Topological periodic cyclic homology of smooth $\mathbb{F}_p$-algebras

Elden Elmanto

The goal of this talk is to use theorems from previous talks to deduce certain calculations of topological periodic cyclic homology of smooth k-algebras, where k is a perfect field of characteristic $p > 0$.

These calculations revolve around the *motivic filtration* constructed in [1] on the topological periodic cyclic homology of a *quasisyntomic ring* A, $\mathrm{TP}(A)$; see [1, Theorem 1.12]. In the present situation, being quasisyntomic means that the cotangent complex of $\mathbb{L}_{A/k}$ has tor-amplitude $[-1, 0]$. We denote by QSyn_k the full subcategory of k-algebras spanned by quasisyntomic k-algebras.

The motivic filtration is a descending filtration defined on the spectrum $\mathrm{TP}(A)$:

$$\mathrm{TP}(A) \cdots \leftarrow \mathrm{Fil}^{-1}\mathrm{TP}(A) \leftarrow \mathrm{Fil}^{0}\mathrm{TP}(A) \leftarrow \mathrm{Fil}^{1}\mathrm{TP}(A) \leftarrow \cdots \mathrm{Fil}^{n}\mathrm{TP}(A) \cdots .$$

By construction it agrees with the double-speed Postnikov filtration of spectra whenever A is *quasiregular semiperfect* [1, Definition 8.8] — this just means that the cotangent complex $\mathbb{L}_{A/k}$ is a flat module concentrated in homological degree 1 and the Frobenius on A is surjective. The first calculation is an identification of the associated graded of the motivic filtration.

Theorem 0.1. *Suppose that A is a smooth k-algebra where k is a perfect field of charecteristic $p > 0$, then there is an equivalence in, $\mathrm{D}(W(k))$, the derived category of $W(k)$-modules*

$$\mathrm{gr}^n\mathrm{TP}(A) \simeq R\Gamma_{\mathrm{crys}}(A/W(k))[2n]. \tag{1}$$

In fact, the associated graded $\mathrm{gr}^0\mathrm{TP}(A)$ identifies with the derived global sections of a certain homotopy sheaf which we now describe. We have a presheaf of commutative $W(k)$-algebras on $\mathrm{QSyn}_k^{\mathrm{op}}$

$$\pi_0\mathrm{TP}(-) : \mathrm{QSyn}_k \to \mathrm{CAlg}_{W(k)}.$$

We endow $\mathrm{QSyn}_k^{\mathrm{op}}$ with the *quasisyntomic topology* where the covers are faithfully flat maps $A \to B$ in QSyn_k such that the cotangent complex $\mathbb{L}_{B/A}$ has tor-amplitude in $[-1, 0]$. Suppose that $A \in \mathrm{QSyn}_k$, then we consider derived global sections of this presheaf restricted to $\mathrm{QSyn}_A := (\mathrm{QSyn}_k)_{/A}$, with respect to the quasisyntomic topology. This is an $\mathbb{E}_\infty$-$W(k)$-algebras which we denote by $R\Gamma_{\mathrm{syn}}(A; \pi_0\mathrm{TP}(-))$ and we have an equivalence

$$\mathrm{gr}^0\mathrm{TP}(-) \simeq R\Gamma_{\mathrm{syn}}(A; \pi_0\mathrm{TP}(-)).$$

This is a consequence of quasisyntomic descent for the presheaf of spectra $\mathrm{TP}(-)$ [1, Corollary 3.3]. Specializing (1) to $n = 0$ we obtain an equivalence of $\mathbb{E}_\infty$-$W(k)$-algebras

$$R\Gamma_{\mathrm{syn}}(A; \pi_0\mathrm{TP}(-)) \simeq R\Gamma_{\mathrm{crys}}(A/W(k)),$$

which is [1, Theorem 1.10].

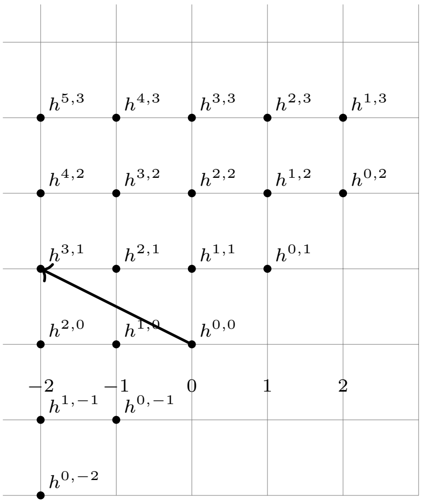
As a result of Theorem 0.1 the spectral sequence obtained from the motivic filtration is of the form

$$E^2_{i,j} = \pi_{i+j}(\mathrm{gr}^{-j}\mathrm{TP}(A)) \cong H^{j-i}_{\mathrm{crys}}(A/W(k)) \Rightarrow \mathrm{TP}_{i+j}(A),$$

where the differentials are of the form

$$d^r : E^r_{i,j} \to E^r_{i-r,j+r-1}.$$

One can think of the graded pieces as the "weight" of the motivic filtration (see §2 for how the Adams operations sort out the weights). Setting $h^{i,j} := \pi_{i+j}(\mathrm{gr}^{-j}\mathrm{TP}(A))$. The spectral sequence then displays as



In the spectral sequence displayed above, the *divided Bott element* discussed in [2, Section 4] lies in the term $h^{0,1}$ with total degree 2. We call this element σ. The next theorem states that the motivic filtration splits after inverting p and, thus, the spectral sequence degenerates at the E_2-page. More precisely:

Theorem 0.2. *Suppose that A is a smooth k-algebra where k is a perfect field of charecteristic $p > 0$, then we have an equivalence of $\mathbb{E}_\infty$-$W(k)$-algebras*

$$\mathrm{TP}(A)[\tfrac{1}{p}] \simeq R\Gamma_{\mathrm{crys}}(A/W(k))[\tfrac{1}{p}][\sigma, \sigma^{-1}]$$

where $|\sigma| = 2$.

The proof of this theorem will exploit the fact that the Adams operations acts by different eigenvalues on each of the associated graded pieces.

1. Proof of Theorem 0.1

Recall that, by [3], the crystalline cohomology of a smooth k-algebra A can be computed as the cohomology of the de Rham-Witt complex $W\Omega_{A/k}$. We first claim that

Proposition 1.1. *For any smooth k-algebra A, the commutative $W(k)$-algebra $W\Omega_{A/k}$ computes the derived global sections of the presheaf $\pi_0\mathrm{TP}(-)|_{\mathrm{QSyn}_A}$.*

Proof. According to [1, Theorem 8.15], for any quasiregular semiperfect k-algebra A, we have an equivalence of commutative $W(k)$-algebras

$$\widehat{LW\Omega}_{A/k} \simeq \pi_0 \mathrm{TP}(A), \tag{2}$$

where $\widehat{LW\Omega}_{A/k}$ is the *Nygaard completed derived de Rham-Witt complex.* By construction, this is the value on A of the left Kan extension of the de Rham-Witt complex along the inclusion of polynomial k-algebras to QSyn_A, and then completed with respect to the Nygaard filtration; see [1, Section 8.1] for details. We claim two properties about the derived de Rham-Witt complex:

(1) the presheaf on $\mathrm{QSyn}_k^{\mathrm{op}}$,

$$\widehat{LW\Omega}_{-/k} : \mathrm{QSyn}_k \to \mathrm{D}(W(k)),$$

is a sheaf for the quasisyntomic topology, and

(2) the restriction of $\widehat{LW\Omega}_{-/k}$ to SmAff_k agrees with $W\Omega_{(-)/k}$.

Let us prove the proposition assuming these two properties. Let A_{perf} be the colimit

$$A \xrightarrow{\phi} A \xrightarrow{\phi} A \cdots ,$$

where ϕ is the Frobenius. Then A_{perf} is quasiregular semiperfect and furthermore the map $A \to A_{\mathrm{perf}}$ is a quasisyntomic cover; the map is faithfully flat using the characterization of regularity via the Frobenius (for a general result see [4], but this fact is an easier exercise in this setting). With this, we get the following string of equivalences in $\mathrm{CAlg}(\mathrm{D}(W(k)))$:

$$\begin{aligned} R\Gamma_{\mathrm{syn}}(A, \pi_0\mathrm{TP}(A)) &\simeq \lim_{\Delta} \pi_0 \mathrm{TP}(A_{\mathrm{perf}}^{\otimes_A \bullet}) \\ &\simeq \lim_{\Delta} \widehat{LW\Omega}_{A_{\mathrm{perf}}^{\otimes_A \bullet}/k} \\ &\simeq \widehat{LW\Omega}_{A/k} \\ &\simeq W\Omega_{A/k}. \end{aligned}$$

We now prove the first of the claimed properties of $\widehat{LW\Omega}_{A/k}$. Taking its mod-p reduction gives an equivalence [1, Theorem 8.14.5] in $\mathrm{D}(k)$

$$\widehat{LW\Omega}_{A/k}/p \simeq \widehat{L\Omega_{A/k}}.$$

where the right hand side is the *Hodge completed derived de Rham complex*, defined by an analogous Kan extension and completion procedure for the deRham complex. Since $\widehat{LW\Omega}_{A/k}$ is p-complete for all $A \in \mathrm{QSyn}_k$, it suffices to check descent after reduction mod-p and thus we need to check descent for the presheaf $\widehat{L\Omega_{(-)/k}}$. This is a consequence of quasisyntomic descent for the cotangent complex, and its exterior powers [1, Theorem 3.1].

To check the second property, we recall that, Zariski-locally, the structure map of a smooth k-algebra A is of the form $k \to k[x_1 \cdots, x_n] \xrightarrow{g} A$ where g is étale. Since the Nygaard completed derived deRham-Witt complex has Zariski descent and its value agrees with the de Rham-Witt complex on polynomial k-algebras, it suffices to check that the derived deRham-Witt satisfies étale base change. This can again be checked after reduction mod p. $\square$

Proposition 1.1 proves the case $n = 0$ of Theorem 0.1. To obtain Theorem 0.1, we use the periodicity of $\mathrm{TP}(A)$ [1, Section 6] to deduce that

$$\mathrm{gr}^n\mathrm{TP}(A) \simeq \mathrm{gr}^0\mathrm{TP}(A)[2n],$$

where the equivalence is given by multiplication by σ^n.

2. Proof of Theorem 0.2

Since any k-algebra is p-complete, we have that $\mathrm{THH}(A) \simeq A^{\otimes \mathbb{T}_p^\wedge}$. Now, $\mathbb{T}_p^\wedge \simeq K(\mathbb{Z}_p, 1)$ and so its space of automorphisms identifies with the units of $\Omega K(\mathbb{Z}_p, 1)$, i.e., the group $\mathbb{Z}_p^\times$. Each $\ell \in \mathbb{Z}_p^\times$, then defines an *Adams operation*

$$\psi^\ell : \mathrm{THH}(A) \simeq A^{\otimes \mathbb{T}_p^\wedge} \overset{(\mathrm{id}_A)^{\otimes \ell\cdot}}{\to} \mathrm{THH}(A) \simeq A^{\otimes \mathbb{T}_p^\wedge}, \tag{3}$$

which is a map of $\mathbb{E}_\infty$-ring spectra, but is *not* $\mathbb{T}$-equivariant for the usual $\mathbb{T}$-action on $\mathrm{THH}(A)$.

We can construct a version of the Adams operation which is $\mathbb{T}$-equivariant after "speeding up" the $\mathbb{T}$-action on the target by multiplication by ℓ. Indeed, consider the self-map $m_\ell : \mathbb{T} \to \mathbb{T}; z \mapsto z^\ell$. For any $\mathbb{T}$-spectrum E we define the $\mathbb{T}$-spectrum E^{reparm_ℓ} where the underlying spectrum is E, and the $\mathbb{T}$-action is informally described by

$$\mathbb{T} \otimes E \overset{m_\ell \otimes \mathrm{id}}{\longrightarrow} \mathbb{T} \otimes E \overset{\mathrm{act}}{\longrightarrow} \mathbb{T} \otimes E.$$

More precisely, restriction along $m_\ell : \mathbb{T} \to \mathbb{T}$ induces a functor $(m_\ell)^* : \mathrm{Sp}^{\mathrm{B}\mathbb{T}} \to \mathrm{Sp}^{\mathrm{B}\mathbb{T}}$. The $\mathbb{T}$-spectrum E^{reparm_ℓ} is defined, as a $\mathbb{T}$-spectrum, as $(m_\ell)^*E$.

In the case of $\mathrm{THH}(A)$, we get the following more explicit description. we denote by $(\mathbb{T}_p^\wedge)^{\mathrm{reparm}}$ the p-complete circle equipped with an action of $\mathbb{T}$ "sped up by ℓ"; the point now is that the map $m_\ell : \mathbb{T}_p^\wedge \to (\mathbb{T}_p^\wedge)^{\mathrm{reparm}_\ell}$ is $\mathbb{T}$-equivariant and thus the map (3)

$$\psi^\ell : \mathrm{THH}(A) \overset{(\mathrm{id})^{\otimes \ell\cdot}}{\longrightarrow} \mathrm{THH}(A)^{\mathrm{reparm}_\ell} \simeq A^{\otimes (\mathbb{T}_p^\wedge)_\ell^{\mathrm{reparm}}},$$

is $\mathbb{T}$-equivariant.

We have the following observation

Lemma 2.1. *Let $\widehat{\mathrm{Sp}}_p$ denote the ∞-category of p-complete spectra and $\ell \in \mathbb{Z}_p^\times$. Then the functor $(m_\ell)^* : (\widehat{\mathrm{Sp}}_p)^{\mathrm{B}\mathbb{T}} \to (\widehat{\mathrm{Sp}}_p)^{\mathrm{B}\mathbb{T}}$ is an equivalence of ∞-categories.*

Proof. For any p-complete spectrum E, the $\mathbb{T}$-action factors uniquely through a $\mathbb{T}_p^\wedge$-action, hence we are left to prove that the induced functor $(m_\ell)^* : (\widehat{\mathrm{Sp}}_p)^{\mathrm{B}\mathbb{T}_p^\wedge} \to (\widehat{\mathrm{Sp}}_p)^{\mathrm{B}\mathbb{T}_p^\wedge}$ is an equivalence of ∞-categories. Since ℓ acts invertibly on $\mathbb{T}_p^\wedge$, we have an inverse operation ℓ^{-1} on $\mathbb{T}_p^\wedge$. The induced functor $(m_{\ell^{-1}})^*$ on $(\widehat{\mathrm{Sp}}_p)^{\mathrm{B}\mathbb{T}_p^\wedge}$ is the inverse to $(m_\ell)^*$.

$\square$

As a result, $\mathrm{THH}(A)^{\mathrm{reparm}_\ell} \simeq \mathrm{THH}(A)$ as $\mathbb{T}$-spectra and thus we get an induced operation

$$\psi^\ell : \mathrm{TP}(A) \xrightarrow{(\psi^\ell)^{t\mathbb{T}}} (\mathrm{THH}(A)^{\mathrm{reparm}_\ell})^{t\mathbb{T}} \simeq \mathrm{TP}(A).$$

We also note that we get Adams operations compatibly on homotopy fixed points, orbits and thus on TC:

$$\psi^\ell : \mathrm{TC}(A) \to \mathrm{TC}(A).$$

Proposition 2.2. [1, Proposition 9.14] *The Adams operation ψ^ℓ acts on $\mathrm{gr}^n\mathrm{TP}(A)$ by multiplication with ℓ^n.*

In particular if we invert p, then we have an isomorphism of $\mathbb{Q}$-vector spaces $(\pi_*\mathrm{gr}^n\mathrm{TP}(A)[\frac{1}{p}])^{\psi^\ell-\ell^n} \cong \pi_*\mathrm{gr}^n\mathrm{TP}(A)[\frac{1}{p}]$. To prove Theorem 0.2 we consider the diagram of spectra

$$\begin{array}{ccc} \oplus_{n\in\mathbf{Z}}(\mathrm{TP}(A)[\frac{1}{p}])^{\psi^\ell-\ell^n} & \longrightarrow & \mathrm{TP}(A)[\frac{1}{p}] \\ \uparrow & & \\ \oplus_{n\in\mathbf{Z}}(\mathrm{Fil}^n\mathrm{TP}(A)[\frac{1}{p}])^{\psi^\ell-\ell^n} & & \\ \downarrow & & \\ \oplus_{n\in\mathbf{Z}}(\mathrm{gr}^n\mathrm{TP}(A)[\frac{1}{p}])^{\psi^\ell-\ell^n} \simeq \oplus_{n\in\mathbf{Z}}(\mathrm{gr}^n\mathrm{TP}(A)[\frac{1}{p}]) & & \end{array} \tag{4}$$

,

where, for any spectrum E with an action of ψ^ℓ, we define

$$E^{\psi^\ell-\ell^n} := \mathrm{fib}(E \xrightarrow{\psi^\ell-\ell^n} E).$$

Since $\pi_*\mathrm{TP}(A)[\frac{1}{p}]$ is a graded $\mathbb{Q}$-vector space, the top horizontal map induces an injection on homotopy groups $\oplus_{n\in\mathbf{Z}}(\pi_*\mathrm{TP}(A)[\frac{1}{p}])^{\psi^\ell-\ell^n} \hookrightarrow \pi_*\mathrm{TP}(A)[\frac{1}{p}]$. It then suffices to prove that vertical arrows are equivalences of spectra, whence we have a map of filtered $\mathbb{Q}$-vector spaces which is an isomorphism on graded pieces. The multiplicative properties of the filtration [1, Theorem 1.12.1] gives us the conclusion of Theorem 0.2.

To check the claimed equivalence, we consider the action of the Adams operation ψ^ℓ on $\mathrm{TP}(A)[\frac{1}{p}]$ as endowing it with the structure as a module over $\mathbb{S}[\psi^\ell]$[1]. The functoriality of the Adams operations tells us that the cofiber sequence of spectra

$$\mathrm{Fil}^n\mathrm{TP}(A)[\tfrac{1}{p}] \to \mathrm{TP}(A)[\tfrac{1}{p}] \to \frac{\mathrm{TP}(A)}{\mathrm{Fil}^n\mathrm{TP}(A)}[\tfrac{1}{p}],$$

[1]This means the spherical monoid algebra of the free monoid on one generator ψ^ℓ. In other words, taking Spec of this derived ring gives us the "flat affine line" over the sphere spectrum.

is a cofiber sequence of $\mathbb{S}[\psi^\ell]$-modules. We first claim that $(\frac{\mathrm{TP}(A)}{\mathrm{Fil}^n\mathrm{TP}(A)}[\frac{1}{p}])^{\psi^\ell-\ell^n}$ is contractible. Indeed, the we have a filtration on $\frac{\mathrm{TP}(A)}{\mathrm{Fil}^n\mathrm{TP}(A)}[\frac{1}{p}]$ given by

$$\{\frac{\mathrm{Fil}^k\mathrm{TP}(A)}{\mathrm{Fil}^n\mathrm{TP}(A)}[\tfrac{1}{p}]\}_{k>n}.$$

Where the associated graded are $\{\mathrm{gr}^k\mathrm{TP}(A)[\frac{1}{p}]\}_{k>n}$. Proposition 2.2 tells us that the action of $\psi^\ell-\ell^n$ on $\mathrm{gr}^k\mathrm{TP}(A)[\frac{1}{p}]$ is homotopic to the action of $\ell^k-\ell^n$ and so is invertible. An induction argument shows that the action of $\psi^\ell-\ell^n$ on $\frac{\mathrm{TP}(A)}{\mathrm{Fil}^n\mathrm{TP}(A)}[\frac{1}{p}]$ is thus invertible and so we have an equivalence of fibers

$$\mathrm{Fil}^n\mathrm{TP}(A)[\tfrac{1}{p}])^{\psi^\ell-\ell^n} \to (\mathrm{TP}(A)[\tfrac{1}{p}])^{\psi^\ell-\ell^n},$$

which tells us that the top vertical arrow of (4) is an equivalence. A similar argument applied to the cofiber sequence

$$\mathrm{Fil}^{n+1}\mathrm{TP}(A)[\tfrac{1}{p}] \to \mathrm{Fil}^n\mathrm{TP}(A)[\tfrac{1}{p}] \to \mathrm{gr}^n\mathrm{TP}(A)[\tfrac{1}{p}]$$

tells us that the bottom vertical arrow is an equivalence.

References

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