SOME NON-TRIVIAL EXAMPLES OF THE BALDWIN-OZSVÁTH-SZABÓ TWISTED SPECTRAL SEQUENCE AND HEEGAARD-FLOER HOMOLOGY OF BRANCHED DOUBLE COVERS

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1. Introduction

The main point of the present project was to compute some non-trivial examples of Baldwin-Ozsváth-Szabó (BOS) cohomology of links [2, 3], and, to apply this to Heegaard-Floer homology of branched double covers via the Baldwin-Ozsváth-Szabó twisted spectral sequence [2]. The examples we consider here are of a different type then what the main focus of recent interest has been, for reasons we shall explain.

Much recent interest has focused on so called L-spaces, which are compact oriented 3-manifolds satisfying

$$rank(\widehat{HF}(Y) = |H_1(Y,\mathbb{Z})|. \tag{1}$$

For example, Lisca and Stipsicz [4, 5] characterized L-spaces which are Seifert-fibered over S^2. For a branched double cover $\Sigma(L)$ of a link L, condition (1) is equivalent to

$$rank(\widehat{HF}(\Sigma(L)) = det(L) \neq 0. \tag{2}$$

Baldwin, Ozsváth and Szabó [2] constructed a twisted variant of their spectral sequence [7] convergent to $\widehat{HF}(\Sigma(L))$, which we will describe below. The point is that the (combinatorially defined) E_3-term of the twisted spectral sequence, which we call Baldwin-Ozsváth-Szabó (briefly BOS) cohomology, is extremely sparse (to the point that one may conjecture it collapses, although that is not known at present). In [3], it was proved that BOS cohomology is an invariant of oriented links, and it was observed that this gives an good method for detecting links which satisfy (2). The reason is that while BOS cohomology is defined as the cohomology of a cochain complex defined over a field of rational functions over $\mathbb{F}_2$ in a number of variables which increases

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with the number of crossings of L (described below), from an algebraic point of view, (2) is "generic behavior" on such complexes. Roughly, "generic behavior" menas that anything that can cancel cancels. For example in a 2-stage complex (given by a single linear map), generic behavior means that the rank of the linear map is maximal allowed by the dimension. As explained in [3] (and also used in this note), there is a mehod for detecting generic behavior in BOS cohomology. Namely, it is possible to set all the variables in the fraction field equal to integral powers of a single variable. If one gets lucky and the E_3-term has rank equal to $det(L)$, it is also true in BOS cohomology, and in $\widehat{HF}$. While calculations in fields of rational functions in many variables are computationally extremely inefficient, calculations in one variable are no problem. Because of this, (2) can be detected by BOS cohomology, and this was used in [3] to find a new weaker condition on links whose branched double covers are L-spaces.

To complement this, in the present project, we wanted to do computations where BOS cohomology behaves non-generically, with some applications to Heegaard-Floer homology. Since there is an algorithm [6] for calculating Heegaard-Floer homology of Seifert-fibered spaces, examples of hyperbolic knots are of most interest. We found that despite the combinatorial definition, it is *extraordinarily difficult* to compute directly non-generically behaved examples of BOS cohomology. This note serves, perhaps, as a case study of the difficulty of such computations. In the end, combining heuristics with computer-assisted methods, we succeeded in computing one example (perhaps one of the smallest ones) in Proposition 3.1 below. The example happens to be a link with 0 determinant, and infinitely many examples of exact computation of BOS cohomology can be deduced using the skein behavior of BOS cohomology (Proposition 4.1 and Corollary 4.2). Since all these examples are links of determinant 0, we do not get an immediate application to $\widehat{HF}$. However, using the Ozsváth-Szabó computation of $\widehat{HF}$ of $T(7,3)$ ([8]) as input, we were able to calculate BOS-cohomology and $\widehat{HF}$ for infinitely many new examples (Theorem 4.3, Theorem 4.5), all but finitely many of which are hyperbolic (Proposition 4.6). The reader should keep in mind that although we have no example of non-collapse of the twisted BOS spectral sequence, BOS cohomology carries more information than $\widehat{HF}$, since non-trivial ranks appear in different degrees; because of this, the BOS calculations are also of independent interest. We consider our examples as a "proof of concept", showing how this method works; it is very likely that many other examples can be calculated in a similar way. At the same time, it is also clear that such examples do not come cheap.

The present paper is organized as follows. In Section 2 we review the preliminaries, i.e. the definition of BOS cohomology, the link invariance of [3] and the BOS spectral sequence to $\widehat{HF}$. In Section 3, we treat the one example of non-trivial BOS cohomology which we were able to compute directly. In Section 4, we treat all the examples derived from this and from what was known about $T(7,3)$ in [8].

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2. Preliminaries

Let us first describe BOS cohomology. Let $\mathcal{D} = \mathcal{D}(L)$ be a non-degenerate projection of a link L, i.e. an embedding of a link in $\mathbb{R}^3$ such that the projection on the xy plane is an immersion with at most finitely many double crossings. Then the faces of the projection can be colored black and white so that no two faces of the same color border the same arc of the projection. Form a planar graph (with possible multiple edges and loops) whose vertices are the faces colored black, and edges are crossings which border two faces colored black. This is called the *black graph* $B(\mathcal{D})$. An edge has *height* 0 if the edge of the black graph, the lower arc of the crossing and the upper arc of the crossing occur in this order clockwise, and *height* 1 otherwise.

The BOS cochain complex $\mathcal{C}_0(\mathcal{D})$ is then formed as follows: Pick one vertex of the black graph as a base point. Let F be a field of rational functions over $\mathbb{F}_2$ on variables corresponding to all vertices of the black graph other than the base point and all bounded faces of the black graph. $\mathcal{C}_0(\mathcal{D})_k$ is the free F-module on all spaning trees T of $B(\mathcal{D})$ whose total height is $2k$; here the total height is defined to be the number of edges of height 1 included in T plus the number of edges of height 0 not included in T. The number k can be a half-integer, but all the possible values of k differ by an integer.

The differential Ψ of $\mathcal{C}_0$ increases total degree by 1. A non-trivial coefficient occurs between a spanning tree T and a spanning tree T' obtained from T by removing one edge e of height 0 and adding one edge f of height 1. The coefficient is of the form

$$\frac{1}{1+\alpha}+\frac{1}{1+\beta},$$

where α is the product of all the variables corresponding to faces enclosed inside the cycle c in $T \cup \{f\}$ provided that e, f and the base point occur counter-clockwise on that cycle, and the product of the inverses of all the variables corresponding to faces enclosed inside of c otherwise; the element β is the product of all the variables corresponding to vertices in the component of $T \smallsetminus \{e\}$ not containing the base point.

It was proved directly in [3] that Ψ is a differential, i.e. that $\Psi \circ \Psi = 0$. A key result of [2] is the following

2.1. **Theorem.** *Let L be a link with non-zero determinant. Then there exists a single-graded spectral sequence*

$$E_3 = H^*(\mathcal{C}_0(\mathcal{D}(L))) \Rightarrow \widehat{HF}(\Sigma(L)) \otimes_{\mathbb{F}_2} F.$$

Moreover, the grading is by total height (i.e. twice the degree), and the spectral sequence is sparse in the sense that the only possible non-zero differentials are of the form d_{4k+2}, $k \in \mathbb{Z}$.

To make BOS cohomology a knot invariant, one must correct by half the number of negative crossings, to take care of Reidemeister 1 moves. For an oriented link, a postive crossing is one where the upper arc of the crossing goes from lower left to upper right and the lower arc goes from lower right to upper left. The other kind of crossing is called a negative crossing. Let

$$\mathcal{C}(\mathcal{D})_k = \mathcal{C}_0(\mathcal{D})_{k+n_-/2}$$

where n_- is the number of negative crossings in $\mathcal{D}$. The differential in $\mathcal{C}$ is defined to be the same as in $\mathcal{C}_0$. In [3], the following was proved:

2.2. **Theorem.** *The numbers*

$$rank H^i(\mathcal{C}(\mathcal{D}))$$

are invariants of oriented links and unoriented knots.

We therefore put

$$H^i_{BOS}(L) = H^i(\mathcal{C}(\mathcal{D}(L)).$$

In this paper, we will work with the unshifted BOS cohomology, i.e. the cohomology of the complex $\mathcal{C}_0(\mathcal{D})$ of a projection $\mathcal{D}$ where

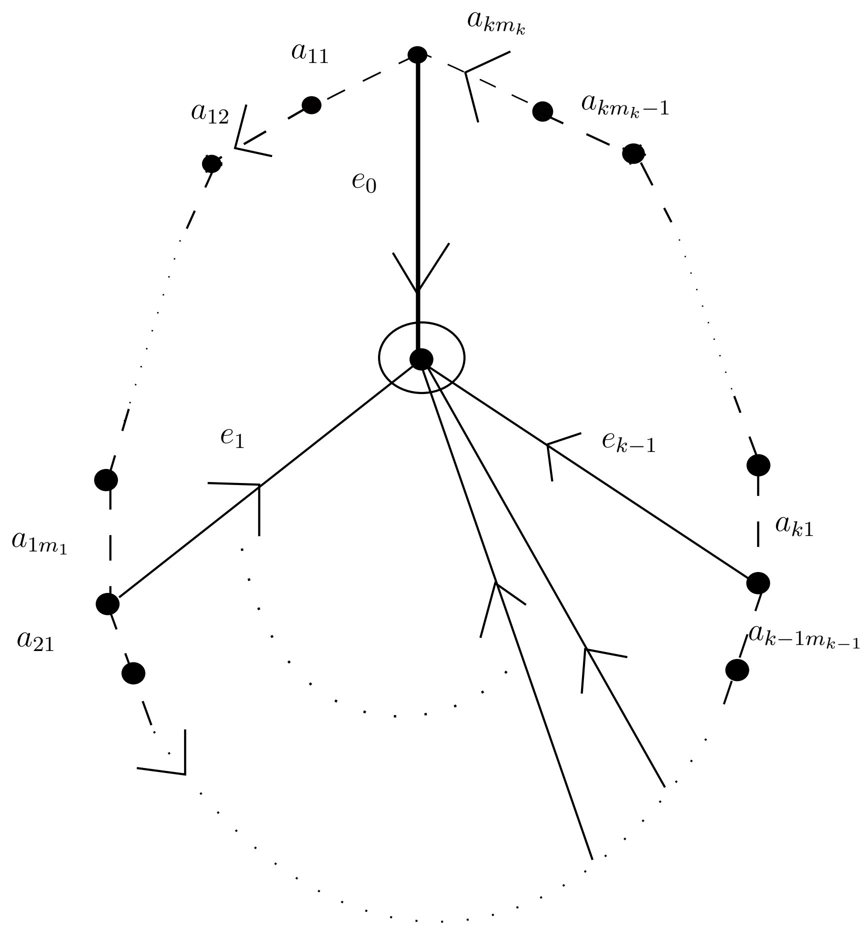

FIGURE 1

generating trees are graded by 1/2 times the total height. This is only a link invariant up to shift, but if $\mathcal{D}^i$ are the i-resolutions of $\mathcal{D}$ along a single edge, $i = 0, 1$, we have a long exact sequence

(3)
$$\ldots \to H^{i-1}(\mathcal{C}_0(\mathcal{D}^0)) \to H^{i-1/2}(\mathcal{C}_0(\mathcal{D}^1)) \to H^i(\mathcal{C}_0(\mathcal{D})) \to H^i(\mathcal{C}_0(\mathcal{D}^0)) \to \ldots$$

Denote by $B_{n_1,\ldots,n_k} = B(\mathcal{D}_{(n_1,\ldots,n_k)})$ the black graph in Figure 1. Denote the corresponding link by $L_{(n_1,\ldots,n_k)}$.

3. The link $L_{(3,3,0)}$

Our first result is the following

3.1. **Proposition.** *We have*

$$rank H^i(\mathcal{C}_0(\mathcal{D}_{(3,3,0)}) = \begin{cases} 1 & \text{if } i = 1,2 \\ 0 & \text{otherwise.} \end{cases}$$

The proof will occupy the remainder of this section. First note that $B_{(3,3,0)}$ has 18 spanning trees in heights 2 and 4 (and none others). The differential is, then, an 18×18 matrix N over a field of rational functions over $\mathbb{F}_2$ with ≥ 6 variables (it can be reduced from 9 to 6 by the Fundamental lemma of [3]). It follows then that

$$rank H^i(\mathcal{C}_0(\mathcal{D}_{(3,3,0)})) = \begin{cases} r & \text{if } i = 1,2 \\ 0 & \text{otherwise} \end{cases}$$

for some number $r = 0, 1, 2, \ldots$, and our statement is equivalent to saying that $r = 1$, i.e. that N has rank 17. This could, in principle, be checked by computer, but it exceeds the computing power of implementations of computer algebra softwares we could find.

Because of this, we simplified the problem as follows. Consider, instead, $\mathcal{D}_{(3,3)}$ (a projection of the knot 8_{19} in Rolfsen's table), and label its vertices and faces as in Figure 2.

Put

$$F_0 = \mathbb{F}_2(x, y_1, y_2, z_1, z_2, v),$$

$$F = F_0(T),$$

$$K = F(Q).$$

(The notation means adjoining algebraically independent variables, i.e. fields of rational functions.) The variables of F_0 are simply the vertex variables which occur in the definition of BOS cohomology; the variables T, Q are related to the face variables by

$$f_1 f_2 = T, \ f_1 = Q.$$

It will also be convenient for our purposes to put

$$A = \frac{1}{1+f_1} = \frac{1}{1+Q}, \quad B = \frac{1}{1+f_2} = \frac{Q}{Q+T}.$$

Denote by $T_{\epsilon,i,j}$, $\epsilon \in \{0,1\}$, $i \in \{1,2\}$, $j \in \{1,2,3\}$ the spanning tree of $B_{(3,3)}$ obtained by omitting the edge e_ϵ and a_{ij}. Denote by $T'_{i,j}$, $i, j \in \{1,2,3\}$ the spannig tree obtained by omitting the edges a_{1i} and a_{2j}.

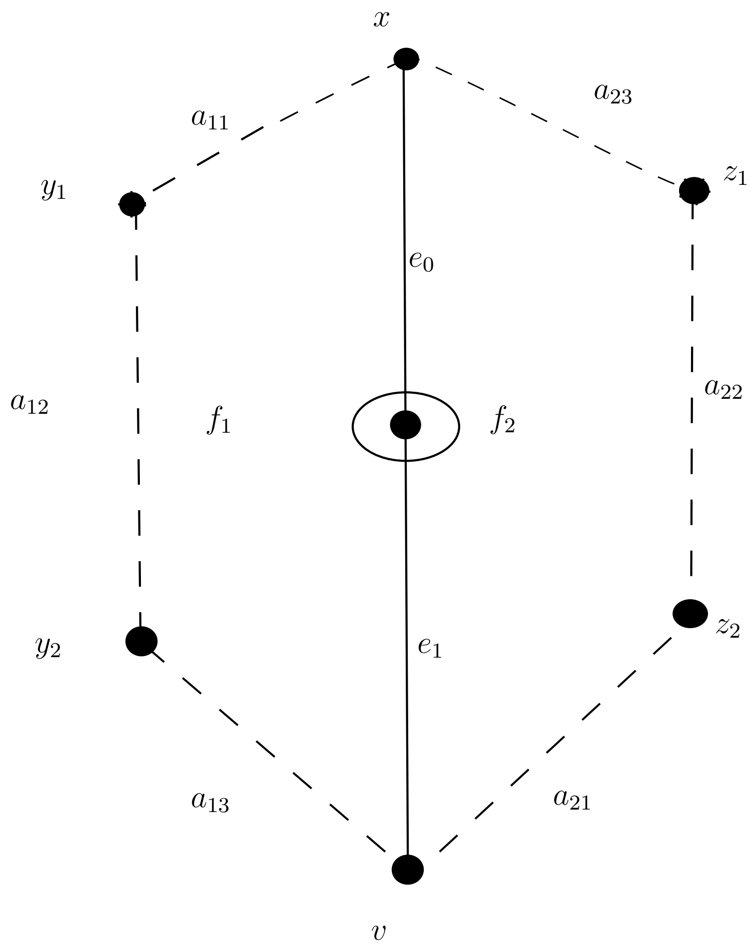

FIGURE 2

It will also be convenient to have matrix rows and columns indexed by simple numbers, so put

$$\begin{aligned} &u_0 = T_{0,2,3}, && u_1 = T_{0,2,2}, && u_2 = T_{0,2,1}, \\ &u_3 = T_{0,1,1}, && u_4 = T_{0,1,2}, && u_5 = T_{0,1,3}, \\ &u_6 = T_{1,2,3}, && u_7 = T_{1,2,2}, && u_8 = T_{1,2,1}, \\ &u_9 = T_{1,1,1}, && u_{10} = T_{1,1,2}, && u_{11} = T_{1,1,3} \end{aligned}$$

(these will correspond to columns) and

$$\begin{aligned} &v_1 = T'_{1,1}, && v_2 = T'_{2,1}, && v_3 = T'_{3,1}, \\ &v_4 = T'_{1,2}, && v_5 = T'_{2,2}, && v_6 = T'_{3.2}, \\ &v_7 = T'_{1,3}, && v_8 = T'_{2,3}, && v_9 = T'_{3,3} \end{aligned}$$

(these will correspond to rows) of the $\{1, \ldots, 9\} \times \{0, \ldots, 11\}$ matrix M of the differential Ψ of $\mathcal{C}_0(\mathcal{D}_{(3,3)})$. The entries of the matrix M are given explicitly by

$$M_{1,0} = A + \frac{1}{1+x}, \qquad M_{2,0} = A + \frac{1}{1+xy_1},$$

$$M_{3,0} = A + \frac{1}{1+xy_1y_2},$$

$$M_{4,1} = A + \frac{1}{1+xz_1}, \qquad M_{5,1} = A + \frac{1}{1+xy_1z_1},$$

$$M_{6,1} = A + \frac{1}{1+xy_1y_2z_1},$$

$$M_{7,2} = A + \frac{1}{1+xz_1z_2}, \qquad M_{8,2} = A + \frac{1}{1+xy_1z_1z_2},$$

$$M_{9,2} = A + \frac{1}{1+xy_1p_2z_1z_2},$$

$$M_{1,3} = B + 1 + \frac{1}{1+x}, \qquad M_{4,3} = B + 1 + \frac{1}{1+xz_1},$$

$$M_{7,3} = B + 1 + \frac{1}{1+xz_1z_2},$$

$$M_{2,4} = B + 1 + \frac{1}{1+xy_1}, \qquad M_{5,4} = B + 1 + \frac{1}{1+xy_1z_1},$$

$$M_{8,4} = B + 1 + \frac{1}{1+xy_1z_1z_2},$$

$$M_{3,5} = B + 1 + \frac{1}{1+xy_1y_2}, \qquad M_{6,5} = B + 1 + \frac{1}{1+xy_1y_2z_1},$$

$$M_{9,5} = B + 1 + \frac{1}{1+xy_1y_2z_1z_2}.$$

Additionally, the entry $M_{i,6+j}$ is obtained from the entry $M_{i,j}$, $i = 0,\dots,5$ by replacing A by $A+1$, $B+1$ by B (to account for a change of orientation) and the summand

$$\frac{1}{1+\zeta}$$

where ζ is any polynomial in x, y_1, y_2, z_1, z_2 by

$$\frac{1}{1+a/\zeta}$$

where

$$a = xy_1y_2z_1z_2v.$$

Unlisted entries $M_{i,j}$ are defined to be 0.

3.2. **Lemma.** *Consider the field $F(Q,Q')$ where Q,Q' are algebraically independent over$F$. Let $\phi : F(Q) \to F(Q,Q')$ be identity on F, and let $\phi(Q) = Q'$. Let V be the intersection of the row space of M with $\langle U_6, \dots, _{11} \rangle$ where U_i denotes the row vector with 1 in the column corresponding to u_i, and 0's in the other columns. Then $dim_{F(Q)}(V) = 3$. Additionally, let w_1, w_2, w_3 be a basis of V consisting of vectors for which there exist different $i_1, i_2, i_3 \in \{6, \dots, 11\}$ such that w_j has i_k'th coordinate equal to δ_j^k, $j,k \in \{1,2,3\}$. Then*

$$r = 3 - rank_{F(Q,Q')} \begin{pmatrix} w_1 - \phi(w_1) \\ w_2 - \phi(w_2) \\ w_3 - \phi(w_3) \end{pmatrix}. \tag{4}$$

Proof: The matrix M is a submatrix of the matrix of differentials of $\mathcal{C}_0(\mathcal{D}_{(3,3,0)}$. More explicitly, we will index things so that M is the $\{1, \dots, 9\} \times \{0, \dots, 11\}$ submatrix of the $\{1, \dots 18\} \times \{0, \dots, 17\}$ matrix N. Again, we will denote the rows of N by v_i and columns by u_j. Explicitly, in $B_{(3,3,0)}$, there are additional spanning trees $T_{2,i,j}$ which are obtained by replacing e_0 by e_2 in $T_{0,i,j}$. These correspond to additional columns

$$\begin{array}{lll} u_{12} = T_{2,2,3}, & u_{13} = T_{2,2,2}, & u_{14} = T_{2,2,1}, \\ u_{15} = T_{2,1,1}, & u_{16} = T_{2,1,2}, & u_{17} = T_{2,1,3}. \end{array}$$

There are also 9 additional spanning trees $T''_{i,j}$ obtained by replacing in $T'_{i,j}$ the edge e_0 by e_2. Let the row v_{9+i}, $i = 1, \dots, 9$, be obtained from the row v_i by replacing T' with T'', thus obtaining 9 additional rows. The additional non-zero entries of N are described as follows: The $(i+9, j+12)$-entry ($i = 1, \dots, 9$, $j = 0, \dots, 11$) is obtained from the (i,j)-entry by replacing A, B with A', B' where, in the field

$$K' = F(Q'),$$

$$A' = \frac{1}{1+Q'},\ B' = \frac{Q'}{Q'+T}.$$

(One has $Q' = f_1 g$, $T/Q' = f_2/g$ where g is the face between e_0 and e_2.)

Now let N_1 resp. N_2 be the $\{1, \dots 9\} \times \{0, \dots, 17\}$ resp. $\{10, \dots, 18\} \times \{0, \dots, 17\}$ submatrices. First note that the rank of each of the matrices N_1 and N_2 is 9 by the calculation of the BOS cohomology of 8_{19} in [3] (it is also verified by the computer-assisted calculation which we will describe below). This implies that the space V defined in the statement of the Lemma has

$$dim_{F(Q)}(V) = 3.$$

Now let w_1, w_2, w_3 be a basis as in the statement of the Lemma. By equality of row and column rank, r is the rank of the $F(Q,Q')$-space of $6-tuples$ $(\alpha_1,\dots,\alpha_6)\in F(Q,Q')^6$ such that

$$\alpha_1 w_1+\alpha_2 w_2+\alpha_3 w_3=\alpha_4\phi(w_1)+\alpha_5\phi(w_2)+\alpha_6\phi(w_3).$$

Obviously, however, by the assumptions about w_i, we must have

$$\alpha_i=\alpha_{3+i},\ i=1,2,3,$$

and the statement follows. □

Now to use the Lema, we first construct explicitly a non-zero element $w\in V$ which is of the form

$$w=\sum_{i=6}^{11}\alpha_i u_i,\ \alpha_i\in F_0.$$

This will show $r\geq 1$. To construct w, let

$$\begin{aligned}X&=V_1+V_2+V_4+V_5,\\ Y&=V_1+V_2+V_7+V_8,\\ Z&=V_1+V_3+V_4+V_6,\\ T&=V_1+V_3+V_7+V_9\end{aligned}$$

where we denote by V_i the i'th row vector of M. One sees immediately from the definition of the row vectors V_i that X,Y,Z,T are linear combinations of the vectors U_j, $j=0,\dots,11$ with coefficients in F_0.

Now putting

$$\begin{array}{ll}p_1=M_{4,1}+M_{5,1} & q_1=M_{2,4}+M_{5,4}\\ p_2=M_{7,2}+M_{8,2} & q_2=M_{2,4}+M_{8,4}\\ p_3=M_{4,1}+M_{6,1} & q_3=M_{3,5}+M_{6,5}\\ p_4=M_{7,2}+M_{9,2} & q_4=M_{3,5}+M_{9,5}\end{array}$$

(p_1 and p_3 are the u_1-coordinates of X,Z respectively, p_3 and p_4 are the u_2-coordinates of Y,T respectively, q_1 and q_2 are the u_4-coordinates of X,Y respectively, and q_3 and q_4 are the u_5-coordinates of Z,T respectively; those are all the non-zero u_1,u_2,u_4,u_5 coordinates of X,Y,Z,T).

Then one verifies by hand that

$$\frac{p_1p_4}{p_2p_3}=\frac{q_1q_4}{q_2q_3}.$$

This means that in the vector

$$w=X+\frac{q_2}{q_1}Y+\frac{p_3}{p_1}Z+\frac{p_4}{p_2}\frac{q_2}{q_1}T,$$

the u_1, u_2, u_4, u_5 coordinates vanish. One then checks, by hand again, that the u_0 and u_3 coordinates vanish as well, thus proving the desired statement about w.

Proving that $r \leq 1$ is done by using Lemma 3.2. We did this as follows: Since we are hoping to detect the absence of a relation at a generic point, it is possible to work at a special point (since a relation absent at a special point cannot occur at the generic point, using the argument made in detail at the end of [3]). Thus, we re-wrote the matrix M over the ring $R = F_0[A, B]$ where $F_0 = \mathbb{F}_2(t)$, setting

$$x = t,\ y_1 = t^2,\ y_2 = T,\ z_1 = t^4,\\ z_2 = t,\ v = t^6,\ a = t^{15}.$$

(The choices of the exponents are arbitrary, with the understanding that too special choices could create unwanted special relations; a field of rational functions in a single variable was chosen because computer algebra systems seem to work much more efficiently in that setting.) We then used Sage to execute manually the Buchberger algorithm for finding a Gröbner basis of $\langle V_1, \dots, V_9 \rangle$ ver the ring R, with lexicographic ordering $u_0 > u_1 > \cdots > u_{11} > A, B$ and degree-lex $A > B$ order in A, B (the latter of which was chosen because Sage naturally uses that ordering when working with $F_0[A, B]$). The main reason we worked manually is to be able to use heuristics (such as identifying the vector w above) for speeding up the algorithm. In 70 easy steps, the Gröbner basis elements we found had leading terms

$$u_0,\ u_1,\ u_2,\ u_5,\ AB^2u_6,\ A^2u_6,\\ u_7,\ Au_8,\ B^2u_8,\ A^2B^2u_9.$$

Note that for our purposes, having a Gröbner basis is actually irrelevant; again, it is merely a tool for performing Gauss elimination over the fraction field of R which, when done by brute force, would exceed the computational power of our current implementation of Sage. We may then get w_1, w_2, w_3 by taking w and our Gröbner basis vectors with leading terms Au_8 and $A^2B^2u_9$ and bringing them to reduced row echelon form, using the substitution

$$A = \frac{1}{1+Q},\ B = \frac{Q}{Q+T},\ T = t^10.$$

Again, the choice of T was arbitrary, hoping to avoid a special relation. As it turns out, when construction the reduced row echelon form, we can actually ignore w, since we already know it results in a zero row.

We used Sage to find by direct computation that $w_2 - \phi(w_2)$, $w_3 - \phi(w_3)$ are linearly independent (this took several minutes), thus concluding that $r = 1$. This concludes the proof of Proposition 3.1.

4. Other links with non-trivial BOS cohomologies and branched double covers with interesting $\widehat{HF}$-homologies

4.1. **Proposition.** *We have*

$$rank(H^i(\mathcal{C}_0(\mathcal{D}_{(3,3,k)}))) = \begin{cases} 1 & for\ i = 1, 2 \\ 0 & else. \end{cases}$$

Proof: We proceed by induction on k. For $k = 0$, this is the statement of Proposition 3.1. Suppose the statement is true for a given k. Consider the long exact sequence for the cohomology of $\mathcal{C}_0(\mathcal{D}_{(3,3,k+1)})$ obtained by resolving the edge $a_{3,k+1}$. Then the 1-resolution is actually an unlink with 2 components, and hence has 0 BOS cohomology. On the other hand, the 0-resolution is $\mathcal{D}_{(3,3,k)}$. Thus, from (3) we obtain

$$H^i(\mathcal{C}_0(\mathcal{D}_{(3,3,k+1)})) \cong H^i(\mathcal{C}_0(\mathcal{D}_{(3,3,k)})),$$

and the induction step is complete. □

4.2. **Corollary.** *We have*

$$rank(H^i(\mathcal{C}_0(\mathcal{D}_{(3,6)}))) = \begin{cases} 1 & for\ i = 1, 2 \\ 0 & else. \end{cases}$$

Proof: Consider the long exact sequence (3) form $\mathcal{D}_{(3,3,3)}$ resolving the edge e_0. The 0-resolution is $\mathcal{D}_{(3,6)}$, the 1-resolution is an unlink with 2 components, hence has trivial BOS cohomology. We conclude that

$$H^i(\mathcal{C}_0(\mathcal{D}_{(3,3,3)})) \cong H^i(\mathcal{C}_0(\mathcal{D}_{(3,6)})),$$

and the statement follows from Proposition 4.1. □

Unfortunately, all the examples of links for which we have computed non-trivial BOS cohomology so far have determinant 0, so we cannot use the BOS spectral sequence to make conclusions about $\widehat{HF}$ of their branched double covers. Consider now the black graph $B_k = B(\mathcal{E}_k)$, $k \geq 2$, depicted in Figure 3. Denote the corresponding link by L_k. This is a knot if $k \geq 3$ is odd and a link with two components when $k \geq 2$ is even.

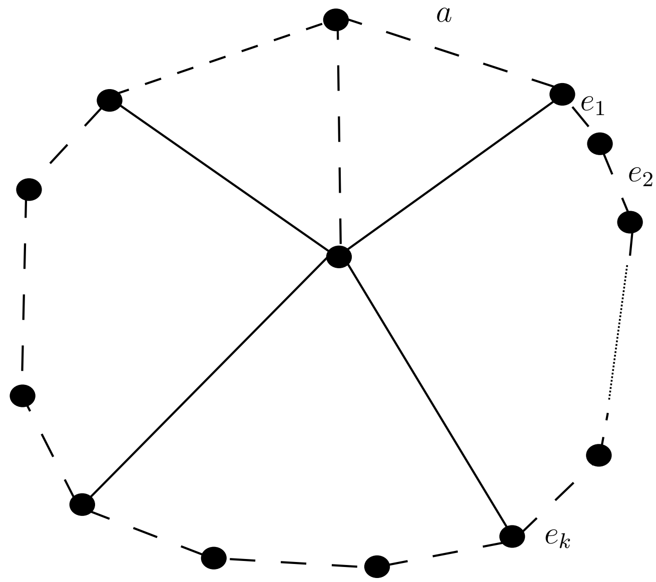

FIGURE 3

4.3. **Theorem.** *For $k \geq 2$,*

$$rankH^i(\mathcal{C}_0(\mathcal{E}_k)) = \begin{cases} 1 & \text{for } i = 3/2, 5/2 \\ k-2 & \text{for } i = 7/2 \\ 0 & \text{else.} \end{cases}$$

Proof: Resolve the projection $\mathcal{E}_k$ at the edge a. The 0-resolution $\mathcal{E}_k^0$ is $\mathcal{D}_{(3,3,k+1)}$ after undoing a single R2 move, thus,

$$H^i(\mathcal{C}_0(\mathcal{D}_{(3,3,k+1)})) \cong H^{i+1/2}(\mathcal{C}_0(\mathcal{E}_k^0)),$$

i.e.

$$rankH^i(\mathcal{C}_0(\mathcal{E}_k^0)) = \begin{cases} 1 & \text{for } i = 3/2, 5/2 \\ 0 & \text{else.} \end{cases}$$

On the other hand, the 1-resolution can be processed as follows: An R3 move combined with undoing a positive (non-height changing) R1 move gives a move shown in Figure 4.

Undoing three R2 moves and three positive R1 moves, we obtain the black graph shown in Figure 5 and, after undoing two R2 moves, we obtain a cycle of $k-2$ height 0 edges when $k > 2$, and an unlink of two components when $k = 2$.

We have then

$$rankH^i(\mathcal{C}_0(\mathcal{E}_k^1)) = \begin{cases} k-2 & \text{for } i = 3 \\ 0 & \text{else.} \end{cases}$$

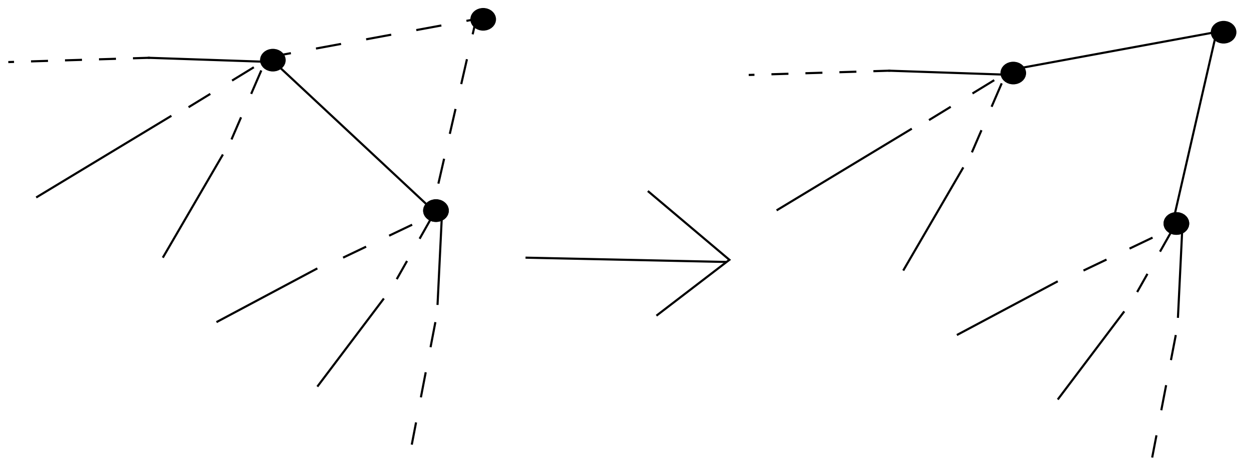

FIGURE 4

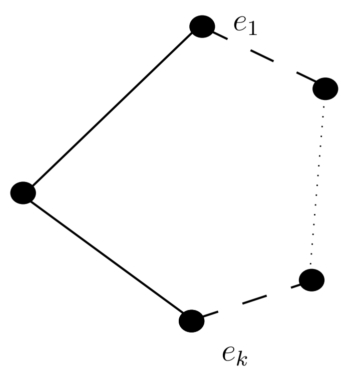

FIGURE 5

For $k = 2$, we are therefore done. For $k > 2$, we are done if we can show the following statement. □

4.4. **Lemma.** *The connecting map*

$$H^i(\mathcal{C}_0(\mathcal{E}_k^0)) \xrightarrow{\delta} H^{i+1/2}(\mathcal{C}_0(\mathcal{E}_k^1))$$

of (3) is 0 *for all* i.

Proof: For $k = 3$, $\mathcal{E}_3$ is actually a projection of the mirror of $T(7,3)$, whose $\widehat{HF}$ has rank 3 by [8]. Therefore, we must have $\delta = 0$ by the Baldwin-Ozsváth-Szabó spectral sequence.

Now our proof will be by induction on $k \geq 3$. Consider, for $k > 3$, the 0-resolutions $\mathcal{E}_k^{(1)}$, $\mathcal{E}_k^{(2)}$ of $\mathcal{E}_k$ at e_{k-1}, e_k. Then every spanning tree of the black graph of $\mathcal{E}_k$ gives rise to a spanning tree of $\mathcal{E}_k^{(i)}$ for $i = 1$

or $i = 2$, and the spanning trees which give rise to both give rise to spanning trees of the 0-resolution $\mathcal{E}_k^{(12)}$ at both e_{k-1}, e_k.

We have, then, a "Mayer-Vietoris exact sequence"

$$0 \longrightarrow mc_0(\mathcal{E}_k) \xrightarrow{\iota} \mathcal{C}_0(\mathcal{E}_k^{(1)}) \oplus \mathcal{C}_0(\mathcal{E}_k^{(2)}) \longrightarrow \mathcal{C}_0(\mathcal{E}_k^{(12)}) \longrightarrow 0. \tag{5}$$

One has, of course,

$$\mathcal{C}_0(\mathcal{E}_k^{(i)}) \cong \mathcal{C}_0(\mathcal{E}_{k-1}),$$
$$\mathcal{C}_0(\mathcal{E}_k^{(12)}) \cong \mathcal{C}_0(\mathcal{E}_{k-2}).$$

Moreover, the maps (5) induce maps of the long exact sequences corresponding to resolution at the edge a. In particular, we obtain a commutative square

$$\begin{array}{ccc} H^i(\mathcal{C}_0(\mathcal{E}_k^0)) & \xrightarrow{\delta} & H^{i+1/2}(\mathcal{C}_0(\mathcal{E}_k^1)) \\ \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} \\ H^i(\mathcal{C}_0(\mathcal{E}_k^{(1)0})) & & H^{i+1/2}(\mathcal{C}_0(\mathcal{E}_k^{(1)1})) \\ \oplus & \xrightarrow{\delta\oplus\delta} & \oplus \\ H^i(\mathcal{C}_0(\mathcal{E}_k^{(2)0})) & & H^{i+1/2}(\mathcal{C}_0(\mathcal{E}_k^{(2)1})). \end{array} \tag{6}$$

By the induction hypothesis, the bottom row satisfies $\delta \oplus \delta = 0$, while the left column of (6) is injective by our computation of the 0-resolutions (the two components omit the $(k-3)$'rd and $(k-2)$'nd summands of $F^{\oplus k}$ where F is the ground field, respectively). Since the left column of (6) is injective, the top row then satisfies $\delta = 0$, as claimed. This concludes the proof of the Lemma and hence the Theorem. □

4.5. **Theorem.** *For $k \geq 3$, we have*

$$rank\widehat{HF}(\Sigma(L_k)) = k,$$

while

$$det(L_k) = k - 2.$$

Proof: The computation of the determinant follows from Theorem 4.3 (since the determinant is, up to sign, the trace of BOS cohomology). Since $det(L_k) \neq 0$, the BOS spectral sequence then applies, with the E_3-term given by Theorem 4.3. By sparsity, no differential is possible, and hence the spectral sequence collapses to E_3 in this case. □

Comment: Note that while Theorems 4.3, 4.5 do not provide examples of non-collapse of the BOS spectral sequence, they exhibit interesting behavior in the sense of an "extension": The BOS cohomology of L_k has non-trivial elements in degrees 3/2 and 7/2, which are congruent modulo 2.

4.6. **Proposition.** *For all but finitely many values of $k > 3$, the link L_k (knot when k is odd) is hyperbolic*

Proof: The moves converting $\mathcal{E}_3$ to the mirror of the standard knot projection of $T(7,3)$ can be made in such a way that the crossing x corresponding to the edge e_2 in Figure 3 is not involved in any Reidemeister move. Form a link M_3 by adding an unknotted link component ℓ to L_3 encircling the crossing x. Using SnapPea, the link M_3 is hyperbolic with volume 6.551743287888. Now L_k for $k > 3$ can be obtained from M_3 by performing hyperbolic Dehn filling on the link component ℓ. Because of this, all but finitely many of the links L_k are hyperbolic by Thurston's theorem [9] (see also [1], Section 3). □

Comment: The only example of $k > 3$ we know for which L_k is not hyperbolic is $k = 5$. The knot L_5 is actually the mirror image of $T(8,3)$. The Jones polynomial of the mirror of L_k, $k \geq 3$, is

$$t^{(k+9)/2}(1 + t^2 - t^9 \frac{1 + t^{k-4}}{1 + t}).$$

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