# PERFECTION IN MOTIVIC HOMOTOPY THEORY

ELDEN ELMANTO AND ADEEL A. KHAN

Abstract. We prove a topological invariance statement for the Morel–Voevodsky motivic homotopy category, up to inverting exponential characteristics of residue fields. When $p > 1$ this implies, in particular, that $\mathbf{SH}[\frac{1}{p}]$ of characteristic $p$ schemes is invariant under passing to perfections. Among other applications we prove Grothendieck–Verdier duality in this context.

Contents



## 1. Introduction

Let S be a scheme and denote by $\mathrm{S}_{\text{ét}}$ its small étale topos. The starting point for this note is Grothendieck's "équivalence remarquable de catégories" [Gro67, Théorème 18.1.2], which asserts that for any nil-immersion $f : \mathrm{S}_0 \hookrightarrow \mathrm{S}$, there is an induced equivalence

$$f^* : \mathrm{S}_{\text{ét}} \to (\mathrm{S}_0)_{\text{ét}}.$$

In fact, Grothendieck further generalized this to a *topological invariance* statement for the small étale topos: for any universal homeomorphism of schemes $f : \mathrm{T} \to \mathrm{S}$, the functor $f^* : \mathrm{S}_{\text{ét}} \to \mathrm{T}_{\text{ét}}$ is an equivalence (see [GR71, Exposé IX, Théorème 4.10], [AGV72, Exposé VIII, Théorème 1.1]).

The large étale topos fails to satisfy nil-invariance. An observation of Morel and Voevodsky [MV99] was that this failure can be repaired by working in the setting of $\mathbf{A}^1$*-invariant* sheaves. Indeed, it is a consequence of the Morel–Voevodsky localization theorem that the stable motivic homotopy category **SH** satisfies nil-invariance (see e.g. [CD12, Proposition 2.3.6(1)]). However, the topological invariance property still fails, at least in positive characteristic (see Remark 2.1.12). Our goal in this paper is to show that topological invariance is in fact true for **SH**, after inverting the exponential characteristic of the base field (Theorem 2.1.1). This also recovers the analogous statement in other related contexts, such as various variants of mixed motives [Ayo14, CD12, CD15, CD16] (see Remark 2.1.2).

A particularly useful consequence of topological invariance is that, for any scheme S of characteristic $p$, $\mathbf{SH}(\mathrm{S})[\frac{1}{p}]$ is invariant under passing to the perfection $\mathrm{S}_{\mathrm{perf}}$ (Corollary 2.1.7). This allows us to remove perfectness hypotheses on the base field in many results, see §3.2. It

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also yields a Grothendieck–Verdier duality statement, following ideas of Cisinski–Déglise [CD15] (Theorem 3.1.1).

## 1.1. **Conventions.**

1.1.1. All schemes will implicitly be assumed to be quasi-compact quasi-separated.

Recall that a morphism of schemes $f : \mathrm{X} \to \mathrm{Y}$ is a *universal homeomorphism* if it induces a homeomorphism on underlying topological spaces after any base change, or equivalently, if it is integral, universally injective, and surjective [Gro67, Corollary 18.12.11].

1.1.2. If S is a scheme of characteristic $p > 0$, i.e., an $\mathbf{F}_p$-scheme, we write $\mathrm{F_S} : \mathrm{S} \to \mathrm{S}$ for the Frobenius endomorphism [Gro77, Exposé XIV=XV]. Recall that S is *perfect* if the Frobenius $\mathrm{F_S} : \mathrm{S} \to \mathrm{S}$ is an isomorphism. Any $\mathbf{F}_p$-scheme S admits a *perfection* $\mathrm{S_{perf}}$, defined as the limit of the tower

$$\cdots \xrightarrow{\mathrm{F_S}} \mathrm{S} \xrightarrow{\mathrm{F_S}} \mathrm{S},$$

see [BS17, Section 3].

1.1.3. Given a scheme S, we denote by $\mathbf{SH}(\mathrm{S})$ the stable $\infty$-category of motivic spectra over S. We will use the language of six operations, see [Hoy14, Appendix C] or [Kha16] for the non-noetherian setting. Any motivic spectrum $\mathrm{E} \in \mathbf{SH}(\mathrm{S})$ represents a cohomology theory on S-schemes, given by the formula

$$\mathrm{E}(\mathrm{X}, \xi) = \mathrm{Maps}_{\mathbf{SH}(\mathrm{S})}(\mathbf{1}_\mathrm{S}, f_* \Sigma^\xi f^*(\mathrm{E}))$$

for any morphism $f : \mathrm{X} \to \mathrm{S}$ and any K-theory class $\xi \in \mathrm{K}(\mathrm{X})$. Similarly, there is a Borel–Moore homology theory

$$\mathrm{E}^{\mathrm{BM}}(\mathrm{X}/\mathrm{S}, \xi) = \mathrm{Maps}_{\mathbf{SH}(\mathrm{S})}(\mathbf{1}_\mathrm{S}, f_* \Sigma^{-\xi} f^!(\mathrm{E})).$$

We refer to [DJK18] for details.

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# 2. Topological invariance

2.1. **Main result and corollaries.** Let $\mathcal{P}$ be a set of prime numbers. For a scheme S, we denote by $\mathbf{SH}(\mathrm{S})[\mathcal{P}^{-1}]$ the localization of $\mathbf{SH}(\mathrm{S})$ at the morphisms $p : \mathrm{E} \to \mathrm{E}$, for $\mathrm{E} \in \mathbf{SH}(\mathrm{S})$ and $p \in \mathcal{P}$. When $\mathcal{P}$ contains a single prime $p$, we write simply $\mathbf{SH}(\mathrm{S})[\frac{1}{p}]$.

**Theorem 2.1.1.** *Let* S *be a scheme and* $\mathcal{P}$ *a set of prime numbers. Suppose that every prime* $q \notin \mathcal{P}$ *is invertible in* $\mathcal{O}_\mathrm{S}$*. Then for any universal homeomorphism* $f : \mathrm{T} \to \mathrm{S}$*, the functor*

$$f^* : \mathbf{SH}(\mathrm{S})[\mathcal{P}^{-1}] \to \mathbf{SH}(\mathrm{T})[\mathcal{P}^{-1}]$$

*is an equivalence.*

*Remark* 2.1.2. The proof of Theorem 2.1.1 will in fact apply to any motivic $\infty$-category of coefficients $\mathbf{D}$ as in [Kha16, Chap. 2, Definition 3.5.2]. See Remark 2.2.7 for details.

We now record some immediate consequences. Taking $\mathcal{P}$ to be the set of all primes, we get:

**Corollary 2.1.3.** *For any universal homeomorphism* $f : \mathrm{T} \to \mathrm{S}$*, the functor*

$$f^* : \mathbf{SH}(\mathrm{S})_{\mathbf{Q}} \to \mathbf{SH}(\mathrm{T})_{\mathbf{Q}}$$

*is an equivalence.*

*Remark* 2.1.4. Recall that for every scheme S, the category $\mathbf{SH}(\mathrm{S})_{\mathbf{Q}}$ admits natural splittings

$$\mathbf{SH}(\mathrm{S})_{\mathbf{Q}} \simeq \mathbf{SH}(\mathrm{S})_{\mathbf{Q},+} \times \mathbf{SH}(\mathrm{S})_{\mathbf{Q},-},$$

see [CD12, Sect. 16.2]. The analogue of Corollary 2.1.3 is known for the plus part $\mathbf{SH}(\mathrm{S})_{\mathbf{Q},+}$, via the identification with Beilinson motives (see Theorems 14.3.3 and 16.2.13 in *op. cit.*). For the minus part, the statement appears to be new.

Taking $\mathcal{P}$ to be a single prime, we get:

**Corollary 2.1.5.** *Let* S *be a scheme of exponential characteristic* $p$*. Then for any universal homeomorphism* $f : \mathrm{T} \to \mathrm{S}$*, the functor*

$$f^* : \mathbf{SH}(\mathrm{S})[\tfrac{1}{p}] \to \mathbf{SH}(\mathrm{T})[\tfrac{1}{p}]$$

*is an equivalence.*

**Corollary 2.1.6.** *For every scheme* S *of characteristic* $p > 0$*, the absolute Frobenius induces an equivalence*

$$\mathrm{F}_{\mathrm{S}}^* : \mathbf{SH}(\mathrm{S})[\tfrac{1}{p}] \to \mathbf{SH}(\mathrm{S})[\tfrac{1}{p}].$$

**Corollary 2.1.7.** *For every scheme* S *of characteristic* $p > 0$*, the canonical morphism* $\mathrm{S}_{\mathrm{perf}} \to \mathrm{S}$ *induces an equivalence*

$$\mathbf{SH}(\mathrm{S})[\tfrac{1}{p}] \to \mathbf{SH}(\mathrm{S}_{\mathrm{perf}})[\tfrac{1}{p}].$$

*Proof.* Follows from Corollary 2.1.6 in view of continuity of **SH** [Hoy14, Proposition C.12(4)]. □

2.1.8. At the level of cohomology and Borel–Moore homology, we have the following reformulation (we consider the case $\mathcal{P} = \{p\}$ for simplicity):

**Corollary 2.1.9.** *Let* S *be a scheme of exponential characteristic* $p$*. Let* $\mathrm{E} \in \mathbf{SH}(\mathrm{S})$ *be a motivic spectrum over* S*. Then we have:*

(i) *For any universal homeomorphism* $f : \mathrm{X} \to \mathrm{Y}$ *of* S*-schemes, the induced maps*

$$f^* : \mathrm{E}(\mathrm{Y}, \xi)[\tfrac{1}{p}] \to \mathrm{E}(\mathrm{X}, f^*(\xi))[\tfrac{1}{p}]$$
$$f_* : \mathrm{E}^{\mathrm{BM}}(\mathrm{X}, f^*(\xi))[\tfrac{1}{p}] \to \mathrm{E}^{\mathrm{BM}}(\mathrm{Y}, \xi)[\tfrac{1}{p}]$$

*are equivalences for every* $\xi \in \mathrm{K}(\mathrm{Y})$*.*

(ii) *The canonical morphism* $f : \mathrm{S}_{\mathrm{perf}} \to \mathrm{S}$ *induces equivalences*

$$f^* : \mathrm{E}(\mathrm{S}, \xi)[\tfrac{1}{p}] \to \mathrm{E}(\mathrm{S}_{\mathrm{perf}}, f^*(\xi))[\tfrac{1}{p}]$$
$$f_* : \mathrm{E}^{\mathrm{BM}}(\mathrm{S}_{\mathrm{perf}}, f^*(\xi))[\tfrac{1}{p}] \to \mathrm{E}^{\mathrm{BM}}(\mathrm{S}, \xi)[\tfrac{1}{p}]$$

*for every* $\xi \in \mathrm{K}(\mathrm{S})$*.*

*Proof.* Note that we have canonical identifications

$$\mathrm{E}(\mathrm{T}, \psi)[\tfrac{1}{p}] \simeq \mathrm{Maps}_{\mathbf{SH}(\mathrm{S})}(\mathbf{1}_{\mathrm{S}}, (\pi_{\mathrm{T}})_* \Sigma^{\psi} \pi_{\mathrm{T}}^*(\mathrm{E}))[\tfrac{1}{p}] \simeq \mathrm{Maps}_{\mathbf{SH}(\mathrm{S})[\frac{1}{p}]}(\mathbf{1}_{\mathrm{S}}, (\pi_{\mathrm{T}})_* \Sigma^{\psi} \pi_{\mathrm{T}}^*(\mathrm{E}))$$

for every $\pi_{\mathrm{T}} : \mathrm{T} \to \mathrm{S}$ and $\psi \in \mathrm{K}(\mathrm{T})$. Therefore the map on cohomology spaces is induced from the natural transformation

$$(\pi_{\mathrm{Y}})_* \Sigma^{\xi} \pi_{\mathrm{Y}}^* \to (\pi_{\mathrm{Y}})_* \Sigma^{\xi} f_* f^* \pi_{\mathrm{Y}}^* \simeq (\pi_{\mathrm{Y}})_* f_* \Sigma^{f^*(\xi)} f^* \pi_{\mathrm{Y}}^* \simeq (\pi_{\mathrm{X}})_* \Sigma^{f^*(\xi)} \pi_{\mathrm{X}}^*.$$

For $f$ as in (i) (resp. (ii)), the unit map $\mathrm{id} \to f_* f^*$ is invertible after inverting $p$ by Theorem 2.1.1 (resp. by Corollary 2.1.7), whence the claim. The proof for Borel–Moore homology is similar, using the fact that the co-unit map $f_* f^! \to \mathrm{id}$ is also invertible in both cases (after inverting $p$). □

*Remark* 2.1.10. Corollary 2.1.9 also holds for the compactly supported variants (cohomology with compact support and relative homology), with the same proofs.

*Example* 2.1.11. Let KGL $\in \mathbf{SH}(\mathbf{F}_p)$ denote the homotopy invariant K-theory spectrum over $\mathrm{Spec}(\mathbf{F}_p)$. For every (possibly singular and non-noetherian) $\mathbf{F}_p$-scheme S, we have functorial equivalences

$$\mathrm{KGL}(\mathrm{S}, 0)[\tfrac{1}{p}] \simeq \mathrm{K}(\mathrm{S})[\tfrac{1}{p}]$$

by [Cis13, Theorem 2.20] and [TT90, Exercise 9.11(h)]. Under these identifications, Corollary 2.1.9 recovers in particular the recent observation of Kelly and Morrow [KM18, Lemma 4.1] that the canonical map

$$\mathrm{K}(\mathrm{S})[\tfrac{1}{p}] \to \mathrm{K}(\mathrm{S}_{\mathrm{perf}})[\tfrac{1}{p}]$$

is an equivalence.

*Remark* 2.1.12. If $p > 1$, then Theorem 2.1.1 is *false* before inverting $p$. Indeed, if $k$ is a non-perfect field, then the functor $\mathrm{F}^* : \mathbf{SH}(k) \to \mathbf{SH}(k)$ is not an equivalence. Supposing the contrary, it would follow that the induced map

$$\mathrm{F}^* : \mathrm{K}(k) \to \mathrm{K}(k)$$

is an equivalence (as in Corollary 2.1.9). However, as the Frobenius map $k \to k$ is by assumption *not* an isomorphism, the map

$$k^\times = \mathrm{K}_1(k) \to k^\times = \mathrm{K}_1(k),$$

which is additively given by multiplication by $p$, is not an isomorphism.

*Remark* 2.1.13. Under the assumptions of Theorem 2.1.1, suppose further that $f$ is of finite type (hence a finite radicial surjection). Then the equivalence $f^* : \mathbf{SH}(\mathrm{S})[\mathcal{P}^{-1}] \to \mathbf{SH}(\mathrm{T})[\mathcal{P}^{-1}]$ is quasi-inverse to $f_* \simeq f_!$. Hence the left adjoint and right adjoints of the latter functor are equivalent. That is, we have an equivalence of functors

$$f^* \simeq f^! : \mathbf{SH}(\mathrm{S})[\mathcal{P}^{-1}] \to \mathbf{SH}(\mathrm{T})[\mathcal{P}^{-1}].$$

2.2. **Proof of Theorem 2.1.1.** In order to prove Theorem 2.1.1, we have to show that the unit and co-unit maps

$$\mathrm{id} \to f_* f^*, \qquad f^* f_* \to \mathrm{id}$$

are both invertible. For the latter, this turns out to be the case before inverting $p$:

**Proposition 2.2.1.** *For any universal homeomorphism $f : \mathrm{T} \to \mathrm{S}$, the co-unit transformation*

$$f^* f_* \to \mathrm{id}_{\mathbf{SH}(\mathrm{T})}$$

*is invertible. In other words, the functor $f_* : \mathbf{SH}(\mathrm{T}) \to \mathbf{SH}(\mathrm{S})$ is fully faithful.*

*Proof.* We argue as in the proof of [CD12, Proposition 2.1.9]. By the proper base change formula, this co-unit is identified with the natural transformation $(\pi_2)_*(\pi_1)^* \to \mathrm{id}$, where $\pi_1$ and $\pi_2$ are the respective projections $\mathrm{T} \times_{\mathrm{S}} \mathrm{T} \to \mathrm{T}$. Since $f$ is a universal homeomorphism, its diagonal $\Delta : \mathrm{T} \to \mathrm{T} \times_{\mathrm{S}} \mathrm{T}$ is a nilpotent closed immersion. Then by the localization theorem (cf. [CD12, Proposition 2.3.6(1)]), $\Delta^*$ is an equivalence. Since $\pi_1$ and $\pi_2$ are retractions of $\Delta$ it follows that we have canonical identifications $\Delta^* \simeq (\pi_\varepsilon)_*$ and $\Delta_* \simeq (\pi_\varepsilon)^*$ for each $\varepsilon \in \{1, 2\}$. In particular, the natural transformation $(\pi_2)_*(\pi_1)^* \to \mathrm{id}$ is identified with the co-unit $\Delta^*\Delta_* \to \mathrm{id}$, which is invertible. □

For the unit map, we will require a more involved argument. We begin by introducing some notation.

*Notation* 2.2.2. Given a unit $a \in \Gamma(\mathrm{S}, \mathcal{O}_{\mathrm{S}})^{\times}$, we write $\langle a \rangle$ for the induced point of $\Omega\mathrm{K}(\mathrm{S})$. For an integer $n \geqslant 0$, we write $n_\epsilon$ for the formal sum

$$n_\epsilon = 1 + \langle -1 \rangle + 1 + \cdots$$

which consists of $n$ terms. We may also regard $n_\epsilon$ as an automorphism of the identity functor $\mathrm{id}_{\mathbf{SH}(\mathrm{S})}$, via the canonical map $\Omega\mathrm{K}(\mathrm{S}) \to \mathrm{Aut}(\mathrm{id}_{\mathbf{SH}(\mathrm{S})})$.

We are grateful to Marc Hoyois for suggesting the following re-interpretation of [EHK+18, Proposition B.1.4].

**Proposition 2.2.3.** *Let* $\mathrm{S}$ *be a scheme,* $\mathrm{P} \in \Gamma(\mathrm{S}, \mathcal{O}_{\mathrm{S}})[x]$ *a monic polynomial of degree* $d$*, and* $\mathrm{T} \subset \mathbf{A}^1_{\mathrm{S}}$ *the closed subscheme cut out by* $\mathrm{P}$*. If* $f : \mathrm{T} \to \mathrm{S}$ *denotes the canonical morphism, then there exists a canonical natural transformation*

$$\mathrm{tr}_f : f_* f^* \to \mathrm{id}_{\mathbf{SH}(\mathrm{S})}$$

*such that the composites*

$$\mathrm{id} \xrightarrow{\mathrm{unit}} f_* f^* \xrightarrow{\mathrm{tr}_f} \mathrm{id}, \qquad f_* f^* \xrightarrow{\mathrm{tr}_f} \mathrm{id} \xrightarrow{\mathrm{unit}} f_* f^*$$

*are homotopic to* $d_\epsilon$ *and* $f_* * d_\epsilon * f^*$*, respectively.*

*Proof.* Note that $f : \mathrm{T} \to \mathrm{S}$ is finite and syntomic. The conormal sheaf $\mathcal{N}_{\mathrm{T}/\mathbf{A}^1_{\mathrm{S}}} \simeq (\mathrm{P})/(\mathrm{P}^2)$ is free of rank 1, and the generator P induces a canonical trivialization $\tau : \mathbf{L}_f \simeq 0$ in $\mathrm{K}(\mathrm{T})$. Therefore, the trace transformation $\mathrm{tr}_f$ of [DJK18] induces a canonical natural transformation

$$f_* f^* \simeq f_* \Sigma^{\mathbf{L}_f} f^* \xrightarrow{\mathrm{tr}_f} \mathrm{id}$$

which we denote again by $\mathrm{tr}_f$. The claim can be reformulated in cohomological terms as the assertion that, for every $\mathrm{E} \in \mathbf{SH}(\mathrm{S})$, the composites

$$\mathrm{E}(\mathrm{S}, 0) \xrightarrow{f^*} \mathrm{E}(\mathrm{T}, 0) \simeq \mathrm{E}(\mathrm{T}, \langle \mathbf{L}_f \rangle) \xrightarrow{f_!} \mathrm{E}(\mathrm{S}, 0), \tag{2.2.4}$$

$$\mathrm{E}(\mathrm{T}, 0) \simeq \mathrm{E}(\mathrm{T}, \langle \mathbf{L}_f \rangle) \xrightarrow{f_!} \mathrm{E}(\mathrm{S}, 0) \xrightarrow{f^*} \mathrm{E}(\mathrm{T}, 0) \tag{2.2.5}$$

are homotopic to multiplication by $d_\epsilon$.

Regarding the assignment $\mathrm{X} \mapsto \mathrm{E}(\mathrm{X}, 0)$ as a presheaf with framed transfers, the first composite is induced by the framed correspondence

$$\begin{array}{ccccc} & & \mathrm{T} & & \\ & {\scriptstyle f,\tau}\swarrow & & \searrow{\scriptstyle f} & \\ \mathrm{S} & & & & \mathrm{S}. \end{array}$$

Therefore the claim follows from [EHK+18, Proposition B.1.4]. The second composite is identified, by the transverse base change property of the trace transformation $\mathrm{tr}_f$ [DJK18, Proposition 2.3.6], with

$$\mathrm{E}(\mathrm{T}, 0) \xrightarrow{\pi_2^*} \mathrm{E}(\mathrm{T} \underset{\mathrm{S}}{\times} \mathrm{T}, 0) \simeq \mathrm{E}(\mathrm{T} \underset{\mathrm{S}}{\times} \mathrm{T}, \langle \mathrm{L}_f \rangle) \xrightarrow{(\pi_1)_!} \mathrm{E}(\mathrm{T}, 0),$$

where $\pi_1$ and $\pi_2$ are the first and second projections of $\mathrm{T} \times_{\mathrm{S}} \mathrm{T}$, respectively. As above, this is induced by the framed correspondence

$$\begin{array}{ccccc} & & \mathrm{T} \times_{\mathrm{S}} \mathrm{T} & & \\ & {\scriptstyle \pi_1, \pi_2^*(\tau)}\swarrow & & \searrow{\scriptstyle \pi_2} & \\ \mathrm{T} & & & & \mathrm{T} \end{array}$$

so the claim follows by another application of [EHK+18, Proposition B.1.4]. □

**Lemma 2.2.6.** *Let* S *be the spectrum of a field* $k$ *of exponential characteristic* $p$*. Then for any power* $q$ *of* $p$*, the canonical map*

$$\mathrm{End}_{\mathbf{SH}(\mathrm{S})}(\mathbf{1}_\mathrm{S}) \to \mathrm{End}_{\mathbf{SH}(\mathrm{S})}(\mathbf{1}_\mathrm{S})[\tfrac{1}{p}] \simeq \mathrm{End}_{\mathbf{SH}(\mathrm{S})[\frac{1}{p}]}(\mathbf{1}_\mathrm{S})$$

*sends* $q_\epsilon \in \mathrm{End}_{\mathbf{SH}(\mathrm{S})}(\mathbf{1}_\mathrm{S})$ *to a unit.*

*Proof.* We only need to consider the case $p > 1$. Using Morel's identification $\mathrm{End}(\mathbf{1}_\mathrm{S}) \simeq \mathrm{GW}(k)$ [Mor04], which has been extended in [BH18, Lemma 10.12], it will suffice to show that the induced element $q_\epsilon \in \mathrm{GW}(k)[\frac{1}{p}]$ is invertible. In view of the cartesian square

$$\begin{array}{ccc} \mathrm{GW}(k) & \xrightarrow{\dim} & \mathbf{Z} \\ \downarrow & & \downarrow \\ \mathrm{W}(k) & \longrightarrow & \mathbf{Z}/2 \end{array}$$

as in [Mor12, (3.1)] (cf. [Bac18, Lemma 17], [KK82, Lemma 1.16]), it will in fact suffice to only check invertibility in $\mathbf{Z}[\frac{1}{p}]$ and in $\mathrm{W}(k)[\frac{1}{p}]$. The former is obvious. For the latter, we first assume that $p$ is odd. In this case, we note that $d_\epsilon = \frac{d-1}{2}h + 1$ and thus $q_\epsilon$ is invertible in $\mathrm{W}(k)$ (without inverting $p$). When $p = 2$, $-1$ is trivially a sum of squares in $k$, so the Witt ring is 2-torsion [MH73, Theorem III.3.6] and the claim follows. □

We are now ready to complete the proof of Theorem 2.1.1:

*Proof of Theorem 2.1.1.* After Proposition 2.2.1 it remains to show that unit map $\mathrm{id}_{\mathbf{SH}(\mathrm{S})} \to f_*f^*$ becomes invertible after inverting the primes in $\mathcal{P}$. By continuity [Hoy14, Proposition C.12(4)] and proper base change, we may use a noetherian approximation argument [TT90, Theorem C.9] to assume that S is noetherian and of finite dimension. Then using [BH18, Proposition A.3] (and proper base change again), we may assume that S is a henselian local scheme (which is still noetherian and finite-dimensional); we denote its closed point by $i : \{s\} \to \mathrm{S}$ and the complement by $j : \mathrm{U} \to \mathrm{S}$. By the localization theorem, the pair of functors $(i^*, j^*)$ is jointly conservative (see [CD12, Section 2.3]). Since U has dimension strictly lower than that of S, we can argue by induction on the dimension of S to reduce to the case where $\mathrm{S} = \{s\}$, i.e., where S is the spectrum of a field $k$. Since $f$ is radicial, it is then induced by a purely inseparable field extension $k \subset \mathrm{K}$. In characteristic zero ($p = 1$), we are already done. Otherwise, by using continuity again, we may assume that the extension $k \subset \mathrm{K}$ is finite, i.e., that $\mathrm{K} = k(\alpha)$ with $\alpha^q \in k$ for $q$ some power of the prime $p$. Now it follows from Proposition 2.2.3 and Lemma 2.2.6 that the unit map $\mathrm{id}_{\mathbf{SH}(\mathrm{S})} \to f_*f^*$ is invertible after inverting $p$. But the assumption implies that $p \in \mathcal{P}$, so the conclusion follows. □

*Remark* 2.2.7. We now explain how the above proof can be generalized to an arbitrary motivic $\infty$-category of coefficients $\mathbf{D}$ as in Remark 2.1.2. First, we recall that the theory of fundamental classes developed in [DJK18] applies in this more general setting. Therefore, for any object $\mathrm{E} \in \mathbf{D}(\mathrm{S})$, the cohomology theory $\mathrm{X} \mapsto \mathrm{E}(\mathrm{X}, 0)$ defines a presheaf with framed transfers on the category of S-schemes. The proof of Proposition 2.2.3 then applies *mutatis mutandis*. To show that Lemma 2.2.6 holds for $\mathbf{D}$, i.e., that $q_\epsilon$ is a unit for every power $q$ of $p$, we argue as follows. Since $\mathbf{D}$ satisfies Nisnevich descent [CD12, Prop. 2.3.8], $\mathbf{A}^1$-invariance, and Thom stability, it follows from the universal property of $\mathbf{SH}$ [Rob15, Corollary 1.2] that there is a canonical monoidal realization functor $\mathbf{SH}(\mathrm{S}) \to \mathbf{D}(\mathrm{S})$. Moreover, the canonical map $\Omega\mathrm{K}(\mathrm{S}) \to \mathrm{Aut}(\mathrm{id}_{\mathbf{D}(\mathrm{S})})$ factors through the induced map $\mathrm{Aut}(\mathrm{id}_{\mathbf{SH}(\mathrm{S})}) \to \mathrm{Aut}(\mathrm{id}_{\mathbf{D}(\mathrm{S})})$, so the claim follows from the universal case of $\mathbf{SH}$. The rest of the proof of Theorem 2.1.1 only relies on the formalism of six operations, which is available for $\mathbf{D}$ [Kha16, Cor. 4.2.3].

## 3. Applications

3.1. **Duality.** Let S be a scheme that is locally of finite type over a field $k$ of exponential characteristic $p$. The structural morphism $\pi : \mathrm{S} \to \mathrm{Spec}(k)$ determines a *duality functor* defined by

$$\mathrm{D_S}(\mathrm{E}) = \underline{\mathrm{Hom}}(\mathrm{E}, \pi^!(\mathbf{1}_k)).$$

To justify this name, we must show that the object $\pi^!(\mathbf{1}_k)$ is *dualizing.* That is:

**Theorem 3.1.1.** *For any compact object* $\mathrm{E} \in \mathbf{SH}(\mathrm{S})$*, the canonical map*

$$\mathrm{E} \to \mathrm{D_S}(\mathrm{D_S}(\mathrm{E}))$$

*is an equivalence in* $\mathbf{SH}(\mathrm{S})[\frac{1}{p}]$.

*Remark* 3.1.2. As remarked in [CD15, Remark 7.4], Theorem 3.1.1 implies the formalism of Grothendieck–Verdier duality for $\mathbf{SH}[\frac{1}{p}]$, for locally of finite type $k$-schemes. In particular, this gives an improvement of [BD15, Theorem 2.4.8].

Theorem 3.1.1 follows from the following statement, analogous to [CD15, Proposition 7.2].

**Proposition 3.1.3.** *The full subcategory of compact objects in* $\mathbf{SH}(\mathrm{S})[\frac{1}{p}]$ *is generated as a thick subcategory by objects of the form* $f_!(\mathbf{1})(n)$*, where* $f : \mathrm{X} \to \mathrm{S}$ *proper,* X *is smooth over a purely inseparable extension of* $k$*, and* $n \in \mathbf{Z}$ *is an integer.*

*Proof.* If $k$ is perfect, the statement is [BD15, Corollary 2.4.7]. In general, the morphism $\varphi : \mathrm{S_{perf}} \simeq \mathrm{S} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k_{\mathrm{perf}}) \to \mathrm{S}$ induces an equivalence $\varphi^* : \mathbf{SH}(\mathrm{S}) \to \mathbf{SH}(\mathrm{S_{perf}})$ by Corollary 2.1.7. If $f : \mathrm{X} \to \mathrm{S_{perf}}$ is a proper morphism with X smooth over $k_{\mathrm{perf}}$, then the composite $\mathrm{X} \to \mathrm{S_{perf}} \to \mathrm{S}$ is as in the statement, so we conclude. □

*Proof of Theorem 3.1.1.* As in the proof of [CD15, Theorem 7.3], this follows immediately from Proposition 3.1.3, and Ayoub's purity theorem for smooth morphisms [Ayo08, Section 1.6]. □

3.2. **Removal of perfectness hypotheses.** Corollary 2.1.7 allows us to immediately drop perfectness hypotheses in many known results, at least after inverting the exponential characteristic. Some examples are listed below.

**Theorem 3.2.1.** *Let* S *be the spectrum of a field* $k$ *of exponential characteristic* $p$*. For any smooth* S*-scheme* X*, the suspension spectrum* $\Sigma^\infty_+(\mathrm{X})$ *is strongly dualizable in* $\mathbf{SH}(\mathrm{S})[\frac{1}{p}]$*. In particular,* $\mathbf{SH}(\mathrm{S})[\frac{1}{p}]$ *is generated under colimits by the strongly dualizable objects.*

Indeed, we can use Corollary 2.1.7 to reduce the case where $k$ is perfect, which is due to Riou, see [LYZ13, Corollary B.2].

*Remark* 3.2.2. Proposition 2.2.3 gives the following refinement of [LYZ13, Lemma B.3]. Suppose $f : \mathrm{Y} \to \mathrm{X}$ is a finite étale morphism of degree $d$ between smooth connected $k$-schemes. Then, up to replacing X by a dense open subset $\mathrm{U} \subseteq \mathrm{X}$ and $f$ by its base change $f_{\mathrm{U}} : \mathrm{Y_U} \to \mathrm{U}$, there are isomorphisms of natural transformations

$$\big(\mathrm{id} \xrightarrow{\text{unit}} f_*f^* \simeq f_\sharp f^* \xrightarrow{\text{counit}} \mathrm{id}\big) \simeq d_\epsilon,$$

$$\big(f_*f^* \simeq f_\sharp f^* \xrightarrow{\text{counit}} \mathrm{id} \xrightarrow{\text{unit}} f_*f^*\mathrm{id}\big) \simeq f_* * d_\epsilon * f^*,$$

where $d_\epsilon$ is as in Notation 2.2.2. To prove this, note there are canonical identifications $\Sigma^{\mathrm{L}_f} \simeq \mathrm{id}$ and $f_* \simeq f_! \simeq f_\sharp$ since $f$ is finite and étale, and the composite

$$f_*f^* \simeq f_\sharp f^* \xrightarrow{\text{counit}} \mathrm{id}$$

is canonically homotopic to the trace transformation $\mathrm{tr}_f : f_* f^* \simeq f_* \Sigma^{\mathrm{L}_f} f^* \to \mathrm{id}$. Replacing X by its generic point, we may assume that $\mathrm{X} = \mathrm{Spec}(k)$. Then $\mathrm{Y} = \mathrm{Spec}(\mathrm{K})$ with $\mathrm{K}/k$ a finite separable field extension, so by the primitive element theorem we are now in the situation of Proposition 2.2.3.

3.2.3. We also have the following variant of Bachmann's conservativity theorem [Bac18].

**Theorem 3.2.4.** *Let $k$ be a field with finite 2-étale cohomological dimension and exponential characteristic $p$. Then the canonical functor*

$$\mathbf{SH}(k)[\tfrac{1}{p}] \to \mathbf{DM}(k; \mathbf{Z}[\tfrac{1}{p}])$$

*is conservative on compact objects.*

*Proof.* Using Corollary 2.1.7 and the analogous result for mixed motives [CD15, Lemma 3.15], we may replace $k$ by its perfection. Then the result is proven in [Bac18, Theorem 16]. □

*Remark* 3.2.5. Using Theorem 3.2.4 we can deduce the Pic-injectivity result of [Bac18, Theorem 18]. This extends Bachmann's results on Po Hu's conjecture on invertibility of the the suspension spectra of affine quadrics to imperfect fields; see *loc. cit* for details.

3.2.6. We can similarly extend the recognition principle for infinite loop spaces [EHK+18, Theorem 3.5.13] to non-perfect fields.

**Theorem 3.2.7.** *Let $k$ be a field of exponential characteristic $p$. Then there are canonical equivalences of symmetric monoidal $\infty$-categories*

$$\mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}}[\tfrac{1}{p}] \simeq \mathbf{SH}^{\mathrm{veff}}(k)[\tfrac{1}{p}],$$
$$\mathbf{SH}^{\mathrm{S}^1,\mathrm{fr}}(k)[\tfrac{1}{p}] \simeq \mathbf{SH}^{\mathrm{eff}}(k)[\tfrac{1}{p}].$$

*Remark* 3.2.8. Theorem 3.2.7 also implies cancellation in the sense of [EHK+18, Theorem 3.5.8] for non-perfect fields, after inverting the exponential characteristic.

**Lemma 3.2.9.** *Let $k$ be a field of exponential characteristic $p$. Then the morphism $f : \mathrm{Spec}(k_{\mathrm{perf}}) \to \mathrm{Spec}(k)$ induces an equivalence*

$$f^* : \mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}}[\tfrac{1}{p}] \to \mathbf{H}^{\mathrm{fr}}(k_{\mathrm{perf}})^{\mathrm{gp}}[\tfrac{1}{p}]$$

*of symmetric monoidal $\infty$-categories.*

*Proof.* We first show fully faithfulness. By continuity, it suffices to show that the Frobenius induces fully faithful functors $\mathrm{F}^* : \mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}}[\frac{1}{p}] \to \mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}}[\frac{1}{p}]$, i.e., that the counit map $\mathrm{id} \to \mathrm{F}_*\mathrm{F}^*$ is an equivalence after inverting $p$. For this it suffices to show that, for every grouplike framed motivic space $\mathcal{F}$ and every smooth $k$-scheme X, the induced map of spaces

$$\mathrm{F}^* : \mathcal{F}(\mathrm{X})[\tfrac{1}{p}] \to \mathcal{F}(\mathrm{F}^{-1}(\mathrm{X}))[\tfrac{1}{p}]$$

is invertible. As in the proof of Proposition 2.2.3, the two composites (2.2.4) and (2.2.5) are now part of the structure of $\mathcal{F}$, so that we can appeal directly to [EHK+18, Proposition B.1.4]. We need to use the analogue of Lemma 2.2.6 for $\mathrm{End}_{\mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}}}(\mathbf{1}_k)$, which holds because there is a canonical equivalence

$$\mathrm{End}_{\mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}}}(\mathbf{1}_k) \to \mathrm{End}_{\mathbf{SH}(k)}(\mathbf{1}_k)$$

by [EHK+18, Theorem 3.5.17].

It remains now to show that $f^*$ is essentially surjective. Since any smooth irreducible $k_{\mathrm{perf}}$-scheme is, up to a universal homeomorphism, the base change of a smooth irreducible $k$-scheme [Sus17, Lemma 1.12], it will suffice to show the following claim: for any universal homeomorphism of smooth schemes over $k_{\mathrm{perf}}$, the induced map in $\mathbf{H}^{\mathrm{fr}}(k_{\mathrm{perf}})^{\mathrm{gp}}[\frac{1}{p}]$ is invertible. By

[EHK$^+$18, Theorem 3.5.13(i)] it suffices to show that the induced map in $\mathbf{SH}(k_{\mathrm{perf}})[\frac{1}{p}]$ is invertible. This follows directly from Theorem 2.1.1. □

*Proof of Theorem 3.2.7.* Note that we need only prove the claim when $p > 1$, and that the second claim follows from the first by stabilization. The equivalence of Corollary 2.1.7 restricts to an equivalence

$$\mathbf{SH}^{\mathrm{veff}}(k)[\tfrac{1}{p}] \to \mathbf{SH}^{\mathrm{veff}}(k_{\mathrm{perf}})[\tfrac{1}{p}]$$

by construction. Combining this with Lemma 3.2.9, we see that we may replace $k$ by its perfection. In that case, the statement is [EHK$^+$18, Theorem 3.5.13(i)]. □

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Københavns Universitet, Institut for Matematiske Fag, Universitetsparken 5 2100 København, Denmark

*E-mail address*: elmanto@math.ku.dk

*URL*: https://www.eldenelmanto.com/

Fakultät für Mathematik, Universität Regensburg, Universitätsstr. 31, 93040 Regensburg, Germany

*E-mail address*: adeel.khan@mathematik.uni-regensburg.de

*URL*: https://www.preschema.com