Topological periodic cyclic homology of smooth $\mathbb{F}_p$-algebras

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The goal of this talk is to use theorems from previous talks to deduce certain calculations of topological periodic cyclic homology of smooth $k$-algebras, where $k$ is a perfect field of characteristic $p > 0$.

These calculations revolve around the motivic filtration constructed in [1] on the topological periodic cyclic homology of a quasisyntomic ring $A$, $\text{TP}(A)$; see [1, Theorem 1.12]. In the present situation, being quasisyntomic means that the cotangent complex of $\mathbb{L}_{A/k}$ has tor-amplitude $[-1, 0]$. We denote by $\text{QSyn}_k$ the full subcategory of $k$-algebras spanned by quasisyntomic $k$-algebras.

The motivic filtration is a descending filtration defined on the spectrum $\text{TP}(A)$:

$$
\text{TP}(A) \cdots \leftarrow \text{Fil}^{-1}\text{TP}(A) \leftarrow \text{Fil}^{0}\text{TP}(A) \leftarrow \text{Fil}^{1}\text{TP}(A) \leftarrow \cdots \text{Fil}^{n}\text{TP}(A) \cdots.
$$

By construction it agrees with the double-speed Postnikov filtration of spectra whenever $A$ is quasiregular semiperfect [1, Definition 8.8] — this just means that the cotangent complex $\mathbb{L}_{A/k}$ is a flat module concentrated in homological degree 1 and the Frobenius on $A$ is surjective. The first calculation is an identification of the associated graded of the motivic filtration.

**Theorem 0.1.** Suppose that $A$ is a smooth $k$-algebra where $k$ is a perfect field of characteristic $p > 0$, then there is an equivalence in, $\text{D}(\mathbb{W}(k))$, the derived category of $\mathbb{W}(k)$-modules

$$
\text{gr}^n\text{TP}(A) \simeq R\Gamma_{\text{crys}}(A/\mathbb{W}(k))[2n].
$$

In fact, the associated graded $\text{gr}^0\text{TP}(A)$ identifies with the derived global sections of a certain homotopy sheaf which we now describe. We have a presheaf of commutative $\mathbb{W}(k)$-algebras on $\text{QSyn}_k^{\text{op}}$

$$
\pi_0\text{TP}(-) : \text{QSyn}_k \rightarrow \text{CAlg}_{\mathbb{W}(k)}.
$$

We endow $\text{QSyn}_k^{\text{op}}$ with the quasisyntomic topology where the covers are faithfully flat maps $A \rightarrow B$ in $\text{QSyn}_k$ such that the cotangent complex $\mathbb{L}_{B/A}$ has tor-amplitude in $[-1, 0]$. Suppose that $A \in \text{QSyn}_k$, then we consider derived global sections of this presheaf restricted to $\text{QSyn}_A := (\text{QSyn}_k)/A$, with respect to the quasisyntomic topology. This is an $\mathbb{E}_\infty$-$\mathbb{W}(k)$-algebras which we denote by $R\Gamma_{\text{syn}}(A; \pi_0\text{TP}(-))$ and we have an equivalence

$$
\text{gr}^0\text{TP}(-) \simeq R\Gamma_{\text{syn}}(A; \pi_0\text{TP}(-)).
$$

This is a consequence of quasisyntomic descent for the presheaf of spectra $\text{TP}(-)$ [1, Corollary 3.3]. Specializing (1) to $n = 0$ we obtain an equivalence of $\mathbb{E}_\infty$-$\mathbb{W}(k)$-algebras

$$
R\Gamma_{\text{syn}}(A; \pi_0\text{TP}(-)) \simeq R\Gamma_{\text{crys}}(A/\mathbb{W}(k)),
$$

which is [1, Theorem 1.10].

As a result of Theorem 0.1 the spectral sequence obtained from the motivic filtration is of the form

$$
E^2_{i, j} = \pi_{i+j}(\text{gr}^{-i}\text{TP}(A)) \cong H^j_{\text{crys}}(A/\mathbb{W}(k)) \Rightarrow \text{TP}_{i+j}(A),
$$

where
where the differentials are of the form
\[ d^r : E^r_{i,j} \to E^r_{i-r,j+r-1}. \]

One can think of the graded pieces as the “weight” of the motivic filtration (see §2 for how the Adams operations sort out the weights). Setting \( h^{i,j} := \pi_{i+j}(\text{gr}^{-j} \text{TP}(A)) \). The spectral sequence then displays as

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In the spectral sequence displayed above, the divided Bott element discussed in [2, Section 4] lies in the term \( h^{0,1} \) with total degree 2. We call this element \( \sigma \). The next theorem states that the motivic filtration splits after inverting \( p \) and, thus, the spectral sequence degenerates at the \( E_2 \)-page. More precisely:

**Theorem 0.2.** Suppose that \( A \) is a smooth \( k \)-algebra where \( k \) is a perfect field of characteristic \( p > 0 \), then we have an equivalence of \( E_\infty \)-\( W(k) \)-algebras
\[
\text{TP}(A)[\frac{1}{p}] \simeq R\Gamma_{\text{cris}}(A/W(k))[\frac{1}{p}][\sigma, \sigma^{-1}]
\]
where \( |\sigma| = 2 \).

The proof of this theorem will exploit the fact that the Adams operations acts by different eigenvalues on each of the associated graded pieces.

1. **Proof of Theorem 0.1**

Recall that, by [3], the crystalline cohomology of a smooth \( k \)-algebra \( A \) can be computed as the cohomology of the de Rham-Witt complex \( W\Omega_A/k \). We first claim that

**Proposition 1.1.** For any smooth \( k \)-algebra \( A \), the commutative \( W(k) \)-algebra \( W\Omega_A/k \) computes the derived global sections of the presheaf \( \pi_0 \text{TP}(-)|_{\mathcal{QSyn}_A} \).
Proof. According to [1, Theorem 8.15], for any quasiregular semiperfect $k$-algebra $A$, we have an equivalence of commutative $W(k)$-algebras

\[(2) \quad \widehat{LW} \Omega_{A/k} \simeq \pi_0 TP(A),\]

where $\widehat{LW} \Omega_{A/k}$ is the Nygaard completed derived de Rham-Witt complex. By construction, this is the value on $A$ of the left Kan extension of the de Rham-Witt complex along the inclusion of polynomial $k$-algebras to $\text{QSyn}_A$, and then completed with respect to the Nygaard filtration; see [1, Section 8.1] for details.

We claim two properties about the derived de Rham-Witt complex:

(1) the presheaf on $\text{QSyn}_k^{\text{op}},$

\[\widehat{LW} \Omega_{-}/k : \text{QSyn}_k \rightarrow D(W(k)),\]

is a sheaf for the quasisyntomic topology, and

(2) the restriction of $\widehat{LW} \Omega_{-}/k$ to $\text{SmAff}_k$ agrees with $W \Omega_{(-)/k}$.

Let us prove the proposition assuming these two properties. Let $A_{\text{perf}}$ be the colimit

\[A \xrightarrow{\phi} A \xrightarrow{\phi} A \cdots,\]

where $\phi$ is the Frobenius. Then $A_{\text{perf}}$ is quasiregular semiperfect and furthermore the map $A \rightarrow A_{\text{perf}}$ is a quasisyntomic cover; the map is faithfully flat using the characterization of regularity via the Frobenius (for a general result see [4], but this fact is an easier exercise in this setting). With this, we get the following string of equivalences in $\text{CAlg}(D(W(k)))$:

\[
\begin{align*}
R \Gamma_{\text{syn}}(A, \pi_0 TP(A)) & \simeq \lim_{\Delta} \pi_0 TP(A_{\text{perf}}) \\
& \simeq \lim_{\Delta} \widehat{LW} \Omega_{A_{\text{perf}}}/k \\
& \simeq \widehat{LW} \Omega_{A/k} \\
& \simeq W \Omega_{A/k}.
\end{align*}
\]

We now prove the first of the claimed properties of $\widehat{LW} \Omega_{A/k}$. Taking its mod-$p$ reduction gives an equivalence [1, Theorem 8.14.5] in $D(k)$

\[
\widehat{LW} \Omega_{A/k}/p \simeq \widehat{L} \Omega_{A/k}.
\]

where the right hand side is the Hodge completed derived de Rham complex, defined by an analogous Kan extension and completion procedure for the deRham complex. Since $\widehat{LW} \Omega_{A/k}$ is $p$-complete for all $A \in \text{QSyn}_k$, it suffices to check descent after reduction mod-$p$ and thus we need to check descent for the presheaf $L \Omega_{(-)/k}$.

This is a consequence of quasisyntomic descent for the cotangent complex, and its exterior powers [1, Theorem 3.1].

To check the second property, we recall that, Zariski-locally, the structure map of a smooth $k$-algebra $A$ is of the form $k \rightarrow k[x_1 \cdots, x_n] \xrightarrow{g} A$ where $g$ is étale. Since the Nygaard completed derived deRham-Witt complex has Zariski descent and its value agrees with the de Rham-Witt complex on polynomial $k$-algebras, it
suffices to check that the derived deRham-Witt satisfies étale base change. This can again be checked after reduction mod $p$.

Proposition 1.1 proves the case $n = 0$ of Theorem 0.1. To obtain Theorem 0.1, we use the periodicity of $\text{TP}(A)$ [1, Section 6] to deduce that

$$\text{gr}^n\text{TP}(A) \simeq \text{gr}^0\text{TP}(A)[2n],$$

where the equivalence is given by multiplication by $\sigma^n$.

2. Proof of Theorem 0.2

Since any $k$-algebra is $p$-complete, we have that $\text{THH}(A) \simeq A^\otimes_{T_p}$. Now, $T_p^\wedge \simeq K(Z_p, 1)$ and so its space of automorphisms identifies with the units of $\Omega K(Z_p, 1)$, i.e., the group $Z_p^\times$. Each $\ell \in Z_p^\times$ then defines an Adams operation

$$(3) \quad \psi^\ell : \text{THH}(A) \xrightarrow{\text{id}} \text{THH}(A) \simeq A^\otimes_{T_p^\wedge},$$

which is a map of $E_\infty$-ring spectra, but is not $T$-equivariant for the usual $T$-action on $\text{THH}(A)$.

We can construct a version of the Adams operation which is $T$-equivariant after “speeding up” the $T$-action on the target by multiplication by $\ell$. Indeed, consider the self-map $m_\ell : T \to T; z \mapsto z^\ell$. For any $T$-spectrum $E$ we define the $T$-spectrum $E_{\text{reparm}}$ where the underlying spectrum is $E$, and the $T$-action is informally described by

$$T \otimes E \xrightarrow{m_\ell \otimes \text{id}} T \otimes E \xrightarrow{\text{act}} T \otimes E.$$ More precisely, restriction along $m_\ell : T \to T$ induces a functor $(m_\ell)^* : \text{Sp}^{BT}_T \to \text{Sp}^{BT}_T$. The $T$-spectrum $E_{\text{reparm}}$ is defined, as a $T$-spectrum, as $(m_\ell)^* E$.

In the case of $\text{THH}(A)$, we get the following more explicit description. We denote by $(T_p^\wedge)_{\text{reparm}}$ the $p$-complete circle equipped with an action of $T$ “sped up by $\ell$”; the point now is that the map $m_\ell : T_p^\wedge \to (T_p^\wedge)_{\text{reparm}}$ is $T$-equivariant and thus the map (3)

$$\psi^\ell : \text{THH}(A) \xrightarrow{(\text{id})^{\otimes \ell}} \text{THH}(A)_{\text{reparm}} \simeq A^\otimes_{(T_p^\wedge)^{\text{reparm}}},$$

is $T$-equivariant.

We have the following observation

**Lemma 2.1.** Let $\widehat{Sp}_p$ denote the $\infty$-category of $p$-complete spectra and $\ell \in Z_p^\times$. Then the functor $(m_\ell)^* : (\widehat{Sp}_p)^{BT}_T \to (\widehat{Sp}_p)^{BT}_T$ is an equivalence of $\infty$-categories.

**Proof.** For any $p$-complete spectrum $E$, the $T$-action factors uniquely through a $T_p^\wedge$-action, hence we are left to prove that the induced functor $(m_\ell)^* : (\widehat{Sp}_p)^{BT}_T \to (\widehat{Sp}_p)^{BT}_T$ is an equivalence of $\infty$-categories. Since $\ell$ acts invertibly on $T_p^\wedge$, we have an inverse operation $\ell^{-1}$ on $T_p^\wedge$. The induced functor $(m_{\ell^{-1}})^* : (\widehat{Sp}_p)^{BT}_T \to (\widehat{Sp}_p)^{BT}_T$ is the inverse to $(m_\ell)^*$.

$\square$
As a result, $\text{THH}(A)^{\text{reparm}} \simeq \text{THH}(A)$ as $\mathbb{T}$-spectra and thus we get an induced operation
\[ \psi^\ell : \text{TP}(A) \overset{(\psi^\ell)^T}{\rightarrow} (\text{THH}(A)^{\text{reparm}})^T \simeq \text{TP}(A). \]
We also note that we get Adams operations compatibly on homotopy fixed points, orbits and thus on $\text{TC}$:
\[ \psi^\ell : \text{TC}(A) \rightarrow \text{TC}(A). \]

**Proposition 2.2.** [1, Proposition 9.14] The Adams operation $\psi^\ell$ acts on $\text{gr}^n \text{TP}(A)$ by multiplication with $\ell^n$. In particular if we invert $p$, then we have an isomorphism of $\mathbb{Q}$-vector spaces
\[ (\pi^* \text{gr}^n \text{TP}(A)[\frac{1}{p}])^{\psi^\ell - \ell^n} \simeq \pi^* \text{gr}^n \text{TP}(A)[\frac{1}{p}]. \]

To prove Theorem 0.2 we consider the diagram of spectra
\[ \oplus_{n \in \mathbb{Z}} (\text{Fil}^n \text{TP}(A)[\frac{1}{p}])^{\psi^\ell - \ell^n} \rightarrow \text{TP}(A)[\frac{1}{p}] \]
\[ \oplus_{n \in \mathbb{Z}} (\text{gr}^n \text{TP}(A)[\frac{1}{p}])^{\psi^\ell - \ell^n} \]
\[ \oplus_{n \in \mathbb{Z}} (\text{Fil}^n \text{TP}(A)[\frac{1}{p}])^{\psi^\ell - \ell^n} \simeq \oplus_{n \in \mathbb{Z}} (\text{gr}^n \text{TP}(A)[\frac{1}{p}]), \]
where, for any spectrum $E$ with an action of $\psi^\ell$, we define
\[ E^{\psi^\ell - \ell^n} := \text{fib}(E \overset{\psi^\ell - \ell^n}{\rightarrow} E). \]
Since $\pi_* \text{TP}(A)[\frac{1}{p}]$ is a graded $\mathbb{Q}$-vector space, the top horizontal map induces an injection on homotopy groups $\oplus_{n \in \mathbb{Z}} (\pi_* \text{TP}(A)[\frac{1}{p}])^{\psi^\ell - \ell^n} \hookrightarrow \pi_* \text{TP}(A)[\frac{1}{p}]$. It then suffices to prove that vertical arrows are equivalences of spectra, whence we have a map of filtered $\mathbb{Q}$-vector spaces which is an isomorphism on graded pieces. The multiplicative properties of the filtration [1, Theorem 1.12.1] gives us the conclusion of Theorem 0.2.

To check the claimed equivalence, we consider the action of the Adams operation $\psi^\ell$ on $\text{TP}(A)[\frac{1}{p}]$ as endowing it with the structure as a module over $S[\psi^\ell]^1$. The functoriality of the Adams operations tells us that the cofiber sequence of spectra
\[ \text{Fil}^n \text{TP}(A)[\frac{1}{p}] \rightarrow \text{TP}(A)[\frac{1}{p}] \rightarrow \text{TP}(A)[\frac{1}{p}]/\text{Fil}^n \text{TP}(A)[\frac{1}{p}], \]

1This means the spherical monoid algebra of the free monoid on one generator $\psi^\ell$. In other words, taking Spec of this derived ring gives us the “flat affine line” over the sphere spectrum.
is a cofiber sequence of $\mathbb{S}[\psi^\ell]$-modules. We first claim that $(\text{Fil}^n_{1p}\text{TP}(A)[\frac{1}{p}])^{\psi^\ell-\ell^n}$ is contractible. Indeed, the we have a filtration on $\text{Fil}^n_{1p}\text{TP}(A)[\frac{1}{p}]$ given by

$$\{\text{Fil}^k\text{TP}(A)[\frac{1}{p}] \mid k>n\}.$$

Where the associated graded are $\{\text{gr}^k\text{TP}(A)[\frac{1}{p}] \mid k>n\}$. Proposition 2.2 tells us that the action of $\psi^\ell-\ell^n$ on $\text{gr}^k\text{TP}(A)[\frac{1}{p}]$ is homotopic to the action of $\ell^k-\ell^n$ and so is invertible. An induction argument shows that the action of $\psi^\ell-\ell^n$ on $\text{Fil}^n_{1p}\text{TP}(A)[\frac{1}{p}]$ is thus invertible and so we have an equivalence of fibers

$$\text{Fil}^n\text{TP}(A)[\frac{1}{p}]^{\psi^\ell-\ell^n} \rightarrow (\text{TP}(A)[\frac{1}{p}])^{\psi^\ell-\ell^n},$$

which tells us that the top vertical arrow of (4) is an equivalence. A similar argument applied to the cofiber sequence

$$\text{Fil}^{n+1}\text{TP}(A)[\frac{1}{p}] \rightarrow \text{Fil}^n\text{TP}(A)[\frac{1}{p}] \rightarrow \text{gr}^n\text{TP}(A)[\frac{1}{p}]$$

tells us that the bottom vertical arrow is an equivalence.

**References**


