

ON MODULES OVER MOTIVIC RING SPECTRA

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ABSTRACT. In this note, we provide an axiomatic framework that characterizes the stable ∞ -categories that are module categories over a motivic spectrum. This is done by invoking Lurie’s ∞ -categorical version of the Barr–Beck theorem. As an application, this gives an alternative approach to Röndigs and Østvær’s theorem relating Voevodsky’s motives with modules over motivic cohomology, and to Garkusha’s extension of Röndigs and Østvær’s result to general correspondence categories, including the category of Milnor–Witt correspondences in the sense of Calmès and Fasel. We also extend these comparison results to regular Noetherian schemes over a field (after inverting the residue characteristic), following the methods of Cisinski and Déglise.

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1. INTRODUCTION

In [36] and [37], Röndigs and Østvær employed the technology of motivic functors developed in [17] to show an important structural result regarding motivic cohomology, namely that there is an equivalence of model categories between motives and modules over motivic cohomology, at least over fields of characteristic zero. In particular, this implies that Voevodsky’s triangulated categories of motives, introduced in [39], is equivalent to the homotopy category of modules over the motivic Eilenberg–MacLane spectra. This result has been extended to bases which are regular schemes over a field in the work of Cisinski–Déglise on integral mixed motives in the equicharacteristic case [8].

These theorems provide pleasant reinterpretations of Voevodsky’s category of motives as modules over a highly structured ring spectrum. The analog in topology is the result that chain complexes over a ring R is equivalent (in an appropriate model categorical sense) to modules over the Eilenberg–Mac Lane spectrum HR . This result was first obtained by Schwede and Shipley in [38] as part of the characterization of stable model categories in *loc. cit.*¹ More recently, Röndigs–Østvær’s result was extended to general categories of correspondences by Garkusha in [15].

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¹We remark that an ∞ -categorical treatment of the Schwede–Shipley results can be found in [29, Theorem 7.1.2.1].

In the present paper, we aim to provide with a robust and conceptually simpler approach to the above results. More precisely, by making use of Lurie's ∞ -categorical version of the Barr–Beck theorem, we derive a characterization of those stable ∞ -categories that are equivalent to a module category over a motivic spectrum. These categories are examples of *motivic module categories* in the sense of Definition 3.1. Examples include $\mathrm{DM}(k)$ in the sense of Voevodsky [32] or $\widetilde{\mathrm{DM}}(k)$ in the sense Déglise–Fasel [16]. Our characterization then reads as follows:

Theorem 1.1 (See Theorem 4.2). *Let k be a field. Suppose that ℓ is a prime which is coprime to the exponential characteristic e of k . If $\mathcal{M}(k)$ a motivic module category on k , then we have an equivalence of presentably symmetric monoidal stable ∞ -categories*

$$\mathcal{M}(k)_{[\frac{1}{e}]} \simeq \mathrm{Mod}_{\gamma_*\gamma^*(\mathbf{1})_{[\frac{1}{e}]}}(\mathrm{SH}(k)).$$

In particular, the associated triangulated categories are equivalent.

In fact, we formulate a parametrized version of motivic module categories and, under further hypotheses, we show that Theorem 1.1 extends to regular schemes over fields (see Theorem 4.5). This proof is essentially borrowed from [8].

The proof method breaks down into three conceptually simple steps:

- (1) Prove that a motivic module category \mathcal{M} is equivalent to the category of modules over some monad on $\mathrm{SH}(k)$.
- (2) Produce a functor from modules over the monad to modules over a corresponding motivic spectrum (Lemma 3.7).
- (3) Determine when this functor is an equivalence.

1.1. Conventions and notation.

- We use the following syntax of higher category theory:
 - Spc is the ∞ -category of ∞ -groupoids (Kan complexes give a concrete model).
 - $\mathrm{Maps}_{\mathcal{C}}(X, Y)$ is the Kan complex of maps between objects X, Y of an ∞ -category \mathcal{C} .
 - We write $\mathrm{Cat}_{\infty, \mathrm{stab}}$ for the the ∞ -category of stable ∞ -categories.
 - Pr^L is the ∞ -category of presentable ∞ -categories and colimit preserving functors (aka left adjoints). This is a symmetric monoidal ∞ -category $\mathrm{Pr}^{L, \otimes}$ for the Lurie tensor product and unit Spc . We let $\mathrm{Pr}_{\mathrm{stab}}^L$ denote the full subcategory of *stable* presentable ∞ -categories.
 - We write $\mathrm{P}(\mathcal{C})$ for the ∞ -category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spc})$ of presheaves on \mathcal{C} . If \mathcal{D} is another ∞ -category, we write $\mathrm{P}(\mathcal{C}, \mathcal{D}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ the ∞ -category of presheaves valued in \mathcal{D} .
 - If \mathcal{C} is an ∞ -category with coproducts, then we write $\mathrm{P}_{\Sigma}(\mathcal{C})$ for the *nonabelian derived category* of \mathcal{C} , that is, the full subcategory of $\mathrm{P}(\mathcal{C})$ which takes coproducts to products. For an exposition see [28, Section 5.5.8] or [41].
 - If $L: \mathcal{C} \rightarrow \mathcal{D}$ is a localization functor, then \mathcal{D} is always assumed to be reflective in \mathcal{C} , i.e., L has a fully faithful right adjoint $i: \mathcal{D} \rightarrow \mathcal{C}$.
- By a *base scheme* we mean a noetherian scheme S of finite dimension. We denote by Sch the category of noetherian schemes.
- If S is a base scheme and $p: X \rightarrow S$ is a smooth S -scheme, we always mean that X is of *finite type* and p is *separated*. An *essentially smooth* S -scheme is a localization of a smooth S -scheme.
- Some conventions in motivic homotopy theory:
 - We denote by \mathbb{T} the thom space of the trivial vector bundle of rank 1 over the base S . We have motivic equivalences $\mathbb{T} \simeq \frac{\mathbf{A}^1}{\mathbf{A}^1 \setminus 0} \simeq \mathbb{P}^1$.
 - We set $\mathbf{S}^{p,q} := (S^1)^{\otimes(p-q)} \otimes \mathbf{G}_m^{\otimes q}$ and $\Sigma^{p,q} M := \mathbf{S}^{p,q} \otimes M$, suitably interpreted in the category of motivic spaces, S^1 -spectra etc. We reserve $\mathbf{1}$ for the motivic sphere spectrum in $\mathrm{SH}(k)$ and write $\Sigma^{p,q} \mathbf{1}$.
 - For a motivic spectrum E over a base scheme S we often just write $E \in \mathrm{SH}(S)$, but sometimes if the need occurs we will write E_S for emphasis.
 - If τ is a topology on Sm_S , we write $H_{\tau}(S)$ (resp. $\mathrm{SH}_{\tau}(S)$) to be the unstable (resp. the \mathbb{T} -stable) motivic homotopy ∞ -category. If $\tau = \mathrm{Nis}$ we will drop the decoration.
- If k is a field, we denote by $e(k)$ or simply e its exponential characteristic.

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2. PRELIMINARIES

2.1. The Barr–Beck–Lurie Theorem. Let us start out by recalling the Barr–Beck–Lurie theorem characterizing modules over a monad, in the setting of ∞ -categories. Let $F : C \rightleftarrows D : G$ be an adjunction. We use the terminology of [25, §3.7]. Then the endofunctor $GF : C \rightarrow C$ is a monad, and the functor $G : D \rightarrow C$ factors as:

$$D \xrightarrow{G^{\text{enh}}} \text{LMod}_{GF}(C) \xrightarrow{u} C,$$

where u is the forgetful functor. Moreover, the functor $G^{\text{enh}} : D \rightarrow \text{LMod}_{GF}(C)$ admits a left adjoint:

$$F^{\text{enh}} : \text{LMod}_{GF}(C) \rightarrow D.$$

2.1.1. The net result is that the adjunction $F : C \rightleftarrows D : G$ factors as

$$\begin{array}{ccc} C & \begin{array}{c} \xrightarrow{F} \\ \xrightarrow{G} \end{array} & D \\ & \begin{array}{c} \searrow u \\ \swarrow G^{\text{enh}} \end{array} & \\ & \text{LMod}_{GF}(C) & \\ & \begin{array}{c} \swarrow \text{Free}_{GF} \\ \searrow F^{\text{enh}} \end{array} & \end{array}$$

Here, the functor $\text{Free}_{GF} : C \rightarrow \text{LMod}_{GF}(C)$ is simply the left adjoint to the functor u appearing in the factorization of G above, and thus deserves to be called the “free GF -module” functor.

2.1.2. The Barr–Beck–Lurie theorem provides a necessary and sufficient conditions for the functor $G^{\text{enh}} : D \rightarrow \text{LMod}_{GF}(C)$ to be an equivalence. Before stating the theorem, recall first that a simplicial object $X_{\bullet} : \Delta^{\text{op}} \rightarrow D$ is *split* if it extends to a split augmented object; in other words it extends to a functor $U : \Delta_{-\infty}^{\text{op}} \rightarrow D$. Here $\Delta_{-\infty}$ is the category whose objects are integers ≥ -1 , and where $\text{Hom}_{\Delta_{-\infty}}(n, m)$ consists of nondecreasing maps $n \cup \{-\infty\} \rightarrow m \cup \{-\infty\}$. Every split augmented simplicial diagram is a colimit diagram so that the map $\text{colim } X_{\bullet} \rightarrow X_{-1}$ is an equivalence. If $G : D \rightarrow C$ is a functor, we say that a simplicial object X_{\bullet} in D is *G-split* if $G \circ X_{\bullet}$ is split.

Theorem 2.1 (Barr–Beck–Lurie [29, Theorem 4.7.4.5]). *Let $G : D \rightarrow C$ be a functor of ∞ -categories admitting a left adjoint $F : C \rightarrow D$. Then the following are equivalent:*

- (1) *The functor G^{enh} and F^{enh} are mutually inverse equivalences.*
- (2) *The functor G^{enh} is conservative, and for any simplicial object $X_{\bullet} : \Delta^{\text{op}} \rightarrow D$ which is G -split, X_{\bullet} admits a colimit in D . Furthermore, any extension $\overline{X}_{\bullet} : (\Delta^{\text{op}})^{\triangleright} \rightarrow D$ is a colimit diagram if and only if $G \circ \overline{X}_{\bullet}$ is.*

Any adjunction (F, G) satisfying the equivalent conditions above is called *monadic*.

2.2. Compact and rigid objects in motivic homotopy theory. We now recall some facts about compact-rigid generation in motivic stable ∞ -categories.

2.2.1. For now we work over an arbitrary base S . Denote by:

- (1) $\text{SH}^{\omega}(S)$ the full subcategory of $\text{SH}(k)$ spanned by the compact objects;
- (2) $\text{SH}^{\text{rig}}(S)$ the full subcategory of $\text{SH}(k)$ spanned by the strongly dualizable objects, and
- (3) $\text{SH}^{\text{Proj}}(S)$ the idempotent completion of the full subcategory of $\text{SH}(k)$ spanned by $\Sigma_{\mathbb{T}}^q \Sigma_{\mathbb{T}}^{\infty} X_{+}$, where X is a smooth projective S -scheme and $q \in \mathbb{Z}$.

Then, as proved in [34], there are inclusions

$$\mathrm{SH}^{\mathrm{proj}}(S) \subseteq \mathrm{SH}^{\mathrm{rig}}(S) \subseteq \mathrm{SH}^{\omega}(S). \quad (2.1)$$

Indeed, the argument in *loc cit.* only relies on the computation of the Spanier-Whitehead dual of the suspension spectrum of a smooth projective S -scheme as the Thom spectrum of its stable normal bundle; this has been established generally by Ayoub in [2].

2.2.2. The ∞ -category $\mathrm{SH}(S)$ is generated under sifted colimits by $\Sigma_{\mathbb{T}}^q \Sigma_{\mathbb{T}}^{\infty} X_+$ where X is an affine smooth scheme over S and $q \in \mathbf{Z}$. Furthermore each generator is a compact object in $\mathrm{SH}(S)$ since Nisnevich sheafification preserves filtered colimits (see, for example, [21, Proposition 6.4] where we set the group of equivariance to be trivial). Hence the ∞ -category $\mathrm{SH}^{\omega}(S)$ is generated under finite colimits and retracts by $\Sigma_{\mathbb{T}}^q \Sigma_{\mathbb{T}}^{\infty} X_+$, $q \in \mathbf{Z}$, where X is affine. From this discussion it is immediate that:

Lemma 2.2. *Let S be a base scheme and let $L: \mathrm{SH}(S) \rightarrow \mathrm{SH}(S)$ be a localization endofunctor. The following are equivalent:*

- (1) *For any smooth affine S -scheme X , the suspension spectrum $L(\Sigma_{\mathbb{T}}^{\infty} X_+)$ is a retract of a $L(\Sigma_{\mathbb{T}}^{\infty} Y_+)$ where Y is a smooth projective S -scheme.*
- (2) *The inclusions (2.1) collapse to equalities:*

$$L(\mathrm{SH}^{\mathrm{proj}}(S)) = L(\mathrm{SH}^{\mathrm{rig}}(S)) = L(\mathrm{SH}^{\omega}(S)). \quad (2.2)$$

Example 2.3. Let k be a field and suppose that ℓ is a prime which is coprime to the exponential characteristic e of k . Let $L_{(\ell)}: \mathrm{SH}(k) \rightarrow \mathrm{SH}(k)$ be the localization endofunctor at ℓ . If k is perfect then, according to [27, Corollary B.2], condition 1 of Lemma 2.2 is satisfied so that $\mathrm{SH}^{\mathrm{proj}}(k)_{(\ell)} = \mathrm{SH}^{\mathrm{rig}}(k)_{(\ell)} = \mathrm{SH}^{\omega}(k)_{(\ell)}$. This immediately implies the equalities

$$\mathrm{SH}^{\mathrm{proj}}(k) \left[\frac{1}{e} \right] = \mathrm{SH}^{\mathrm{rig}}(k) \left[\frac{1}{e} \right] = \mathrm{SH}^{\omega}(k) \left[\frac{1}{e} \right]. \quad (2.3)$$

The above equalities are extended to the case of arbitrary fields in [13]. More precisely, [13] shows that as long as the endomorphism given by multiplication by e is invertible, the ∞ -categories $\mathrm{SH}(k_{\mathrm{perf}})$ and $\mathrm{SH}(k)$ are equivalent.

2.2.3. For a general base scheme S , it is not known if the inclusions in (2.1) collapse to equalities even after applying localizations at a prime ℓ or inversions. Hence it is useful to record when it does:

Definition 2.4. Let $L: \mathrm{SH}(S) \rightarrow \mathrm{SH}(S)$ be a localization endofunctor. We say that $L(\mathrm{SH}(S))$, or simply L , has *compact-rigid generation* if any of the equivalent conditions of Lemma 2.2 are satisfied.

Hence Example 2.3 tells us that $\mathrm{SH}(k)_{(\ell)}$ and $\mathrm{SH}(k) \left[\frac{1}{e} \right]$ have compact-rigid generation.

2.3. Premotivic categories and adjunctions. Lastly, we recall Cisinski and Déglise's notion of a *premotivic category* [7]. Suppose that \mathcal{S} is a full subcategory of the category Sch of noetherian schemes, and let \mathcal{P} denote a class of admissible morphisms [7, §1.0]. In fact, the only example we care about is when \mathcal{P} is the class of smooth morphisms. As in [7, §1] (see [9, Appendix A] for a more succinct discussion) a functor

$$\mathcal{M}: \mathcal{S}^{\mathrm{op}} \rightarrow \mathrm{Cat}_{\infty}$$

is called a \mathcal{P} -premotivic category over \mathcal{S} if for each morphism $f: T \rightarrow S$ in \mathcal{S} , the functor $f^*: \mathcal{M}(S) \rightarrow \mathcal{M}(T)$ admits a right adjoint f_* . If f is admissible, it admits a left adjoint $f_{\#}$. We refer the reader to the thesis of Khan [26] for a detailed discussion of this notion in the ∞ -categorical setting. If the context is clear, we simply refer to \mathcal{M} as a premotivic category. We may also speak of premotivic ∞ -categories taking values in other (large) ∞ -categories such as $\mathrm{Cat}_{\infty}^{\otimes}$, $\mathrm{Cat}_{\infty, \mathrm{stab}}$ or Pr^L .

2.3.1. We also have the appropriate notion of an adjunction between premotivic categories (see [7, Definition 1.4.6], [9, Definition A.1.7]). Suppose that $\mathcal{M}, \mathcal{M}'$ are premotivic categories, then a *premotivic adjunction* is a transformation $\gamma^*: \mathcal{M} \rightarrow \mathcal{M}'$ such that

- (1) for each $S \in \mathcal{S}$, the functor $\gamma_S^*: \mathcal{M}(S) \rightarrow \mathcal{M}'(S)$ admits a right adjoint γ_{S*} .
- (2) For each morphism $f: T \rightarrow S \in \mathcal{S}$, the canonical transformation $f_{\#} \gamma_S^* \rightarrow \gamma_T^* f_{\#}$ is an equivalence.

Furthermore we say that a premotivic adjunction γ^* is a *localization of premotivic categories* (or, simply, a *localization*) if for each $S \in \mathcal{S}$ the functor γ_{S*} is fully faithful, i.e., a localization in the sense of [28, Definition 5.2.7.2]. Furthermore we say that a localization of premotivic categories is *smashing* if γ_{S*} preserves colimits. Further suppose that \mathcal{M} takes values in $\text{Cat}_\infty^\otimes$. In particular, the functors f^* are strongly symmetric monoidal. Then a localization is *symmetric monoidal* if for any $S \in \mathcal{S}$ then if $E \in \mathcal{M}(S)$ is L -local, then for any $F \in \mathcal{M}(S)$, $E \otimes F$ is L -local as well. This last condition implies that the symmetric monoidal structure on $\mathcal{M}(S)$ descends to one in subcategory of L -local objects and the localization functor is strongly symmetric monoidal [29, Proposition 2.2.1.9].

2.3.2. We recall two conditions on \mathcal{M} which will be relevant to us later. Let $S \in \mathcal{S}$ be a scheme. Suppose that $i: Z \rightarrow S$ is a closed subscheme, and let $j: U \rightarrow S$ be its open complement.

Definition 2.5. We say that a premotivic category \mathcal{C} satisfies (Loc_i) if the following holds:

- (a) The counit $i^*i_* \rightarrow \text{id}$ is an equivalence,
- (b) The pair (j^*, i^*) is conservative.

We say that \mathcal{C} satisfies (Loc) if $\mathcal{C}(\emptyset) \simeq *$, and \mathcal{C} satisfies (Loc_i) for any closed immersion i .

We formulate the second property with the assumption that \mathcal{M} takes values in stable ∞ -categories. Let $c = (c_i)_{i \in I}$ be a collection of Cartesian sections of \mathcal{M} (the only case we consider are $\{\Sigma^{p,q}\mathbf{1}\}_{p,q \in \mathbb{Z}}$). We denote by $\mathcal{M}_c(S) \subseteq \mathcal{M}(S)$ the smallest thick subcategory² of $\mathcal{M}(S)$ which contains $f_{\#}f^*c_{i,X}$ for any smooth morphism $f: T \rightarrow S$. Following [8, Definition 2.3], we call objects in $\mathcal{M}_c(S)$ *c-constructible*. We say that \mathcal{M} is *c-generated* if for all $X \in \mathcal{S}$ the stable ∞ -category $\mathcal{M}(S)$ is generated by $\mathcal{M}_c(S)$ under all small colimits.

Definition 2.6. Suppose that $\mathcal{A} \subseteq \mathcal{S}^{\Delta^1}$ is a collection of morphisms in \mathcal{S} . We say that \mathcal{A} is *continuous with respect to \mathcal{F}* if for any diagram $X: I \rightarrow \mathcal{S}$ where I is a cofiltered diagram and the transition maps are in \mathcal{F} and the limit $X := \lim_i X_i$ exists in \mathcal{S}' the canonical map

$$\mathcal{M}_c(X') \rightarrow \lim_I \mathcal{M}_c(X_i)$$

is an equivalence.

3. MOTIVIC MODULE CATEGORIES

We now formulate the notion of *motivic module categories*. By [2], [7] we have the premotivic category $\text{SH}|_{\mathcal{S}}: \mathcal{S} \rightarrow \text{Pr}_{\text{stab}}^{L,\otimes}$.

Definition 3.1. Let \mathcal{S} be a full subcategory of Sch and suppose that $L: \text{SH}|_{\mathcal{S}} \rightarrow L(\text{SH})|_{\mathcal{S}}$ is a localization which is symmetric monoidal in the sense of §2.3. We then define the following:

- (1) If $S \in \mathcal{S}$, then an *L-local motivic module category on S* is a presentably symmetric monoidal stable ∞ -category $\mathcal{M}(S)$ equipped with an adjunction

$$\gamma_S^*: L(\text{SH}(S)) \rightleftarrows \mathcal{M}(S) : \gamma_{S*}$$

such that the right adjoint γ_{S*} is conservative and preserves sifted colimits.

- (2) An *L-local motivic module category over \mathcal{S}* (or, simply, a *motivic module category* if the context is clear) is a premotivic category \mathcal{M} valued in presentably symmetric monoidal stable ∞ -categories,

$$\mathcal{M}: \mathcal{S}^{\text{op}} \rightarrow \text{Pr}_{\text{stab}}^{L,\otimes}$$

along with a premotivic adjunction

$$\gamma^*: L(\text{SH})|_{\mathcal{S}} \rightarrow \mathcal{M}; \quad S \mapsto (\gamma_S^*: L(\text{SH}(S)) \rightarrow \mathcal{M}(S)),$$

which evaluates to an L -local motivic module category $\mathcal{M}(S)$ on S for each $S \in \mathcal{S}$.

If L is the identity functor, then we simply say that \mathcal{M} is a *motivic module category*. When the localization L is clear, we will denote a motivic module category by a pair $(\text{SH}|_{\mathcal{S}}, \mathcal{M})$. Moreover, if the scheme S is implicitly understood, we may drop the S from the notation $(\gamma_S^*, \gamma_{S*})$.

²Recall that if \mathcal{C} is a stable ∞ -category, a subcategory $\mathcal{D} \subseteq \mathcal{C}$ is called *thick* if it is a stable subcategory and contains all retracts.

Remark 3.2. A motivic module category often arises from a premotivic adjunction

$$\gamma^* : \mathrm{SH}|_{\mathcal{S}} \rightarrow \mathcal{M}$$

which factors as $\bar{\gamma}^* : L(\mathrm{SH})|_{\mathcal{S}} \rightarrow \mathcal{M}$. For each $S \in \mathcal{S}$, the functor $\bar{\gamma}_S^* : L(\mathrm{SH}(S))|_{\mathcal{S}} \rightarrow \mathcal{M}(S)$ automatically has a right adjoint. If L is given by smashing with a spectrum, then γ_S^* preserves sifted colimits if and only if $\bar{\gamma}_S^*$ does.

In §3.2 we will give a way to construct motivic module categories using very general inputs.

Lemma 3.3. *Let $S \in \mathcal{S}$, and let $\mathbf{1}_S \in \mathrm{SH}(S)$ denote the motivic sphere spectrum over S . If \mathcal{M} is an L -local motivic module category, then the spectrum $L\gamma_*\gamma^*(\mathbf{1}_S) \in \mathrm{SH}(S)$ is an \mathcal{E}_∞ -ring spectrum.*

Proof. As γ_* is lax symmetric monoidal, it follows that γ_* preserves \mathcal{E}_∞ -algebras. Since γ^* is symmetric monoidal, $\gamma^*(\mathbf{1}_S)$ is the unit object in \mathcal{M} and is thus an \mathcal{E}_∞ -algebra. As L is symmetric monoidal, we conclude that $\gamma_*\gamma^*(\mathbf{1}_S)$ is an \mathcal{E}_∞ -ring spectrum. \square

3.0.1. The Barr–Beck–Lurie theorem ensures that a motivic module category on S is always equivalent to modules over a monad, as the following lemma records. In the next subsection, we will investigate when we can further enhance this equivalence to modules over a motivic \mathcal{E}_∞ -ring spectrum.

Lemma 3.4. *If $\mathcal{M}(S)$ is a motivic module category on S , then the induced adjunction*

$$\gamma^{*,\mathrm{enh}} : \mathrm{LMod}_{\gamma_*\gamma^*}(L(\mathrm{SH}(S))) \rightleftarrows \mathcal{M}(S) : \gamma_*^{\mathrm{enh}}$$

is an equivalence of ∞ -categories.

Proof. By assumption, the conditions of Theorem 2.1 are satisfied. \square

3.1. Motivic module categories versus categories of modules. The following definition will be essential in relating a motivic module category to a category modules over a motivic \mathcal{E}_∞ -ring spectrum.

Definition 3.5. Let \mathcal{M} be an L -local motivic module category over \mathcal{S} and let $S \in \mathcal{S}$. We say that the pair $(\mathrm{SH}|_{\mathcal{S}}, \mathcal{M})$ admits the projection formula at S if the transformation

$$\gamma_*\gamma^*(\mathbf{1}_S) \otimes (-) \rightarrow \gamma_*\gamma^*$$

is an equivalence in $L(\mathrm{SH}(S))$. If $(\mathrm{SH}|_{\mathcal{S}}, \mathcal{M})$ admits the projection formula at any $S \in \mathcal{S}$, we say that $(\mathrm{SH}|_{\mathcal{S}}, \mathcal{M})$ admits the projection formula.

Theorem 3.6. *Let \mathcal{M} is an L -local motivic module category over \mathcal{S} . Suppose that $S \in \mathcal{S}$ is a scheme such that $(\mathrm{SH}|_{\mathcal{S}}, \mathcal{M})$ admits the projection formula at S . Then there is an equivalence of presentably symmetric monoidal stable ∞ -categories*

$$\mathcal{M}(S) \simeq \mathrm{Mod}_{\gamma_*\gamma^*(\mathbf{1}_S)}(L(\mathrm{SH}(S))).$$

Moreover, if $(\mathrm{SH}|_{\mathcal{S}}, \mathcal{M})$ admits the projection formula, then we have an equivalence of premotivic categories

$$\mathcal{M} \simeq \mathrm{Mod}_{\gamma_*\gamma^*(\mathbf{1})}(L(\mathrm{SH}(-))).$$

3.1.1. In light of Lemma 3.4, we can prove Theorem 3.6 by means of relating modules over monad $\gamma_*\gamma^*$ with modules over the motivic spectrum $\gamma_*\gamma^*(\mathbf{1}_S)$. Thus, given $S \in \mathcal{S}$ our task is to formulate a relationship between the two ∞ -categories

$$\mathrm{LMod}_{\gamma_*\gamma^*}(\mathrm{SH}(S)) \quad \text{and} \quad \mathrm{LMod}_{\gamma_*\gamma^*(\mathbf{1}_S) \otimes (-)}(\mathrm{SH}(S)).$$

To do so, it suffices produce a map of monads

$$c : \gamma_*\gamma^*(\mathbf{1}_S) \otimes (-) \rightarrow \gamma_*\gamma^*,$$

which will induce a functor

$$c^* : \mathrm{LMod}_{\gamma_*\gamma^*(\mathbf{1}_S) \otimes (-)}(\mathrm{SH}(S)) \rightarrow \mathrm{LMod}_{\gamma_*\gamma^*}(\mathrm{SH}(S)).$$

For this, we appeal to a general lemma.

Lemma 3.7. *Let \mathcal{C}, \mathcal{D} be monoidal ∞ -categories and suppose that we have an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ such that F is symmetric monoidal (so that G is lax symmetric monoidal). Then there is a map of monads*

$$c : GF(1) \otimes (-) \rightarrow GF, \quad (3.1)$$

which gives rise to a commutative diagram of adjunctions

$$\begin{array}{ccc}
 \mathcal{C} & \begin{array}{c} \xrightarrow{\text{Free}_{GF}} \\ \xrightarrow{u} \end{array} & \text{LMod}_{GF}(\mathcal{C}) \\
 \begin{array}{c} \downarrow GF(1) \otimes (-) \\ \downarrow u \end{array} & & \begin{array}{c} \downarrow c_* \\ \downarrow c^* \end{array} \\
 \text{LMod}_{GF(1) \otimes (-)}(\mathcal{C}) & & \text{LMod}_{GF}(\mathcal{C})
 \end{array}$$

Proof. Since F is monoidal and G is lax monoidal, the functor GF is lax monoidal. Hence $GF(1)$ is an algebra object of \mathcal{C} , and thus $GF(1) \otimes (-)$ is indeed a monad. We construct the map of monads $c : GF(1) \otimes (-) \rightarrow GF(-)$ by letting c be the composite of the following maps of monads

$$\begin{aligned}
 GF(1) \otimes (-) &\simeq (GF(1) \otimes (-)) \circ \text{id} \\
 &\xrightarrow{\text{id} \circ \epsilon} (GF(1) \otimes (-)) \circ GF(-) \\
 &\xrightarrow{\mu} G(F(1) \otimes F(-)) \\
 &\simeq GF.
 \end{aligned}$$

Here ϵ is the unit of the adjunction (F, G) . The transformation ϵ is a map of monads via the triangle identities, and the map $\text{id} \circ \epsilon$ is a map of monads since we are \circ -tensoring two maps of monads. The map μ is given by the lax monoidal structure of G ; more precisely, we note that the endofunctor $G(A \otimes F(-))$ is a monad for any algebra object A , and so $G(F(1) \otimes F(-))$ is in particular a monad. We have a canonical equivalence of monads

$$(GF(1) \otimes (-)) \circ GF(-) \simeq GF(1) \otimes GF(-).$$

The lax structure of G then provides a morphism of endofunctors

$$GF(1) \otimes GF(-) \rightarrow G(F(1) \otimes F(-)) \simeq GF(-),$$

and the lax structure also verifies that this is a map of monads. This gives rise to a functor $c_* : \text{LMod}_{GF}(\mathcal{C}) \rightarrow \text{LMod}_{GF(1) \otimes (-)}(\mathcal{C})$, which has a left adjoint by the adjoint functor theorem.

To obtain the desired factorizations, we note that we have the following commutative diagram of forgetful functors

$$\begin{array}{ccc}
 \mathcal{C} & \begin{array}{c} \xleftarrow{u} \\ \xleftarrow{u} \end{array} & \text{LMod}_{GF}(\mathcal{C}) \\
 & & \downarrow c_* \\
 & & \text{LMod}_{GF(1) \otimes (-)}(\mathcal{C})
 \end{array}$$

Thus the left adjoints also commute. □

3.1.2. We can now apply Lemma 3.7 to prove Theorem 3.6.

Proof of Theorem 3.6. We claim that the adjunction of Lemma 3.7,

$$c^* : \text{LMod}_{\gamma_* \gamma^*(\mathbf{1}_S)}(\text{SH}(S)) \rightleftarrows \text{LMod}_{\gamma_* \gamma^*}(\text{SH}(S)) : c_*$$

is an equivalence. By the construction in the proof of Lemma 3.7, the above adjunction arises from a map of monads given by $c : \gamma_* \gamma^*(\mathbf{1}_S) \otimes (-) \rightarrow \gamma_* \gamma^*$. Since $(\text{SH}|_{\mathcal{S}, \mathcal{M}})$ satisfies the projection formula, we conclude that the adjunction (c^*, c_*) is an equivalence.

Now, note that Theorem 2.1 and Lemma 3.7 are phrased for \mathcal{E}_1 -algebras and left modules. However, as $\gamma_* \gamma^*(\mathbf{1})$ is an \mathcal{E}_∞ -ring spectrum by Lemma 3.3, the ∞ -categories of left and right $\gamma_* \gamma^*(\mathbf{1})$ -modules coincide. We thus conclude that there is a natural equivalence

$$\text{Mod}_{\gamma_* \gamma^*(\mathbf{1}_S)}(\text{SH}(S)) \simeq \mathcal{M}(S)$$

of ∞ -categories, which carries $\gamma_*\gamma^*(\mathbf{1}_S)$ to the unit object $\gamma^*(\mathbf{1}_S)$ of $\mathcal{M}(S)$. Finally, if \mathcal{M} satisfies the projection formula at any $S \in \mathcal{S}$, then the naturality of the above equivalence furnishes the equivalence of premotivic categories $\mathcal{M} \simeq \text{Mod}_{\gamma_*\gamma^*(\mathbf{1})}(\text{SH}(-))$. \square

Remark 3.8. In fact, the above reduction can be achieved using a more refined version of Lurie’s Barr–Beck theorem [29, Proposition 4.8.5.8].

Remark 3.9. We were also informed by Niko Naumann that the above result is a consequence of [31, Proposition 5.29].

In Section 4 we will provide examples for which the hypotheses of Theorem 3.6 are satisfied.

3.2. Correspondence categories. The prime examples of motivic module categories are built from various notions of correspondences. In this section we will give an axiomatization of ∞ -categories which behave like the category of framed correspondences as in [11]; Suslin–Voevodsky’s category of finite correspondences [42], [32, Chapters 1 and 2]; Calmès and Fasel’s Milnor–Witt correspondences [6, 16]; Grothendieck–Witt correspondences [14]; and, more recently, the categories of correspondences studied in [12] and [10]. These examples will be discussed in §3.3. To begin with, consider the discrete category Sch_{S+} , whose objects are S -schemes of the form X_+ and morphisms which preserve the base point. We consider the subcategory $\text{Sm}_{S+} \subseteq \text{Sch}_{S+}$ spanned by smooth S -schemes of finite type. We will use heavily the *nonabelian derived ∞ -category* $\text{P}_\Sigma(\mathcal{C})$ associated to an ∞ -category \mathcal{C} with finite products; more detailed treatments of this construction can be found in [5, Chapter 1] and [28, 5.5.8].

Definition 3.10. A *correspondence category* (over a base scheme S) is a preadditive ∞ -category \mathcal{C} equipped with a *graph functor*

$$\gamma_{\mathcal{C}}: \text{Sm}_{S+} \rightarrow \mathcal{C} \quad (3.2)$$

such that the following hold:

- (1) The functor $\gamma_{\mathcal{C}}$ is essentially surjective and preserves finite coproducts³, so that we get an induced functor

$$\gamma_*: \text{P}_\Sigma(\mathcal{C}) \rightarrow \text{P}(\text{Sm}_S); \quad \mathcal{F} \mapsto \mathcal{F} \circ \gamma_{\mathcal{C}}.$$

- (2) The composite functor

$$\text{Sm}_{S+} \rightarrow \mathcal{C} \rightarrow \text{P}_\Sigma(\mathcal{C}) \xrightarrow{\gamma_*} \text{P}_\Sigma(\text{Sm}_{S+}), \quad (3.3)$$

has a right lax Sm_{S+} -linear structure. We abusively denote the composite (3.3) by $\gamma_{\mathcal{C}}(-)$; the context will always make it clear.

The ∞ -category of correspondence categories CorrCat is defined as a full subcategory of $\text{PreAdd}_{\infty, \text{Sm}_{S+}/}$,⁴ the (large) ∞ -category of small preadditive ∞ -categories and functors that preserves finite coproducts equipped with a finite coproduct-preserving functor from Sm_{S+} .

3.2.1. We begin with a couple of clarifying remarks and an example.

Remark 3.11. Informally, the Sm_{S+} -linear structure on $\gamma_{\mathcal{C}}(-)$ encodes for any $X, Y \in \text{Sm}_S$ maps

$$X_+ \otimes \gamma_{\mathcal{C}}(Y_+) \rightarrow \gamma_{\mathcal{C}}(X_+ \otimes Y_+)$$

in $\text{P}_\Sigma(\text{Sm}_{S+}) \simeq \text{P}_\Sigma(\text{Sm}_S)_*$, which are subject to various compatibilities. For example, if $f: X_+ \rightarrow Z_+$ is a map in Sm_{S+} then we have a 2-cell witnessing the commutativity of

$$\begin{array}{ccc} X_+ \otimes \gamma_{\mathcal{C}}(Y_+) & \longrightarrow & \gamma_{\mathcal{C}}(X_+ \otimes Y_+) \\ f \otimes \text{id} \downarrow & & \downarrow \gamma_{\mathcal{C}}(f \otimes \text{id}) \\ Z_+ \otimes \gamma_{\mathcal{C}}(Y_+) & \longrightarrow & \gamma_{\mathcal{C}}(Z_+ \otimes Y_+). \end{array}$$

³Including the empty coproduct, so that the $\gamma_{\mathcal{C}}$ also preserves the base point of Sm_{S+} .

⁴“Officially” it is the pullback of ∞ -categories: $\text{PreAdd} \times_{\text{Cat}_{\text{II}}} \{\text{Sm}_{S+}\}$ where Cat_{II} denotes ∞ -categories with finite coproducts and finite coproducts-preserving functors.

Similarly, if $g: Y_+ \rightarrow Z_+$ is a map in Sm_{S_+} then we have a 2-cell witnessing the commutativity of

$$\begin{array}{ccc} X_+ \otimes \gamma_C(Y_+) & \longrightarrow & \gamma_C(X_+ \otimes Y_+) \\ \mathrm{id} \otimes g \downarrow & & \downarrow \gamma_C(\mathrm{id} \otimes g) \\ X_+ \otimes \gamma_C(Z_+) & \longrightarrow & \gamma_C(X_+ \otimes Z_+). \end{array}$$

These cells are required satisfy an infinite list of coherences.

Remark 3.12. The Sm_{S_+} -linearity assumption will be satisfied if \mathcal{C} has a symmetric monoidal structure and the functor γ_C is symmetric monoidal. In more detail, we denote by $\mathrm{CorrCat}^{\otimes}$ the ∞ -category of preadditive ∞ -categories with a symmetric monoidal structure such that the functor $\gamma: \mathrm{Sm}_{S_+} \rightarrow \mathcal{C}$ is symmetric monoidal, essentially surjective and preserves finite coproducts. There is a forgetful functor $\mathrm{CorrCat}^{\otimes} \rightarrow \mathrm{CorrCat}$; part 2 of Definition 3.10 is obtained from the strong symmetric monoidality of γ_C . This is the case in the examples considered in this paper, but we include it as an axiom to clarify proofs of certain properties.

Example 3.13. Let $\mathrm{Corr}_S^{\mathrm{clopen}}$ denote the discrete category whose objects are smooth S -schemes and morphisms are $X \leftarrow Y \rightarrow Z$ such that $X \leftarrow Y$ is a summand inclusion. There is an equivalence of categories $\mathrm{Sm}_{S_+} \xrightarrow{\simeq} \mathrm{Corr}_S^{\mathrm{clopen}}$ which takes a morphism $f: X_+ \rightarrow Y_+$ to $X \leftarrow f^{-1}(Y) \rightarrow Y$. This graph functor witnesses $\mathrm{Corr}_S^{\mathrm{clopen}}$ as a correspondence category.

3.2.2. We start with some elementary properties of a correspondence category.

Proposition 3.14. *Let \mathcal{C} be a preadditive ∞ -category equipped with an essential surjection*

$$\gamma_C: \mathrm{Sm}_k \rightarrow \mathcal{C}$$

which preserves coproducts. Consider the induced functor

$$\gamma_{C*}: \mathrm{P}_{\Sigma}(\mathcal{C}) \rightarrow \mathrm{P}_{\Sigma}(\mathrm{Sm}_S); \quad \mathcal{F} \mapsto \mathcal{F} \circ \gamma_C.$$

Then the following hold

- (1) *The ∞ -category $\mathrm{P}_{\Sigma}(\mathcal{C})$ is presentable and preadditive.*
- (2) *The functor γ_{C*} preserves sifted colimits.*
- (3) *The functor γ_{C*} is conservative.*

Proof. Presentability of $\mathrm{P}_{\Sigma}(\mathcal{C})$ is [28, Proposition 5.5.8.10.1], while P_{Σ} applied to a preadditive ∞ -category is again preadditive by [20, Corollary 2.4]. The functor γ_{C*} preserves sifted colimits since sifted colimits are computed pointwise (a direct consequence of [28, 5.5.8.4.10] parts 4 and 5), while γ_{C*} is conservative since γ_C is essentially surjective. \square

3.2.3. The composite of γ_C with Yoneda functor $\mathrm{Sm}_{S_+} \xrightarrow{\gamma_C} \mathcal{C} \xrightarrow{y} \mathrm{P}_{\Sigma}(\mathcal{C})$ has a canonical sifted colimit-preserving extension $\gamma_C^*: \mathrm{P}_{\Sigma}(\mathrm{Sm}_{S_+}) \rightarrow \mathrm{P}_{\Sigma}(\mathcal{C})$. It is easy to check that γ_{C*} is the right adjoint to γ_C^* and thus γ_C^* preserves all small colimits. As a result, we have an adjunction

$$\gamma_C^*: \mathrm{P}_{\Sigma}(\mathrm{Sm}_{S_+}) \rightleftarrows \mathrm{P}_{\Sigma}(\mathcal{C}) : \gamma_{C*}. \quad (3.4)$$

It is also easy to promote the Sm_{S_+} -linear structure in axiom 3 of a correspondence category to a $\mathrm{P}_{\Sigma}(\mathrm{Sm}_{S_+})$ -linear structure so that the functor

$$\gamma_{C*} \circ \gamma_C^*: \mathrm{P}_{\Sigma}(\mathrm{Sm}_{S_+}) \rightarrow \mathrm{P}_{\Sigma}(\mathrm{Sm}_{S_+}).$$

extends to a right lax $\mathrm{P}_{\Sigma}(\mathrm{Sm}_{S_+})$ -linear functor.

3.2.4. Now we would like to do motivic homotopy theory on \mathcal{C} . Recall that if $X, Y \in \mathrm{P}_{\Sigma}(\mathrm{Sm}_{S_+})$, then X is \mathbf{A}^1 -homotopy equivalent to Y if there are maps $f: X \rightarrow Y, g: Y \rightarrow X$ and \mathbf{A}^1 -homotopies $H: \mathbf{A}_+^1 \otimes X \rightarrow X, H': \mathbf{A}_+^1 \otimes Y \rightarrow Y$ from gf and fg to the respective identity morphisms. Any \mathbf{A}^1 -homotopy equivalence is an $L_{\mathbf{A}^1}$ -equivalence [33, Lemma 3.6].

Lemma 3.15. *The functor $\gamma_C: \mathrm{P}_{\Sigma}(\mathrm{Sm}_{S_+}) \rightarrow \mathrm{P}_{\Sigma}(\mathrm{Sm}_{S_+})$ preserves \mathbf{A}^1 -homotopy equivalences.*

Proof. Suppose that we have a homotopy $H: \mathbf{A}_+^1 \otimes X_+ \rightarrow Y$ between maps $f, g: X \rightarrow Y$. We obtain, using the right lax-structure for the first arrow,

$$\mathbf{A}_+^1 \otimes \gamma_C(X) \rightarrow \gamma_C(\mathbf{A}^1 \times X) \rightarrow \gamma_C(Y),$$

a homotopy between $\gamma_C(f)$ and $\gamma_C(g)$. \square

Lemma 3.16. *The functor $\gamma_C: P_\Sigma(\mathbf{Sm}_{S_+}) \rightarrow P_\Sigma(\mathbf{Sm}_{S_+})$ preserves $L_{\mathbf{A}^1}$ -equivalences.*

Proof. By definition the class of $L_{\mathbf{A}^1}$ -equivalences is the strong saturation, in the sense [28, Proposition 5.5.4.5], of the maps in $P_\Sigma(\mathbf{Sm}_{S_+})$ by the (Yoneda image of) \mathbf{A}^1 -projections $\pi_X: (\mathbf{A}^1 \times X)_+ \simeq \mathbf{A}_+^1 \otimes X_+ \rightarrow X_+$ for $X \in \mathbf{Sm}_S$. According to [5, Lemma 2.10] the class of $L_{\mathbf{A}^1}$ -equivalences is then generated under 2-out-of-3 and sifted colimits by maps of the form $\pi_X \amalg \text{id}_{Y_+}$ where $Y \in \mathbf{Sm}_S$.

Since π_X is an \mathbf{A}^1 -homotopy equivalence $\gamma_C(\pi_X)$, and thus the morphism

$$\gamma_C(\pi_X \amalg \text{id}_{Y_+}) \simeq \gamma_C(\pi_X) \amalg \gamma_C(\text{id}_{Y_+}),$$

is an \mathbf{A}^1 -homotopy equivalence by Lemma 3.15 and the assumption that γ_C preserves coproducts. The functor γ_C clearly preserves the 2-out-of-3-property. Lastly, the functor γ_C preserves sifted colimits by definition and sifted colimits are computed valuably in $P_\Sigma(\mathbf{Sm}_{S_+})^{\Delta^1}$. Hence we conclude that γ_C preserves $L_{\mathbf{A}^1}$ -equivalences. \square

3.2.5. Now, we take into account a topology that we might want to put on \mathbf{Sm}_{S_+} , namely, the topology of *coproduct decomposition*. This is a topology on \mathbf{Sm}_{S_+} defined by a cd-structure, denoted by \amalg , generated by squares

$$\begin{array}{ccc} S & \longrightarrow & U_+ \\ \downarrow & & \downarrow \\ V_+ & \longrightarrow & X_+ \end{array}$$

where U and V are summands of X and $U \amalg V = X$. Sheaves with respect to the topology generated by this cd structure is exactly the nonabelian derived category on C . In other words we have

$$\text{Shv}_{\amalg}(\mathbf{Sm}_{S_+}) \simeq P_\Sigma(\mathbf{Sm}_{S_+})$$

by [5, Lemma 2.4]. Hence all topologies τ considered in this paper satisfies $\text{Shv}_\tau(\mathbf{Sm}_{S_+}) \subseteq P_\Sigma(\mathbf{Sm}_{S_+})$.

Definition 3.17. Let τ be a topology on \mathbf{Sm}_S . Let C be a correspondence category with a graph functor $\gamma_C: \mathbf{Sm}_{S_+} \rightarrow C$. Then C is *compatible with τ* if for every τ -sieve $U \hookrightarrow X$ in \mathbf{Sm}_S , the natural map

$$\gamma_C(U_+) \rightarrow \gamma_C(X_+)$$

is an L_τ -equivalence in $P_\Sigma(\mathbf{Sm}_{S_+})$.

Lemma 3.18. *Suppose that C is a correspondence category which is compatible with τ . Then the functor $\gamma_C: P_\Sigma(\mathbf{Sm}_{S_+}) \rightarrow P_\Sigma(\mathbf{Sm}_{S_+})$ preserves L_τ -equivalences.*

Proof. By definition the class of L_τ -equivalences is the strong saturation, in the sense of [28, Proposition 5.5.4.5], of the maps in $P_\Sigma(\mathbf{Sm}_{S_+})$ by the (Yoneda image of the) maps $i_+: U_+ \hookrightarrow X_+$ where $X \in \mathbf{Sm}_S$ and i is a τ -sieve. According to [5, Lemma 2.10], the class of L_τ -equivalences is then generated under 2-out-of-3 and sifted colimits by maps of the form $\pi_X \amalg \text{id}_{Y_+}$ for $Y \in \mathbf{Sm}_S$. By the same reasoning as in Proposition 3.16 we need only check that $\gamma_C(U_+) \rightarrow \gamma_C(X_+)$ is an L_τ -equivalence which is true by hypothesis. \square

From now on, we make the following assumption on the topologies we discuss:

- The topology τ is at least as fine as the Nisnevich topology and is compatible in the sense of Definition 3.17.

3.2.6. If C is a correspondence category, then we can construct its unstable motivic homotopy ∞ -category in the usual way, as we now do. We consider two full subcategories of $P_\Sigma(C)$ spanned by objects \mathcal{F} satisfying the usual conditions:

- ($P_{\mathbf{A}^1}(C)$) The presheaf $\mathcal{F} \circ \gamma_C : \mathrm{Sm}_S^{\mathrm{op}} \rightarrow \mathrm{Spc}$ is \mathbf{A}^1 -invariant. We denote the ∞ -category spanned by such \mathcal{F} 's by $P_{\mathbf{A}^1}(C)$.
- ($\mathrm{Shv}_\tau(C)$) The presheaf $\mathcal{F} \circ \gamma_C : \mathrm{Sm}_S^{\mathrm{op}} \rightarrow \mathrm{Spc}$ is a τ -sheaf. We denote the ∞ -category spanned by such \mathcal{F} 's by $\mathrm{Shv}_\tau(C)$.

Now since $P_\Sigma(C)$ is preadditive by Proposition (3.14) we have a canonical equivalence $\mathrm{CMon}(P_\Sigma(C)) \simeq P_\Sigma(C)$. The ∞ -category of *unstable C -motives*, denoted by $H_\tau(C)$, is then defined as $P_{\mathbf{A}^1}(C) \cap \mathrm{Shv}_\tau(C) \subseteq P_\Sigma(C)$. As usual we have localization functors $L_\tau^C : P_\Sigma(C) \rightarrow \mathrm{Shv}_\tau(C)$, $L_{\mathbf{A}^1}^C : P_\Sigma(C) \rightarrow P_{\mathbf{A}^1}(C)$ and $L_{\mathrm{mot},\tau}^C : P_\Sigma(C) \rightarrow H_\tau(C)$. From the construction of these localizations and the assumption on τ , the adjunction (3.5) descends to an adjunction

$$\gamma_C^* : H_\tau(\mathrm{Sm}_{S+}) \simeq H_\tau(S)_* \rightleftarrows H_\tau(C) : \gamma_{C*} \quad (3.5)$$

Lemma 3.19. *The ∞ -category $H_\tau(C)$ is preadditive. Hence we have a canonical equivalence $\mathrm{CMon}(H_\tau(C)^\times) \simeq H_\tau(C)$.*

Proof. The ∞ -category $H_\tau(C)$ is closed under finite products by checking that the conditions (Htpy) and (τ -Desc) are preserved under taking products which are computed pointwise. The statement follows since $P_\Sigma(C)$ is preadditive by Proposition 3.14. \square

Definition 3.20. The ∞ -category of *effective C -motives* $H_\tau(C)^{\mathrm{gp}}$ is defined to be the full subcategory of $H_\tau(C)$ spanned by the grouplike objects, in the sense of [20, Definition 1.2].

3.2.7. The next proposition captures the main property of categories of correspondences from the point of view of motivic homotopy theory.

Proposition 3.21. *Suppose that C is a correspondence category which is compatible with τ . Then the functor*

$$\gamma_{C*} : H_\tau(C) \rightarrow H_\tau(S)_*$$

preserves sifted colimits and is conservative. Furthermore, $H_\tau(C)$ is canonically an $H(S)_$ -module.*

Proof. For the first claim it suffices, after Proposition 3.14, to check that

$$\gamma_{C*} : P_\Sigma(C) \rightarrow P_\Sigma(\mathrm{Sm}_{S+}) \simeq P_\Sigma(\mathrm{Sm}_S)_*$$

sends $L_{\mathrm{mot},\tau}^C$ -equivalences to $L_{\mathrm{mot},\tau}$ -equivalences. This holds by Lemma 3.21 and Lemma 3.16. The assertion that $H_\tau(C)$ is an $H(S)_*$ -module follows from the right lax structure of γ_{C*} . \square

Remark 3.22. If τ is a topology finer than the Nisnevich topology, then the fully faithful functor $H_\tau(S)_* \rightarrow H(S)_*$ need not preserve colimits. Hence the composite $H_\tau(C) \rightarrow H_\tau(S)_*$ need not preserve colimits.

From now on, if $\tau = \mathrm{Nis}$, we drop the decoration τ from $H_\tau(C)$ and $H_\tau(S)_*$ and so forth.

3.2.8. From the above point of view, we see that γ_{C*} is very close to preserving all colimits—we need only show that it preserves finite coproducts. The universal way to enforce this is to take commutative monoid objects on both sides with respect to Cartesian monoidal structures. We can do this for $H_\tau(S)_*$ since it has finite products, and $\mathrm{CMon}(H_\tau(C)^\times) \simeq H_\tau(C)$ since it is preadditive [20, Proposition 2.3]. We remark that the symmetric monoidal structure on $P_\Sigma(\mathrm{Sm}_{S+})$ given by Day convolution is not Cartesian ⁵.

To see this, consider the left adjoint to γ_{C*} , that is,

$$\gamma_C^* : H_\tau(S)_* \rightarrow H_\tau(C),$$

⁵Although the symmetric monoidal structure on $P_\Sigma(\mathrm{Sm}_S)$ given by Day convolution is, and the natural sifted-colimit preserving functor $P_\Sigma(\mathrm{Sm}_S) \rightarrow P_\Sigma(\mathrm{Sm}_{S+})$ is symmetric monoidal.

which preserves all small colimits. According to the universal property of \mathbf{CMon} [20, Corollary 4.9] we obtain an essentially unique functor $\gamma_C^* : \mathbf{CMon}(\mathbf{H}_\tau(S)_*^\times) \rightarrow \mathbf{H}_\tau(C)$ since $\mathbf{H}_\tau(C)$ is preadditive by Proposition 3.14.1. This functor admits a right adjoint $\gamma_{C*} : \mathbf{H}_\tau(C) \rightarrow \mathbf{CMon}(\mathbf{H}_\tau(S)_*)$ which fits into a commutative diagram

$$\begin{array}{ccc} & \mathbf{CMon}(\mathbf{H}_\tau(S)_*) & \\ & \nearrow \gamma_C^* & \downarrow \\ \mathbf{H}_\tau(C) & \xrightarrow{\gamma_{C*}} & \mathbf{H}_\tau(S)_*. \end{array} \quad (3.6)$$

In other words, the functor γ_{C*} factors through the forgetful functor $\mathbf{CMon}(\mathbf{H}_\tau(S)_*) \rightarrow \mathbf{H}_\tau(S)_*$.

Proposition 3.23. *Suppose that C is a correspondence category which is compatible with τ . Then the functor*

$$\gamma_{C*} : \mathbf{H}_\tau(C) \rightarrow \mathbf{CMon}(\mathbf{H}_\tau(S)_*^\times)$$

preserves all small colimits and is conservative.

Proof. By the diagram (3.6), the functor γ_{C*} preserves sifted colimits as the horizontal arrow preserves sifted colimit by Proposition 3.21 and the vertical arrow preserves sifted colimit as a special case of [20, Proposition B.4]. Since it is a right adjoint it preserves finite products, but since its domain and codomain are preadditive it preserves finite coproducts as well and we are done by [5, Lemma 2.8]. The conservativity statement follows from Proposition 3.21 and the fact that the forgetful functor from commutative monoid objects is conservative. \square

3.2.9. \mathbb{T} -stability. Now we introduce a more refined notion than simply \otimes -inverting \mathbb{T} . This is inspired by the treatment of [30, Appendix C] on prestable ∞ -categories.

Definition 3.24. Let C be an $\mathbf{H}(S)_*$ -module in \mathbf{Cat}_∞ . Then C is \mathbb{T} -prestability if the endofunctor

$$\mathbb{T} \otimes (-) : C \rightarrow C \quad (3.7)$$

is fully faithful. The ∞ -category C is \mathbb{T} -stable if the endofunctor (3.7) is invertible.

Remark 3.25. The notion of a \mathbb{T} -stable ∞ -category is a familiar one in motivic homotopy theory. In fact, \mathbb{T} -prestability is too—it is inspired by *cancellation theorems* in the sense of [40] which asserts that $\mathbf{DM}^{\text{eff}}(k; \mathbf{Z})$ is a \mathbb{T} -prestability for any perfect field k . The analogous statement hold for Milnor–Witt motivic cohomology as proved in [19]. The results of [10] give a framework for cancellation theorems. For the ∞ -category of framed motivic spaces, cancellation holds as well [11, Theorem 3.5.8], which in turn relies on the cancellation theorem of Ananyevskiy, Garkusha and Panin [1]. Moreover, for any base scheme S , the subcategory $\mathbf{SH}(S)^{\text{eff}} \subseteq \mathbf{SH}(S)$ of effective motivic spectra is \mathbb{T} -prestability.

3.2.10. The thesis of Robalo [35] provides a way to invert \mathbb{T} for any $\mathbf{H}(C)_*$ -module and obtain a symmetric monoidal stable ∞ -category—in fact one that is a module over $\mathbf{SH}(S)$. We define the stable ∞ -category of C -motives simply by

$$\mathbf{SH}_\tau(C) := \mathbf{H}_\tau(C)[\mathbb{T}^{\otimes -1}],$$

with notation as in *loc. cit* Definition 2.6. We then have the basic adjunction

$$\Sigma_{\mathbb{T}, C}^\infty : \mathbf{H}_\tau(C) \rightleftarrows \mathbf{SH}_\tau(C) : \Omega_{\mathbb{T}, C}^\infty.$$

The following summarizes the basic properties of $\mathbf{SH}_\tau(C)$:

Proposition 3.26. *If C is correspondence category, then*

- (1) *The ∞ -category $\mathbf{SH}_\tau(C)$ is a stable presentably symmetric monoidal ∞ -category, and*
- (2) *is generated under sifted colimits by objects of the form $\{\mathbb{T}^{\otimes n} \otimes \Sigma_{\mathbb{T}, C}^\infty X\}_{n \in \mathbf{Z}, X \in C}$.*
- (3) *The ∞ -category $\mathbf{SH}_\tau(C)$ is computed as the colimit in $\mathbf{Mod}_{\mathbf{H}(\mathbf{Sm}_S)_*}(\mathbf{Pr}^L)$ of*

$$\mathbf{H}_\tau(C) \xrightarrow{\mathbb{T}^{\otimes -}} \mathbf{H}_\tau(C) \xrightarrow{\mathbb{T}^{\otimes -}} \mathbf{H}_\tau(C) \xrightarrow{\mathbb{T}^{\otimes -}} \dots \quad (3.8)$$

- (4) *The functor*

$$\gamma_{C*} : \mathbf{SH}_\tau(C) \rightarrow \mathbf{SH}_\tau(\mathbf{Sm}_S)$$

is conservative and preserves colimits.

Proof. Stability follows from the standard equivalence $\mathbb{T} \simeq \mathbf{G}_m \otimes S^1$ in $\mathrm{SH}(S)$, which persists on modules over $\mathrm{SH}(S)$. The second assertion follows from the third via [28, Lemma 6.3.3.7] and the fact that $H_\tau(C)$ is generated under sifted colimits by representables which are smooth affine by the argument of [26, Proposition 2.2.9] (which works for any topology τ finer than Nis), while the third comes from [35, Corollary 2.22]. The last assertion follows from Proposition 3.23. \square

3.2.11. The last part of Proposition 3.27 is the main point of our axiomatization: the adjunction $\mathrm{SH}_\tau(S) \rightleftarrows \mathrm{SH}_\tau(C)$ is monadic. In particular, if $\tau = \mathrm{Nis}$, then $\mathrm{SH}(S) \rightleftarrows \mathrm{SH}(C)$ is monadic.

3.2.12. *From categories of correspondences to motivic module categories.* Suppose that we have a functor

$$C : \mathcal{S}^{\mathrm{op}} \rightarrow \mathrm{CorrCat}^{\otimes}$$

which carries a morphism of schemes $f : T \rightarrow S$ to $f^* : C_S \rightarrow C_T$. By naturality of the preceding constructions ⁶ we obtain a functor

$$\mathrm{SH}_\tau \circ C : \mathcal{S}^{\mathrm{op}} \rightarrow \mathrm{Pr}_{\mathrm{stab}}^{L, \otimes}$$

equipped with a transformation $\mathrm{SH}|_{\mathcal{S}} \rightarrow \mathrm{SH}_\tau \circ C$. We impose an additional assumption on C , inspired by [7, Lemma 9.3.7]:

- For each $p : T \rightarrow S$, a smooth morphism in \mathcal{S} , the functor p^* admits a left adjoint $p_\#$ such that the transformation $p_\# \gamma_{C_T} \rightarrow \gamma_{C_S} p_\#$ is an equivalence.

In this case, we say that C is *adequate*.

3.2.13. We employ the following additional notation: if $L : \mathrm{SH}(S) \rightarrow \mathrm{SH}(S)$ is a localization, denote by $L(\mathrm{SH}_\tau(C_S))$ the subcategory of $\mathrm{SH}_\tau(C_S)$ spanned by objects X such that $\gamma_{C_*} X$ is L -local. Since γ_{C_*} preserves limits, the inclusion $L(\mathrm{SH}_\tau(C_S)) \hookrightarrow \mathrm{SH}_\tau(C_S)$ is closed under limits and there is a localization functor (by the adjoint functor theorem)

$$L_{C_S} : \mathrm{SH}_\tau(C_S) \rightarrow L(\mathrm{SH}_\tau(C_S))$$

rendering the following diagram commutative (since their right adjoints commute):

$$\begin{array}{ccc} \mathrm{SH}(S) & \xrightarrow{\gamma_{C_S}^*} & \mathrm{SH}_\tau(C_S) \\ L \downarrow & & \downarrow L_{C_S} \\ L(\mathrm{SH}(S)) & \xrightarrow{\gamma_{C_S}^*} & L(\mathrm{SH}_\tau(C_S)). \end{array}$$

Proposition 3.27. *If $C : \mathcal{S}^{\mathrm{op}} \rightarrow \mathrm{CorrCat}^{\otimes}$ is adequate, then the following hold:*

- (1) *We have premotivic adjunctions $\mathrm{SH}|_{\mathcal{S}} \rightleftarrows \mathrm{SH}_\tau \circ C$.*
- (2) *If L is a smashing and symmetric monoidal localization of $\mathrm{SH}|_{\mathcal{S}}$, then we have a premotivic adjunction $L(\mathrm{SH})|_{\mathcal{S}} \rightleftarrows L(\mathrm{SH} \circ C)$.*
- (3) *If τ is a topology such that for each $S \in \mathcal{S}$, the functor $L(\mathrm{SH}_\tau(S)) \rightarrow L(\mathrm{SH}(S))$ preserves sifted colimits, then the premotivic adjunction $L(\mathrm{SH})|_{\mathcal{S}} \rightleftarrows L(\mathrm{SH}_\tau \circ C)$ is a motivic module category (in particular, this holds, when $\tau = \mathrm{Nis}$).*

Proof. The proof of (1) follows as in the case of Grothendieck abelian categories [7, Corollary 10.3.11] and Voevodsky's $C = \mathrm{Corr}$ (in the sense of [7, §9]) we give only the main points. Since C is adequate, we get that the equivalence $p_\# \gamma_{C_T} \rightarrow \gamma_{C_S} p_\#$ persists on the level of \mathbb{T} -stabilizations. What we need to verify, just as in [7, Proposition 10.3.9] is that the transformation $L_\tau \gamma_{C_*} \simeq \gamma_{C_*} L_\tau$ is an equivalence on the unstable level, i.e., the “forgetful functor $H_\tau \circ C \rightarrow H|_{\mathcal{S}}$ preserves τ -local objects and this is given by Lemma 3.18 under the standing assumption that C is compatible with τ . The next two statements are then immediate from the definition of motivic module categories and the last statement of Proposition 3.27. \square

⁶The most non-trivial of which is the universal property of \mathbb{T} -stabilization for which we can appeal to [5, Lemma 4.1].

3.3. Examples. We now discuss examples of the above constructions and results.

Example 3.28. Consider a Cartesian section of $\mathrm{SH} \rightarrow \mathcal{S}$, taking values in motivic \mathcal{E}_∞ -ring spectra. Then the transformation $E \otimes (-) : \mathrm{SH}|_{\mathcal{S}} \rightarrow \mathrm{Mod}_E$ furnishes the first examples of motivic module categories. We can also consider further localizations of the premotivic category Mod_E , such as in [18] where $\mathcal{S} = \mathrm{Sch}_{\mathbb{Z}[\frac{1}{2}]}$ the localization functor is given by the composite of ℓ -completion and étale localization, and E is MGL; see *loc. cit* for more details where results in this paper is used to describe the ∞ -category of modules over étale cobordism.

Example 3.29. Consider a localization $L : \mathrm{SH}|_{\mathcal{S}} \rightarrow L(\mathrm{SH}|_{\mathcal{S}})$. Then if L is smashing $L(\mathrm{SH}|_{\mathcal{S}})$ is a motivic module category. Examples of these smashing localizations are given by *periodization of elements*; we refer the reader to [22, Section 3] for an extensive discussion in our context. For example, a theorem of Bachmann [3] proves that periodizing the element ρ yields real étale localization. If $x : \Sigma^{p,q}\mathbf{1} \rightarrow \mathbf{1}$, then the results of [22, §3] (or apply [3, Lemma 15]) tells us that $\mathbf{1}[x^{-1}]$ is an \mathcal{E}_∞ -ring and the projection formula holds. Thus, the category of x -periodic motivic spectra are modules over $\mathbf{1}[x^{-1}]$.

Example 3.30. The first example of a category of correspondences is Voevodsky’s category of correspondences Corr_S in the sense of [32, Appendix 1A] [7, §9]; this is defined any Noetherian scheme S [7, §9.1]. When S is essentially smooth over a base field, the category Milnor-Witt correspondences $\widetilde{\mathrm{Corr}}_S$ of Calmès and Fasel [6] is defined and is also a category of correspondences. Over a field (whence both categories are defined), these categories are generalized by Garkusha’s axioms in [15]. When defined, these categories are adequate in the sense of §3.2.12. All these are examples of categories of correspondences, and thus gives rise motivic module categories.

Example 3.31. Let k be a field. For any $S \in \mathrm{Sm}_k$, and any good cohomology theory A on Sm_S in the sense of [10, 2], [10, 3.2] defines an adequate category of correspondences Corr^A on Sm_S .

Example 3.32. The ∞ -category of framed correspondences of [11] is another example of a category of correspondences and is defined for any qcqs scheme S . The main theorem of [23] asserts that the corresponding motivic module category is equivalent to $\mathrm{SH}(S)$, relying on the “recognition principle” of [11].

Example 3.33. If $E \in \mathrm{SH}(S)$ is a homotopy associative ring spectrum, then [12] defines a category of *finite E-correspondences*. These are discrete categories enriched in spaces. These are again motivic module categories and $\mathrm{SH}(C)$ in this paper corresponds to $\mathrm{DM}^E(S)$ in *loc. cit*.

4. THE CASE OF REGULAR SCHEMES OVER A FIELD

4.1. The case of fields. We now aim to investigate when the hypotheses of Theorem 3.6 are satisfied. One way to ensure that the projection formula holds is to use the following computation to reduce to the case of compact-rigid generation:

Lemma 4.1. *Suppose that we have an adjunction of symmetric monoidal ∞ -categories*

$$F : C \rightleftarrows D : G,$$

such that F is strong symmetric monoidal. Let $1 \in C$ denote the unit object of C . If $E \in C$ is a strongly dualizable object, then the map $c : GF(1) \otimes E \rightarrow GF(E)$ is an equivalence.

Proof. This follows from a standard computation: let $E' \in C$ be arbitrary, then we have a string of equivalences

$$\begin{aligned} \mathrm{Maps}_C(E', GF(1) \otimes E) &\simeq \mathrm{Maps}_C(E' \otimes E^\vee, GF(1)) \\ &\simeq \mathrm{Maps}_D(F(E' \otimes E^\vee), F(1)) \\ &\simeq \mathrm{Maps}_D(F(E') \otimes F(E)^\vee, F(1)) \\ &\simeq \mathrm{Maps}_D(F(E'), F(E)) \\ &\simeq \mathrm{Maps}_C(E', GF(E)), \end{aligned}$$

which shows the claim. □

4.1.1. Thus, if $\mathrm{SH}(-)$ is generated by strongly dualizable objects, it follows that the projection formula holds:

Theorem 4.2. *Let k be a field. Suppose that ℓ is a prime which is coprime to the exponential characteristic e of k and \mathcal{M} is a motivic module category on k . Then we have the following equivalences of presentably symmetric monoidal stable ∞ -categories:*

$$L_{(\ell)}(\mathcal{M}(k)) \simeq \mathrm{Mod}_{L_{(\ell)}\gamma_*\gamma^*(\mathbf{1}_S)}(\mathrm{SH}(k)),$$

and

$$\mathcal{M}(k)_{[\frac{1}{e}]} \simeq \mathrm{Mod}_{\gamma_*\gamma^*(\mathbf{1})_{[\frac{1}{e}]}}(\mathrm{SH}(k)).$$

Proof. After Theorem 3.6, we need to verify the appropriate projection formulas. By assumption, the functor γ_* preserves sifted colimits and thus the functors $\gamma_*\gamma^*(\mathbf{1}_S) \otimes (-)$ and $\gamma_*\gamma^*(-)$ do as well. Since Lemma 4.1 verifies the projection formula for strongly dualizable objects in $\mathrm{SH}(k)_{(\ell)}$, we will be done if we can prove that the second inclusion of (2.1) $\mathrm{SH}^{\mathrm{rig}}(k)_{(\ell)} \subseteq \mathrm{SH}^\omega(k)_{(\ell)}$ is an equality, i.e., $\mathrm{SH}(k)_{(\ell)}$ is in fact generated by sifted colimits by strongly dualizable objects. This follows by Example 2.3, which also verifies the theorem for the e -inverted case. \square

4.1.2. We now obtain the following extension of [37, Theorem 1], [24, Theorem 5.8], [15, Theorem 5.3], [4, Lemma 5.3]:

Corollary 4.3. *Let k be a field with exponential characteristic e and let $\gamma_C: \mathrm{Sm}_k \rightarrow \mathcal{C}$ be a correspondence category. Then there is an equivalence of presentably symmetric monoidal stable ∞ -categories*

$$\mathrm{SH}(\mathcal{C})_{[\frac{1}{e}]} \simeq \mathrm{Mod}_{\gamma_{C*}\gamma_C^*(\mathbf{1})_{[\frac{1}{e}]}}(\mathrm{SH}(k)).$$

4.2. **The case $\mathcal{S} = \mathrm{Reg}_k$.** Following [8], we can extend the previous result to the category Reg_k of finite dimensional noetherian schemes that are regular over a field, provided that we impose some additional assumptions on \mathcal{M} . For the rest of this section, we will therefore assume that \mathcal{M} is a motivic module category which in addition satisfies the following property:

- The premotivic category \mathcal{M} satisfies the localization (Definition 2.5) and continuity (Definition 2.6).

Lemma 4.4. *Suppose that $f: T \rightarrow S$ is a morphism in Reg_k . In the following cases, the transformation*

$$f^*\gamma_* \rightarrow \gamma_*f^*$$

is an equivalence:

- (1) *The map f is an inverse limit*

$$f = \varprojlim_{\alpha} f_{\alpha} T_{\alpha} \rightarrow S,$$

where the transition maps $f_{\alpha\beta}: T_{\alpha} \rightarrow T_{\beta}$ are dominant, affine and smooth.

- (2) *The map f is a closed immersion and*

$$S \simeq \varprojlim_{\alpha} S_{\alpha},$$

where each S_{α} is a smooth, separated k -scheme of finite type with flat affine transition maps.

Proof. Under the continuity and localization assumption on \mathcal{M} , the proof in [8, Lemma 3.20] for the case of $\mathcal{M} = \mathrm{DM}(-, R)$ applies verbatim. \square

4.2.1. We now have the following extension of Theorem 4.2.

Theorem 4.5. *Let k be a field of exponential characteristic e , and let \mathcal{M} be a motivic module category on Reg_k . Then the functor $\gamma^*: \mathrm{SH} \rightarrow \mathcal{M}$ induces a canonical equivalence*

$$\mathrm{Mod}_{\gamma_*\gamma^*(\mathbf{1})_{[\frac{1}{e}]}}(\mathrm{SH}(-)) \rightarrow \mathcal{M}_{[\frac{1}{e}]}$$

of premotivic categories on Reg_k .

Proof. After Theorem 3.6, our goal is to verify that $(\mathrm{SH}|_{\mathrm{Reg}_k, \mathcal{M}})$ satisfies the projection formula. Suppose that $S \in \mathrm{Reg}_k$, and let $E \in \mathrm{SH}(S)$. We claim that the map

$$\gamma_* \gamma^*(\mathbf{1}_S) \otimes E \rightarrow \gamma_* \gamma^*(E) \quad (4.1)$$

is an equivalence. We follow closely the logic of [8, Theorem 3.1].

First, assume that S is an essentially smooth scheme over a field. For each $x \in S$, we write S_x for the localization of S at x . Then the family of functors

$$\{\mathrm{SH}(S) \rightarrow \mathrm{SH}(S_x)\}$$

is conservative by [7, Proposition 4.3.9]. Hence we are reduced to proving that the map (4.1) is an equivalence in the case that S is furthermore *local*. In this case, let $i: x \hookrightarrow S_x$ be the closed point and write $j: U_x \hookrightarrow$ for the open complement which, by our assumption on S , has dimension $< \dim S$. We consider the following commutative diagram, where the rows are cofiber sequences:

$$\begin{array}{ccccc} j_!(j^* \gamma_* \gamma^*(\mathbf{1}_S) \otimes j^* E) & \longrightarrow & \gamma_* \gamma^*(\mathbf{1}_S) \otimes E & \longrightarrow & i_*(i^* \gamma_* \gamma^*(\mathbf{1}_S) \otimes i^* E) \\ \downarrow & & \downarrow & & \downarrow \\ j_! j^* \gamma_* \gamma^*(E) & \longrightarrow & \gamma_* \gamma^*(E) & \longrightarrow & i_* i^* \gamma_* \gamma^*(E) \\ \downarrow f_1 & & \downarrow = & & \downarrow f_2 \\ j_! \gamma_* \gamma^* j^* E & \longrightarrow & \gamma_* \gamma^* E & \longrightarrow & i_* \gamma_* \gamma^* i^* E. \end{array} \quad (4.2)$$

Now,

- The left vertical composite is an equivalence because (1) j_* commutes with γ_* since j is smooth, and (2) by the induction hypothesis.
- The right vertical composite is an equivalence using (1) Lemma 4.4.2 and (2) the case of fields, Theorem 4.2

Hence, it remains to show that f_1 and f_2 are equivalences.

- The map f_1 is an equivalence because j_* commutes with γ_* , again because j is smooth.
- That f_2 is an equivalence follows from Lemma 4.4.2.

Now, following the “General case” of [8], we explain how the bootstrap to regular k -schemes work. By continuity (appealing to [7, Proposition 4.3.9] again), we may again assume that S is a *Henselian local* regular k -scheme. As explained in *loc. cit.*, there is a sequence of regular Noetherian k -schemes

$$T \xrightarrow{f} S' \xrightarrow{q} S$$

such that the following hold:

- the scheme S' has infinite residue field and the functor $q^*: \mathrm{SH}(S)[\frac{1}{e}] \rightarrow \mathrm{SH}(S')[\frac{1}{e}]$ is conservative.
- The scheme T is the ∞ -*goflement* of $\Gamma(S', \mathcal{O}'_S)$ [8, Definition 3.21] and the functor $f^*: \mathrm{SH}(S')[\frac{1}{e}] \rightarrow \mathrm{SH}(T)[\frac{1}{e}]$ is conservative.
- Both f, q satisfy the hypotheses of Lemma 4.4.1 and thus f^*, q^* commutes with γ_* .

Hence, to check that the map (4.1) is an equivalence it suffices to check that it is an equivalence after $(qf)^*$. Since T is, by construction, the spectrum of a filtered union of its smooth subalgebras we invoke continuity of SH to conclude. \square

4.2.2. Lastly, we have the following class of examples of motivic module categories for which localization and continuity holds. Recall that any motivic \mathcal{E}_∞ -ring spectrum E gives rise the motivic module category DM^E [12].

Proposition 4.6. *Let $\mathcal{S} \subseteq \mathrm{Sch}_S$. Then, for any homotopy associative ring spectrum $E \in \mathrm{SH}(S)$, premotivic category $\mathrm{DM}^E: \mathcal{S}^{\mathrm{op}} \rightarrow \mathrm{Cat}_\infty$ satisfies continuity for dominant affine morphisms.*

Proof. We first claim the analog of [7, Proposition 9.3.9] for E -correspondences. Let $(X_i)_{i \in I}$ be a cofiltered diagram of separated S -schemes of finite type with affine dominant transition morphisms. Let $X = \varprojlim_i X_i$,

which is assumed to exist in Sch_S and is assumed to be Noetherian. Then we claim that for any separated S -scheme Y of finite type, the map

$$\text{colim}_{i \in I^{\text{op}}} \text{Corr}_S^E(X_i, Y) \rightarrow \text{Corr}_S^E(X, Y) \quad (4.3)$$

is an equivalence.

To do so, we use the dual of [11, Lemma 4.1.26]. Denote by c_{X_i} (resp. c_X) be the filtered poset of reduced subschemes of $X_i \times_S Y$ (resp. $X \times_S Y$) which are finite and universally open over X_i (resp. X). We denote by $\text{Sub}(c_X)$ the poset of full sub-posets of c_X and we have functor $K: I \rightarrow \text{Sub}(c_X)$, $i \mapsto K_i = c_{X_i}$ where c_{X_i} is regarded as a full sub-poset in the obvious way. We have a functor $E^{\text{BM}}(-/X): c_X \rightarrow \text{Spc}$ and, by continuity of SH, restricts to the functor $E^{\text{BM}}(-/X_i): c_{X_i} \rightarrow \text{Spc}$.

Hence, the map (4.3) is, by [12, Definition 4.1.1], equal to the map

$$\text{colim}_{I^{\text{op}}} \text{colim}_{c_{X_i}} E^{\text{BM}}(Z_i/X_i) \rightarrow \text{colim}_{Z \in c_X} E^{\text{BM}}(Z/X),$$

which we claim is an equivalence. The hypotheses of [11, Lemma 4.1.26] follows easily (under the hypotheses that the transition maps are affine and dominant) by [7, Proposition 8.3.9, 8.3.6]. Hence the desired claim follows. The rest of the proof follows as in the case of DM from [7, Theorem 11.1.24]. \square

Proposition 4.7. *Let k be a field and let $E \in \text{SH}(k)$ be a homotopy associative ring spectrum, then the premotivic category $\text{DM}^E: \mathcal{S}^{\text{op}} \rightarrow \text{Cat}_\infty$ satisfies Loc_i whenever i is a closed immersion of regular schemes.*

Proof. Since DM^E is constructed from Nisnevich local objects, it is Nisnevich separated. By [7, Proposition 6.3.14], it has the weak localization property, i.e., it has Loc_i for any closed immersions with smooth retractions. Arguing as in [7, Corollary 6.3.15], it has the localization property with respect to any closed immersion between smooth schemes. The rest of the argument then follows as in [8, Proposition 3.12], which uses the continuity results established in Proposition 4.7 as above. \square

4.2.3. From this we conclude:

Corollary 4.8. *Let k be a field, $E \in \text{SH}(k)$ a homotopy associative ring spectrum, then we have a canonical equivalence of premotivic categories on Reg_k*

$$\text{DM}^E\left[\frac{1}{e}\right] \simeq \text{Mod}_{\gamma_*\gamma^*(\mathbf{1})\left[\frac{1}{e}\right]}.$$

Example 4.9. In [12], it is proved that $\text{DM}^{\text{HZ}}(S) \simeq \text{DM}(S)$ (resp. $\text{DM}^{\text{HZ}}(S) \simeq \widetilde{\text{DM}}(S)$) whenever S is essentially smooth over a Dedekind domain (resp. essentially smooth over a perfect field) [12, Proposition 4.3.8] (resp. [12, Proposition 4.3.19]). By the continuity result of Proposition 4.7 we can enhance the comparison results for DM to regular schemes over fields. While $\widetilde{\text{DM}}(S)$ is not defined outside of smooth schemes over fields, Corollary 4.8 promotes the comparison results between $\widetilde{\text{DM}}$ and modules over HZ of [15] and [4] at least to smooth schemes over fields. We contend, however, that $\text{DM}^{\text{HZ}}(S)$ is a decent definition for $\widetilde{\text{DM}}(S)$ in general.

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