lifting property to the case at hand, where $Z \times I^1 \rightarrow B$ is a nullhomotopy from the map $Z \rightarrow B$ to the map $Z \rightarrow \{b\} \subseteq B$, we see that the initial map $Z \rightarrow Y$ is homotopic to a map landing in the actual fiber of $Y \rightarrow B$. This fiber is X . \square

Theorem 2.11. *If X is simply connected (X is path-connected and $\pi_1(X) = 0$), then the Postnikov tower of X is a tower of principal fibrations. In particular, for each $i \geq 1$ there is a ladder of homotopy fiber sequences*

$$\begin{array}{ccc} K(\pi_i X, i) & \longrightarrow & X[i] \\ & & \downarrow \\ & & X[i-1] \xrightarrow{k_{i-1}} K(\pi_i X, i+1). \end{array}$$

*Specifically, $p_i : X[i] \rightarrow X[i-1]$ is the homotopy fiber of k_{i-1} and $K(\pi_i X, i) \rightarrow X[i]$ is the homotopy fiber of p_i . The map k_{i-1} is the $(i-1)$ st **k -invariant** of X . It represents a cohomology class in $H^{i+1}(X[i-1], \pi_i X)$.*

Corollary 2.12. *Let X be a simply connected space. For each $f : Y \rightarrow X[i-1]$, there is a uniquely determined class $f^*k_{i-1} \in H^{i+1}(Y, \pi_i X)$. The map f lifts if and only if $f^*k_{i-1} = 0$.*

Proof. This follows from Theorem 2.11 and Lemma 2.10, together with the fact that Eilenberg-MacLane spaces represent cohomology classes. \square

A more complicated, equivariant version of the theory applies in the non-principal case (so that X is in particular not simply connected), but we ignore that for now as it is unnecessary in the topological applications we have in mind below. It is explained briefly in Section 8.

2.2 Representability of topological vector bundles

Definition 2.13. Let G be a topological group. Recall that a **G -torsor** on a space X is a space $p : Y \rightarrow X$ over X together with a (left) group action $a : G \times Y \rightarrow Y$ such that

1. $p(a(g, y)) = p(y)$ (the action preserves fibers), and
2. the natural map $G \times Y \rightarrow Y \times_X Y$ given by $(g, y) \mapsto (a(g, y), y)$ is an isomorphism.

Torsors for G are also called **principal G -bundles**.

Example 2.14. The **trivial** G -torsor on X is $G \times X \rightarrow X$, with the projection map as the structure map.

Given G -torsors $p : Y \rightarrow X$ and $p' : Y' \rightarrow X'$, a morphism of G -torsors $(f, g) : (Y, p, X) \rightarrow (Y', p', X')$ is a map $f : Y \rightarrow Y'$ of G -spaces (i.e., compatible with the G -action) together with a map $g : X \rightarrow X'$ such that $g(p(y)) = p'(f(y))$. Given a map $g : X \rightarrow X'$ and a G -torsor $p' : Y' \rightarrow X'$, there is a uniquely determined G -torsor structure on $Y = X \times_{X'} Y'$. The projection map $f : Y \rightarrow X$ makes (f, g) into a morphism of G -torsors. We will write g^*Y' for the pull back bundle.

Definition 2.15. A G -torsor $Y \rightarrow X$ is **locally trivial** if there is an open cover $\{g_i : U_i \rightarrow X\}_{i \in I}$ of X such that g_i^*Y is isomorphic as a G -torsor to $G \times U_i$ for each i . The subcategory of G -torsors on a fixed base X is naturally a groupoid. We write $\mathbf{Bun}_G(X)$ for the full subcategory of locally trivial G -torsors on X . The set of isomorphism classes of $\mathbf{Bun}_G(X)$ will be denoted $\text{Tors}_G(X)$.

Example 2.16. Let $L \rightarrow X$ be a complex line bundle. The fibers are in particular 1-dimensional complex vector spaces. Let $Y = L - s(X)$, where $S : X \rightarrow L$ is the 0-section. There is a natural action of the topological (abelian) group \mathbb{C}^* on Y given simply by scalar multiplication in the fibers. In this case, Y becomes a principal \mathbb{C}^* -bundle on X .

Theorem 2.17 (Steenrod). *Let G be a topological group. Then, there is a connected space BG with a G -torsor γ_G such that the natural pullback map $[X, BG] \rightarrow \text{Tors}_G(X)$ sending $f : X \rightarrow BG$ to $f^*\gamma_G$ is an isomorphism for all paracompact Hausdorff spaces X . Moreover, $\Omega BG \simeq G$.*

Proof. See Husemoller [Hus75, Theorem 4.12.2]. The existence of BG can be proved in greater generality as the representability of certain functors satisfying Mayer-Vietoris and homotopy invariance properties. The total space of γ_G is a contractible space with a free G -action. Hence, $G \rightarrow \gamma_G \rightarrow BG$ is a fiber sequence. It follows that $\Omega BG \simeq G$. \square

Remark 2.18. Any CW complex is paracompact Hausdorff. Any differentiable manifold is paracompact and Hausdorff, as is the underlying topological space associated to a separated complex algebraic variety.

Example 2.19. If A is an abelian group, then $K(A, n)$ can be given the structure of a topological abelian group. In this case, $BK(A, n)$ is a $K(A, n + 1)$ -space. Indeed, $\Omega BK(A, n) \simeq K(A, n)$.

Definition 2.20. Let $p : Y \rightarrow X$ be a G -torsor, and let F be a space with a (left) G -action. There is then a left G -action on $Y \times F$, the diagonal action. Let F_Y denote the quotient $(Y \times F)/G$. There is a natural map $F_Y \rightarrow Y/G \cong X$. The fibers of this map are all isomorphic to F . The space $F_Y \rightarrow X$ is called the F -**bundle** associated to Y .

Example 2.21. Let $p : Y \rightarrow X$ be a locally trivial $\mathrm{GL}_n(\mathbb{C})$ -torsor. Let $\mathrm{GL}_n(\mathbb{C})$ act on \mathbb{C}^n by matrix multiplication. Then, $\mathbb{C}_Y^n \rightarrow X$ is a vector bundle. In fact, this association gives a natural bijection

$$\mathrm{Tors}_{\mathrm{GL}_n(\mathbb{C})}(X) \cong \mathrm{Vect}_n^{\mathrm{top}}(X).$$

Corollary 2.22. *If X is a paracompact Hausdorff space, then there is a natural bijection*

$$[X, \mathrm{BGL}_n(\mathbb{C})] \cong \mathrm{Vect}_n^{\mathrm{top}}(X).$$

In the case of $\mathrm{GL}_n(\mathbb{C})$ we can construct a more explicit version of $\mathrm{BGL}_n(\mathbb{C})$ by using Grassmannians. Let $\mathrm{Gr}_n(\mathbb{C}^{n+k})$ denote the Grassmannian of n -plane bundles in \mathbb{C}^{n+k} , and let $\mathrm{Gr}_n = \mathrm{colim}_k \mathrm{Gr}_n(\mathbb{C}^{n+k})$ denote the colimit. Over each $\mathrm{Gr}_n(\mathbb{C}^{n+k})$ there is a canonical $\mathrm{GL}_n(\mathbb{C})$ -bundle given by the Stiefel manifold $V_n(\mathbb{C}^{n+k})$, the space of n linearly independent vectors in \mathbb{C}^{n+k} . The map sending a set of linearly independent vectors to the subspace they span gives a surjective map

$$V_n(\mathbb{C}^{n+k}) \rightarrow \mathrm{Gr}_n(\mathbb{C}^{n+k}).$$

There is a natural free action of $\mathrm{GL}_n(\mathbb{C})$ on $V_n(\mathbb{C}^{n+k})$, and $V_n(\mathbb{C}^{n+k}) \rightarrow \mathrm{Gr}_n(\mathbb{C}^{n+k})$ is a locally trivial $\mathrm{GL}_n(\mathbb{C})$ -torsor with this action.

Lemma 2.23. *The space $V_n(\mathbb{C}^{n+k})$ is $2k$ -connected for $n \geq 1$.*

Proof. Consider the map $V_n(\mathbb{C}^{n+k}) \rightarrow \mathbb{C}^{n+k} - \{0\}$ sending a set of linearly independent vectors to the last vector. The fibers are all isomorphic to $V_{n-1}(\mathbb{C}^{n+k-1})$. If $n = 2$, the fiber is thus $V_1(\mathbb{C}^{1+k}) \cong \mathbb{C}^{1+k} - \{0\} \simeq S^{1+2k}$, which is certainly $2k$ -connected. Therefore, by induction, we can assume that $V_{n-1}(\mathbb{C}^{n+k-1})$ is $2k$ -connected, and since $\mathbb{C}^{n+k} - \{0\} \simeq S^{2n+2k-1}$ is $2n + 2k - 2$ -connected, it follows that $V_n(\mathbb{C}^{n+k})$ is $2k$ -connected for $n \geq 1$. \square

As a result, the colimit $V_n = \mathrm{colim}_k V_n(\mathbb{C}^{n+k})$ is a contractible $\mathrm{GL}_n(\mathbb{C})$ -torsor over Gr_n . Hence, $\mathrm{Gr}_n \simeq \mathrm{BGL}_n(\mathbb{C})$. There is a fairly easy way to see why Grassmannians should control $\mathrm{GL}_n(\mathbb{C})$ -torsors on X , or equivalently vector bundles. Let $p : E \rightarrow X$ be a complex vector bundle of rank n . Suppose that E is trivial on a finite cover $\{U_i\}_{i=1}^m$ of X . Let $s_i : X \rightarrow [0, 1]$ be a partition of unity subordinate to $\{U_i\}$, so that the support of each s_i is contained in U_i , and $s_1 + \cdots + s_m = 1_X$. Choose trivializations $t_i : \mathbb{C}^n \times U_i \rightarrow E|_{U_i}$. Now, we can define $g : E \rightarrow \mathbb{C}^{nm}$ by $g = \bigoplus_{i=1}^m g_i$, where $g_i = (s_i \circ p) \cdot (p_1 \circ t_i^{-1})$; outside U_i , $g_i = 0$. Now, the **Gauss map** g clearly defines a map $X \rightarrow \mathrm{Gr}_n(\mathbb{C}^{nm})$. The entire formalism of vector bundles can be based on these Gauss maps. See Husemoller [Hus75, Chapter 3].

We conclude the section with two remarks. First, the classifying space construction is functorial in homomorphisms of topological groups. That is, if there is a map of topological groups $G \rightarrow H$, then there is an induced map $BG \rightarrow BH$. The corresponding map $\mathrm{Tors}_G(X) \rightarrow \mathrm{Tors}_H(X)$ is an example of the fiber bundle construction. Indeed, since G acts on H , we can apply this construction to produce an H -torsor from a G -torsor.

Example 2.24. The most important example for us will be the determinant map $BGL_n(\mathbb{C}) \rightarrow BGL_1(\mathbb{C})$, which gives the determinant map $\text{Vect}_n^{\text{top}}(X) \rightarrow \text{Vect}_1^{\text{top}}(X)$. The fiber of this map is just as important. Indeed, because

$$1 \rightarrow SL_n(\mathbb{C}) \rightarrow GL_n(\mathbb{C}) \rightarrow GL_1(\mathbb{C}) \rightarrow 1$$

is an exact sequence of topological groups, the sequence $BSL_n(\mathbb{C}) \rightarrow BGL_n(\mathbb{C}) \rightarrow BGL_1(\mathbb{C})$ turns out to be a homotopy fiber sequence. Hence, $[X, BSL_n(\mathbb{C})]$ classifies topological complex vector bundles on X with trivial determinant.

The second remark is that one often works with BU_n rather than $BGL_n(\mathbb{C})$. The natural inclusion $U_n \rightarrow GL_n(\mathbb{C})$ of the unitary matrices into all invertible complex matrices is a homotopy equivalence (using polar decomposition). Hence, $BU_n \rightarrow BGL_n(\mathbb{C})$ is also a homotopy equivalence. Philosophically, this corresponds to the fact that any complex vector bundle on a paracompact Hausdorff space admits a Hermitian metric.

2.3 Topological line bundles

Using the Grassmannian description of $BGL_1(\mathbb{C})$, we find that $\text{colim}_n \text{Gr}_1(\mathbb{C}^n) \simeq BGL_1(\mathbb{C})$. Of course, $\text{Gr}_1(\mathbb{C}^n) \simeq \mathbb{C}P^{n-1}$. Hence, $\mathbb{C}P^\infty \simeq BGL_1(\mathbb{C})$.

Lemma 2.25. *The infinite complex projective space $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$.*

Proof. Indeed, since $GL_1(\mathbb{C}) \cong \mathbb{C}^* \simeq S^1$ and $\Omega\mathbb{C}P^\infty \simeq GL_1(\mathbb{C})$, we find that $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$. \square

As a result we may describe the set of line bundles on X in terms of a cohomology group:

Corollary 2.26. *The natural map $\text{Vect}_1^{\text{top}}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z})$ is a bijection for any paracompact Hausdorff space.*

2.4 Rank 2 bundles in low dimension

Recall that the cohomology of the infinite Grassmannian is

$$H^*(\text{Gr}_n, \mathbb{Z}) \cong H^*(BGL_n(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n],$$

where $|c_i| = 2i$. We will also need Bott's computation [Bot58, Theorem 5] of the homotopy groups of Gr_n in the stable range. If $i \leq 2n + 1$, then

$$\pi_i \text{Gr}_n \cong \begin{cases} \mathbb{Z} & \text{if } i \leq 2n \text{ is even,} \\ 0 & \text{if } i \leq 2n \text{ is odd,} \\ \mathbb{Z}/n! & \text{if } i = 2n + 1. \end{cases}$$

When $i > 2n + 1$ much less is known about the homotopy groups of Gr_n (except when $n = 1$). Playing the computation of the cohomology rings off of these homotopy groups gives a great deal of insight into the low stages of the Postnikov tower of Gr_n .

Lemma 2.27. *The map $\text{Gr}_n \rightarrow \text{Gr}_n[3] \simeq \text{Gr}_n[2] \simeq K(\mathbb{Z}, 2)$ is precisely $c_1 \in H^2(\text{Gr}_n, \mathbb{Z})$.*

Proof. Indeed, this map is induced by the determinant map $GL_n(\mathbb{C}) \rightarrow GL_1(\mathbb{C})$. \square

We are in particular interested in the following part of the Postnikov tower of Gr_2 :

$$\begin{array}{ccccc} K(\mathbb{Z}/2, 5) & \longrightarrow & \text{Gr}_2[5] & & \\ & & \downarrow & & \\ K(\mathbb{Z}, 4) & \longrightarrow & \text{Gr}_2[4] & \xrightarrow{k_4} & K(\mathbb{Z}/2, 6) \\ & & \downarrow & & \\ & & \text{Gr}_2[3] \simeq K(\mathbb{Z}, 2) & \xrightarrow{k_3} & K(\mathbb{Z}, 5) \end{array}$$

Note that because $\mathrm{Gr}_2[3] \simeq \mathrm{Gr}_2[2]$ there is no obstruction to lifting a map $X \rightarrow \mathrm{Gr}_2[2]$ to a map $X \rightarrow \mathrm{Gr}_2[3]$ if X is a CW complex.

Lemma 2.28. *The k -invariant k_3 is nullhomotopic. Hence,*

$$\mathrm{Gr}_2[4] \simeq K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4).$$

Moreover, this equivalence may be chosen so that the composition $\mathrm{Gr}_2 \rightarrow \mathrm{Gr}_2[4] \rightarrow K(\mathbb{Z}, 4)$ is $c_2 \in H^4(\mathrm{Gr}_2, \mathbb{Z})$.

Proof. The class $k_3 \in H^5(K(\mathbb{Z}, 2), \mathbb{Z})$ vanishes simply because the cohomology of $K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$ is concentrated in even degrees. This gives the splitting claimed (we lift the identity map $K(\mathbb{Z}, 2) \rightarrow \mathrm{Gr}_2[2]$ up the Postnikov tower). Consider the map $\mathrm{Gr}_2 \rightarrow K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4)$ classified by the pair (c_1, c_2) in the cohomology of Gr_2 . By definition, (c_1, c_2) factors through the functorial Postnikov section $\mathrm{Gr}_2 \rightarrow \mathrm{Gr}_2[4]$. It is enough to check that the induced map $\mathrm{Gr}_2[4] \rightarrow K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4)$ is a weak equivalence. We have already seen that it is an isomorphism on π_2 . We have a map of fiber sequences

$$\begin{array}{ccccc} \mathrm{BSL}_2[4] & \longrightarrow & \mathrm{Gr}_2[4] & \longrightarrow & \mathrm{Gr}_1 \\ \downarrow & & \downarrow & & \downarrow \\ K(\mathbb{Z}, 4) & \longrightarrow & K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4) & \longrightarrow & K(\mathbb{Z}, 2), \end{array}$$

and the outside vertical arrows are weak equivalences by the Hurewicz isomorphism theorem. This proves the lemma. \square

We can now classify rank 2 vector bundles on 4-dimensional spaces.

Proposition 2.29. *Let X be a 4-dimensional space having the homotopy type of a CW complex. Then, the natural map*

$$\mathrm{Vect}_2^{\mathrm{top}}(X) \rightarrow H^2(X, \mathbb{Z}) \times H^4(X, \mathbb{Z})$$

is a bijection.

Proof. The previous lemma shows that $[X, \mathrm{Gr}_2[4]] \rightarrow H^2(X, \mathbb{Z}) \times H^4(X, \mathbb{Z})$ is a bijection. The obstruction to lifting a given map $f : X \rightarrow \mathrm{Gr}_2[4]$ to $\mathrm{Gr}_2[5]$ is a class $f^*k_4 \in H^6(X, \mathbb{Z}/2) = 0$. Similarly, the choice of lifts is bijective to a quotient of $H^5(X, \mathbb{Z}/2)$, and this group is 0. Hence, for every such f there is a unique lift to $\mathrm{Gr}_2[5]$, and then the same reasoning gives a unique lift to $\mathrm{Gr}_2[m]$ for all $m \geq 5$. Since Gr_2 is the limit of its Postnikov tower, the proposition follows. \square

If $\dim X = 5$, the situation is similar but more complicated. To state the theorem let us recall that a cohomology operation is a natural transformation of functors $H^i(-, R) \rightarrow H^j(-, R')$; by Yoneda such a map is classified by an element of $[K(i, R), K(j, R')] \simeq H^j(K(i, R), R')$.

Proposition 2.30. *If X is a 5-dimensional space having the homotopy type of a CW complex, then the map $\mathrm{Vect}_2^{\mathrm{top}}(X) \rightarrow H^2(X, \mathbb{Z}) \times H^4(X, \mathbb{Z})$ is surjective, and the choice of lifts is parametrized by $H^5(X, \mathbb{Z}/2)/\mathrm{im}(H^3(X, \mathbb{Z}) \rightarrow H^5(X, \mathbb{Z}/2))$, where the map $H^3(X, \mathbb{Z}) \rightarrow H^5(X, \mathbb{Z}/2)$ is a certain non-zero cohomology operation.*

Proof. Consider the fiber sequence $K(\mathbb{Z}/2, 5) \rightarrow \mathrm{Gr}_2[5] \rightarrow \mathrm{Gr}_2[4]$. As above, $[X, \mathrm{Gr}_2[4]]$ is classified by the 1st and 2nd Chern classes. On a 5-dimensional space, once a lift to $\mathrm{Gr}_2[5]$ is specified, there is a unique lift all the way to Gr_2 , just as in the proof of the previous proposition. The obstructions to finding a lift from $\mathrm{Gr}_2[4]$ to $\mathrm{Gr}_2[5]$ are in $H^6(X, \mathbb{Z}/2)$, and

hence all lift. Recall that to any fiber sequence there is an associated long exact sequence of fibrations. See [Hat02, Section 4.3]. Extending to the left a little bit, in our cases this is

$$\Omega\mathrm{Gr}_2[4] \rightarrow K(\mathbb{Z}/2, 5) \rightarrow \mathrm{Gr}_2[5] \rightarrow \mathrm{Gr}_2[4].$$

However, $K(\mathbb{Z}/2, 5) \rightarrow \mathrm{Gr}_2[5] \rightarrow \mathrm{Gr}_2[4]$ is principal, so it extends to the right one term as well:

$$\Omega\mathrm{Gr}_2[4] \rightarrow \Omega K(\mathbb{Z}/2, 6) \rightarrow \mathrm{Gr}_2[5] \rightarrow \mathrm{Gr}_2[4] \rightarrow K(\mathbb{Z}/2, 6),$$

where $\Omega K(\mathbb{Z}/2, 6) \simeq K(\mathbb{Z}/2, 5)$. It follows that there is an exact sequence of pointed sets

$$\mathrm{H}^1(X, \mathbb{Z}) \times \mathrm{H}^3(X, \mathbb{Z}) \rightarrow \mathrm{H}^5(X, \mathbb{Z}/2) \rightarrow \mathrm{Vect}_2^{\mathrm{top}}(X) \rightarrow \mathrm{H}^2(X, \mathbb{Z}) \times \mathrm{H}^4(X, \mathbb{Z}),$$

which is surjective on the right. Moreover, the map $\mathrm{H}^1(X, \mathbb{Z}) \times \mathrm{H}^3(X, \mathbb{Z}) \rightarrow \mathrm{H}^5(X, \mathbb{Z}/2)$ is a group homomorphism because it is induced by taking loops of a map. There is an action of $\mathrm{H}^5(X, \mathbb{Z}/2)$ on $\mathrm{Vect}_2^{\mathrm{top}}(X)$ such that two rank 2 vector bundles on X have the same Chern classes if and only if they are in the same orbit of $\mathrm{H}^5(X, \mathbb{Z}/2)$. There are no cohomology operations $\mathrm{H}^1(X, \mathbb{Z}) \rightarrow \mathrm{H}^5(X, \mathbb{Z}/2)$, since $\mathrm{H}^5(S^1, \mathbb{Z}/2) = 0$. However, there is a cohomology operation $\mathrm{H}^3(X, \mathbb{Z}) \rightarrow \mathrm{H}^5(X, \mathbb{Z}/2)$, often denoted Sq_2^2 . Note that this class is precisely Ωk_4 . That is, since we have $k_4 : K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4) \rightarrow K(\mathbb{Z}/2, 6)$, the loop space is $K(\mathbb{Z}, 1) \times K(\mathbb{Z}, 3) \rightarrow K(\mathbb{Z}/2, 5)$. One can check, using the Postnikov tower and cohomology of $\mathrm{BSL}_2(\mathbb{C})$ that this class Ωk_4 is precisely the unique non-zero element of $\mathrm{H}^5(K(\mathbb{Z}, 3), \mathbb{Z}/2) \cong \mathbb{Z}/2$ by the next lemma. \square

Lemma 2.31. *The k -invariant $k_4 : \mathrm{Gr}_2[4] \rightarrow K(\mathbb{Z}/2, 6)$ is non-trivial.*

Proof. It is enough to show that the corresponding k -invariant $\mathrm{BSL}_2(\mathbb{C})[4] \rightarrow K(\mathbb{Z}/2, 6)$ is non-trivial. Note that $\mathrm{BSL}_2(\mathbb{C}) \rightarrow \mathrm{BSL}_2(\mathbb{C})[4] \simeq K(\mathbb{Z}, 4)$ is a 5-equivalence and that $\mathrm{BSL}_2(\mathbb{C}) \rightarrow \mathrm{BSL}_2(\mathbb{C})[5]$ is a 6-equivalence. It follows that $\mathrm{H}^6(\mathrm{BSL}_2(\mathbb{C})[5], \mathbb{Z}/2) = 0$ since $\mathrm{H}^*(\mathrm{BSL}_2(\mathbb{C}), \mathbb{Z}/2) \cong \mathbb{Z}/2[c_2]$. On the other hand, $\mathrm{H}^6(\mathrm{BSL}_2(\mathbb{C})[4], \mathbb{Z}/2) \cong \mathrm{H}^6(K(\mathbb{Z}/4), \mathbb{Z}/2) \cong \mathbb{Z}/2$. If the extension $K(\mathbb{Z}/2, 5) \rightarrow \mathrm{BSL}_2(\mathbb{C})[5] \rightarrow \mathrm{BSL}_2(\mathbb{C})[4]$ were split, the cohomology of $\mathrm{BSL}_2(\mathbb{C})[4]$ would inject into the cohomology of $\mathrm{BSL}_2(\mathbb{C})[5]$. Since this does not happen, we see that k_4 is non-zero. \square

Exercise 2.32. Describe the obstruction class $k_4 \in \mathrm{H}^6(X, \mathbb{Z}/2)$ by computing the cohomology of $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4)$ and finding k_4 .

Finally, if $\dim X = 6$, there is a similar picture, except that there is an obstruction to realizing a given pair of Chern classes, and there is an additional choice of lift.

Example 2.33. Let X be the 6-skeleton of $\mathrm{BGL}_3(\mathbb{C})$. There is a universal rank 3 vector bundle E on X with Chern classes $c_i(E) = c_i \in \mathrm{H}^{2i}(X, \mathbb{Z})$ for $i = 1, 2, 3$. On the other hand, we can ask if there is a rank 2 bundle F on X with Chern classes $c_i(F) = c_i$ for $i = 1, 2$. This is the universal example where the obstruction above is nonzero and demonstrates the *incompressibility* of Grassmannians.

One can use the fact that $\mathrm{SU}_2 \cong \mathrm{SO}_3$, which is itself isomorphic to the 3-sphere, to find that $\pi_6\mathrm{Gr}_2 \cong \pi_5S^3 = \mathbb{Z}/2$. This leads to the following description of rank 2-bundles on a 6-dimensional CW complex.

Proposition 2.34. *Let X be a 6-dimensional space with the homotopy type of a CW complex. The map $\mathrm{Vect}_2^{\mathrm{top}}(X) \rightarrow \mathrm{H}^2(X, \mathbb{Z}) \times \mathrm{H}^4(X, \mathbb{Z})$ has image precisely those pairs (c_1, c_2) such that $k_4(c_1, c_2) = 0$ in $\mathrm{H}^6(X, \mathbb{Z})$. If $k_4(c_1, c_2) = 0$, the set of lifts to $\mathrm{Gr}_2[5]$ is parameterized by a quotient of $\mathrm{H}^5(X, \mathbb{Z}/2)$ as above. Each lift then lifts to Gr_2 , and the set of lifts from $\mathrm{Gr}_2[5]$ to Gr_2 is parameterized by a quotient of $\mathrm{H}^6(X, \mathbb{Z}/2)$.*

2.5 Rank 3 bundles in low dimension

Proposition 2.35. *Suppose that X is a 5-dimensional CW complex. Then, the natural map*

$$(c_1, c_2) : \text{Vect}_3^{\text{top}}(X) \rightarrow \mathbb{H}^2(X, \mathbb{Z}) \times \mathbb{H}^4(X, \mathbb{Z})$$

is an isomorphism.

Proof. Indeed, $\text{Gr}_3[4] \simeq K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4)$, just as for $\text{Gr}_2[4]$. But, this time, $\pi_5 \text{Gr}_3 = 0$. Hence, the next interesting problem is to lift from $\text{Gr}_2[4]$ to $\text{Gr}_2[6]$. The obstructions are in $\mathbb{H}^7(X, \mathbb{Z})$, and hence vanish. The lifts of a given map to $\text{Gr}_3[4]$ are a quotient of $\mathbb{H}^6(X, \mathbb{Z}) = 0$. \square

As a consequence, one sees immediately from the last section that every rank 3 vector bundle E on a 5-dimensional CW complex splits as $E_0 \oplus \mathbb{C}$ for some rank 2 vector bundle E_0 . This is one example of a more general phenomenon we leave to the reader to discover.

Proposition 2.36. *If X is a 6-dimensional closed real orientable manifold, then*

$$(c_1, c_2, c_3) : \text{Vect}_3^{\text{top}}(X) \rightarrow \mathbb{H}^2(X, \mathbb{Z}) \times \mathbb{H}^4(X, \mathbb{Z}) \times \mathbb{H}^6(X, \mathbb{Z})$$

is a injection with image the triples with c_3 an even multiple of a generator of $\mathbb{H}^6(X, \mathbb{Z}) \cong \mathbb{Z}$.

Proof. As above, once we have constructed a map $X \rightarrow \text{Gr}_3[6]$, there is a unique lift to Gr_3 . Given a map $X \rightarrow \text{Gr}_3[4] \simeq K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4)$, the obstruction to lifting to $\text{Gr}_3[6]$ is a class in $\mathbb{H}^7(X, \mathbb{Z}) = 0$ since X is 6-dimensional. We have again an exact sequence of pointed sets

$$\mathbb{H}^1(X, \mathbb{Z}) \times \mathbb{H}^3(X, \mathbb{Z}) \rightarrow \mathbb{H}^6(X, \mathbb{Z}) \rightarrow \text{Vect}_3^{\text{top}}(X) \rightarrow \mathbb{H}^2(X, \mathbb{Z}) \times \mathbb{H}^4(X, \mathbb{Z}).$$

The map on the left is induced from a map $K(\mathbb{Z}, 1) \times K(\mathbb{Z}, 3) \rightarrow K(\mathbb{Z}, 6)$ which is Ωk_5 , where k_5 is the k -invariant $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4) \rightarrow K(\mathbb{Z}, 7)$. In particular, the image in $\mathbb{H}^6(X, \mathbb{Z})$ consists of torsion classes. But, $\mathbb{H}^6(X, \mathbb{Z}) \cong \mathbb{Z}$ by hypothesis. One can check that the composition $K(\mathbb{Z}, 6) \rightarrow \text{Gr}_3[6] \xrightarrow{c_3} K(\mathbb{Z}, 6)$ is multiplication by 2. This completes the proof. \square

In general, understanding vector bundles of a fixed dimension becomes more and more difficult as the dimension of the base space increases. The systematic approach to this kind of problem uses cohomology and Serre spectral sequences to determine Postnikov extensions one step a time. For an overview, see [Tho66].

3 The construction of the \mathbb{A}^1 -homotopy category

The first definitions of \mathbb{A}^1 -homotopy theory were given in [MV99] when the base scheme S is noetherian of finite Krull dimension. An equivalent homotopy theory was constructed by Dugger [Dug01a], and we will follow Dugger's definition, but with the added generality of allowing S to be quasi-compact and quasi-separated using Lurie's Nisnevich topology [Lur16, Section A.2.4]. We use model categories for the construction, but in the Section 4, where we give many properties of the homotopy theory, we emphasize the model-independence of the proofs.

3.1 Model categories

Model categories are a technical framework for working up to homotopy. The axioms guarantee that certain category-theoretic localizations exist without enlarging the ambient set-theoretic universe and that it is possible in some sense to compute the hom-sets in the localization. The theory generalizes the use of projective or injective resolutions in the construction of derived categories of rings or schemes.

References for this material include Quillen's original book on the theory [Qui67], Dwyer-Spalinski [DS95], Goerss-Jardine [GJ99], and Goerss-Schemmerhorn [GS07]. For consistency, we refer the reader where possible to [GJ99]. However, unlike some of these references, we assume that the category underlying M has all small limits and colimits. This is satisfied immediately in all cases of interest to us.

Definition 3.1. Let M be a category with all small limits and colimits. A model category structure on M consists of three classes W, C, F of morphisms in M , called **weak equivalences**, **cofibrations**, and **fibrations**, subject to the following set of axioms.

M1 Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ two composable morphisms in M , if any two of $g \circ f$, f , and g are weak equivalences, then so is the third.

M2 Each class W, C, F is closed under retracts.

M3 Given a diagram

$$\begin{array}{ccc} Z & \longrightarrow & E \\ \downarrow i & \nearrow & \downarrow p \\ X & \longrightarrow & B \end{array}$$

of solid arrows, a dotted arrow can be found making the diagram commutative if either

- (a) p is an **acyclic fibration** ($p \in W \cap F$) and i is a cofibration, or
- (b) i is an **acyclic cofibration** ($i \in W \cap C$) and p is a fibration.

(In particular, cofibrations i have the **left lifting property** with respect to acyclic fibrations, while fibrations p have the **right lifting property** with respect to acyclic cofibrations.)

M4 Any map $X \rightarrow Z$ in M admits two factorizations $X \xrightarrow{f} E \xrightarrow{p} Z$ and $X \xrightarrow{i} Y \xrightarrow{g} Z$, such that f is an acyclic cofibration, p is a fibration, i is a cofibration, and g is an acyclic fibration.

Remark 3.2. In practice, a model category is determined by only W and either C or F . Indeed, C is precisely the class of maps in M having the left lifting property with respect to acyclic fibrations. Similarly, F consists of exactly those maps in M having the right lifting property with respect to acyclic cofibrations. The reader can prove this fact using the axioms or refer to [DS95, Proposition 3.13]. However, some caution is required. While one often sees model categories specified in the literature by just fixing W and either C or F , it usually has to be checked that these really do give M a model category structure.

Remark 3.3. Many authors strengthen **M4** to assume the existence of *functorial* factorizations. This is satisfied in all model categories of relevance for this paper by [Hov99, Section 2.1] as they are all cofibrantly generated.

Exercise 3.4. Let A be an associative ring. Consider $\text{Ch}_{\geq 0}(A)$, the category of non-negatively graded chain complexes of right A -modules. Since limits and colimits of chain complexes are computed degree-wise, $\text{Ch}_{\geq 0}(A)$ is closed under all small limits and colimits. Let W be the class of quasi-isomorphisms, i.e., those maps $f : M_{\bullet} \rightarrow N_{\bullet}$ of chain complexes such that $H_n(f) : H_n(M_{\bullet}) \rightarrow H_n(N_{\bullet})$ is an isomorphism for all $n \geq 0$. Let F be the class of maps of chain complexes which are surjections in positive degrees. Describe the class C of maps satisfying the left lifting property with respect to $F \cap W$. Prove that W, C, F is a model category structure on $\text{Ch}_{\geq 0}(A)$.

Definition 3.5. A model category M has an initial object \emptyset and a final object $*$, since it is closed under colimits and limits. An object X of M is **fibrant** if $X \rightarrow *$ is a fibration, and X is **cofibrant** if $\emptyset \rightarrow X$ is a cofibration. Given an object X of M , an acyclic fibration

$QX \rightarrow X$ such that QX is cofibrant is called a **cofibrant replacement**. Similarly, if $X \rightarrow RX$ is an acyclic fibration with RX fibrant, then RX is called a **fibrant replacement** of X . These replacements always exist, by applying **M4** to $\emptyset \rightarrow X$ or $X \rightarrow *$.

Example 3.6. In $\text{Ch}_{\geq 0}(A)$, let M be a right A -module (viewed as a chain complex concentrated in degree zero). A projective resolution $P_{\bullet} \rightarrow M$ is an example of a cofibrant replacement. Indeed, such a resolution is an acyclic fibration. Moreover, the map $0 \rightarrow P_{\bullet}$ is a cofibration, since the cokernel is projective in each degree.

Example 3.7. Let sSets be the category of simplicial sets. This is the category of functors $\Delta^{\text{op}} \rightarrow \text{Sets}$, where Δ is the category of finite non-empty ordered sets. (For details, see [GJ99].) There is a geometric realization functor $\text{sSets} \rightarrow \text{Spc}$, which sends a simplicial set X_{\bullet} to a space $|X_{\bullet}|$. Let W denote the class of weak homotopy equivalences in sSets , i.e., those maps $f : X_{\bullet} \rightarrow Y_{\bullet}$ such that $|f| : |X_{\bullet}| \rightarrow |Y_{\bullet}|$ is a weak homotopy equivalence. Let C denote the class of level-wise monomorphisms. If F is the class of maps having the right lifting property with respect to acyclic cofibrations, then sSets together with W, C, F is a model category. In sSets , every object is cofibrant. The fibrant objects are the **Kan complexes**, namely those simplicial sets having a filling property for all horns. See [GJ99, Section I.3].

Definition 3.8. A model category M is **pointed** if the natural map $\emptyset \rightarrow *$ is an isomorphism. Examples of pointed model categories include $\text{Ch}_A^{\geq 0}$, which is pointed by the 0 object, and sSets_{*} , the category of *pointed* simplicial sets.

Now, we come to the main reason why model categories have been so successful in encoding homotopical ideas: the homotopy category of a model category.

Definition 3.9. Let M be a category and W a class of morphisms in M . The localization of M by W , if it exists, is a category $M[W^{-1}]$ with a functor $L : M \rightarrow M[W^{-1}]$ such that

1. $L(w)$ is an isomorphism for every $w \in W$,
2. every functor $F : M \rightarrow N$ having the property that $F(w)$ is an isomorphism for all $w \in W$ factors uniquely through L in the sense that there is a functor $G : M[W^{-1}] \rightarrow N$ and a natural isomorphism of functors $G \circ L \simeq F$, and
3. for any category N , the functor $\text{Fun}(M[W^{-1}], N) \rightarrow \text{Fun}(M, N)$ induced by composition with $L : M \rightarrow M[W^{-1}]$ is fully faithful.

The localization of M by W , if it exists, is unique up to categorical equivalence.

In general, there is no reason that a localization of M by W should exist much less be useful. The fundamental problem is that in attempting to concretely construct the morphisms in $M[W^{-1}]$, for example by hammock localization (hat piling), one discovers size issues, where it might be necessary to enlarge the universe in order to obtain a category: the morphisms sets in a category must be actual sets, not proper classes.

Theorem 3.10 ([Qui67]). *Let M be a model category with class of weak equivalences W . Then, the localization $M[W^{-1}]$ exists. It is called the homotopy category of M , and we will denote it by $\text{Ho}(M)$.*

Recipe 3.11. It is generally difficult to compute $[X, Y] = \text{Hom}_{\text{Ho}(M)}(X, Y)$ given two objects $X, Y \in M$. We give a recipe. Replace X by a weakly equivalent cofibrant object QX , and Y by a weakly equivalent fibrant object RY . Then, $[X, Y] = \text{Hom}_M(QX, RY) / \sim$, where \sim is an equivalence relation on $\text{Hom}_M(QX, RY)$ generalizing homotopy equivalence (see [GJ99, Section II.1]). See [DS95, Proposition 5.11] for a proof that this construction does indeed compute the set of maps in the homotopy category.

Remark 3.12. In many cases, every object of M might be cofibrant, in which case one just needs to replace Y by RY and compute the homotopy classes of maps. This is for example the case in sSets .

Remark 3.13. In Goerss-Jardine [GJ99, Section II.1], the homotopy category $\mathrm{Ho}(M)$ is itself defined to be the category of objects of M that are both fibrant and cofibrant, with maps given by $\mathrm{Hom}_{\mathrm{Ho}(M)}(A, B) = \mathrm{Hom}(A, B) / \sim$. Given an arbitrary X in M it is possible to assign to X a fibrant-cofibrant object RQX as follows. First, take, via **M4**, a factorization $\emptyset \rightarrow QX \rightarrow X$ where QX is cofibrant $QX \rightarrow X$ is a weak equivalence. Now, take a factorization $QX \rightarrow RQX \rightarrow *$ of the canonical map $QX \rightarrow *$ in which $QX \rightarrow RQX$ is an acyclic cofibration and $RQX \rightarrow *$ is a fibration. In particular, RQX is fibrant. Since compositions of cofibrations are cofibrations, RQX is also cofibrant. Moreover, if $f : X \rightarrow Y$ is a morphism, then it is possible using **M3** to (non-uniquely) assign to f a morphism $RQf : RQX \rightarrow RQY$ such that one gets a well-defined functor $M \rightarrow \mathrm{Ho}(M)$ (i.e., after enforcing \sim).

Remark 3.14. In practice, we will work with simplicial model category structures, for which there exist objects $QX \times \Delta^1$, where Δ^1 is the standard 1-simplex (so that $|\Delta^1| = I^1$). In this case, the equivalence relation \sim is precisely that of (left) homotopy classes of maps. See Definition 3.16.

Exercise 3.15. For chain complexes, the equivalence relation \sim is precisely that of chain homotopy equivalence. (See [Wei94, Section 1.4].) Using the recipe above, compute

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Ch}_{\geq 0}(\mathbb{Z}))}(\mathbb{Z}/p, \mathbb{Z}[1]),$$

where $\mathbb{Z}/p[1]$ denotes the chain complex with \mathbb{Z}/p placed in degree 1 and zeros elsewhere.

3.2 Mapping spaces

We will now explain simplicial model categories since we will need to discuss mapping spaces. For details, we refer the reader to [GJ99, II.2-3]. If X and Y are simplicial sets, then we may define the **simplicial mapping space** $\mathrm{map}_{\mathrm{sSets}}(X, Y)$ as the simplicial set with n -simplices given by

$$\mathrm{map}_{\mathrm{sSets}}(X, Y)_n := \mathrm{Hom}_{\mathrm{sSets}}(X \times \Delta^n, Y).$$

This simplicial set fits into a tensor-hom adjunction given by

$$\mathrm{Hom}_{\mathrm{sSets}}(Z \times X, Y) \cong \mathrm{Hom}_{\mathrm{sSets}}(Z, \mathrm{map}_{\mathrm{sSets}}(X, Y)).$$

Indeed, from this adjunction we may deduce the formula for $\mathrm{map}(X, Y)_n$ by evaluating at $Z = \Delta^n$.

Abstracting these formulas, one arrives at the axioms for a **simplicial category** [GJ99, II Definition 2.1]. A simplicial category is a category M equipped with

1. a **mapping space functor**: $\mathrm{map} : M^{\mathrm{op}} \times M \rightarrow \mathrm{sSets}$, written $\mathrm{map}_M(X, Y)$,
2. an **action** of sSets , $M \times \mathrm{sSets} \rightarrow M$, written $X \otimes S$, and
3. an **exponential**, $\mathrm{sSets}^{\mathrm{op}} \times M \rightarrow M$, written X^S for an object $X \in M$ and a simplicial set S

subject to certain compatibilities. The most important are that

$$- \otimes X : \mathrm{sSets} \rightleftarrows C : \mathrm{map}_M(X, -)$$

should be an adjoint pair of functors and that $\mathrm{Hom}_M(X, Y) \cong \mathrm{map}(X, Y)_0$ for all $X, Y \in M$.

Suppose that M is a simplicial category simultaneously equipped with a model structure. We would like the simplicial structure above to play well with the model structure. For example, if $i : A \rightarrow X$ is a cofibration, we expect $\mathrm{map}_M(Y, A) \rightarrow \mathrm{map}_M(Y, X)$ to be a fibration (and hence induce long exact sequences in homotopy groups) for any object Y as is the case in simplicial sets.

Definition 3.16. Suppose that M is a model category which is also a simplicial category. Then M satisfies **SM7**, and is called a **simplicial model category**, if for any cofibration $i : A \rightarrow X$ and any fibration: $p : E \rightarrow B$ the map of simplicial sets (induced by the functoriality of map)

$$\text{map}_M(X, E) \rightarrow \text{map}_M(A, E) \times_{\text{map}_M(A, B)} \text{map}_M(X, B)$$

is a fibration of simplicial sets which is moreover a weak equivalence if either i or p is.

Exercise 3.17. Show that in a simplicial model category M , if $A \rightarrow X$ is a cofibration, then for any object Y , the natural map $\text{map}_M(Y, A) \rightarrow \text{map}_M(Y, X)$ is a fibration of simplicial sets.

Another feature of simplicial model categories is the fact that one may define a concept of homotopy that is more transparent than in an ordinary model category (where one defines left and right homotopies, see [DS95]). Suppose that $A \in M$ is a cofibrant object, then we say that two morphisms $f, g : A \rightarrow X$ are homotopic if there is a morphism: $H : A \otimes \Delta^1 \rightarrow X$ such that

$$\begin{array}{ccc} A \amalg A & \xrightarrow{d_1 \amalg d_0} & A \otimes \Delta^1 \\ \downarrow f \amalg g & \swarrow H & \downarrow \\ X & & X \end{array}$$

commutes. Write $f \sim g$ if f and g are homotopic.

Exercise 3.18. Prove that \sim is an equivalence relation on $\text{Hom}_M(A, X)$ when A is cofibrant.

In 3.11 we stated a recipe for calculating $[X, Y]$, the hom-sets in $\text{Ho}(M)$. We replace X by a weakly equivalent cofibrant object QX , and Y by a weakly equivalent fibrant object RY . Then, we claimed that $[X, Y] = \text{Hom}_M(QX, RY) / \sim$ where \sim was an unspecified equivalence relation. For a simplicial model category, this equivalence relation can be taken to be the one just given. The fact the this is well defined is checked in [GJ99, Proposition 3.8].

3.3 Bousfield localization of model categories

One way of creating new model categories from old is via Bousfield localization. The underlying category remains the same, while the class of weak equivalences is enlarged. To describe these localizations, we first need to consider a class of functors between model categories that are well-adapted to their homotopical nature.

Definition 3.19. Consider a pair of adjoint functors

$$F : M \rightleftarrows N : G$$

between model categories M and N . The pair is called a **Quillen pair**, or a pair of Quillen functors, if one of the following equivalent conditions is satisfied:

- F preserves cofibrations and acyclic cofibrations;
- G preserves fibrations and acyclic fibrations.

In this case, F is also called a **left Quillen functor**, and G a **right Quillen functor**.

Quillen pairs provide a sufficient framework for a pair of adjoint functors on model categories to descend to a pair of adjoint functors on the homotopy categories.

Proposition 3.20. *Suppose that $F : M \rightleftarrows N : G$ is a pair of Quillen functors. Then, there are functors $\mathbf{L}F : M \rightarrow \text{Ho}(N)$ and $\mathbf{R}G : N \rightarrow \text{Ho}(M)$, each of which takes weak equivalences to isomorphisms, such that there is an induced adjunction $\mathbf{L}F : \text{Ho}(M) \rightleftarrows \text{Ho}(N) : \mathbf{R}G$ between homotopy categories.*

Proof. See [DS95, Theorem 9.7]. \square

Remark 3.21. The familiar functors from homological algebra all arise in this way, so $\mathbf{L}F$ is called the left derived functor of F , while $\mathbf{R}G$ is the right derived functor of G . There is a recipe for computing the value of the derived functors on an arbitrary object X of M and Y of N . Specifically, $\mathbf{L}F(X)$ is weakly equivalent to $F(QX)$ where QX a cofibrant replacement of X . Similarly, $\mathbf{R}G(Y)$ is weakly equivalent to $G(RY)$ where RY is a fibrant replacement of Y .

Remark 3.22. It follows from the previous remark that when a functorial cofibrant replacement functor $Q : M \rightarrow M$ exists, then we can factor $\mathbf{L}F : M \rightarrow \mathrm{Ho}(N)$ through $M \xrightarrow{Q} M \xrightarrow{F} N \rightarrow \mathrm{Ho}(N)$. As mentioned above, this is the case for all model categories in this paper. As such, we will often abuse notation and write $\mathbf{L}F$ for the functor $F \circ Q : M \rightarrow N$.

Definition 3.23. A **Quillen equivalence** is a Quillen pair $F : M \rightleftarrows N : G$ such that $\mathbf{L}F : \mathrm{Ho}(M) \rightleftarrows \mathrm{Ho}(N) : \mathbf{R}G$ is an inverse equivalence.

Definition 3.24. Let M be a simplicial model category with class of weak equivalences W . Suppose that I is a set of maps in M . An object X of M is **I -local** if it is fibrant and if for all $i : A \rightarrow B$ with $i \in I$, the induced morphism on mapping spaces $i^* : \mathrm{map}_M(B, X) \rightarrow \mathrm{map}_M(A, X)$ is a weak equivalence (of simplicial sets). A morphism $f : A \rightarrow B$ is an **I -local weak equivalence** if for every I -local object X , the induced morphism on mapping spaces $f^* : \mathrm{map}_M(B, X) \rightarrow \mathrm{map}_M(A, X)$ is a weak equivalence. Let W_I be the class of all I -local weak equivalences. By using **SM7**, $W \subseteq I$.

Let F_I denote the class of maps satisfying the right lifting property with respect to W_I -acyclic cofibrations ($W_I \cap C$). If (W_I, C, F_I) is a model category structure on M , we call this the **left Bousfield localization** of M with respect to I .

To distinguish between the model category structures on M , we will write $L_I M$ for the left Bousfield model category structure on M . We will only write $L_I M$ when the classes of morphisms defined above do define a model category structure.

When it exists, the Bousfield localization of M with respect to I is universal with respect to Quillen pairs $F : M \rightleftarrows N : G$ such that $\mathbf{L}F(i)$ is a weak equivalence in N for all $i \in I$.

Exercise 3.25. Show that if it exists, then the identity functors $\mathrm{id}_M : M \rightleftarrows M : \mathrm{id}_M$ induce a Quillen pair between M (on the left) and $L_I M$.

We want to quote an important theorem asserting that in good cases the left Bousfield localization of a model category with respect to a set of morphisms exists. Some conditions, which we now define, are needed on the model category.

Definition 3.26. A model category M is **left proper** if pushouts of weak equivalences along cofibrations are weak equivalences.

Note that this is a condition about how weak equivalences and cofibrations behave with respect to ordinary categorical pushouts. Model categories in which all objects are cofibrant are left proper [Lur09, Proposition A.2.4.2].

The next condition we need is for M to be **combinatorial**. This definition, due to Jeff Smith, is rather technical, so we leave it to the interested reader to refer to [Lur09, Definition A.2.6.1]. Recall that a category is **presentable** if it has all small colimits and is κ -compactly generated for some regular cardinal κ . For details, see the book of Adámek-Rosicky [AR94], although note that they call this condition *locally presentable*. We keep Lurie's terminology for the sake of consistency. The most important thing to know about combinatorial model categories for the purposes of this paper is that they are presentable as categories.

Exercise 3.27. Show that the model category structure on $\mathrm{Ch}_{\geq 0}(A)$ of Exercise 3.4 is left proper.

Theorem 3.28. *If M is a left proper and combinatorial simplicial model category and I is a set of morphisms in M , then the left Bousfield localization $L_I M$ exists and inherits a simplicial model category structure from M .*

Proof. This is [Lur09, Proposition A.3.7.3]. \square

We refer to [Hir03, Proposition 3.4.1] for the next result, which identifies the fibrant objects in the Bousfield localization.

Proposition 3.29. *If M is a left proper simplicial model category and I is a set of maps such that $L_I M$ exists as a model category, then the fibrant objects of $L_I M$ are precisely the I -local objects of M .*

Exercise 3.30. Consider the model category structure given in Exercise 3.4 on $\text{Ch}_{\geq 0}(\mathbb{Z})$. It is not hard to show that this is a simplicial model category using the Dold-Kan correspondence (see [GJ99]). Let I be the set of all morphisms between chain complexes of finitely generated abelian groups inducing isomorphisms on *rational* homology groups. Then, $L_I \text{Ch}_{\geq 0}(\mathbb{Z})$ is Quillen equivalent to $\text{Ch}_{\geq 0}(\mathbb{Q})$ with the model category structure of Exercise 3.4. Show that every rational homology equivalence is an isomorphism in $\text{Ho}(L_I \text{Ch}_{\geq 0}(\mathbb{Z}))$.

Exercise 3.31. Construct a category of \mathbb{Q} -local spaces, by letting I be a set of maps $f : X \rightarrow Y$ of simplicial sets such that $H_*(f, \mathbb{Q})$ is an isomorphism.

3.4 Simplicial presheaves with descent

Let C be an essentially small category. Let $\text{sPre}(C)$ denote the category of functors $X : C^{\text{op}} \rightarrow \text{sSets}$. This is the category of **simplicial presheaves** on C , and there is a Yoneda functor $h : C \rightarrow \text{sPre}(C)$. Bousfield and Kan [BK72] defined a model category structure on $\text{sPre}(C)$, the **projective model category structure**, which has a special universal property highlighted by Dugger [Dug01a]: it is the initial model category into which C embeds. Consider the following classes of morphisms in $\text{sPre}(C)$:

- objectwise weak equivalences: those maps $w : X \rightarrow Y$ such that $w(V) : X(V) \rightarrow Y(V)$ is a weak equivalence of simplicial sets for all objects V of C ,
- objectwise fibrations, and
- projective cofibrations, those maps having the left lifting property with respect to acyclic objectwise fibrations.

Proposition 3.32. *The category of simplicial presheaves with the weak equivalences, fibrations, and cofibrations as above is a left proper combinatorial simplicial model category.*

Proof. The reader can find a proof in [Lur09, Proposition A.2.8.2]. See [Lur09, Remark A.2.8.4] for left properness. \square

Suppose now that C has a Grothendieck topology τ . Let U_{\bullet} be an object of $\text{sPre}(C)$ (so that each U_n is a presheaf of *sets* on C), and suppose that there is a map $U \rightarrow V$, where V is a representable object. We call $U \rightarrow V$ a **hypercover** if each U_n is a coproduct of representables, the induced map $U_0 \rightarrow V$ is a τ -cover, and each $U^{\Delta^n} \rightarrow U^{\partial \Delta^n}$ is a τ -cover in degree 0. For details about hypercovers, see [AM69, Section 8]. Except for the definition of the τ -local category below, we will not need hypercovers in the rest of the paper. The reader may safely just imagine these to be Čech complexes.

The standard example of a hypercover is the **Čech** complex $\check{U} \rightarrow V$ associated to a τ -cover $U \rightarrow V$. So, $\check{U}_n = U \times_V \cdots \times_V U$, the product of U with itself $n + 1$ times over V . Roughly speaking, a hypercover looks just like a Čech complex, except that one is allowed to refine the Čech simplicial object by iteratively taking covers of the fiber products.

Theorem 3.33. *The Bousfield localization of $\mathrm{sPre}(C)$ with respect to the class of hypercovers*

$$\check{U}_\bullet \rightarrow V$$

exists. We will denote this model category throughout the paper by $L_\tau\mathrm{sPre}(C)$.

Proof. By Theorem 3.28, we only have to remark that there is up to isomorphism only a set of τ -hypercovers since C is small. \square

Remark 3.34. We will refer to τ -local objects and τ -local weak equivalences for the I -local notions when I is the class of morphisms in the theorem. In the τ -local model category $L_\tau\mathrm{sPre}(C)$, an object V of C (viewed as the functor it represents) is equivalent to the Čech complex of any τ -covering. Since $\mathrm{sPre}(C)$ with its projective model category structure is left proper, the fibrant objects of $L_\tau\mathrm{sPre}(C)$ are precisely the τ -local objects by Proposition 3.29. Hence, the fibrant objects of $L_\tau\mathrm{sPre}(C)$ are precisely the presheaves of Kan complexes X such that

$$X(V) \rightarrow \mathrm{holim}_\Delta X(U) \tag{1}$$

is a weak equivalence for every τ -hypercover $U \rightarrow V$. In other words, the fibrant objects are the **homotopy sheaves of spaces**.

There is another, older definition of the homotopy theory of τ -homotopy sheaves due to Joyal and Jardine. It is useful to know that it is Quillen equivalent to the one given above.

Definition 3.35. Let X be an object of $\mathrm{sPre}(C)$, V an object of C , and $x \in X(V)$ a basepoint. We can define a presheaf of sets (or groups or abelian groups) $\pi_n(X, x)$ on C/V , the category of objects in C over V , by letting

$$\pi_n(X, x)(U) = \pi_n(X(U), f^*(x))$$

for $g : U \rightarrow V$ an object of C/V . Let $\pi_n^\tau(X, x)$ be the sheafification of $\pi_n(X, x)$ in the τ -topology restricted to C/V . These are the **τ -homotopy sheaves** of X .

Let W_τ denote the class of maps $s : X \rightarrow Y$ in $\mathrm{sPre}(C)$ such that $s_* : \pi_n^\tau(X, x) \rightarrow \pi_n^\tau(Y, s(x))$ is an isomorphism for all V and all basepoints $x \in X(V)$. Jardine proved that together with W_τ , the class of objectwise cofibrations determines a model category structure $\mathrm{sPre}_J(C)$ on $\mathrm{sPre}(C)$.

Theorem 3.36 (Dugger-Hollander-Isaksen [DHI04]). *The identity functor $L_\tau\mathrm{sPre}(C) \rightarrow \mathrm{sPre}_J(C)$ is a Quillen equivalence.*

Example 3.37. A τ -sheaf of sets on C , when viewed as a presheaf of simplicial sets, is in particular fibrant. It follows that when τ is subcanonical (i.e., every representable presheaf is in fact a sheaf) the Yoneda embedding $C \rightarrow \mathrm{sPre}(C)$ factors through the category of fibrant objects for the τ -local model category on $\mathrm{sPre}(C)$. Thus, there is a fully faithful Yoneda embedding $C \rightarrow \mathrm{Ho}(L_\tau\mathrm{sPre}(C))$.

3.5 The Nisnevich topology

In this section S denotes a quasi-compact and quasi-separated scheme. We denote by Sm_S the category of finitely presented smooth schemes over S . Recall that while all smooth schemes U over S are locally of finite presentation by definition, saying that $U \rightarrow S$ is finitely presented means in addition to local finite presentation that the morphism is quasi-compact and quasi-separated. Note that Sm_S is an essentially small category because smooth implies locally of finite presentation and because S is quasi-compact and quasi-separated.

Definition 3.38 (Lurie [Lur16, Section A.2.4]). The Nisnevich topology on Sm_S is the topology generated by those finite families of étale morphisms $\{p_i : U_i \rightarrow X\}_{i \in I}$ such that there is a finite sequence $\emptyset \subseteq Z_n \subseteq Z_{n-1} \subseteq \cdots \subseteq Z_1 \subseteq Z_0 = X$ of finitely presented closed subschemes of X such that

$$\prod_{i \in I} p_i^{-1}(Z_m - Z_{m+1}) \rightarrow Z_m - Z_{m+1}$$

admits a section for $0 \leq m \leq n - 1$.

Remark 3.39. The referee pointed out that Hoyois has proved in a preprint [Hoy16] that this definition is equivalent (for S quasi-compact and quasi-separated) to the original definition of Nisnevich [Nis89], which says that an étale cover $U \rightarrow X$ is Nisnevich if it is surjective on k -points for all fields k .

Exercise 3.40. Show that when S is noetherian of finite Krull dimension, then a finite family of étale morphisms $\{p_i : U_i \rightarrow X\}_{i \in I}$ is a Nisnevich cover if and only if for each point $x \in X$ there is an index $i \in I$ and a point $y \in U_i$ over x such that the induced map $k(x) \rightarrow k(y)$ is an isomorphism. This is the usual definition of a Nisnevich cover, as used for example by [MV99].

Example 3.41. Let k be a field of characteristic different than 2 and $a \in k$ a non-zero element. We cover \mathbb{A}^1 by the Zariski open immersion $\mathbb{A}^1 - \{a\} \rightarrow \mathbb{A}^1$ and the étale map $\mathbb{A}^1 - \{0\} \rightarrow \mathbb{A}^1$ given by $x \mapsto x^2$. This étale cover is Nisnevich if and only if a is a square in k .

Exercise 3.42. Zariski covers are in particular Nisnevich covers. For example, we will use later the standard cover of \mathbb{P}^1 by two copies of \mathbb{A}^1 .

Of particular importance in the Nisnevich topology are the so-called elementary distinguished squares.

Definition 3.43. A pullback diagram

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

of S -schemes in Sm_S is an **elementary distinguished (Nisnevich) square** if i is a Zariski open immersion, p is étale, and $p^{-1}(X - U) \rightarrow (X - U)$ is an isomorphism of schemes where $X - U$ is equipped with the reduced induced scheme structure.

The proof of the following lemma is left as an easy exercise for the reader.

Lemma 3.44. *In the notation above, $\{i : U \rightarrow X, p : V \rightarrow X\}$ is a Nisnevich cover of X .*

Example 3.45. If a is a square in Example 3.41, then we obtain a Nisnevich cover which does not come from an elementary distinguished square. However, if we remove one of the square roots of a from $\mathbb{A}^1 - \{0\}$, then we do obtain an elementary distinguished square.

Exercise 3.46. Let p be a prime, let $X = \text{Spec } \mathbb{Z}_{(p)}$, and let $V = \text{Spec } \mathbb{Z}_{(p)}[i] \rightarrow X$, where $i^2 + 1 = 0$. Let $U = \text{Spec } \mathbb{Q} \rightarrow X$. Then, $\{U, V\}$ is an étale cover of $\text{Spec } \mathbb{Z}_{(p)}$ for all odd p . It is Nisnevich if and only if in addition $p \equiv 1 \pmod{4}$.

Example 3.47. Let $X = \text{Spec } R$, where R is a discrete valuation ring with field of fractions K . Suppose that $p : V \rightarrow X$ is an étale map where V is the spectrum of another discrete valuation ring S . Then, the square

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

with $i : U = \text{Spec } K \rightarrow X$ is an elementary distinguished square if and only if the inertial degree of $R \rightarrow S$ is 1.

Definition 3.48. The **Nisnevich-local model category** $L_{\text{Nis}}\text{sPre}(\text{Sm}_S)$ will be denoted simply by Spc_S , and the fibrant objects of Spc_S will be called **spaces**. So, a space is a presheaf of Kan complexes on Sm_S satisfying Nisnevich hyperdescent in the sense that the arrows (1) are weak equivalences for Nisnevich hypercovers.

Warning 3.49. There are three candidates for the \mathbb{A}^1 -homotopy theory over S . One is the \mathbb{A}^1 -localization of the Joyal-Jardine Nisnevich-local model structure [Jar87]. The other is that used by [AHW15a], which imposes descent only for covers. Finally, we impose descent for all hypercovers. When S is noetherian of finite Krull dimension, all three definitions are Quillen equivalent. In all cases, our definition is equivalent to the Joyal-Jardine definition, by the main result of [DHI04].

Notation 3.50. If X and Y are presheaves of simplicial sets on Sm_S , we will write $[X, Y]_{\text{Nis}}$ for the set of Nisnevich homotopy classes of maps from X to Y , which is the hom-set from X to Y in the homotopy category of $L_{\text{Nis}}\text{sPre}(\text{Sm}_S)$. The pointed version is written $[X, Y]_{\text{Nis}, \star}$. When necessary, we will write $[X, Y]_s$ for the homotopy classes of maps from X to Y in $\text{sPre}(\text{Sm}_S)$, and similarly we write $[X, Y]_{s, \star}$ for the homotopy classes of pointed maps.

Notation 3.51. We will write L_{Nis} for the left derived functor of the identity functor $\text{sPre}(\text{Sm}_S) \rightarrow L_{\text{Nis}}\text{sPre}(\text{Sm}_S)$. Thus, L_{Nis} is computed by taking a cofibrant replacement functor with respect to the Nisnevich-local model category structure on $\text{sPre}(\text{Sm}_S)$.

Example 3.52. Given a scheme X essentially of finite presentation over S , we abuse notation and also view X as the presheaf it represents on Sm_S . So, if Y is a finitely presented smooth S -scheme, then $X(Y) = \text{Hom}_S(Y, X)$. Since the Nisnevich topology is subcanonical, it is an easy exercise to see that X is fibrant. Indeed, homotopy limits of discrete spaces are just computed as limits of the underlying sets of components. We will discuss homotopy limits and colimits further in Section 4.

The next proposition is a key tool for practically verifying Nisnevich fibrancy for a given presheaf of simplicial sets on Sm_S .

Proposition 3.53. *Suppose that S is a noetherian scheme of finite Krull dimension. A simplicial presheaf F on Sm_S is Nisnevich-fibrant if and only if for every elementary distinguished square*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

the natural map

$$F(X) \rightarrow F(V) \times_{F(U \times_X V)} F(U)$$

is a weak equivalence of simplicial sets and $F(\emptyset)$ is a final object.

Presheaves possessing the property in the proposition are said to satisfy the **Brown-Gersten property** [BG73] or the **excision** property, although Brown and Gersten studied the Zariski analog.

Proof. Let us first indicate the references for this theorem. The proof in [MV99, Section 3.1] applies to sheaves of sets (i.e. sheaves valued in discrete simplicial sets); to deduce the simplicial version, one uses the techniques in [BG73].

One way the reader can get straight to the case of simplicial presheaves is via the following argument. The Nisnevich topology is generated by a **cd-structure**, a collection of squares in Sm_S stable under isomorphism; the cd-structure corresponding to the Nisnevich

topology is given by the elementary distinguished squares. This observation amounts to [Voe10b, Proposition 2.17, Remark 2.18]. In [Voe10a, Section 2], Voevodsky gives conditions on a category C equipped with a cd-structure for when the sheaf condition on a presheaf of sets on C coincides with the excision condition with respect to the cd-structure. For a proof of the corresponding claim for simplicial presheaves (and hyperdescent), one can refer to [AHW15a, Theorem 3.2.5].

For another, direct approach see [Dug01b]. \square

Let $\mathrm{Sm}_S^{\mathrm{Aff}}$ denote the full subcategory of Sm_S consisting of (absolutely) affine schemes. A presheaf X on $\mathrm{Sm}_S^{\mathrm{Aff}}$ satisfies **affine Nisnevich excision** if it satisfies excision for the cd-structure on $\mathrm{Sm}_S^{\mathrm{Aff}}$ consisting of cartesian squares

$$\begin{array}{ccc} \mathrm{Spec} R'_f & \longrightarrow & \mathrm{Spec} R' \\ \downarrow & & \downarrow \\ \mathrm{Spec} R_f & \longrightarrow & \mathrm{Spec} R, \end{array}$$

where $\mathrm{Spec} R' \rightarrow \mathrm{Spec} R$ is étale, $f \in R$, and $R/(f) \cong R'/(f)$. An important result of [AHW15a] says that the topology generated by the affine Nisnevich cd-structure is the same as the Nisnevich topology restricted to $\mathrm{Sm}_S^{\mathrm{Aff}}$.

3.6 The \mathbb{A}^1 -homotopy category

To define the \mathbb{A}^1 -homotopy category, we perform a further left Bousfield localization of $L_{\mathrm{Nis}}\mathrm{sPre}(\mathrm{Sm}_S)$. As above, S denotes a quasi-compact and quasi-separated scheme.

Definition 3.54. Let I be the class of maps $\mathbb{A}^1 \times_S X \rightarrow X$ in $L_{\mathrm{Nis}}\mathrm{sPre}(\mathrm{Sm}_S)$ as X ranges over all objects of Sm_S . Since Sm_S is essentially small, we can choose a subset $J \subseteq I$ containing maps $\mathbb{A}^1 \times_S X \rightarrow X$ as X ranges over a representative of each isomorphism class of Sm_S .

The **\mathbb{A}^1 -homotopy theory** of S is the left Bousfield localization $L_{\mathbb{A}^1}L_{\mathrm{Nis}}\mathrm{sPre}(\mathrm{Sm}_S)$ of $L_{\mathrm{Nis}}\mathrm{sPre}(\mathrm{Sm}_S)$ with respect to J . Its homotopy category will be called the **\mathbb{A}^1 -homotopy category** of S . Let $\mathrm{Spc}_S^{\mathbb{A}^1}$ be $L_{\mathbb{A}^1}L_{\mathrm{Nis}}\mathrm{sPre}(\mathrm{Sm}_S)$. Fibrant objects of $\mathrm{Spc}_S^{\mathbb{A}^1}$ will be called **\mathbb{A}^1 -spaces** or **\mathbb{A}^1 -local spaces**. Note that the simplicial presheaf underlying any \mathbb{A}^1 -space is in particular a space in the sense that it is fibrant in Spc_S . The homotopy category of $\mathrm{Spc}_S^{\mathbb{A}^1}$ will always be written as $\mathrm{Ho}(\mathrm{Spc}_S^{\mathbb{A}^1})$, and usually functoriality or naturality statements will be made with respect to the homotopy category.

Proposition 3.55. *The Bousfield localization $\mathrm{Spc}_S^{\mathbb{A}^1} = L_{\mathbb{A}^1}L_{\mathrm{Nis}}\mathrm{sPre}(\mathrm{Sm}_S)$ exists.*

Proof. The simplicial structure, left properness, and combinatoriality are inherited by Spc_S from $\mathrm{sPre}(\mathrm{Sm}_S)$, and hence by Theorem 3.28 the Bousfield localization exists. \square

Notation 3.56. If X and Y are presheaves of simplicial sets on Sm_S , we will write $[X, Y]_{\mathbb{A}^1}$ for the set of \mathbb{A}^1 -homotopy classes of maps from X to Y , which is the hom-set from X to Y in the homotopy category of $L_{\mathbb{A}^1}L_{\mathrm{Nis}}\mathrm{sPre}(\mathrm{Sm}_S)$. The pointed version is written $[X, Y]_{\mathbb{A}^1, \star}$.

Notation 3.57. We will write $L_{\mathbb{A}^1}$ for the left derived functor of the identity functor $L_{\mathrm{Nis}}\mathrm{sPre}(\mathrm{Sm}_S) \rightarrow L_{\mathbb{A}^1}L_{\mathrm{Nis}}\mathrm{sPre}(\mathrm{Sm}_S)$. Thus, $L_{\mathbb{A}^1}L_{\mathrm{Nis}}$ is computed by taking a cofibrant replacement functor with respect to the \mathbb{A}^1 -local model category structure on $\mathrm{sPre}(\mathrm{Sm}_S)$.

Remark 3.58. It is common to call an \mathbb{A}^1 -space, an \mathbb{A}^1 -local space, and indeed the fibrant objects of $\mathrm{Spc}_S^{\mathbb{A}^1}$ are \mathbb{A}^1 -local. In fact, a simplicial presheaf X in $\mathrm{sPre}(\mathrm{Sm}_S)$ is \mathbb{A}^1 -local, i.e., a fibrant object of $\mathrm{Spc}_S^{\mathbb{A}^1}$, if it

1. takes values in Kan complexes (so that it is fibrant in $\mathrm{sPre}(\mathrm{Sm}_S)$),

2. satisfies Nisnevich hyperdescent (so that it is fibrant in Spc_S), and
3. if $X(U) \rightarrow X(\mathbb{A}^1 \times_S U)$ is a weak equivalence of simplicial sets for all U in Sm_S .

Exercise 3.59. Construct the pointed version $\mathrm{Spc}_{S,\star}^{\mathbb{A}^1}$ of $\mathrm{Spc}_S^{\mathbb{A}^1}$, the homotopy theory of pointed \mathbb{A}^1 -spaces. We will have occasion to use this pointed version as well as the Quillen adjunction

$$\mathrm{Spc}_S^{\mathbb{A}^1} \rightleftarrows \mathrm{Spc}_{S,\star}^{\mathbb{A}^1},$$

which sends a presheaf of spaces X to the pointed presheaf of spaces X_+ obtained by adding a disjoint basepoint.

Definition 3.60. The weak equivalences in $\mathrm{Spc}_S^{\mathbb{A}^1}$ are called \mathbb{A}^1 -**weak equivalences** or \mathbb{A}^1 -**local weak equivalences**.

Here is an expected class of \mathbb{A}^1 -weak equivalences.

Definition 3.61. Let $f, g : X \rightarrow Y$ be maps of simplicial presheaves. We say that f and g are \mathbb{A}^1 -homotopic if there exists a map $H : F \times \mathbb{A}^1 \rightarrow G$ such that $H \circ (\mathrm{id}_F \times i_0) = f$ and $H \circ (\mathrm{id}_F \times i_1) = g$. A map $g : F \rightarrow G$ is an \mathbb{A}^1 -homotopy equivalence if there exists morphisms $h : G \rightarrow F$ and that $h \circ g$ and $g \circ h$ are \mathbb{A}^1 -homotopic to id_F and id_G respectively.

Exercise 3.62. Show that if $p : E \rightarrow X$ is a vector bundle in Sm_S , then p is an \mathbb{A}^1 -homotopy equivalence.

Exercise 3.63. Show that any \mathbb{A}^1 -homotopy equivalence $f : F \rightarrow G$ is an \mathbb{A}^1 -weak equivalence. Note that there are many more \mathbb{A}^1 -weak equivalences.

4 Basic properties of \mathbb{A}^1 -algebraic topology

This long section is dedicated to outlining the basic facts that form the substrate of the unstable motivic homotopy theorists' work. Examples and basic theorems abound, and we hope that it provides a helpful user's manual. Most non-model category theoretic results below are due to Morel and Voevodsky [MV99].

Throughout this section we fix a quasi-compact and quasi-separated base scheme S , and we study the model category

$$\mathrm{Spc}_S^{\mathbb{A}^1} = \mathrm{L}_{\mathbb{A}^1} \mathrm{L}_{\mathrm{Nis}} \mathrm{Pre}(\mathrm{Sm}_S).$$

4.1 Computing homotopy limits and colimits through examples

An excellent source for the construction of homotopy limits or colimits is the exposition of Dwyer and Spalinski [DS95]. We start with an example from ordinary homotopy theory. Consider the following morphism of pullback diagrams of topological spaces:

$$(\star \rightarrow S^1 \leftarrow \star) \rightarrow (\star \rightarrow S^1 \leftarrow P_\star S^1),$$

where $P_\star S^1$ is the path space of S^1 consisting of paths beginning at the basepoint of S^1 . This diagram is a homotopy equivalence in each spot. However, the pullback of the first is just a point, while the pullback of the second is the loop space ΩS^1 , which is homotopy equivalent to the discrete space \mathbb{Z} . This example illustrates that some care is needed when forming the homotopically correct notion of pullback.

Similarly, consider the maps of pushout diagrams

$$(\star \leftarrow S^0 \rightarrow D^1) \rightarrow (\star \leftarrow S^0 \rightarrow \star),$$

where D^1 is the 1-disk. Again, this map is a homotopy equivalence in each place. But, the pushout in the first case is S^1 , and in the second case it is just a point. Again, care is required in order to compute the *correct* pushout.

The key in these examples is that $P_*S^1 \rightarrow S^1$ is a fibration, while $S^0 \rightarrow D^1$ is a cofibration. By uniformly replacing pullback diagrams with pullback diagrams where the maps are fibrations, and then taking the pullback, one obtains a *homotopy-invariant* notion of pullback, the homotopy pullback. Similarly, by replacing pushout diagrams with homotopy equivalent diagrams in which the morphisms are cofibrations, one obtains homotopy pushouts.

Definition 4.1. A **homotopy pullback** diagram in a model category M is a pullback diagram

$$\begin{array}{ccc} c & \longrightarrow & d \\ \downarrow & & \downarrow \\ e & \longrightarrow & f \end{array}$$

in M where at least one of $e \rightarrow f$ or $d \rightarrow f$ is a fibration in M . Given a pullback diagram $e \rightarrow f \leftarrow d$, the **homotopy pullback** is the pullback of either the diagram $e' \rightarrow f \leftarrow d$ or $e \rightarrow f \leftarrow d'$ where $e' \rightarrow f$ (resp $d' \rightarrow f$) is the fibrant replacement via **M4** of $e \rightarrow f$ (resp. $d \rightarrow f$).

Exercise 4.2. Show that homotopy pullbacks are independent up to weak equivalence of any choices made.

To put this notion on a more precise footing, we make the following construction.

Proposition 4.3 ([Lur09, Proposition A.2.8.2]). *Let M be a combinatorial model category and I a small category. The pointwise weak equivalences and pointwise fibrations determine a model category structure on M^I called the **projective model category structure**, which we will denote by M_{proj}^I . The pointwise weak equivalences and pointwise cofibrations determine a model category structure on M^I called the **injective model category structure**, which we will denote by M_{inj}^I .*

We have already seen the projective model category structure in our discussion of presheaves of spaces on a small category. These two model categories on M^I can be used to compute homotopy limits and homotopy colimits.

Lemma 4.4. *The functor $\Delta : M \rightarrow M^I$ taking $m \in M$ to the constant functor $I \rightarrow M$ on m admits both a left and a right adjoint.*

Proof. Note that the category M is presentable by the definition of a combinatorial model category. This means that M has all small colimits and is λ -compactly generated for some regular cardinal λ . By the adjoint functor theorem [AR94, Theorem 1.66]¹, it suffices to prove that Δ is accessible, preserves limits, and preserves small colimits. However, accessibility of Δ simply means that it commutes with κ -filtered colimits for some regular cardinal κ . Since we will show that it commutes with all small colimits, accessibility is an immediate consequence.

To prove that Δ commutes with small limits, let $y \cong \lim_k y_k$ be a limit in M . Consider

¹This gives the criterion for the existence of a left adjoint for a functor between locally presentable categories; it is somewhat easier to prove that a functor between locally presentable categories which preserves small colimits is a left adjoint. There is a good discussion of these issues on the nLab.

an object $x : I \rightarrow M$ of M^I . Then,

$$\begin{aligned} \mathrm{hom}_{M^I}(x, \Delta(y)) &\cong \mathrm{eq} \left(\prod_{i \in I} \mathrm{hom}_M(x(i), y(i)) \rightrightarrows \prod_{f \in \mathrm{Ar}(I)} \mathrm{hom}_M(x(i), y(j)) \right) \\ &\cong \mathrm{eq} \left(\prod_{i \in I} \lim_k \mathrm{hom}_M(x(i), y_k(j)) \rightrightarrows \prod_{f \in \mathrm{Ar}(I)} \lim_k \mathrm{hom}_M(x(i), y_k(j)) \right) \\ &\cong \lim_k \mathrm{eq} \left(\prod_{i \in I} \mathrm{hom}_M(x(i), y_k(j)) \rightrightarrows \prod_{f \in \mathrm{Ar}(I)} \mathrm{hom}_M(x(i), y_k(j)) \right) \\ &\cong \lim_k \mathrm{hom}_{M^I}(x, \Delta(y_k)), \end{aligned}$$

using the fact that small limits commute with small limits and hence in particular equalizers and small products. It follows that $\Delta(y) \cong \lim_k \Delta(y_k)$, as desired. The proof that Δ preserves small colimits is left as an exercise. \square

Exercise 4.5. Show that Δ preserves small colimits.

Definition 4.6. We will call the right adjoint to Δ the limit functor \lim_I , while the left adjoint is the colimit functor colim_I .

Lemma 4.7. *The pairs of adjoint functors*

$$\Delta : M \rightleftarrows M_{\mathrm{inj}}^I : \lim_I$$

and

$$\mathrm{colim}_I : M_{\mathrm{proj}}^I \rightleftarrows M : \Delta$$

are Quillen pairs.

Proof. Note that Δ preserves pointwise weak equivalences, pointwise fibrations, and pointwise cofibrations. \square

Definition 4.8. We will write holim_I for $\mathbf{R}\lim_I$ and $\mathrm{hocolim}_I$ for $\mathbf{L}\mathrm{colim}_I$, and call these the homotopy limit and homotopy colimit functors.

Exercise 4.9. Let I be the small category $\bullet \leftarrow \bullet \rightarrow \bullet$, which classifies pushouts. To compute the homotopy pushout $x \leftarrow y \rightarrow z$ in M , we must take an cofibrant replacement $x' \leftarrow y' \rightarrow z'$ in M_{proj}^I , and then we can compute the categorical pushout of the new diagram. Describe the cofibrant objects of M^I . Show that the homotopy pushout can be computed as the pushout of $x' \leftarrow y' \rightarrow z'$ where x' and y' are cofibrant and $y' \rightarrow z'$ is a cofibration. Show however that such diagrams are not in general cofibrant in M_{proj}^I .

Proposition 4.10. *Right derived functors of right Quillen functors commute with homotopy limits and left derived functors of left Quillen functors commute with homotopy colimits.*

Proof. We prove the result for right Quillen functors and homotopy limits. Suppose that we have a Quillen adjunction:

$$F : M \rightleftarrows N : G.$$

It is easy to check that this induces a Quillen adjunction $F^I : M_{\mathrm{inj}}^I \rightleftarrows N_{\mathrm{inj}}^I : G^I$. Indeed, it is enough to check that F^I preserves cofibrations and acyclic cofibrations, but these are defined pointwise in the injective model category structure, so the fact that F is a left Quillen functor implies that F^I is as well. Consider the following diagram

$$\begin{array}{ccc} M_{\mathrm{inj}}^I & \xrightarrow{F^I} & N_{\mathrm{inj}}^I \\ \uparrow \Delta & & \uparrow \Delta \\ M & \xrightarrow{F} & N \end{array}$$

of left Quillen functors. This diagram commutes on the level of underlying categories; picking appropriate fibrant replacements to compute the right adjoints, the right derived versions of the functors commute which induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Ho}(M_{\mathrm{inj}}^I) & \xrightarrow{\mathbf{L}F^I} & \mathrm{Ho}(N_{\mathrm{inj}}^I) \\ \uparrow \mathbf{L}\Delta & & \uparrow \mathbf{L}\Delta \\ \mathrm{Ho}(M) & \xrightarrow{\mathbf{L}F} & \mathrm{Ho}(N) \end{array}$$

of left adjoints on the level of homotopy categories. This means means that the diagram

$$\begin{array}{ccc} \mathrm{Ho}(M_{\mathrm{inj}}^I) & \xleftarrow{\mathbf{R}G^I} & \mathrm{Ho}(N_{\mathrm{inj}}^I) \\ \mathbf{R}\mathrm{holim}_I \downarrow & & \downarrow \mathbf{R}\mathrm{holim}_I \\ \mathrm{Ho}(M) & \xleftarrow{\mathbf{R}G} & \mathrm{Ho}(N) \end{array}$$

of right adjoints commutes. □

We are now in a position to give examples.

Exercise 4.11. One should be careful when trying to commute homotopy limits or colimits using the above proposition — the functors must be derived. Construct an example using a morphism of commutative rings $R \rightarrow S$, the functor $\otimes_R S : \mathrm{Ch}_R^{\geq 0} \rightarrow \mathrm{Ch}_S^{\geq 0}$, and the mapping cone of an R -module $M \rightarrow N$ thought of as chain complexes concentrated in a single degree to show that preservation of homotopy colimits fail if $\otimes_R S$ is not derived. Hint: see the example of mapping cones worked out in Example 4.12.

Example 4.12. Let A be an associative ring, and consider $\mathrm{Ch}_A^{\geq 0}$, the category of non-negatively graded chain complexes equipped with the projective model category structure. Let M be an A -module viewed as a chain complex concentrated in degree 0, and let N_\bullet be a chain complex. The actual pushout of a map $M \rightarrow N_\bullet$ along $M \rightarrow 0$ is just the cokernel of the map of complexes. If $N_\bullet = 0$, this cokernel is zero. However, by the recipe above, we should replace 0 with a quasi-isomorphic fibrant model P_\bullet with a map $M \rightarrow P_\bullet$ that is a cofibration. A functorial choice turns out to be the cone on the identity of M . This is the complex $M \xrightarrow{\mathrm{id}_M} M$ with M placed in degrees 1 and 0. This time, when we take the cokernel, we get the complex $M[1]$. This confirms what everyone wants: that $M \rightarrow 0 \rightarrow M[1]$ should be a distinguished triangle in the derived category of A , which is what is needed to have long exact sequences in homology.

Let us now turn to examples in \mathbb{A}^1 -homotopy theory. The following proposition gives a way of constructing many examples of homotopy pushouts in the category Spc_S and is a consequence of the characterization of fibrant objects in Spc_S .

Proposition 4.13. *If S is a noetherian scheme of finite Krull dimension, then an elementary distinguished (Nisnevich) square*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X, \end{array}$$

in Sm_S thought of as a diagram of simplicial presheaves is a homotopy pushout in Spc_S .

Proof. Since the Nisnevich topology is subcanonical (it is coarser than the étale topology which is subcanonical) we may regard these squares as diagrams in Spc_S via the Yoneda

embedding (or, rather, its simplicial analogue — we think of schemes as sheaves of discrete simplicial sets). Let X be a space, i.e., a fibrant object of Spc_S . Proposition 3.53 tells us that applying X to an elementary distinguished square gives rise to a homotopy pullback square. This verifies the universal property for a homotopy pushout. \square

One problem with the category of schemes, as mentioned above, is that it lacks general colimits, even finite colimits. In particular, general quotient spaces do not exist in Sm_S .

Definition 4.14. For the purposes of this paper, the **quotient** X/Y of a map $X \rightarrow Y$ of schemes in Sm_S is always defined to be the homotopy cofiber of the map in $\mathrm{Spc}_S^{\mathbb{A}^1}$. Recall that the homotopy cofiber is the homotopy pushout of $\star \leftarrow X \rightarrow Y$. Note that since localization is a left adjoint, this definition agrees up to homotopy with the \mathbb{A}^1 -localization of the homotopy cofiber computed in Spc_S by Proposition 4.10.

Example 4.15. Proposition 4.13 implies that in the situation of an elementary distinguished square, the natural map

$$\frac{V}{U \times_X V} \rightarrow \frac{X}{U}$$

is an \mathbb{A}^1 -local weak equivalence. To see this, we see that Proposition 4.13 gives a Nisnevich local weak equivalence of the cofibers of the top and bottom horizontal arrows; since $L_{\mathbb{A}^1}$ is a left adjoint, we see that it preserves cofibers and thus gives rise to the desired \mathbb{A}^1 -local weak equivalence.

Example 4.16. A particularly important example of a quotient or homotopy cofiber is the **suspension** of a pointed object X in $\mathrm{Spc}_{S,\star}^{\mathbb{A}^1}$. This is simply the homotopy cofiber of $X \rightarrow \star$, or in other words, the homotopy pushout of the diagram

$$\begin{array}{ccc} X & \longrightarrow & \star \\ \downarrow & & \\ \star & & \end{array}$$

which we denote by ΣX . See Section 4.6 for one use of the construction.

4.2 \mathbb{A}^1 -homotopy fiber sequences and long exact sequences in homotopy sheaves

Definition 4.17. Let $X \rightarrow Y$ be a map of pointed objects in a model category. The **homotopy fiber** F is the homotopy pullback of $\star \rightarrow Y \leftarrow X$. In general, if $F \rightarrow X \rightarrow Y$ is a sequence of spaces and if F is weak equivalent to the homotopy fiber of $X \rightarrow Y$, then we call $F \rightarrow X \rightarrow Y$ a homotopy fiber sequence.

Recall that in ordinary algebraic topology, given a homotopy fiber sequence

$$F \rightarrow X \rightarrow Y$$

of pointed spaces, there is a long exact sequence

$$\cdots \pi_{n+1}Y \rightarrow \pi_n F \rightarrow \pi_n X \rightarrow \pi_n Y \rightarrow \pi_{n-1} F \rightarrow \cdots$$

of homotopy groups, where we omit the basepoint for simplicity. Exactness should be carefully interpreted for $n = 0, 1$, when these are only pointed sets or not-necessarily-abelian groups. For details, consult Bousfield and Kan [BK72, Section IX.4.1].

Definition 4.18. The **Nisnevich homotopy sheaf** $\pi_n^{\mathrm{Nis}}(X)$ of a pointed object X of Spc_S is the Nisnevich sheafification of the presheaf

$$U \mapsto [S^n \wedge U_+, X]_{\mathrm{Nis},\star}$$

Definition 4.19. The \mathbb{A}^1 -homotopy sheaf $\pi_n^{\mathbb{A}^1}(X)$ of a pointed object X of $\text{Spc}_S^{\mathbb{A}^1}$ is the Nisnevich sheafification of the presheaf

$$U \mapsto [S^n \wedge U_+, X]_{\mathbb{A}^1, \star}.$$

Exercise 4.20. Show that if X is weakly equivalent to $L_{\mathbb{A}^1}L_{\text{Nis}}X$, where X is a pointed simplicial presheaf, then the natural map $\pi_n^{\text{Nis}}(X) \rightarrow \pi_n^{\mathbb{A}^1}(X)$ is an isomorphism of Nisnevich sheaves.

The following result is a good illustration of the theory we have developed so far.

Proposition 4.21. *Let $F \rightarrow X \rightarrow Y$ be a homotopy fiber sequence in $\text{Spc}_S^{\mathbb{A}^1}$. Then, there is a natural long exact sequence*

$$\cdots \rightarrow \pi_{n+1}^{\mathbb{A}^1}Y \rightarrow \pi_n^{\mathbb{A}^1}F \rightarrow \pi_n^{\mathbb{A}^1}X \rightarrow \pi_n^{\mathbb{A}^1}Y \rightarrow \cdots$$

of Nisnevich sheaves.

Proof. The forgetful functor $\text{Spc}_S^{\mathbb{A}^1} \rightarrow \text{sPre}(\text{Sm}_S)$ is a right adjoint, and hence it preserves homotopy fiber sequences. It follows from the fact that fibrations are defined as object-wise fibrations that there is a natural long exact sequence of homotopy presheaves. Since sheafification, and in particular Nisnevich sheafification, is exact [TS14, Tag 03CN], the claim follows. \square

Remark 4.22. We caution the reader that although the functor: $L_{\mathbb{A}^1}L_{\text{Nis}} : \text{sPre}(\text{Sm}_S) \rightarrow \text{Spc}_S^{\mathbb{A}^1}$ preserves homotopy colimits, it is not clear that resulting homotopy colimit diagram in $\text{Spc}_S^{\mathbb{A}^1}$ possess any exactness properties. To be more explicit, let $i : \text{Spc}_S^{\mathbb{A}^1} \rightarrow \text{sPre}(\text{Sm}_S)$ be the forgetful functor. Suppose that we have a homotopy cofiber sequence: $X \rightarrow Y \rightarrow Z$ in $\text{sPre}(\text{Sm}_S)$, then it is not clear that $iL_{\mathbb{A}^1}L_{\text{Nis}}(Z)$ is equivalent to the cofiber of $iL_{\mathbb{A}^1}L_{\text{Nis}}(X) \rightarrow iL_{\mathbb{A}^1}L_{\text{Nis}}(Y)$ since we are composing a Quillen left adjoint with a Quillen right adjoint. Consequently, long exact sequences which arise out of cofiber sequences (such as mapping into Eilenberg-MacLane spaces which produces the long exact sequences in ordinary cohomology) in $\text{sPre}(\text{Sm}_S)$ will not apply to this situation.

4.3 The $\text{Sing}^{\mathbb{A}^1}$ -construction

While the process of Nisnevich localization, which produces objects of Spc_S , is familiar from ordinary sheaf theory, the localization $L_{\mathbb{A}^1} : \text{Spc}_S \rightarrow \text{Spc}_S^{\mathbb{A}^1}$ is more difficult to grasp concretely. This section describes one model for the localization functor $L_{\mathbb{A}^1}$.

Consider the cosimplicial scheme Δ^\bullet where

$$\Delta^n = \text{Spec } k[x_0, \dots, x_n]/(x_0 + \dots + x_n = 1)$$

with the face and degeneracy maps familiar from the standard topological simplex. The scheme Δ^n is a closed subscheme of \mathbb{A}^{n+1} isomorphic to \mathbb{A}^n , the i th coface map $\partial_j : \Delta^n \rightarrow \Delta^{n+1}$ is defined by setting $x_j = 0$, and the i th codegeneracy $\sigma_i : \Delta^n \rightarrow \Delta^{n-1}$ is given by summing the i th and $i+1$ st coordinates.

Definition 4.23. Let X be a simplicial presheaf. We define the simplicial presheaf $\text{Sing}^{\mathbb{A}^1} X := |X(- \times \Delta^\bullet)|$. This gives the **singular construction** functor

$$\text{Sing}^{\mathbb{A}^1} : \text{sPre}(\text{Sm}_S) \rightarrow \text{sPre}(\text{Sm}_S).$$

We will also write $\text{Sing}^{\mathbb{A}^1}$ for the restriction of the singular construction to $\text{Spc}_S \subseteq \text{sPre}(\text{Sm}_S)$.

Remark 4.24. Since geometric realizations do not commute in general with homotopy limits, there is no reason to expect $\text{Sing}^{\mathbb{A}^1}$ to preserve the property of being Nisnevich-local. At heart this is the reason for both the subtlety and the depth of motivic homotopy theory.

Proof. Let $\Phi = \text{L}_{\text{Nis}} \text{Sing}^{\mathbb{A}^1}$, so that the theorem claims that $\Phi^{\circ\mathbb{N}} \simeq \text{L}_{\text{Nis}}$. We first argue that $(\text{L}_{\text{Nis}} \text{Sing}^{\mathbb{A}^1})^{\circ\mathbb{N}} X$ is fibrant in $\text{Spc}_S^{\mathbb{A}^1}$ for any X in $\text{sPre}(\text{Sm}_S)$. We must simply check that it is Nisnevich and \mathbb{A}^1 -local. To check that it is Nisnevich local, write

$$\Phi^{\circ\mathbb{N}}(X) \simeq \text{hocolim}_{n \rightarrow \infty} (\text{L}_{\text{Nis}} \text{Sing}^{\mathbb{A}^1})^{\circ n}(X),$$

a filtered homotopy colimit of Nisnevich local presheaves of spaces. It hence suffices to show that the forgetful functor $\text{Spc}_S \rightarrow \text{sPre}(\text{Sm}_S)$ preserves filtered homotopy colimits. However, since the sheaf condition is checked on the finite homotopy limits induced from the elementary distinguished squares by Proposition 3.53, and since filtered homotopy limits commute with finite homotopy limits, the result is immediate. At this point we must be honest and point out that the main reference we know for the commutativity of finite homotopy limits and filtered homotopy colimits, namely [Lur09, Proposition 5.3.3.3], is for ∞ -categories rather than model categories. However, since homotopy limits and colimits in combinatorial simplicial model categories (such as all model categories in this paper) agree with the corresponding ∞ -categorical limits and colimits by [Lur09, Section 4.2.4], this should be no cause for concern.

To check that $(\text{L}_{\text{Nis}} \text{Sing}^{\mathbb{A}^1})^{\circ\mathbb{N}} X$ is \mathbb{A}^1 -local, note that we can write

$$\Phi^{\circ\mathbb{N}} X \simeq \text{hocolim}_{n \rightarrow \infty} (\text{Sing}^{\mathbb{A}^1} \text{L}_{\text{Nis}})^{\circ n} (\text{Sing}^{\mathbb{A}^1} X),$$

a filtered homotopy colimit of \mathbb{A}^1 -invariant presheaves by Theorem 4.25. But, filtered homotopy colimits of \mathbb{A}^1 -invariant presheaves are \mathbb{A}^1 -invariant. Since $\Phi^{\circ\mathbb{N}}(X)$ is Nisnevich local and \mathbb{A}^1 -invariant, it is \mathbb{A}^1 -local.

Thus, we have seen that $\Phi^{\circ\mathbb{N}}$ does indeed take values in the fibrant objects of $\text{Spc}_S^{\mathbb{A}^1}$. Finally, we claim that it suffices to show that $\Phi \simeq \text{L}_{\text{Nis}} \text{Sing}^{\mathbb{A}^1}$ preserves \mathbb{A}^1 -local weak equivalences. Indeed, if this is the case, then so does $\Phi^{\circ\mathbb{N}}$, which will show that

$$\Phi^{\circ\mathbb{N}}(X) \simeq \Phi^{\circ\mathbb{N}}(\text{L}_{\mathbb{A}^1} \text{L}_{\text{Nis}} X) \simeq \text{L}_{\mathbb{A}^1} \text{L}_{\text{Nis}} X,$$

since it is clear that $X \simeq \Phi(X)$ when X is \mathbb{A}^1 -local. For the remainder of the proof, write $\text{map}(-, -)$ for the mapping spaces in $\text{sPre}(\text{Sm}_S)$. We want to show that

$$\text{map}(\Phi(X), Y) \simeq \text{map}(X, Y)$$

for all \mathbb{A}^1 -local objects Y of $\text{sPre}(\text{Sm}_S)$. But,

$$\begin{aligned} \text{map}(\Phi(X), Y) &\simeq \text{map}_{\text{Spc}_S}(\text{L}_{\text{Nis}} \text{Sing}^{\mathbb{A}^1} X, Y) \\ &\simeq \text{map}(\text{Sing}^{\mathbb{A}^1} X, Y) \end{aligned}$$

since Y is in particular Nisnevich local. As the singular construction functor $\text{Sing}^{\mathbb{A}^1}$ is a homotopy colimit, it commutes with homotopy colimits. Since $X \simeq \text{hocolim}_{U \rightarrow X} U$, where the colimit is over maps from smooth S -schemes U , it follows that it is enough to show that

$$\text{map}(\text{Sing}^{\mathbb{A}^1} U, X) \simeq \text{map}(U, Y)$$

for U a smooth S -scheme and Y an \mathbb{A}^1 -local presheaf. To prove this, it is enough in turn to show that

$$\text{map}(U(- \times \mathbb{A}^n), Y) \simeq \text{map}(U, Y),$$

where $U(- \times \mathbb{A}^n)$ is the presheaf of spaces $V \mapsto U(V \times \mathbb{A}^n)$. Note that because there is an S -point of \mathbb{A}^n , the representable presheaf U is a retract of $U(- \times \mathbb{A}^n)$, so it suffices to show that

$$\pi_0 \text{map}(U(- \times \mathbb{A}^n), Y) \cong \pi_0 \text{map}(U, Y),$$

or even just that the map

$$\pi_0 \text{map}(U, Y) \rightarrow \pi_0 \text{map}(U(- \times \mathbb{A}^n), Y)$$

induced by an S -point of \mathbb{A}^n is a surjection. Now, $U(- \times \mathbb{A}^n) \simeq \text{hocolim}_{V \times \mathbb{A}^n \rightarrow U} V \times \mathbb{A}^n$, so

$$\pi_0 \text{map}(U(- \times \mathbb{A}^n), Y) \cong \pi_0 \lim_{V \times \mathbb{A}^n \rightarrow U} \text{map}(V \times \mathbb{A}^n, Y) \cong \pi_0 \lim_{V \times \mathbb{A}^n \rightarrow U} \text{map}(V, Y),$$

the last weak equivalence owing to the fact that Y is \mathbb{A}^1 -local. This limit can be computed as $\lim_{V \times \mathbb{A}^n \rightarrow U} \pi_0 \text{map}(V, Y)$ since π_0 commutes with all colimits (being left adjoint to the inclusion of discrete spaces in all spaces). Picking an S -point of \mathbb{A}^n gives a compatible family

$$\lim_{V \rightarrow U} \pi_0 \text{map}(V, Y) \cong \pi_0 \text{map}(U, Y),$$

giving a section of the natural map $\pi_0 \text{map}(U, Y) \rightarrow \lim_{V \times \mathbb{A}^n \rightarrow U} \pi_0 \text{map}(V, Y)$. \square

From this description, we get a number of non-formal consequences.

Corollary 4.28. *The \mathbb{A}^1 -localization functor commutes with finite products.*

Proof. Both $\text{Sing}^{\mathbb{A}^1}$ (being a sifted colimit) and L_{Nis} have this property. For L_{Nis} the fact is clear because it is the left adjoint of a geometric morphism of ∞ -topoi and hence left exact (see [Lur09]), while for $\text{Sing}^{\mathbb{A}^1}$ we refer to [ARV10]. Alternatively, it is easy to check directly that the singular construction commutes with finite products and it is shown in [MV99, Theorem 1.66] that Nisnevich localization commutes with finite products. (Note that since finite products and finite homotopy products agree, it is easy to transfer the Morel-Voevodsky proof along the Quillen equivalences necessary to bring it over to our model for $\text{Spc}_S^{\mathbb{A}^1}$.) \square

The corollary is important in proving that certain functors which are symmetric monoidal on the level of presheaves, remain symmetric monoidal after \mathbb{A}^1 -localization.

Definition 4.29. If $X \in \text{sPre}(\text{Sm}_S)$, then X is \mathbb{A}^1 -**connected** if the canonical map $X \rightarrow S$ induces an isomorphism of sheaves $\pi_0^{\mathbb{A}^1} X \rightarrow \pi_0^{\mathbb{A}^1} S = \star$. We say that X is **naively- \mathbb{A}^1 -connected** if the canonical map $\text{Sing}^{\mathbb{A}^1} X \rightarrow S$ induces an isomorphism $\pi_0^{\text{Nis}} \text{Sing}^{\mathbb{A}^1} X \rightarrow \pi_0^{\text{Nis}} S = \star$.

Corollary 4.30 (Unstable \mathbb{A}^1 -connectivity theorem). *Suppose that X is a simplicial presheaf on Sm_S . The canonical morphism $X \rightarrow L_{\mathbb{A}^1} L_{\text{Nis}} X$ induces an epimorphism $\pi_0^{\text{Nis}} X \rightarrow \pi_0^{\text{Nis}} L_{\mathbb{A}^1} L_{\text{Nis}} X = \pi_0^{\mathbb{A}^1} X$. Hence, if $\pi_0^{\text{Nis}} X = \star$, then X is \mathbb{A}^1 -connected.*

Proof. By Theorem 3.36, it follows that $X \rightarrow L_{\text{Nis}} X$ induces isomorphisms on homotopy sheaves $\pi_0^{\text{Nis}} X \rightarrow \pi_0^{\text{Nis}} L_{\text{Nis}} X$. Hence, using the fact that sheafification preserves epimorphisms and Theorem 4.27, it suffices to show that $\pi_0 X(U) \rightarrow \pi_0 \text{Sing}^{\mathbb{A}^1} X(U)$ is surjective for all $X \in \text{sPre}(\text{Sm}_S)$ and all $U \in \text{Sm}_S$. To do so, we note that $\pi_0 \text{Sing}^{\mathbb{A}^1} X(U)$ is calculated as π_0 of the bisimplicial set $X_\bullet(U \times \Delta^\bullet)$. This is in turn calculated as the coequalizer of the diagram

$$\pi_0 X(U \times_S \mathbb{A}^1) \rightrightarrows \pi_0 X(U),$$

where the maps are induced by Δ^\bullet and thus we get the desired surjection. \square

Consequently, to determine if a simplicial presheaf is \mathbb{A}^1 -connected, it suffices to calculate its sheaf of “naive” \mathbb{A}^1 -connected components. We will use this observation later to prove that SL_n is \mathbb{A}^1 -connected.

Corollary 4.31. *If $X \in \text{sPre}(\text{Sm}_S)$, then the natural morphism $\pi_0^{\text{Nis}} \text{Sing}^{\mathbb{A}^1}(X) \rightarrow \pi_0^{\mathbb{A}^1} X$ is an epimorphism. Hence if X is naively \mathbb{A}^1 -connected, then it is \mathbb{A}^1 -connected.*

Proof. Since the natural map $X \rightarrow \text{Sing}^{\mathbb{A}^1} X$ is an \mathbb{A}^1 -local weak equivalence by Theorem 4.25, we deduce that $L_{\mathbb{A}^1} L_{\text{Nis}} X \simeq L_{\mathbb{A}^1} L_{\text{Nis}} \text{Sing}^{\mathbb{A}^1} X$, so we may apply Theorem 4.30 to $\text{Sing}^{\mathbb{A}^1} X$ to get the desired conclusion. \square

4.4 The sheaf of \mathbb{A}^1 -connected components

The 0-th \mathbb{A}^1 -homotopy sheaf, or the **sheaf of \mathbb{A}^1 -connected components**, admits a simple interpretation: it is the Nisnevich sheafification of the presheaf $U \mapsto [U_+, X]_{\mathbb{A}^1} \simeq [U_+, L_{\mathbb{A}^1} L_{\text{Nis}} X]_s$. With this description, we may perform some calculations whose results deviate from our intuition from topology.

Definition 4.32. Let X be an S -scheme. We say that X is **\mathbb{A}^1 -rigid** if $L_{\mathbb{A}^1} X \simeq X$ in Spc_S . Concretely, this condition amounts to saying that $X(U \times_S \mathbb{A}_S^1) \simeq X(U)$ for any finitely presented smooth S -scheme U .

Exercise 4.33. Let k be a field. Prove that the following k -schemes are all \mathbb{A}^1 -rigid:

1. \mathbb{G}_m ;
2. smooth projective k -curves of positive genus;
3. abelian varieties.

In fact, if S is a reduced scheme of finite Krull dimension, show that \mathbb{G}_m is rigid in $\text{Spc}_S^{\mathbb{A}^1}$.

Proposition 4.34. *Let X be an \mathbb{A}^1 -rigid S -scheme. Then $\pi_0^{\mathbb{A}^1}(X) \simeq X$ as Nisnevich sheaves, and $\pi_n^{\mathbb{A}^1}(X) = 0$ for $n > 0$.*

Proof. The homotopy set $U \mapsto [U, X]_{\mathbb{A}^1} \simeq [U, X]_s = \pi_0(\text{map}_{\text{Spc}_S}(U, X))$ is equivalent to the set of S -scheme maps from U to X as U and X are discrete simplicial sets. Hence this presheaf is equivalent to the presheaf represented by X which is already a Nisnevich sheaf on Sm_S . Now, $[S^n \wedge U_+, X]_s = [S^n, \text{map}_{\text{Spc}_S}(U_+, X)]_{\text{sSets}}$, which is trivial since the target is a discrete simplicial set. Since X is \mathbb{A}^1 -rigid we see that $[S^n \wedge U_+, X]_s \cong [S^n \wedge U_+, X]_{\mathbb{A}^1}$, and thus the sheafification is also trivial. \square

Exercise 4.35. Let $\text{Sm}_S^{\mathbb{A}^1} \hookrightarrow \text{Sm}_S$ be the full subcategory spanned by \mathbb{A}^1 -rigid schemes. Then the natural functor $\text{Sm}_S^{\mathbb{A}^1} \rightarrow \text{Spc}_S^{\mathbb{A}^1}$ which is the composite of $L_{\mathbb{A}^1} : \text{Spc}_S \rightarrow \text{Spc}_S^{\mathbb{A}^1}$ and the Yoneda embedding is fully faithful. In other words, two \mathbb{A}^1 -rigid schemes are isomorphic as schemes if and only if they are \mathbb{A}^1 -equivalent.

4.5 The smash product and the loops-suspension adjunction

Let us begin with some recollection about smash products in simplicial sets. Let $(X, x), (Y, y)$ be two pointed simplicial sets, then we can form the smash product $(X, x) \wedge (Y, y)$ which is defined to be the pushout:

$$\begin{array}{ccc} (X, x) \vee (Y, y) & \longrightarrow & (X, x) \times (Y, y) \\ \downarrow & & \downarrow \\ \star & \longrightarrow & (X, x) \wedge (Y, y) \end{array}$$

The functor $- \wedge (X, x)$ is then a left Quillen endofunctor on the category of simplicial sets by the following argument: if $(Z, z) \rightarrow (Y, y)$ is a cofibration of simplicial sets, then we note that $(X, x) \wedge (Z, z) \rightarrow (X, x) \wedge (Y, y)$ is cofibration since cofibrations are stable under pushouts. The case of acyclic cofibrations is left to the reader. The right adjoint to $- \wedge (X, x)$ is given by the pointed mapping space $\text{map}_\star(X, -)$ which is given by the formula:

$$\text{map}_\star(X, Y)_n \cong \text{Hom}_{\text{sSets}_\star}(X \wedge \Delta_+^n, Y).$$

To promote the smash product to the level of simplicial presheaves, we first take the pointwise smash product, i.e., if $(X, x), (Y, y)$ are objects in $\text{sPre}(\text{Sm}_S)_\star$, then we form the smash product $(X, x) \wedge (Y, y)$ as the simplicial presheaf:

$$U \mapsto (X, x)(U) \wedge (Y, y)(U).$$

An analogous pointwise formula is used for the pointed mapping space functor.

Proposition 4.36. *The Quillen adjunction $(X, x) \wedge - : \text{sPre}(\text{Sm}_S)_\star \rightarrow \text{sPre}(\text{Sm}_S)_\star : \text{map}_\star(X, -)$ descends to a Quillen adjunction: $(X, x) \wedge - : \text{Spc}^{\mathbb{A}^1}(S)_\star \rightarrow \text{Spc}^{\mathbb{A}^1}(S)_\star : \text{map}_\star(X, -)$ and thus there are natural isomorphisms categories:*

$$[(X, x) \wedge^{\mathbf{L}} (Z, z), (Y, y)]_{\mathbb{A}^1} \simeq [(X, x), \mathbf{Rmap}_\star((X, x), (Y, y))]_{\mathbb{A}^1}$$

for $X, Y, Z \in \text{Spc}_S^{\mathbb{A}^1}$.

Proof. The question of whether a monoidal structure defined on the underlying category of a model category descends to a Quillen adjunction with mapping spaces as its right adjoint is answered in the paper of Schwede and Shipley [SS00]. The necessary conditions are checked in [DR03, Section 2.1]. \square

Now, recall that the suspension of (X, x) is calculated either as the homotopy cofiber of the canonical morphism $(X, x) \rightarrow \star$ or, equivalently, as $S^1 \wedge (X, x)$ (check this!), while the loop space is calculated as the homotopy pullback of the diagram

$$\begin{array}{ccc} \Omega(X, x) & \longrightarrow & \star \\ \downarrow & & \downarrow \\ \star & \longrightarrow & (X, x) \end{array}$$

or, equivalently, as $\text{map}_\star(S^1, (Y, y))$. Consequently:

Corollary 4.37. *For any objects $(X, x), (Y, y) \in \text{Spc}^{\mathbb{A}^1}(\text{Sm}_S)_\star$, there is an isomorphism*

$$[\mathbf{L}\Sigma(X, x), (Y, y)]_{\mathbb{A}^1} \simeq [(X, x), \mathbf{R}\Omega(Y, y)]_{\mathbb{A}^1}.$$

4.6 The bigraded spheres

We will now delve into some calculations in \mathbb{A}^1 -homotopy theory. More precisely, these are calculations in the pointed category $\text{Spc}_{S, \star}^{\mathbb{A}^1}$. We use the following conventions for base points of certain schemes which will play a major role in the theory.

1. \mathbb{A}^1 is pointed by 1
2. \mathbb{G}_m is pointed by 1.
3. \mathbb{P}^1 is pointed by ∞ .
4. X_+ denotes X with a disjoint base point for a space X .

In particular, we only write pointed objects as (X, x) when the base points are not the ones indicated above. We also note that the forgetful functors $\text{Spc}_{S, \star} \rightarrow \text{Spc}_S$ and $\text{Spc}_{S, \star}^{\mathbb{A}^1} \rightarrow \text{Spc}_S^{\mathbb{A}^1}$ preserve and detect weak equivalences. Hence when the context is clear, we will say Nisnevich or \mathbb{A}^1 -weak equivalence as opposed to *pointed* Nisnevich or \mathbb{A}^1 -weak equivalence.

Remark 4.38. In many cases, base points of schemes are negotiable in the sense that there is an explicit pointed \mathbb{A}^1 -local weak equivalence between (X, x) and (X, y) for two base points x, y . For example (\mathbb{P}^1, ∞) is \mathbb{A}^1 -equivalent in the pointed category to (\mathbb{P}^1, x) for any other point $x \in \mathbb{P}^1$ via an explicit \mathbb{A}^1 -homotopy.

Of course, if one takes a cofiber of pointed schemes (or even simplicial presheaves), the cofiber is automatically pointed: if $X \rightarrow Y \rightarrow X/Y$ is a cofiber sequence, then X/Y is pointed by the image of Y .

The first calculation one encounters in \mathbb{A}^1 -homotopy theory is the following.

Lemma 4.39. *In $\mathrm{Spc}_{S,\star}^{\mathbb{A}^1}$, there are \mathbb{A}^1 -weak equivalences $\Sigma(\mathbb{G}_m, 1) \simeq (\mathbb{P}^1, \infty) \simeq \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$.*

Proof. Consider the distinguished Nisnevich square

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & (\mathbb{P}^1, 1) \end{array}$$

in Sm_S . By Proposition 4.13, this can be viewed as a homotopy pushout in $\mathrm{Spc}_{S,\star}$ as well. Since the localization functor $\mathrm{Spc}_{S,\star} \rightarrow \mathrm{Spc}_{S,\star}^{\mathbb{A}^1}$ is a Quillen left adjoint, it commutes with homotopy colimits, and in particular with homotopy pushouts. Therefore, when viewed in $\mathrm{Spc}_{S,\star}^{\mathbb{A}^1}$ the square above is a homotopy pushout. However, since $\mathbb{A}^1 \simeq \star$ in the \mathbb{A}^1 -homotopy theory, it follows that $\Sigma\mathbb{G}_m \simeq (\mathbb{P}^1, \infty)$ (by contracting both copies of \mathbb{A}^1 and noting that $(\mathbb{P}^1, 1) \simeq (\mathbb{P}^1, \infty)$) or $\Sigma\mathbb{G}_m \simeq \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$ (by contracting one of the copies of \mathbb{A}^1). \square

The above calculation justifies the idea that in \mathbb{A}^1 -homotopy theory there are two kinds of circles: the simplicial circle S^1 and the ‘‘Tate’’ circle \mathbb{G}_m . The usual convention (which matches up with the grading in motivic cohomology) is to define

$$S^{1,1} = \mathbb{G}_m,$$

and

$$S^{1,0} = S^1.$$

Consequently, by the lemma, we have an \mathbb{A}^1 -weak equivalence $S^{2,1} \simeq \mathbb{P}^1$.

Now, given a pair a, b of non-negative integers satisfying $a \geq b$, we can define $S^{a,b} = \mathbb{G}_m^{\wedge b} \wedge (S^1)^{\wedge(a-b)}$. In general, there is no known nice description of these motivic spheres. However, the next two results give two important classes of exceptions.

Proposition 4.40. *In $\mathrm{Spc}_{S,\star}^{\mathbb{A}^1}$, there are \mathbb{A}^1 -weak equivalences $S^{2n-1,n} \simeq \mathbb{A}^n - \{0\}$ for $n \geq 1$.*

Proof. The case $n = 1$ is Lemma 4.39; we need to do the $n = 2$ case and then perform induction. Specifically, the claim for $n = 2$ says that $\mathbb{A}^2 - \{0\} \simeq S^1 \wedge (\mathbb{G}_m^{\wedge 2})$. First, observe that we have a homotopy push-out diagram:

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \times \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{G}_m \times \mathbb{A}^1 & \longrightarrow & \mathbb{A}^2 - \{0\}, \end{array}$$

from which we conclude that $\mathbb{A}^2 - \{0\}$ is calculated as the homotopy push-out of

$$\mathbb{G}_m \leftarrow \mathbb{G}_m \times \mathbb{G}_m \rightarrow \mathbb{G}_m.$$

On the other hand we may calculate this homotopy push-out using the diagram

$$\begin{array}{ccccc} \star & \longleftarrow & \star & \longrightarrow & \star \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{G}_m & \longleftarrow & \mathbb{G}_m \vee \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{G}_m & \longleftarrow & \mathbb{G}_m \times \mathbb{G}_m & \longrightarrow & \mathbb{G}_m. \end{array}$$

Taking the homotopy push-out across the horizontal rows gives us $\star \leftarrow \star \rightarrow \mathbb{A}^2 - \{0\}$; taking this homotopy push-out gives back $\mathbb{A}^2 - \{0\}$. On the other hand, taking the homotopy push-out across the vertical rows gives us $\star \leftarrow \mathbb{G}_m \wedge \mathbb{G}_m \rightarrow \star$ which calculates the homotopy push-out $S^1 \wedge (\mathbb{G}_m \wedge \mathbb{G}_m)$.

Let us now carry out the induction. We have a distinguished Nisnevich square

$$\begin{array}{ccc} \mathbb{A}^{n-1} - \{0\} \times \mathbb{G}_m & \longrightarrow & \mathbb{A}^n \times \mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathbb{A}^{n-1} - \{0\} \times \mathbb{A}^1 & \longrightarrow & \mathbb{A}^n - \{0\}, \end{array}$$

from which we conclude that $\mathbb{A}^n - \{0\}$ is calculated as the homotopy push-out of

$$\mathbb{A}^{n-1} - \{0\} \leftarrow \mathbb{A}^{n-1} - \{0\} \times \mathbb{G}_m \rightarrow \mathbb{G}_m.$$

Hence we can set-up an analogous diagram:

$$\begin{array}{ccccc} \star & \longleftarrow & \star & \longrightarrow & \star \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{A}^{n-1} - \{0\} & \longleftarrow & \mathbb{A}^{n-1} - \{0\} \vee \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{A}^{n-1} - \{0\} & \longleftarrow & \mathbb{A}^{n-1} - \{0\} \times \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \end{array}$$

to conclude as in the base case that $S^1 \wedge ((\mathbb{A}^{n-1} - \{0\}) \wedge \mathbb{G}_m) \simeq \mathbb{A}^n - \{0\}$. \square

Corollary 4.41. *In $\mathrm{Spc}_{S, \star}^{\mathbb{A}^1}$ there are \mathbb{A}^1 -weak equivalences $\mathbb{A}^n / \mathbb{A}^n - \{0\} \simeq S^n \wedge \mathbb{G}_m \simeq S^{2n, n}$ for $n \geq 1$.*

Proof. The homotopy cofiber of the inclusion $\mathbb{A}^n - 0 \hookrightarrow \mathbb{A}^n$ is calculated as the homotopy pushout

$$\begin{array}{ccc} \mathbb{A}^n - \{0\} & \longrightarrow & \mathbb{A}^n \\ \downarrow & & \downarrow \\ \star & \longrightarrow & \mathbb{A}^n / \mathbb{A}^n - \{0\}. \end{array}$$

In $\mathrm{Spc}_S^{\mathbb{A}^1}$, this cofiber can be calculated as the homotopy pushout

$$\begin{array}{ccc} \mathbb{A}^n - \{0\} & \longrightarrow & \star \\ \downarrow & & \downarrow \\ \star & \longrightarrow & \mathbb{A}^n / \mathbb{A}^n - \{0\}. \end{array}$$

Therefore $\mathbb{A}^n / \mathbb{A}^n - \{0\} \simeq S^1 \wedge (\mathbb{A}^n - \{0\}) \simeq S^1 \wedge (S^{2n-1, n})$ by Proposition 4.40. \square

Remark 4.42. In [ADF], the authors study the question of when the motivic sphere $S^{a, b}$ is \mathbb{A}^1 -weak equivalent to a smooth scheme. Proposition 4.40 shows that this is the case for $S^{2n-1, n} \simeq \mathbb{A}^n - \{0\}$. Asok, Doran, and Fasel prove that it is also the case for $S^{2n, n}$, which they show is \mathbb{A}^1 -weak equivalent to the affine quadric with coordinate ring

$$k[x_1, \dots, x_n, y_1, \dots, y_n, z] / \left(\sum_i x_i y_i - z(1+z) \right)$$

when $S = \mathrm{Spec} k$ for a commutative ring k . They also show that $S^{a, b}$ is *not* \mathbb{A}^1 -weak equivalent to a smooth affine scheme if $a > 2b$. Conjecturally, the only motivic spheres $S^{a, b}$ admitting smooth models are those above, when $(a, b) = (2n-1, n)$ or $(a, b) = (2n, n)$.

Remark 4.43. If we impose only Nisnevich descent rather than Nisnevich hyperdescent, the results in this section remain true. This might provide one compelling reason to do so. For details, see [AHW15a].

4.7 Affine and projective bundles

Proposition 4.44. *Let $p : E \rightarrow X$ be a Nisnevich-locally trivial affine space bundle. Then, $E \rightarrow X$ is an \mathbb{A}^1 -weak equivalence.*

Proof. Pick a Zariski cover $\mathcal{U} := \{U_\alpha\}$ of X that trivializes E . Suppose that $\check{C}(U)_\bullet$ is the Čech nerve of the cover, then we have a weak equivalence

$$\mathrm{hocolim}_{\Delta^{op}} \check{C}(U)_\bullet \simeq X$$

in Spc_S and an \mathbb{A}^1 -weak equivalence

$$\mathrm{hocolim}_{\Delta^{op}} \check{C}(U)_\bullet \times_X \mathbb{A}^n \simeq E$$

in $\mathrm{Spc}_S^{\mathbb{A}^1}$. But now, we have an levelwise- \mathbb{A}^1 -weak equivalence of simplicial objects

$$\check{C}(U)_\bullet \times_X \mathbb{A}^n \rightarrow \check{C}(U)_\bullet.$$

Hence, the homotopy colimits are equivalent by construction. \square

Note that the above proposition covers a larger class of morphisms than just vector bundles $p : E \rightarrow X$. For these, the homotopy inverse of the projection map is the zero section as per Exercise 3.62.

We obtain immediate applications of this proposition in the form of certain presentations of $\mathbb{A}^n - 0$ in terms of a homogeneous space and an affine scheme in the \mathbb{A}^1 -homotopy category. We leave the proofs to the reader.

Corollary 4.45. *Let $n \geq 2$, and let $\mathrm{SL}_n \rightarrow \mathbb{A}^n - \{0\}$ be the map defined by taking the last column of a matrix in SL_n . In $\mathrm{Spc}_S^{\mathbb{A}^1}$, this map factors through the cofiber $\mathrm{SL}_n \rightarrow \mathrm{SL}_n / \mathrm{SL}_{n-1}$, and the map $\mathrm{SL}_n / \mathrm{SL}_{n-1} \rightarrow \mathbb{A}^n - \{0\}$ is an \mathbb{A}^1 -weak equivalence.*

Corollary 4.46. *Let S be the spectrum of a field, and give \mathbb{A}^{2n} the coordinates $x_1, \dots, x_n, y_1, \dots, y_n$. Consider the quadric $Q_{2n-1} = V(x_1y_1 + \dots + x_ny_n = 1)$. The map $Q_{2n-1} \rightarrow \mathbb{A}^n - \{0\}$ induced by the projection to the x -coordinates is an \mathbb{A}^1 -weak equivalence.*

Note that some authors might write Q_{2n} for what we have written Q_{2n-1} .

Now we will use the above proposition to deduce results about projective bundles. We will then recover yet another presentation of the spheres $S^{2n,n}$. Furthermore we will also introduce an important construction on vector bundles that we will encounter later.

Definition 4.47. If $\nu : E \rightarrow X$ is a vector bundle, then the Thom space $\mathrm{Th}(\nu)$ of E (sometimes also written $\mathrm{Th}(E)$) is defined as the cofiber

$$E/(E - X),$$

where the embedding of X into E is given by the zero section.

The Thom space construction plays a central role in algebraic topology and homotopy theory, and is intimately wrapped up in computations of the bordism ring for manifolds and in the representation of homology classes by manifolds [Tho54].

Example 4.48. Let S be a base scheme, then $\mathbb{A}_S^n \rightarrow S$ is a trivial vector bundle over S . The Thom space of the trivial rank n vector bundle is then by definition $\frac{\mathbb{A}^n}{\mathbb{A}^n - \{0\}}$. From Proposition 4.41 we conclude that Thom space in this case is given by $S^{2n,n}$.

In topology, one has a weak homotopy equivalence: $\mathbb{C}\mathbb{P}^n/\mathbb{C}\mathbb{P}^{n-1} \simeq S^{2n}$, thanks to the standard cell decomposition of projective space. One of the benefits of having this decomposition is that for a suitable class of generalized cohomology theories, the complex orientable theories, there exists a theory of Chern classes similar to the theory in ordinary cohomology. We would like a similar story in \mathbb{A}^1 -homotopy theory, and this indeed exists.

Exercise 4.49. Let $E \rightarrow X$ be a trivial rank n vector bundle, then there is an \mathbb{A}^1 -weak equivalence: $\mathrm{Th}(E) \simeq \mathbb{P}^{1 \wedge n} \wedge X_+$. Hint: use Corollary 4.41.

Proposition 4.50. *Suppose that $E \rightarrow X$ is a vector bundle and $\mathbb{P}(E) \rightarrow \mathbb{P}(E \oplus \mathcal{O})$ is the closed embedding at infinity. Then, there is an \mathbb{A}^1 -weak equivalence*

$$\frac{\mathbb{P}(E \oplus \mathcal{O})}{\mathbb{P}(E)} \rightarrow \mathrm{Th}(E).$$

Proof. Throughout, X is identified with its zero section for ease of notation. Observe that we have a morphism $X \rightarrow E \rightarrow \mathbb{P}(E \oplus \mathcal{O})$ where the first map is the closed embedding of X via the zero section and the second map is the embedding complementary to the embedding $\mathbb{P}(E) \rightarrow \mathbb{P}(E \oplus \mathcal{O})$ at infinity. We also identify X in $\mathbb{P}(E \oplus \mathcal{O})$ via this embedding. Hence, there is an elementary distinguished square

$$\begin{array}{ccc} E - X & \longrightarrow & \mathbb{P}(E \oplus \mathcal{O}) - X \\ \downarrow & & \downarrow \\ E & \longrightarrow & \mathbb{P}(E \oplus \mathcal{O}), \end{array}$$

which means that we have an weak equivalence of simplicial presheaves:

$$\mathrm{Th}(E) \simeq \frac{\mathbb{P}(E \oplus \mathcal{O})}{\mathbb{P}(E \oplus \mathcal{O}) - X}.$$

We have a map $\frac{\mathbb{P}(E \oplus \mathcal{O})}{\mathbb{P}(E)} \rightarrow \frac{\mathbb{P}(E \oplus \mathcal{O})}{\mathbb{P}(E \oplus \mathcal{O}) - X}$ because $\mathbb{P}(E)$ avoids the embedding of X described above. This is the map that we want to be an \mathbb{A}^1 -weak equivalence, so it suffices to prove that we have an \mathbb{A}^1 -weak equivalence $\mathbb{P}(E) \rightarrow \mathbb{P}(E \oplus \mathcal{O}) - X$. The lemma below shows that the map is indeed the zero section of an affine bundle, and so we are done by Proposition 4.44. \square

Lemma 4.51. *Let X be a scheme, $p : E \rightarrow X$ a vector bundle with s its zero section. Consider the open embedding $j : E \rightarrow \mathbb{P}(E \oplus \mathcal{O}_X)$ and its closed complement $i : \mathbb{P}(E) \rightarrow \mathbb{P}(E \oplus \mathcal{O}_X)$. In this case, there is a morphism*

$$q : \mathbb{P}(E \oplus \mathcal{O}_X) \setminus j(s(X)) \rightarrow \mathbb{P}(E)$$

such that $q \circ i = \mathrm{id}$ and q is an \mathbb{A}^1 -bundle over $\mathbb{P}(E)$.

Proof. Recall that to give a morphism $T \rightarrow \mathbb{P}(E)$ over X , one must give a morphism $h : T \rightarrow X$ and a surjection $h^*(E) \rightarrow \mathcal{L}$ where \mathcal{L} is a line bundle on T . Now $\bar{p} : \mathbb{P}(E \oplus \mathcal{O}_X) \rightarrow X$ has a universal line bundle \mathcal{L}_{univ} and a universal quotient map $\bar{p}^*(E \oplus \mathcal{O}_X) \simeq \bar{p}^*(E) \oplus \mathcal{O}_{\mathbb{P}(E \oplus \mathcal{O}_X)} \rightarrow \mathcal{L}_{univ}$. Restricting to the first factor gives us a map $\bar{p}^*(E) \rightarrow \mathcal{L}_{univ}$ and hence a rational map $t : \mathbb{P}(E \oplus \mathcal{O}_X) \dashrightarrow \mathbb{P}(E)$. Over a point of X , t is given by projection onto E -coordinates. Hence, this map is well-defined away from $j(s(X))$, so we get a morphism $q : \mathbb{P}(E \oplus \mathcal{O}_X) \setminus j(s(X)) \rightarrow \mathbb{P}(E)$; by construction $q \circ i = \mathrm{id}_{\mathbb{P}(E)}$. To check the last claim, since it is local on the base, one may assume that $E \cong \mathcal{O}_X^{n+1}$, so we are looking at $\mathbb{P}_X^{n+1} \setminus X \rightarrow \mathbb{P}^n$. In coordinates X embeds as $[0 : \dots : 0 : 1]$, and the map is projection onto the first n coordinates, which is an \mathbb{A}^1 -bundle. \square

Corollary 4.52. *There are \mathbb{A}^1 -weak equivalences $\mathbb{P}^n/\mathbb{P}^{n-1} \simeq S^{2n,n}$ for $n \geq 1$ when S is noetherian of finite Krull dimension.*

5 Classifying spaces in \mathbb{A}^1 -homotopy theory

One of the main takeaways from Section 2 is that one can go very far using homotopical methods to study topological vector bundles on CW complexes. The key inputs in this technique are the existence of the Postnikov tower and knowledge of the homotopy groups of the classifying spaces BGL_n in low degrees. In this section, we will give a sampler of the techniques involved in accessing the \mathbb{A}^1 -homotopy sheaves of the classifying spaces BGL_n . In the end will identify a “stable” range for these homotopy sheaves, which will naturally lead us to a discussion of algebraic K -theory in the next section.

As usual, S is a quasi-compact and quasi-separated unless stated otherwise.

5.1 Simplicial models for classifying spaces

Definition 5.1. Let τ be a topology (typically this will be Zariski, Nisnevich or étale) on Sm_S , and let G a τ -sheaf of groups. A τ - G -torsor over $X \in \mathrm{Sm}_S$ is the data of a τ -sheaf of sets P on Sm_S , a right action $a : P \times G \rightarrow P$ of G on P , and a G -equivariant morphism $\pi : P \rightarrow X$ (where X has the trivial G -action) such that

1. the morphism $(\pi, a) : P \times G \rightarrow P \times_X P$ is an isomorphism, and
2. there exists a τ -cover $\{U_i \rightarrow X\}_{i \in I}$ of X such that $U_i \times_X P \rightarrow U_i$ has a section for all $i \in I$.

Let G be a τ -sheaf of groups. Consider the simplicial presheaf EG described section-wise in the following way: $EG_n(U) =: G(U)^{\times n+1}$ with the usual faces and degeneracies. We write $E_\tau G$ as a fibrant replacement in the model category $L_\tau(\mathrm{sPre}(\mathrm{Sm}_S))$.

Proposition 5.2. *There is a weak equivalence $E_\tau G \simeq \star$ in $L_\tau \mathrm{sPre}(\mathrm{Sm}_S)$.*

Proof. The fact that each $EG(U)$ is contractible is standard: the diagonal morphism: $G(U) \rightarrow G(U) \times G(U)$ produces an extra degeneracy. See Goerss and Jardine [GJ99, Lemma III.5.1]. Thus, $EG \rightarrow \star$ is a weak equivalence in $\mathrm{sPre}(\mathrm{Sm}_S)$. Since localization (τ -sheafification) preserves weak equivalences, it follows that $E_\tau G$ is contractible. \square

There is a right G -action on EG by letting G act on the last coordinate in each simplicial degree. The level-wise quotient is the simplicial presheaf we christen BG . We write $B_\tau G$ for a fibrant replacement in the model category $L_\tau \mathrm{sPre}(\mathrm{Sm}_S)$. We would like to make sense of $B_\tau G$ as a simplicial presheaf classifying τ - G -torsors.

Definition 5.3. Let $\mathrm{BTors}_\tau(G)$ be the simplicial presheaf which assigns to $U \in \mathrm{Sm}_S$ the nerve of the groupoid of G -torsors on U and to a morphism $f : U' \rightarrow U$ a map of simplicial presheaves $\mathrm{BTors}_\tau(G)(U) \rightarrow \mathrm{BTors}_\tau(G)(U')$ induced by pullback.

Remark 5.4. The above definition is valid by the work of Hollander [Hol08, Section 3]; the functor that assigns to U the nerve of the groupoid of G -torsors over U does not have strictly functorial pullbacks and thus one needs to appeal to some rectification procedure.

The following proposition is well known.

Proposition 5.5. *The simplicial presheaf $\mathrm{BTors}_\tau(G)$ is τ -local.*

Proof. This follows from the local triviality condition and the fact that we can construct τ - G -torsors by gluing; see, for example, [Vis05]. \square

Let $U \in \mathrm{Sm}_S$, we denote by $H_\tau^1(U, G)$ be the (non-abelian) cohomology set of τ - G -bundles on U . More precisely, we set

$$H_\tau^1(U, G) = \pi_0(\mathrm{BTors}_\tau(G)(U)).$$

Proposition 5.6. *Let G be a τ -sheaf of groups, then there is a natural weak equivalence*

$$B_\tau G \rightarrow \mathrm{BTors}_\tau(G).$$

Hence, for all $U \in \mathrm{Sm}_S$, there is a natural isomorphism $\pi_0(B_\tau G(U)) \cong H_\tau^1(U; G)$ and a natural weak equivalence $\mathbf{R}\Omega B_\tau G(U) \simeq G(U)$.

Proof. A proof is given in [MV99, section 4.1], we also recommend [AHW15b, Lemma 2.2.2] and the references therein. Let us sketch the main ideas. To define a map to $\mathrm{BTors}_\tau(G)$, we can first define a map $BG \rightarrow B\mathrm{Tors}_\tau(G)$ of presheaves and then use the fact that the target is τ -local to get a map $B_\tau G \rightarrow B\mathrm{Tors}_\tau(G)$. The former map is given by sending the unique vertex of $BG(U)$ to the trivial G -torsor over U . Since G -torsors with respect to τ are τ -locally trivial, we conclude that the map must be a τ -local weak equivalence. The fact that $B_\tau G$ is fibrant is by definition, and for $B_\tau G$ it follows from [AHW15b, Lemma 2.2.2]. The second part of the assertion then follows by definition, and the standard fact that loop space of the nerve of a groupoid is homotopy equivalent to the automorphism group of a fixed object. \square

Many interesting objects in algebraic geometry, such as Azumaya algebras and the associated PGL_n -torsors, are only étale locally trivial. The classifying spaces of these torsors are indeed objects of \mathbb{A}^1 -homotopy theory as we shall explain. We can consider $\mathrm{Sm}_{S,\acute{\mathrm{e}}\mathrm{t}}$, the full subcategory of the big étale site over S spanned by smooth S -schemes. Completely analogous to the Nisnevich case, one can develop étale- \mathbb{A}^1 -homotopy theory by the formula $\mathrm{Spc}_{S,\acute{\mathrm{e}}\mathrm{t}}^{\mathbb{A}^1} = \mathrm{L}_{\mathbb{A}^1} \mathrm{L}_{\acute{\mathrm{e}}\mathrm{t}} \mathrm{sPre}(\mathrm{Sm}_S)$.

Theorem 5.7. *The morphism of sites: $\pi : \mathrm{Sm}_{S,\acute{\mathrm{e}}\mathrm{t}} \rightarrow \mathrm{Sm}_{S,\mathrm{Nis}}$ induced by the identity functor induces a Quillen pair*

$$\pi^* : \mathrm{Spc}_S^{\mathbb{A}^1} \rightleftarrows \mathrm{Spc}_{S,\acute{\mathrm{e}}\mathrm{t}}^{\mathbb{A}^1} : \pi_*$$

and hence an adjunction

$$\mathbf{L}\pi^* : \mathrm{Ho}(\mathrm{Spc}_S^{\mathbb{A}^1}) \rightleftarrows \mathrm{Ho}(\mathrm{Spc}_{S,\acute{\mathrm{e}}\mathrm{t}}^{\mathbb{A}^1}) : \mathbf{R}\pi_*$$

on the level of homotopy categories.

Proof. Since our categories are constructed via Bousfield-localization of $\mathrm{sPre}(\mathrm{Sm}_S)$, the universal property tells us that to define a Quillen pair

$$\pi^* : \mathrm{Spc}_S^{\mathbb{A}^1} \rightleftarrows \mathrm{Spc}_{S,\acute{\mathrm{e}}\mathrm{t}}^{\mathbb{A}^1} : \pi_*$$

it suffices to define a Quillen pair:

$$\pi^* : \mathrm{sPre}(\mathrm{Sm}_S) \rightleftarrows \mathrm{Spc}_{S,\acute{\mathrm{e}}\mathrm{t}}^{\mathbb{A}^1} : \pi_*$$

such that $\pi^*(i)$ is a weak equivalence for i belonging to the class of Nisnevich hypercovers and \mathbb{A}^1 -weak equivalences. However, the model category $\mathrm{Spc}_{S,\acute{\mathrm{e}}\mathrm{t}}^{\mathbb{A}^1}$ is also constructed via Bousfield localization, so we use the Quillen pair from this Bousfield localization. But, it is clear that the identity functor $\pi^* : \mathrm{sPre}(\mathrm{Sm}_S) \rightarrow \mathrm{sPre}(\mathrm{Sm}_S)$ takes Nisnevich hypercovers to étale hypercovers and the morphisms $X \times_S \mathbb{A}^1 \rightarrow X$ to $X \times_S \mathbb{A}^1$. Hence, the Quillen pair exists by the universal property of Bousfield localization. \square

Proposition 5.8. *There are natural isomorphisms of Nisnevich sheaves $\pi_0^{\mathrm{Nis}}(\mathbf{R}\pi_* B_{\acute{\mathrm{e}}\mathrm{t}} G) \simeq H_{\acute{\mathrm{e}}\mathrm{t}}^1(-; G)$ and $\pi_1^{\mathrm{Nis}}(\mathbf{R}\pi_* B_{\acute{\mathrm{e}}\mathrm{t}} G) \simeq G$, where the étale sheaf of groups G is considered as a Nisnevich sheaf.*

Proof. By adjunction, $[U, \mathbf{R}\pi_* B_{\acute{\mathrm{e}}\mathrm{t}} G]_{\mathbb{A}^1} \simeq [\mathbf{L}\pi^* U, B_{\acute{\mathrm{e}}\mathrm{t}} G]_{\mathbb{A}^1} \cong H_{\acute{\mathrm{e}}\mathrm{t}}^1(U, G)$. To see the π_1 -statement, we note that $\mathbf{R}\pi_*$ is a right Quillen functor and hence commutes with homotopy limits. Since the loop space is calculated via a homotopy limit, we have that $\Omega \mathbf{R}\pi_* B_{\acute{\mathrm{e}}\mathrm{t}} G \simeq \mathbf{R}\pi_* \Omega B_{\acute{\mathrm{e}}\mathrm{t}} G \simeq \mathbf{R}\pi_* G$, as desired. \square

Example 5.9. Let $G = \mathrm{GL}_n, \mathrm{SL}_n$ or Sp_{2n} ; these are the special groups in the sense of Serre. In this case, any étale- G -torsor is also a Zariski-locally trivial and hence a Zariski- G -torsor (or a Nisnevich- G -torsor). One way to say this in our language is to consider the Quillen adjunction

$$\pi^* : \mathrm{L}_{\mathrm{Nis}}(\mathrm{sPre}(\mathrm{Sm}_S)) \rightleftarrows \mathrm{L}_{\mathrm{ét}}(\mathrm{sPre}(\mathrm{Sm}_S)) : \mathbf{R}\pi_*$$

Then there is a unit map $\mathrm{B}_{\mathrm{Nis}}G \rightarrow \mathbf{R}\pi_*\pi^*\mathrm{B}_{\mathrm{ét}}G$, which is an weak equivalence in the cases above.

5.2 Some calculations with classifying spaces

We are now interested in the \mathbb{A}^1 -homotopy sheaves of classifying spaces. The first calculation is a direct consequence of the unstable- \mathbb{A}^1 -0-connectivity theorem. We work over an arbitrary Noetherian base in this section, unless specified otherwise.

Proposition 5.10. *If G is a Nisnevich sheaf of groups, then $\pi_0^{\mathbb{A}^1}(BG) = \star$.*

Proof. By Theorem 4.30, it suffices to prove that $\pi_0^{\mathrm{Nis}}(BG)$ is trivial. Note that this is the sheafification of the functor $U \mapsto \mathrm{H}_{\mathrm{Nis}}^1(U, G)$. The claim follows from the fact that we are considering G -torsors which are Nisnevich-locally trivial. \square

Remark 5.11. Let G be an étale sheaf of groups. If we replace BG by $\mathbf{R}\pi_*\mathrm{B}_{\mathrm{ét}}G$, then the above result will *not* hold unless étale G -torsors are also Nisnevich locally trivial. This is not the case for example for PGL_n . For more about $\mathrm{B}_{\mathrm{ét}}\mathrm{PGL}_n$, see [Aso13, Corollary 3.16].

In order to proceed further, we need a theorem of Asok-Hoyois-Wendt [AHW15b].

Theorem 5.12 ([AHW15b]). *If $X \rightarrow Y \rightarrow Z$ is a fiber sequence in $\mathrm{sPre}(\mathrm{Sm}_S)$ such that Z satisfies affine Nisnevich excision and $\pi_0(Z)$ satisfies affine \mathbb{A}^1 -invariance, then $X \rightarrow Y \rightarrow Z$ is an \mathbb{A}^1 -fiber sequence.*

Corollary 5.13. *If $\mathrm{H}_{\mathrm{Nis}}^1(-, G)$ is \mathbb{A}^1 -invariant, then the sequence $G \rightarrow EG \rightarrow BG$ is an \mathbb{A}^1 -fiber sequence.*

From now on to the end of this section, we will need the base scheme to be a field (although we can do better — see the discussions in [AHW15b]) in order to utilize \mathbb{A}^1 -invariance of various cohomology sets and apply Theorem 5.12 above. As a first example, we let T be a split torus over a field k .

Proposition 5.14. *Let T be a split torus over a field k . If $P \rightarrow X$ is a T -torsor with a k -point $x : \mathrm{Spec} k \rightarrow P$, then we have a short exact sequence*

$$1 \rightarrow \pi_1^{\mathbb{A}^1}(P, x) \rightarrow \pi_1^{\mathbb{A}^1}(X, x) \rightarrow T.$$

Proof. We need to check that $\pi_0(BT)$ is \mathbb{A}^1 -invariant. Recall that a split torus over a field simply means that it is isomorphic over k to products of \mathbb{G}_m , and so $\pi_0(BT) \cong \mathrm{Pic}(-)^{\oplus n}$ where n is the number of copies of \mathbb{G}_m . Therefore it is indeed \mathbb{A}^1 -invariant on smooth k -schemes. This shows that T is an \mathbb{A}^1 -rigid scheme over k , hence $\pi_0^{\mathbb{A}^1}(T) \simeq T$ and the higher homotopy groups are zero by Proposition 4.34, giving us the short exact sequence above. \square

Remark 5.15. The result is true in greater generality for not-necessarily-split tori with some assumptions on the base field, see [Aso11] for details.

5.3 BGL and BSL

In our classification of vector bundles, on affine schemes, we need to calculate the homotopy sheaves of BGL_n . We use the machinery above to highlight two features of this calculation. First, just like in topology, we may reduce the calculation of homotopy sheaves of BGL_n to that of BSL_n , save for π_1 . Secondly, the \mathbb{A}^1 -homotopy sheaves of BSL_n stabilize: for each i , $\pi_i^{\mathbb{A}^1}(\mathrm{BSL}_n)$ is independent of the value of n as n tends to ∞ .

Proposition 5.16. *Let S be a regular noetherian affine scheme of finite Krull dimension, and suppose that the Bass-Quillen conjecture holds for smooth schemes of finite presentation over S . The space SL_n in $\mathrm{Spc}_S^{\mathbb{A}^1}$ is \mathbb{A}^1 -connected and BSL_n is \mathbb{A}^1 -1-connected, i.e. $\pi_1^{\mathbb{A}^1}(\mathrm{BSL}_n) = \star$.*

Proof. We show that the sheaf $\pi_0^{\mathbb{A}^1}(\mathrm{SL}_n)$ is trivial by showing that the stalks of $\pi_0^{\mathbb{A}^1}(\mathrm{SL}_n)$ are trivial. To show this it suffices by Theorem 4.31 to show that for any henselian local ring R ,

$$[\mathrm{Spec} R, \mathrm{Sing}^{\mathbb{A}^1}(\mathrm{SL}_n)]_s = \star$$

(i.e. the set of naive \mathbb{A}^1 -homotopy classes is trivial), where we view $\mathrm{Spec} R$ as an object of $\mathrm{Spc}_S^{\mathbb{A}^1}$ via the functor of points it represents.

In fact we will prove the above claim for R , any local ring. We want to connect any matrix $M \in \mathrm{SL}_n(R)$ to the identity via a chain of naive \mathbb{A}^1 -homotopies. Let \mathfrak{m} be the maximal ideal of R , and let $k = R/\mathfrak{m}$ be the residue field. The subgroup $E_n(k) \subseteq \mathrm{SL}_n(k)$ generated by the elementary matrices is actually all of $\mathrm{SL}_n(k)$, so we can write \bar{M} , the image of M in $\mathrm{SL}_n(k)$, as a product of elementary matrices. Recall that an elementary matrix in $\mathrm{SL}_n(k)$ is the identity matrix except for a single off-diagonal entry. Since we can lift each of these to $\mathrm{SL}_n(R)$, we can write $M = EN$, where E is a product of elementary matrices in $\mathrm{SL}_n(R)$ and

$$N = I_n + P,$$

where $P = (p_{ij}) \in M_n(\mathfrak{m})$ is a matrix with entries in \mathfrak{m} . Note that the condition that $N \in \mathrm{SL}_n(R)$ means that we can solve for p_{11} . Indeed,

$$1 = \det(N) = (1 + p_{11})|C_{11}| - p_{12}|C_{12}| + \cdots + (-1)^n p_{1n}|C_{1n}|,$$

where C_{ij} is the ij th minor of N . Each p_{1r} is in \mathfrak{m} for $2 \leq r \leq n$. Hence, $1 - n = (1 + p_{11})|C_{11}|$, where $n \in \mathfrak{m}$. Since $1 - n$ and $1 + p_{11}$ are units, $|C_{11}|$ must be a unit in R as well. Thus, we can solve

$$p_{11} = \frac{1 - n}{|C_{11}|} - 1.$$

Now, define a new matrix $Q = (q_{ij})$ in $M_n(\mathfrak{m}[t])$ by $q_{ij} = tp_{ij}$ unless $(i, j) = (1, 1)$, in which case set q_{11} so that $\det(1 + Q) = 1$, using the formula above. Then, we see that $Q(0) = I_n$, while $Q(1) = P$. It follows that $1 + Q$ defines an explicit homotopy from I_n to $N = I_n + P$. It follows that M is \mathbb{A}^1 -homotopic to a product of elementary matrices. Since each elementary matrix is \mathbb{A}^1 -homotopic to I_n , we have proved the claim.

Now, by Theorem 5.12, $\mathrm{SL}_n \rightarrow \mathrm{ESL}_n \rightarrow \mathrm{BSL}_n$ is an \mathbb{A}^1 -fiber sequence due to the fact that SL_n -torsors are \mathbb{A}^1 -invariant on smooth affine schemes (since GL_n -torsors are \mathbb{A}^1 -invariant on smooth affine schemes). Therefore we have an exact sequence:

$$\pi_1^{\mathbb{A}^1}(\mathrm{ESL}_n) \rightarrow \pi_1^{\mathbb{A}^1}(\mathrm{BSL}_n) \rightarrow \pi_0^{\mathbb{A}^1}(\mathrm{SL}_n).$$

The left term is \star since ESL_n is simplicially (and hence \mathbb{A}^1 -)contractible and the right term is a singleton due to the first part of this proposition. \square

Exercise 5.17. Prove the following statements when S satisfies the hypotheses of the previous theorem. For $i > 2$, $\pi_i^{\mathbb{A}^1}(\mathrm{BG}_m) = 0$. For $i = 1$, the sheaf of groups $\pi_i^{\mathbb{A}^1}(\mathrm{BG}_m) \cong \mathbb{G}_m$. Finally, $\pi_0^{\mathbb{A}^1}(\mathrm{BG}_m) = \star$. Hint: use the \mathbb{A}^1 -rigidity of \mathbb{G}_m and Theorem 5.12.

Proposition 5.18. *Let S be a regular noetherian affine scheme of finite Krull dimension, and suppose that the Bass-Quillen conjecture holds for smooth schemes of finite presentation over S . For $i > 1$, the map $\mathrm{SL}_n \rightarrow \mathrm{GL}_n$ induces an isomorphism $\pi_i^{\mathbb{A}^1}(\mathrm{BSL}_n) \rightarrow \pi_i^{\mathbb{A}^1}(\mathrm{BGL}_n)$.*

Proof. By Theorem 5.12, the sequence $\mathrm{BSL}_n \rightarrow \mathrm{BGL}_n \rightarrow \mathrm{BG}_m$ induces a long exact sequence of \mathbb{A}^1 -homotopy sheaves and the result for $i > 1$ follows from the above proposition above. However we note that the case of $\pi_1^{\mathbb{A}^1}$ is different: we have an exact sequence $\pi_1^{\mathbb{A}^1}(\mathrm{BSL}_n) \rightarrow \pi_1^{\mathbb{A}^1}(\mathrm{BGL}_n) \rightarrow \pi_1^{\mathbb{A}^1}(\mathrm{BG}_m) \rightarrow \pi_0^{\mathbb{A}^1}(\mathrm{BSL}_n)$; The groups on the right are zero by Proposition 5.10, and the group on the left is zero by Proposition 5.16. \square

Recall from Corollary 4.45 that we have an \mathbb{A}^1 -weak equivalence: $\mathrm{SL}_{n+1}/\mathrm{SL}_n \rightarrow \mathbb{A}^{n+1} - \{0\}$ for $n \geq 1$. Moreover, $\mathbb{A}^{n+1} - \{0\}$ is \mathbb{A}^1 -weak equivalent to $(S^1)^{\wedge n} \wedge \mathbb{G}_m^{\wedge n+1}$. Our intuition from topology suggests therefore that $\mathrm{SL}_{n+1}/\mathrm{SL}_n$ should be $(n-1)$ - \mathbb{A}^1 -connected. This is indeed the case but it relies on a difficult theorem of Morel, the unstable \mathbb{A}^1 -connectivity theorem [Mor12, Theorem 6.38]. That theorem uses an \mathbb{A}^1 -homotopy theoretic version of Hurewicz theorem and of \mathbb{A}^1 -homology sheaves, which are defined not by pointwise sheafification but instead using the so-called \mathbb{A}^1 -derived category.

We may apply Theorem 5.12 to the fiber sequence of simplicial presheaves: $\mathrm{SL}_{n+1}/\mathrm{SL}_n \rightarrow \mathrm{BSL}_n \rightarrow \mathrm{BSL}_{n+1}$ to see that this is also an \mathbb{A}^1 -fiber sequence. We have thus proved the following important stability result.

Theorem 5.19 (Stability). *Let S be a regular noetherian affine scheme of finite Krull dimension, and suppose that the Bass-Quillen conjecture holds for smooth schemes of finite presentation over S . Let $i > 0$ and $n \geq 1$. The morphism*

$$\pi_i^{\mathbb{A}^1}(\mathrm{BSL}_n) \rightarrow \pi_i^{\mathbb{A}^1}(\mathrm{BSL}_{n+1})$$

is an epimorphism if $i \leq n$ and an isomorphism if $i \leq n - 1$.

Setting $\mathrm{GL} = \mathrm{colim}_{n \rightarrow \infty} \mathrm{GL}_n$ and similarly for SL , we obtain the following corollary.

Corollary 5.20. *Let S be a regular noetherian affine scheme of finite Krull dimension, and suppose that the Bass-Quillen conjecture holds for smooth schemes of finite presentation over S . For $i \geq 2$, we have $\pi_i^{\mathbb{A}^1}(\mathrm{BSL}) \simeq \pi_i^{\mathbb{A}^1}(\mathrm{BGL})$.*

6 Representing algebraic K -theory

One reason to contemplate the \mathbb{A}^1 -homotopy category is the fact that many invariants of schemes are \mathbb{A}^1 -invariant; one important example is algebraic K -theory, at least for regular schemes. The goal of this section is to prove the representability of algebraic K -theory in \mathbb{A}^1 -homotopy theory and identify its representing space when the base scheme S is regular and noetherian. One important consequence is an identification of the \mathbb{A}^1 -homotopy sheaves of the classifying spaces BGL_n 's in the stable range, which plays a crucial role in the classification of algebraic vector bundles via \mathbb{A}^1 -homotopy theory. Indeed, it turns out that the relationship between algebraic K -theory and these classifying spaces is just like what happens in topology — the latter assembles into a representing space for the former. This is perhaps a little surprising as one way to define the algebraic K -theory of rings is via the complicated $+$ -construction which alters the homotopy type of $\mathrm{BGL}_n(R)$ rather drastically. The key insight is that the $\mathrm{Sing}^{\mathbb{A}^1}$ construction is an alternative to the $+$ -construction in nice cases, which leads to the identification of the representing space in $\mathrm{Spc}^{\mathbb{A}^1}(S)_*$.

Throughout this section, we let S be a fixed *regular* noetherian scheme of finite Krull dimension. An argument using Weibel's homotopy invariant K -theory will yield a similar result over an arbitrary noetherian base, but for homotopy K -theory; for details see [Cis13].

6.1 Representability of algebraic K -theory

The first thing we note is that representability of algebraic K -theory in the \mathbb{A}^1 -homotopy category itself is a formal consequence of basic properties of algebraic K -theory.

Proposition 6.1. *Let S be a regular noetherian scheme of finite Krull dimension. Then, the K -theory space functor \mathcal{K} is a fibrant object of $\mathrm{Spc}_{S,\star}^{\mathbb{A}^1}$. In particular, there are natural isomorphisms*

$$K_i(X) \cong [\Sigma_+^i X, \mathcal{K}]_{\mathbb{A}^1}$$

for all finitely presented smooth S -schemes X and all $i \geq 0$.

Proof. It is enough to show that \mathcal{K} is an \mathbb{A}^1 -local object of $\mathrm{sPre}(\mathrm{Sm}_S)_\star$. For this, we must show that \mathcal{K} is both a Nisnevich-local object and satisfies \mathbb{A}^1 -homotopy invariance. The first property follows from [TT90, Proposition 6.8]. The second property is proved in [TT90, Theorem 10.8] for the K -theory spectra. Since $\mathbf{R}\Omega^\infty$ is a Quillen right adjoint, it preserves homotopy limits, and hence \mathcal{K} also satisfies descent. \square

Therefore algebraic K -theory is indeed representable in $\mathrm{Spc}_{S,\star}^{\mathbb{A}^1}$ by an object which we denote by \mathcal{K} . This argument is purely formal. Next, we need to get a better grasp of the representing object \mathcal{K} . To do so, we need some review on H -spaces.

Definition 6.2. Let X be a simplicial set. We say that X is an H -space if it has a map $m : X \times X \rightarrow X$ and a point $e \in X$ which is a homotopy identity, that is, the maps $m(e, -), m(-, e) : X \rightarrow X$ are homotopic to the identity map.

Exercise 6.3. Prove that the fundamental group of any H -space is always abelian.

Definition 6.4. Let X be a homotopy commutative and associative H -space. A group completion of X is an H -space Y together with an H -map $X \rightarrow Y$ such that

1. $\pi_0(X) \rightarrow \pi_0(Y)$ is a group completion of the abelian monoid $\pi_0(X)$, and
2. for any commutative ring R , the homomorphism $H_*(X; R) \rightarrow H_*(Y; R)$ is a localization of the graded commutative ring $H_*(X; R)$ at the multiplicative subset $\pi_0(X) \subset H_0(X, R)$.

We denote by X^{gp} the group completion of X .

There is a simple criterion for checking if a commutative and associative H -space is indeed its own group completion.

Definition 6.5. Let X be an H -space. We say that X is group-like if the monoid $\pi_0(X)$ is a group.

The following proposition is standard. See [MS75] for example. A specific model of the group completion of X is ΩBX when X is homotopy commutative and associative [Seg74].

Proposition 6.6. *Let X be a homotopy commutative and associative H -space, then the group completion of X is unique up to homotopy and further, if X is group-like, then X is weakly equivalent to its own group-completion.*

Example 6.7. Let R be an associative ring. We have maps $m : \mathrm{GL}_n(R) \times \mathrm{GL}_m(R) \rightarrow \mathrm{GL}_{m+n}(R)$ defined by block sum. This map is a group homomorphism and thus induces a map

$$m : \left(\prod_{n \geq 0} \mathrm{BGL}_n(R) \right)^{\times 2} \rightarrow \prod_{n \geq 0} \mathrm{BGL}_n(R).$$

One easily checks that this is indeed a homotopy associative and homotopy commutative H -space.

Remark 6.8. On the other hand we have the group $\mathrm{GL}(R) = \mathrm{colim} \mathrm{GL}_d(R)$ where the transition maps are induced by adding a single entry “1” at the bottom right corner. We can take $\mathrm{BGL}(R)$, the classifying space of R . This space is *not* an H -space: its fundamental group is $\mathrm{GL}(R)$ which is not an abelian group. For it to have any chance of being an H -space we need to perform the $+$ -construction of Quillen which kills off a perfect normal subgroup of the fundamental group of a space and does not alter homology. For details see [Wei13, Section IV.1]. One key property of the $+$ -construction that we will need is the following theorem of Quillen.

Theorem 6.9 (Quillen). *Let R be an associative ring with unit, the map $i : \mathrm{BGL}(R) \rightarrow \mathrm{BGL}(R)^+$ is universal for maps into H -spaces. In other words for each map $f : \mathrm{BGL}(R) \rightarrow H$ where H is an H -space, there is a map $g : \mathrm{BGL}(R)^+ \rightarrow H$ such that $f \simeq g \circ i$ and the induced map on homotopy groups is independent of g .*

Proof. See [Wei13, Section IV.1 Theorem 1.8] and the references therein. \square

Having this construction, the two spaces we discussed are intimately related.

Theorem 6.10 (Quillen). *Let R be an associative ring with unit, then the group completion of $\coprod \mathrm{BGL}_n(R)$ is weakly equivalent to $\mathbb{Z} \times \mathrm{BGL}(R)^+$.*

See [Wei13] for a proof. The plus construction alters the homotopy type of a space rather drastically. There are other models for the plus construction like Segal’s $\Omega\mathrm{B}$ construction mentioned above. The $\mathrm{Sing}^{\mathbb{A}^1}$ -construction turns out to provide another model, as we explain in the next section.

6.2 Applications to representability

The following theorem was established by Morel and Voevodsky [MV99], although a gap was pointed out by Schlichting and Tripathi [ST15], who also provided a fix.

Theorem 6.11. *Let S be a regular noetherian scheme of finite Krull dimension. The natural map $\mathbb{Z} \times \mathrm{BGL} \rightarrow \mathcal{K}$ in $\mathrm{Spc}_S^{\mathbb{A}^1}$ is an \mathbb{A}^1 -local weak equivalence.*

There is an \mathbb{A}^1 -local weak equivalence

$$\left(\coprod_n \mathrm{BGL}_n \right)^{\mathrm{gp}} \simeq \mathcal{K}$$

from Theorem 6.10 since the $+$ -construction is one way to obtain $\mathcal{K}(R)$ by Quillen. (Note that sheafification takes care of the fact that there might be non-trivial finitely generated projective R -modules.) Hence, to prove the theorem, we must construct an \mathbb{A}^1 -weak equivalence

$$\mathbb{Z} \times \mathrm{BGL} \simeq \left(\coprod_n \mathrm{BGL}_n \right)^{\mathrm{gp}}.$$

We have already mentioned that BGL is not an H -space, so that group completion will not formally lead to a weak equivalence. It is rather the $\mathrm{Sing}^{\mathbb{A}^1}$ -construction which leads to an H -space structure on the \mathbb{A}^1 -localization of $\mathbb{Z} \times \mathrm{BGL}$.

Lemma 6.12. *If R is a commutative ring, then $\mathrm{Sing}^{\mathbb{A}^1} \mathrm{BGL}(R)$ is an H -space.*

Proof. See [Wei13, Exercise IV.11.9]. \square

Proposition 6.13. *If R is a commutative ring, the natural map*

$$\mathrm{Sing}^{\mathbb{A}^1} \mathrm{BGL}(R) \rightarrow \mathrm{Sing}^{\mathbb{A}^1} \mathrm{BGL}(R)^+$$

is a weak equivalence.

Proof. The map is a homology equivalence since each $\mathrm{BGL}(\Delta_R^n) \rightarrow \mathrm{BGL}(\Delta_R^n)^+$ is a homology equivalence. Since both sides are group-like H -spaces they are nilpotent, so the fact that the map is a homology equivalence implies that it is a weak homotopy equivalence. \square

Proof of Theorem 6.11. More precisely, we claim that we have a weak equivalence of simplicial presheaves $\mathrm{L}_{\mathbb{A}^1} \mathrm{L}_{\mathrm{Nis}}(\mathbb{Z} \times \mathrm{BGL}) \rightarrow \mathcal{K}$. Since both simplicial presheaves are, in particular, Nisnevich local we need only check on stalks. Therefore we need only check that $\mathrm{Sing}^{\mathbb{A}^1}(\mathbb{Z} \times \mathrm{BGL}(R)) \simeq \mathcal{K}(R)$ for R a regular noetherian local ring because after this, further application of $\mathrm{Sing}^{\mathbb{A}^1}$ does not change the stalk by \mathbb{A}^1 -homotopy invariance. but this follows from our work above since $\mathcal{K}(R) \simeq \mathrm{K}_0(R) \times \mathrm{BGL}(R)^+$. Since \mathcal{K} is \mathbb{A}^1 -invariant on smooth affine schemes, the natural map

$$|\mathrm{K}_0(R) \times \mathrm{BGL}(\Delta_R^\bullet)^+| \rightarrow \mathrm{K}_0(R) \times \mathrm{BGL}(R)^+$$

is a weak equivalence, and the result follows from Quillen. \square

Remark 6.14. One can further prove that BGL is represented in $\mathrm{Sp}c_{S, \star}^{\mathbb{A}^1}$ by the Grassmanian schemes. In order to do this, Morel and Voevodsky used an elegant model for classifying spaces in [MV99], also considered by Totaro [Tot99].

As a corollary, we get a calculation of the stable range of the \mathbb{A}^1 -homotopy sheaves of BGL_n and BSL_n .

Corollary 6.15. *Let $i > 1$ and $n \geq 1$. Then if $i \leq n - 1$, we have isomorphisms*

$$\pi_i^{\mathbb{A}^1} \mathrm{BSL}_n \cong \pi_i^{\mathbb{A}^1} (\mathrm{BGL}_n) \cong \mathrm{K}_i.$$

Proof. This follows from the stable range results in Theorem 5.19 and Corollary 5.20. \square

7 Purity

In this section, we prove the purity theorem. The theorem has its roots in the following theorem from étale cohomology: suppose that k is an algebraically closed field with characteristic prime to an integer n and $Z \hookrightarrow X$ is a regular closed immersion of k -varieties. Suppose further that Z is of pure codimension c in X . Then, for any locally constant sheaf of \mathbb{Z}/n -modules \mathcal{F} there is a canonical isomorphism

$$g : \mathrm{H}_{\mathrm{ét}}^{r-2c}(Z, \mathcal{F}(-n)) \rightarrow \mathrm{H}_Z^r(X, \mathcal{F}),$$

the **purity isomorphism**. Here $\mathrm{H}_Z^r(X, -)$ is the étale cohomology of X with supports on Z , which is characterized as the group fitting into the long exact sequence

$$\cdots \rightarrow \mathrm{H}_Z^r(X, \mathcal{F}) \rightarrow \mathrm{H}_{\mathrm{ét}}^r(X, \mathcal{F}) \rightarrow \mathrm{H}_{\mathrm{ét}}^r(X - Z, \mathcal{F}) \rightarrow \mathrm{H}_Z^{r+1}(X, \mathcal{F}) \rightarrow \cdots .$$

Substituting the isomorphism above into the long exact sequence we obtain the **Gysin sequence**

$$\cdots \rightarrow \mathrm{H}_{\mathrm{ét}}^{r-2c}(Z; \mathcal{F}(-c)) \rightarrow \mathrm{H}_{\mathrm{ét}}^r(X; \mathcal{F}) \rightarrow \mathrm{H}_{\mathrm{ét}}^r(X - Z; \mathcal{F}) \rightarrow \mathrm{H}_{\mathrm{ét}}^{r+1-2c}(Z; \mathcal{F}(-c)) \rightarrow \cdots .$$

The Gysin sequence is extremely useful for calculation: the naturality of the long exact sequence and purity isomorphism leads to calculations of Frobenius weights of smooth varieties U by embedding them into a smooth projective variety $U \hookrightarrow X$ whose complement is often a normal crossing divisor [Del75].

In topology, the Gysin sequence is also available and is deduced in the following way. Suppose that $Z \hookrightarrow X$ is a closed immersion of smooth manifolds of (real) codimension c and

ν_Z is the normal bundle of Z in X . The tubular neighborhood theorem identifies the Thom space of ν_Z with the cofiber of $X - Z \rightarrow X$, i.e. there is a weak homotopy equivalence

$$\mathrm{Th}(\nu_Z) \simeq \frac{X}{X - Z}.$$

One then proves that there is an isomorphism

$$\tilde{H}^{i-c}(Z; k) \rightarrow \tilde{H}^i(\mathrm{Th}(\nu_Z); k)$$

in reduced singular cohomology with coefficients in a field k . In fact, this last isomorphism is true if we replace ordinary singular cohomology with any complex-oriented cohomology theory [May99]. Therefore, the crucial step is identifying the cofiber $\frac{X}{X-Z}$ with the Thom space of the normal bundle. In this light, the purity theorem in \mathbb{A}^1 -homotopy theory may be interpreted as a kind of tubular neighborhood theorem.

We will now prove the crucial purity theorem of Morel and Voevodsky [MV99, Theorem 2.23]. We benefited from unpublished notes of Asok and from the exposition of Hoyois in [Hoy15, Section 3.5] in the equivariant case. We follow the latter closely below. The discussion in this section is valid for S a quasi-compact quasi-separated base scheme S .

Definition 7.1. A **smooth pair** over a scheme S is a closed embedding $i : Z \hookrightarrow X$ of finitely presented smooth S -schemes. We will often write such a pair as (X, Z) , omitting reference to the map i . The smooth pairs over S form a category $\mathrm{Sm}_S^{\mathrm{pairs}}$ in which the morphisms $(X, Z) \rightarrow (X', Z')$ are pullback squares

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & X'. \end{array}$$

A morphism $f : (X, Z) \rightarrow (X', Z')$ of smooth pairs is **Nisnevich** if $f : X \rightarrow X'$ is étale and if $f^{-1}(Z') \rightarrow Z'$ is an isomorphism.

We will need following local characterization of smooth pairs.

Proposition 7.2. *Let $i : Z \rightarrow X$ be a smooth pair over a quasi-compact and quasi-separated scheme S . Assume that the codimension of i is c along Z . Then, there is a Zariski cover $\{U_i \rightarrow X\}_{i \in I}$ and a set of étale morphisms $\{U_i \rightarrow \mathbb{A}_S^{n_i}\}_{i \in I}$ such that the smooth pair $U_i \times_X Z \rightarrow U_i$ is isomorphic to the pullback of the inclusion of a linear subspace $\mathbb{A}_S^{n_i-c} \rightarrow \mathbb{A}_S^{n_i}$ for all $i \in I$.*

Proof. See [Gro63, Théorème II.4.10]. □

Certain moves generate all smooth pairs, which lets one prove statements for all smooth pairs by checking them locally, checking that they transport along Nisnevich morphisms of smooth pairs, and checking that they hold for zero sections of vector bundles.

Lemma 7.3. *Suppose that \mathbf{P} is a property of smooth pairs over a quasi-compact and quasi-separated scheme S satisfying the following conditions:*

1. *if (X, Z) is a smooth pair and if $\{U_i \rightarrow X\}$ is a Zariski cover such that \mathbf{P} holds for $(U_{i_1} \times_X \cdots \times_X U_{i_n}, Z \times_X U_{i_1} \times_X \cdots \times_X U_{i_n})$ for all tuples $i_1, \dots, i_n \in I$, then \mathbf{P} holds for (X, Z) ;*
2. *if $(V, Z) \rightarrow (X, Z)$ is a Nisnevich morphism of smooth pairs, then \mathbf{P} holds for (V, Z) if and only if \mathbf{P} holds for (X, Z) ;*
3. *\mathbf{P} holds for all smooth pairs of the form (\mathbb{A}_Z^n, Z) .*

Then, **P** holds for all smooth pairs over S .

Proof. By (1), it suffices to check that **P** is true Zariski-locally on X . Pick a Zariski cover $\{U_i \rightarrow X\}$ satisfying the conclusion of Proposition 7.2. Thus, the problem is reduced to showing that if $(X, Z) \rightarrow (\mathbb{A}^n, \mathbb{A}^m)$ is a map of smooth pairs with $X \rightarrow \mathbb{A}^n$ étale, then **P** holds for (X, Z) . Indeed, all pairs $(U_{i_1} \times_X \cdots \times_X U_{i_n}, Z \times_X U_{i_1} \times_X \cdots \times_X U_{i_n})$ have this form by our choice of cover. The rest of the argument follows [MV99, Lemma 2.28]. Form the fiber product $X \times_{\mathbb{A}^n} (Z \times_S \mathbb{A}^c)$, where $c = n - m$, and $Z \times_S \mathbb{A}^c \rightarrow \mathbb{A}^n$ is the product of the maps $Z \rightarrow \mathbb{A}^m$ and $\mathbb{A}^c \xrightarrow{\text{id}} \mathbb{A}^c$. Since $Z \rightarrow \mathbb{A}^m$ is étale, we see that $Z \times_{\mathbb{A}^n} Z \subseteq X \times_{\mathbb{A}^n} (Z \times_S \mathbb{A}^c)$ is the disjoint union of Z and some closed subscheme W . Let $U = X \times_{\mathbb{A}^n} (Z \times_S \mathbb{A}^c) - W$. The projection maps induce Nisnevich maps of pairs $(U, Z) \rightarrow (X, Z)$ and $(U, Z) \rightarrow (Z \times_S \mathbb{A}^c, Z)$. By (3), **P** holds for $(Z \times_S \mathbb{A}^c, Z)$ and hence for (U, Z) by (2), and hence for (X, Z) by (2) again. \square

Definition 7.4. A morphism $(X, Z) \rightarrow (X', Z')$ of smooth pairs over S is **weakly excisive** if the induced square

$$\begin{array}{ccc} Z & \longrightarrow & X/(X - Z) \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & X'/(X' - Z') \end{array}$$

is homotopy cocartesian in $\text{Spc}_S^{\mathbb{A}^1}$.

The following exercise is used in the proof of the purity theorem.

Exercise 7.5 ([Hoy15, Lemma 3.19]). Let $(X, Z) \xrightarrow{f} (X', Z') \xrightarrow{g} (X'', Z'')$ be composable morphisms of smooth pairs over S . Prove the following statements.

1. If f is weakly excisive, then g is weakly excisive if and only if $g \circ f$ is weakly excisive.
2. If g and $g \circ f$ are weakly excisive, and if $g : Z' \rightarrow Z''$ is an \mathbb{A}^1 -local weak equivalence, then f is weakly excisive.

Finally, we come to the purity theorem itself.

Theorem 7.6 (Purity theorem [MV99, Theorem 2.23]). *Let $Z \hookrightarrow X$ be a closed embedding in Sm_S where S is quasi-compact and quasi-separated. If $\nu_Z : \mathbb{N}_X Z \rightarrow Z$ is the normal bundle to Z in X , then there is an \mathbb{A}^1 -local weak equivalence*

$$\frac{X}{X - Z} \rightarrow \text{Th}(\nu_Z)$$

which is natural in $\text{Ho}(\text{Spc}_S^{\mathbb{A}^1})$ for smooth pairs (X, Z) over S .

Proof. First, we construct the map. Consider the construction

$$\mathbb{D}_Z X = \text{Bl}_{Z \times_S \{0\}}(X \times_S \mathbb{A}^1) - \text{Bl}_{Z \times_S \{0\}}(X \times_S \{0\}),$$

which is natural in smooth pairs (X, Z) . The fiber of $\mathbb{D}_Z X \rightarrow \mathbb{A}^1$ at $\{0\}$ is the complement $\mathbb{P}(\mathbb{N}_Z X \oplus \mathcal{O}_Z) - \mathbb{P}(\mathbb{N}_Z X)$, which is naturally isomorphic to the vector bundle $\mathbb{N}_Z X$. Hence, by taking the zero section at $\{0\}$, we get a closed embedding $Z \times_S \mathbb{A}^1 \rightarrow \mathbb{D}_Z X$. The fiber at $\{0\}$ of $(\mathbb{D}_Z X, Z \times_S \mathbb{A}^1)$ is $(\mathbb{N}_Z X, Z)$, while the fiber at $\{1\}$ is (X, Z) . Thus, there are morphisms of smooth pairs

$$(X, Z) \xrightarrow{i_1} (\mathbb{D}_Z X, Z \times_S \mathbb{A}^1) \xleftarrow{i_0} (\mathbb{N}_Z X, Z),$$

and it is enough to prove that i_1 and i_0 are weakly excisive for all smooth pairs (X, Z) . Indeed, in that case there are natural \mathbb{A}^1 -weak equivalences $X/(X - Z) \simeq \mathbb{D}_Z X/(\mathbb{D}_Z X - Z \times_S \mathbb{A}^1) \simeq \mathbb{N}_Z X/(\mathbb{N}_Z X - Z) = \text{Th}(\nu_Z)$ because the cofiber of $Z \rightarrow Z \times_S \mathbb{A}^1$ is contractible.

Let \mathbf{P} hold for the smooth pair (X, Z) if and only if i_0 and i_1 are excisive. We show that \mathbf{P} satisfies conditions (1)-(3) of Lemma 7.3.

Let $\{U_i \rightarrow X\}_{i \in I}$ be a Zariski cover of X , and let $(U_{i_1, \dots, i_n}, Z_{i_1, \dots, i_n}) \rightarrow (X, Z)$ be the induced morphisms of smooth pairs. For Suppose that \mathbf{P} holds for each $(U_{i_1, \dots, i_n}, Z_{i_1, \dots, i_n})$. Then, there is a diagram

$$\begin{array}{ccc} |Z_\bullet| & \xrightarrow{i_1} & |U_\bullet / (U_\bullet - Z_\bullet)| \\ \downarrow & & \downarrow \\ |Z_\bullet \times_S \mathbb{A}^1| & \xrightarrow{i_1} & |D_{Z_i} U_i / (D_{Z_i} U_i - Z_i \times_S \mathbb{A}^1)| \end{array}$$

of geometric realizations. However, this is the geometric realization of a simplicial cocartesian square by hypothesis, so it is itself cocartesian. The same argument works for i_0 , so we see that \mathbf{P} satisfies (1).

Consider a Nisnevich morphism $(V, Z) \rightarrow (X, Z)$ of smooth pairs, and consider the diagram

$$\begin{array}{ccccc} (V, Z) & \xrightarrow{i_1} & (D_Z V, Z \times_S \mathbb{A}^1) & \xleftarrow{i_0} & (N_Z V, Z) \\ \downarrow & & \downarrow & & \downarrow \\ (X, Z) & \xrightarrow{i_1} & (D_Z X, Z \times_S \mathbb{A}^1) & \xleftarrow{i_0} & (N_Z X, Z). \end{array}$$

We leave it as an easy exercise to the reader to show using Exercise 7.5 that (2) will follow if the vertical arrows are all weakly excisive. But, the vertical maps are all Nisnevich morphisms. So, it is enough to check that Nisnevich morphisms $(V, Z) \rightarrow (X, Z)$ of smooth pairs are weakly excisive. Let U be the complement of Z in X . By hypothesis, the diagram

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

is an elementary distinguished square, and hence a homotopy cocartesian square in $\mathrm{Spc}_S^{\mathbb{A}^1}$ by Proposition 4.13. In particular, the cofiber of $V / (U \times_X V) \rightarrow X / U$ is contractible. Since the cofiber of $Z \rightarrow Z$ is obviously contractible, this proves $(V, Z) \rightarrow (X, Z)$ is weakly excisive.

To complete the proof, we just have to show that (3) holds. Of course, in the situation (\mathbb{A}_Z^n, Z) of (3) we can prove the main result of the theorem quite easily. However, the structure of the proof requires us to check weak excision for i_0 and i_1 . For this we can immediately reduce to the case where $Z = S$, which we omit from the notation for the rest of the proof. The blowup $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n \times \mathbb{A}^1)$ is the total space of an \mathbb{A}^1 -bundle over \mathbb{P}^n , and the image of $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$ in \mathbb{P}^n is a hyperplane $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$. Hence, there is a morphism of pairs

$$(D_{\{0\}}(\mathbb{A}^n), \{0\} \times \mathbb{A}^1) \xrightarrow{f} (\mathbb{A}^n, \{0\}).$$

Since $D_{\{0\}}(\mathbb{A}^n) \rightarrow \mathbb{A}^n$ is the total space of an \mathbb{A}^1 -bundle, this morphism is weakly excisive. The composition of f with i_1 is the identity on $(\mathbb{A}^n, \{0\})$ and hence is weakly excisive as well. By Exercise 7.5(2), it follows that i_1 is weakly excisive. Similarly, $f \circ i_0$ is the identity on $(N_{\{0\}} \mathbb{A}^n, \{0\}) \cong (\mathbb{A}^n, \{0\})$, so i_0 is weakly excisive, again by Exercise 7.5(2). \square

8 Vista: classification of vector bundles

In this section we give a brief summary of how to use the theory developed above to give a straightforward proof of the classification of vector bundles on smooth affine curves and

surfaces. We must understand how to compute \mathbb{A}^1 -homotopy classes of maps to BGL_n so we can apply the Postnikov obstruction approach to the classification problem.

Recall from Theorem 4.27 that the \mathbb{A}^1 -localization functor above may be calculated as a transfinite composite of L_{Nis} and $\mathrm{Sing}^{\mathbb{A}^1}$. This process is rather unwieldy. However, things get better if this process stops at a finite stage, in particular suppose that \mathcal{F} was already a Nisnevich-local presheaf and suppose that one can somehow deduce that $\mathrm{Sing}^{\mathbb{A}^1} X$ was already Nisnevich local, then one can conclude that $L_{\mathbb{A}^1} X \simeq \mathrm{Sing}^{\mathbb{A}^1} X$ and therefore, using our formulae for mapping spaces in the \mathbb{A}^1 -homotopy category, one concludes that

$$[U, X]_{\mathbb{A}^1} \cong [U, L_{\mathbb{A}^1} X]_s \simeq \pi_0 \mathrm{Sing}^{\mathbb{A}^1} X(U).$$

In general, this does not work.

Remark 8.1. Work of Balwe-Hogadi-Sawant [BHS15] constructs explicit smooth projective varieties X over \mathbb{C} for which $\mathrm{Sing}^{\mathbb{A}^1} X$ is *not* Nisnevich-local, so extra conditions must be imposed to calculate the \mathbb{A}^1 -homotopy classes of maps naively. However, there is often an intimate relation between naive \mathbb{A}^1 -homotopies and genuine ones: Cazanave constructs in [Caz12] a monoid structure on $\pi_0 \mathrm{Sing}^{\mathbb{A}^1}(\mathbb{P}_k^1)$ and proves that the map $\pi_0 \mathrm{Sing}^{\mathbb{A}^1}(\mathbb{P}_k^1) \rightarrow [\mathbb{P}_k^1, \mathbb{P}_k^1]_{\mathbb{A}^1}$ is group completion, with the group structure on the target induced by the \mathbb{A}^1 -weak equivalence $\mathbb{P}^1 \simeq S^1 \wedge G_m$.

Exercise 8.2. Even for fields, one can show that the sets of isomorphism classes of vector bundles over the simplest non-affine scheme \mathbb{P}_k^1 are not \mathbb{A}^1 -invariant. Construct (e.g. write down explicit transition functions) a vector bundle over $\mathbb{P}^1 \times_k \mathbb{A}^1$ that restricts to $\mathcal{O}(0) \oplus \mathcal{O}(0)$ on $\mathbb{P}^1 \times \{1\}$ and $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ on $\mathbb{P}^1 \times \{0\}$ for a counter-example.

Remark 8.3. In [AD08], examples are given of smooth \mathbb{A}^1 -contractible varieties with families of non-trivial vector bundles of any given dimension. Were vector bundles to be representable in $\mathrm{Spc}^{\mathbb{A}^1}$, such pathologies could not occur. These varieties are non-affine.

As the exercise and remark show, the only hope for computing vector bundles as \mathbb{A}^1 -homotopy classes of maps to BGL_n is to restrict to affine schemes, but even there it is not at all obvious that this is possible, as the map $\mathrm{BGL}_n \rightarrow L_{\mathbb{A}^1} \mathrm{BGL}_n$ is not a simplicial weak equivalence. Remarkably, despite this gulf, Morel and later Asok-Hoyois-Wendt showed that for smooth affine schemes one can compute vector bundles in this way. In fact, this follows from a much more general and formal result, which we now explain.

We say that a presheaf F of sets on Sm_S satisfies **affine \mathbb{A}^1 -invariance** if the pullback maps $F(U) \rightarrow F(U \times_S \mathbb{A}^1)$ are isomorphisms for all finitely presented smooth affine S -schemes U . Note that we say that an S -scheme is affine if $U \rightarrow \mathrm{Spec} \mathbb{Z}$ is affine, so that $U = \mathrm{Spec} R$ for some commutative ring R .

Theorem 8.4 ([AHW15a]). *Let S be a quasi-compact and quasi-separated scheme. Suppose that X is a simplicial presheaf on Sm_S . Assume that $\pi_0(X)$ is affine \mathbb{A}^1 -invariant and that X satisfies affine Nisnevich excision. For all affine schemes U in Sm_S , the canonical map*

$$\pi_0(X)(U) \rightarrow [U, X]_{\mathbb{A}^1}$$

is an isomorphism.

Sketch Proof. The key homotopical input to this theorem is the π_* -Kan condition, which ensures that homotopy colimits of simplicial diagram commutes over pullbacks [BF78]. This condition was first used in this area by Schlichting [Sch15]. It provides a concrete criterion to check if the functor $\mathrm{Sing}^{\mathbb{A}^1}(F)$ restricted to smooth affine schemes is indeed Nisnevich local (to make this argument precise, the key algebro-geometric input is the equivalence between the Nisnevich cd-structure and the affine Nisnevich cd-structure defined above [AHW15a, Proposition 2.3.2] when restricted to affine schemes).

More precisely, for any elementary distinguished square

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & Y \end{array}$$

we have a homotopy pullback square

$$\begin{array}{ccc} X(Y \times \mathbb{A}^n) & \longrightarrow & X(V \times \mathbb{A}^n) \\ \downarrow & & \downarrow \\ X(U \times \mathbb{A}^n) & \longrightarrow & X(U \times_X V \times \mathbb{A}^n) \end{array}$$

of simplicial sets for all $n \geq 0$.

The π_* -Kan condition applies with the hypothesis that $\pi_0(X)$ satisfies affine \mathbb{A}^1 -invariance and we may conclude that taking $\text{hocolim}_{\Delta^{\text{op}}}$ of the above squares preserve pullbacks and therefore we conclude that $\text{Sing}^{\mathbb{A}^1}(X)$ satisfies affine Nisnevich excision.

Applying the above proposition, we have that for any affine U ,

$$\text{Sing}^{\mathbb{A}^1}(X)(U) \rightarrow \text{L}_{\text{Nis}} \text{Sing}^{\mathbb{A}^1}(X)(U)$$

is a weak equivalence. Since the left hand side is \mathbb{A}^1 -invariant, we conclude that the right hand side is \mathbb{A}^1 -invariant; since being \mathbb{A}^1 -invariant and Nisnevich local may be tested on affine schemes (by [AHW15a, Proposition 2.3.2]), we conclude that $\text{L}_{\text{Nis}} \text{Sing}^{\mathbb{A}^1}(X) \simeq \text{L}_{\mathbb{A}^1}(X)$. Taking π_0 of the weak equivalence above gets us the desired claim. \square

Corollary 8.5 (Affine representability of vector bundles). *Let S be a regular noetherian affine scheme of finite Krull dimension, and suppose that the Bass-Quillen conjecture holds for smooth schemes of finite presentation over S . In this case, the natural map $\text{Vect}_r(U) \rightarrow [U, \text{BGL}_r]_{\mathbb{A}^1}$ is an isomorphism for all $U \in \text{Sm}_S^{\text{Aff}}$ and all $r \geq 0$.*

The Jouanolou-Thomason homotopy lemma states that, up to \mathbb{A}^1 -homotopy, we may replace a smooth scheme with an affine one.

Theorem 8.6 ([Jou73] and [Wei89]). *Given a smooth separated scheme U over a regular noetherian affine scheme S , there exists an affine vector bundle torsor $\tilde{U} \rightarrow U$ such that \tilde{U} is affine.*

Proof. The point is that U is quasi-compact and quasi-separated and hence admits an ample family of line bundles (so U is divisorial) by [71, Proposition II.2.2.7]. The theorem now follows from [Wei89, Proposition 4.4]. \square

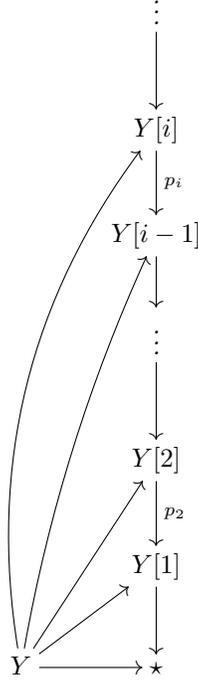
This theorem lets us compute in some sense $[U, \text{BGL}_n]_{\mathbb{A}^1}$ for any $U \in \text{Sm}_S$, but it is not known at the moment what kind of objects these are on U .

One of the main features of \mathbb{A}^1 -localization is the ability to employ topological thinking in algebraic geometry, if one is willing to work \mathbb{A}^1 -locally. The homotopy sheaves $\pi_i^{\mathbb{A}^1}(X)$ are sometimes computable using input from both homotopy theory and algebraic geometry. At the same time, many algebro-geometric problems are inherently not \mathbb{A}^1 -local in nature so one only gets an actual algebro-geometric theorem under certain conditions, as in Theorem 8.13 below. Let us first start with a review of Postnikov towers in \mathbb{A}^1 -homotopy theory. Our main reference is [AF14], which in turn uses [Mor12], [MV99] and [GJ99].

Let G be a Nisnevich sheaf of groups and M a Nisnevich sheaf of abelian groups on which G acts (a G -module). In this case, G acts on the Eilenberg-MacLane sheaf $K(A, n)$, from which we may construct $K^G(A, n) := EG \times^G K(A, n)$. The first projection gives us a map $K^G(A, n) \rightarrow G$.

Of primary interest is the Nisnevich sheaf of groups $\pi_1^{\mathbb{A}^1}(Y)$ for some pointed \mathbb{A}^1 -connected space Y . In this case, $\pi_1^{\mathbb{A}^1}(Y)$ acts on the higher homotopy sheaves $\pi_n^{\mathbb{A}^1}(Y)$ where $n \geq 2$.

Theorem 8.7. *Let Y be a pointed \mathbb{A}^1 -connected space. There exists a commutative diagram of pointed \mathbb{A}^1 -connected spaces*



such that

1. $Y[1] \simeq B\pi_1^{\mathbb{A}^1}(Y)$,
2. $\pi_j^{\mathbb{A}^1} Y[i] = 0$ for $j > i$,
3. the map $\pi_j^{\mathbb{A}^1} Y \rightarrow \pi_j^{\mathbb{A}^1} Y[i]$ is an isomorphism of $\pi_1^{\mathbb{A}^1} Y$ -modules for $1 \leq j \leq i$,
4. the $K(\pi_i^{\mathbb{A}^1} Y, i)$ -bundle $Y[i] \rightarrow Y[i-1]$ is a **twisted principal fibration** in the sense that there is a map

$$k_i : Y[i-1] \rightarrow K^{\pi_1^{\mathbb{A}^1} Y}(\pi_i^{\mathbb{A}^1} Y, i+1)$$

such that $Y[i]$ is obtained as the pullback

$$\begin{array}{ccc} Y[i] & \longrightarrow & B_{\text{Nis}}\pi_1^{\mathbb{A}^1} Y \\ \downarrow & & \downarrow \\ Y[i-1] & \longrightarrow & K^{\pi_1^{\mathbb{A}^1} Y}(\pi_i^{\mathbb{A}^1} Y, i+1), \end{array}$$

5. and $Y \rightarrow \lim_i Y[i]$ is an \mathbb{A}^1 -weak equivalence.

The tower, which can be made functorial in Y , is called the \mathbb{A}^1 -**Postnikov tower** of Y .

Proof. This is left as an exercise, which basically amounts to Nisnevich sheafifying and \mathbb{A}^1 -localizing the usual Postnikov tower. For an extensive discussion, see [AF14, Section 6] and the references therein. \square

The point of the Postnikov tower is to make it possible to classify maps from X to Y by constructing maps inductively, i.e., starting with a map $X \rightarrow Y[1]$, lifting it to $X \rightarrow Y[2]$ while controlling the choices of lifts, and so on.

Theorem 8.8. *Let S be a quasi-compact quasi-separated base scheme, and let X be a smooth noetherian S -scheme of Krull dimension at most d . Suppose that (Y, y) is a pointed \mathbb{A}^1 -connected space. The natural map*

$$[X, Y]_{\mathbb{A}^1} \rightarrow [X, Y[i]]_{\mathbb{A}^1}$$

is an isomorphism for $i \geq d$ and a surjection for $i = d - 1$.

Proof. The first obstruction to lifting a map $X \rightarrow Y[i]$ to $X \rightarrow Y$ is the obstruction to lifting it to $X \rightarrow Y[i + 1]$. This is classified by the k -invariant, and hence a class in $H_{\text{Nis}, \pi_1^{\mathbb{A}^1} Y}^{i+2}(X, \pi_{i+1}^{\mathbb{A}^1} Y) = [X, K^{\pi_1^{\mathbb{A}^1} Y}(\pi_{i+1}^{\mathbb{A}^1} Y, i + 2)]_{\mathbb{A}^1}$. One important feature of the theory intervenes at this point: the equivariant cohomology group $H_{\text{Nis}, \pi_1^{\mathbb{A}^1} Y}^{i+2}(X, \pi_{i+1}^{\mathbb{A}^1} Y)$ can be identified with an ordinary Nisnevich cohomology group of a twisted form $(\pi_{i+1}^{\mathbb{A}^1} Y)_\lambda$ of $\pi_{i+1}^{\mathbb{A}^1} Y$ in Nisnevich sheaves on X :

$$H_{\text{Nis}, \pi_1^{\mathbb{A}^1} Y}^{i+2}(X, \pi_{i+1}^{\mathbb{A}^1} Y) \cong H_{\text{Nis}}^{i+2}(X, (\pi_{i+1}^{\mathbb{A}^1} Y)_\lambda).$$

See [Mor12, Appendix B]. This group vanishes if $i + 2 > d$, or $i + 1 \geq d$, since the Nisnevich cohomological dimension of X is at most d by hypothesis. Thus, the map in the theorem is a surjection for $i \geq d - 1$. The set of lifts, by the \mathbb{A}^1 -fiber sequence induced from Theorem 8.7(4), is a quotient of $H_{\text{Nis}, \pi_1^{\mathbb{A}^1} Y}^{i+1}(X, \pi_{i+1}^{\mathbb{A}^1} Y)$, which vanishes for the same reason as above if $i + 1 > d$, or $i \geq d$. This completes the proof. \square

We have an immediate consequence of the existence of the Postnikov towers as follows.

Proposition 8.9. *If E is a rank $n > d$ vector bundle on a smooth affine d -dimensional variety X , then E splits off a trivial direct summand.*

Proof. Using Proposition 5.18 and Theorem 5.19, we see that $\pi_i^{\mathbb{A}^1} \text{BGL}_d \rightarrow \pi_i^{\mathbb{A}^1} \text{BGL}_n$ is an isomorphism for $i \leq d - 1$, and a surjection for $i = d$. By the representability theorem, E is represented by a map $X \rightarrow \text{BGL}_n$ in the \mathbb{A}^1 -homotopy category. Compose this map with $\text{BGL}_n \rightarrow \text{BGL}_n[d]$ to obtain $g : X \rightarrow \text{BGL}_n[d]$. Note that E is uniquely determined by g by Theorem 8.8. It suffices to lift g to a map $h : X \rightarrow \text{BGL}_d[d]$. The fiber of $\text{BGL}_d[d] \rightarrow \text{BGL}_n[d]$ is a $K(A, d)$ -space for some Nisnevich sheaf A with an action of $\mathbb{G}_m = \pi_1^{\mathbb{A}^1} \text{BGL}_d$. It follows that the obstruction to lifting g through $\text{BGL}_d[d] \rightarrow \text{BGL}_n[d]$ is a class of $H_{\text{Nis}, \mathbb{G}_m}^{d+1}(X, A) = 0$. \square

Remark 8.10. As a Nisnevich sheaf of spaces, BGL_n is a $K(\pi, 1)$ -sheaf in the sense that it has only one non-zero homotopy group. For the purposes of obstruction theory and classification theory this is not terribly useful as choosing a lift to the first stage of the Nisnevich-local Postnikov tower of BGL_n is equivalent to specifying a vector bundle. The process of \mathbb{A}^1 -localization mysteriously acts as a prism that separates the single homotopy sheaf into an entire sequence (spectrum) of homotopy sheaves, allowing a finer step-by-step analysis.

Proposition 8.11. *The first few \mathbb{A}^1 -homotopy sheaves of BGL_2 are*

$$\begin{aligned} \pi_0^{\mathbb{A}^1} \text{BGL}_2 &= \star, \\ \pi_1^{\mathbb{A}^1} \text{BGL}_2 &\cong \mathbb{G}_m, \\ \pi_2^{\mathbb{A}^1} \text{BGL}_2 &\cong \text{K}_2^{\text{MW}}, \end{aligned}$$

where K_2^{MW} denotes the second unramified Milnor-Witt sheaf.

Proof. The \mathbb{A}^1 -connectivity statement $\pi_0^{\mathbb{A}^1} \mathrm{BGL}_2 = \star$ follows from the fact that vector bundles are Zariski and hence Nisnevich locally trivial. The fact that $\pi_1 \mathrm{BGL}_2 \cong \mathbb{G}_m$ follows from the stable range result that gives $\pi_1 \mathrm{BGL}_2 \cong \pi_1 \mathrm{BGL}_\infty \cong K_1 \cong \mathbb{G}_m$, where the last two isomorphisms are explained in Section 5.3. The last follows from the \mathbb{A}^1 -fiber sequence

$$\mathbb{A}^2 - \{0\} \rightarrow \mathrm{BGL}_1 \rightarrow \mathrm{BGL}_2,$$

the fact that BGL_1 is a $K(\mathbb{G}_m, 1)$ -space, and Morel's result [Mor12, Theorem 6.40], which says that $\pi_1^{\mathbb{A}^1} \mathbb{A}^2 - \{0\} \cong K_2^{\mathrm{MW}}$. \square

Now, for any smooth scheme X and any line bundle \mathcal{L} on X , there is an exact sequence of Nisnevich sheaves

$$0 \rightarrow \mathbb{I}^3(\mathcal{L}) \rightarrow K_2^{\mathrm{MW}}(\mathcal{L}) \rightarrow K_2 \rightarrow 0, \quad (2)$$

where the first and second terms are the \mathcal{L} -twisted forms (see [Mor12]). The sheaf $K_2^{\mathrm{MW}}(\mathcal{L})$ controls the rank 2 vector bundles on X with determinant \mathcal{L} .

If X is a smooth affine surface, there is a bijection $[X, \mathrm{BGL}_2]_{\mathbb{A}^1} \rightarrow [X, \mathrm{BGL}_2[2]]_{\mathbb{A}^1}$, from which it follows that the rank 2 vector bundles on X with determinant \mathcal{L} are classified by a quotient of

$$H_{\mathrm{Nis}, \mathbb{G}_m}^2(X, K_2^{\mathrm{MW}}) \cong H_{\mathrm{Nis}}^2(X, K_2^{\mathrm{MW}}(\mathcal{L})).$$

In fact, we will see that the quotient is all of $H_{\mathrm{Nis}}^2(X, K_2^{\mathrm{MW}}(\mathcal{L}))$.

Lemma 8.12. *If X is a smooth affine surface over a quadratically closed field k , then $H_{\mathrm{Nis}}^n(X, \mathbb{I}^3(\mathcal{L})) = 0$ for $n \geq 2$.*

Proof. This is [AF14, Proposition 5.2]. \square

It follows from the lemma and the exact sequence (2) that the space of lifts is a quotient of $H_{\mathrm{Nis}}^2(X, K_2) \cong \mathrm{CH}^2(X)$, where the isomorphism is due to Quillen [Qui73] in the Zariski topology and Thomason-Trobaugh [TT90] in the Nisnevich topology. Now, looking at

$$\mathbb{G}_m(X) \cong [X, K(\mathbb{G}_m, 0)]_{\mathbb{A}^1} \rightarrow [X, K^{\mathbb{G}_m}(K_2^{\mathrm{MW}}, 2)]_{\mathbb{A}^1} \rightarrow [X, \mathrm{BGL}_2[2]]_{\mathbb{A}^1} \rightarrow [X, \mathrm{BGL}_2[1]]_{\mathbb{A}^1} \cong \mathrm{Pic}(X),$$

we see that the map $[X, K^{\mathbb{G}_m}(K_2^{\mathrm{MW}}, 2)]_{\mathbb{A}^1} \rightarrow [X, \mathrm{BGL}_2[2]]_{\mathbb{A}^1}$ is injective because every element of $\mathbb{G}_m(X)$ extends to an automorphism of the vector bundle classified by $X \rightarrow \mathrm{BGL}_2$.

It follows that the map

$$\mathrm{Vect}_2(X) \rightarrow \mathrm{CH}^1(X) \times H_{\mathrm{Nis}}^2(X, K_2)$$

is a bijection. Thus, we have sketched a proof of the following theorem.

Theorem 8.13. *Let X be a smooth affine surface over a quadratically closed field. Then, the map*

$$(c_1, c_2) : \mathrm{Vect}_2(X) \rightarrow \mathrm{CH}^1(X) \times \mathrm{CH}^2(X),$$

induced by taking the first and second Chern classes, is a bijection.

Remark 8.14. To see that the natural maps involved are the Chern classes, as claimed, refer to [AF14, Section 6].

The fact that the theorem holds over quadratically closed fields is stronger than the previous results in this direction, which had been obtained without \mathbb{A}^1 -homotopy theory.

Asok and Fasel have carried this program much farther in several papers, for instance showing in [AF14] that $\mathrm{Vect}_2(X) \cong \mathrm{CH}^1(X) \times \mathrm{CH}^2(X)$ when X is a smooth affine *three-fold* over a quadratically closed field. This theorem, which is outside the stable range, is much more difficult.

9 Further directions

In most of the exercises below, none of which are supposed to be easy, it will be useful to bear in mind the universal properties of L_{Nis} and $L_{\mathbb{A}^1}$.

Exercise 9.1. Use the formalism of model categories to construct topological and étale realization functors out of the \mathbb{A}^1 -homotopy category $\text{Spc}_S^{\mathbb{A}^1}$. Dugger’s paper [Dug01a] on universal homotopy theories may come in handy. This problem is studied specifically in [DI04] and [DI08].

Exercise 9.2. Show that topological realization takes the motivic sphere $S^{a,b}$ where $a \leq b$ to the topological sphere S^a .

Exercise 9.3. Prove that complex topological K -theory is representable in $\text{Spc}_S^{\mathbb{A}^1}$.

Exercise 9.4. Let \mathbb{R} denote the field of real numbers. Construct a realization functor from $\text{Spc}_{\mathbb{R}}^{\mathbb{A}^1}$ to the homotopy theory of $\mathbb{Z}/2$ -equivariant topological spaces. Again, see [DI04].

Exercise 9.5. Construct a realization functor from $\text{Spc}_S^{\mathbb{A}^1}$ to Voevodsky’s category $\text{DM}(S)$ of (big) motives over S . It will probably be necessary to search the literature for a model category structure for $\text{DM}(S)$.

Exercise 9.6. Show that the realization functor from $\text{Spc}_S^{\mathbb{A}^1}$ to Voevodsky’s category factors through the *stable* motivic homotopy category obtained from $\text{Spc}_S^{\mathbb{A}^1}$ by stabilizing with respect to $S^{2,1} \simeq \mathbb{P}^1$.

Exercise 9.7. Ayoub [Ayo07] has constructed a 6-functors formalism for stable motivic homotopy theory. Construct some functors between $\text{Spc}_S^{\mathbb{A}^1}$ and $\text{Spc}_U^{\mathbb{A}^1}$ when U is open in S and between $\text{Spc}_S^{\mathbb{A}^1}$ and $\text{Spc}_Z^{\mathbb{A}^1}$ when Z is closed in S .

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