VOEVODSKY'S SLICE CONJECTURES VIA HILBERT SCHEMES

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ABSTRACT. Using recent development in motivic infinite loop space theory, we offer short and conceptual reproofs of some conjectures of Voevodsky's on the slice filtration using the birational geometry of Hilbert schemes. The original proofs were due to Marc Levine using very different methods, namely, the homotopy coniveau tower.

CONTENTS

1. INTRODUCTION

One major application of motivic homotopy theory is Voevodsky's construction of the Atiyah-Hirzebruch spectral sequence from motivic cohomology (which are Bloch's higher Chow groups [Blo86] of algebraic cycles up to reindexing [Voe02b]) converging to algebraic K-theory. While other constructions of this spectral sequence were proposed prior to motivic homotopy theory, Voevodsky's approach is arguably the cleanest and most definitive — we refer to [Lev18, §2] for a survey. This spectral sequence is obtained via the *slice filtration* constructed in [Voe02a], [Voe02c], which is a functorial filtration that one associates to a motivic spectrum E:

$$\cdots \to f_q E \to f_{q-1} E \to \cdots f_0 E \to \cdots E.$$

The associated graded spectra are denoted

$$s_q E := \operatorname{cofib}(f_{q+1} E \to f_q E),$$

and are called the *q-th slice* of E. Letting $E = \mathrm{KGL}$, the motivic spectrum representing algebraic K-theory, one obtains the desired spectral sequence.

While the construction of this filtration is formal, the identification of the spectral sequence (in other words, the associated graded) hinged on the next two conjectures [Voe02c] which are now theorems of Levine's[1].

Conjecture 1.1. [Lev08, Theorem 10.5.1], [Voe02c, Conjecture 2] *Let k be a perfect field and $\mathbb{1}_k$ denote the motivic sphere spectrum. Then $s_0 \mathbb{1}_k$ canonically identifies with the spectrum representing motivic cohomology.*

Conjecture 1.2. [Lev08, Theorem 9.0.3], [Voe02c, Conjecture 3] *Let k be a perfect field. The functor*

$$\omega^\infty : \mathcal{SH}(k) \to \mathcal{SH}^{S^1}(k),$$

respects the slice filtration.

We will recall the definition of the slice filtrations on $\mathcal{SH}(k)$ and $\mathcal{SH}^{S^1}(k)$ in the main text.

The purpose of this paper is to give an independent, short and conceptual proof of Conjecture 1.2 and a simplification of Levine's proof of Conjecture 1.1, *assuming motivic infinite loop space theory* [EHK+17, EHK+19, BEH+19] as developed by the authors and Hoyois, Khan, Sosnilo and Yakerson. In fact a proof of Conjecture 1.2 along these lines was already envisioned by Voevodksy in [Voe02c], although our proof proceeds via rather different methods.

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[1]Conjecture 1.1 was initially proved, in characteristic zero only, by Voevodsky [Voe04].

The proof of Conjecture 1.2 is independent because we make no reference to Levine's proof. It is short, given the length of this paper. Finally, it is conceptual because we can reformulate both conjectures as relatively elementary statements about the birational geometry of certain Hilbert schemes. Indeed, motivic infinite loop space theory furnishes for us geometric models for the infinite loop space of the motivic sphere spectrum and, in fact, the suspension spectrum of any smooth k-variety as framed Hilbert schemes [EHK$^+$17, §5.1]. That slices have something to do with the birational geometry of varieties is already well-known in the literature [KS16, Pel14]. In lieu of proving Conjecture 1.1 as stated, we identify $s_0(\mathbb{1})$ with a certain explicit framed suspension spectrum. While this characterizes the spectrum uniquely, the relationship with higher Chow groups is not clear from this perspective. On the other hand, this simplifies the proof of Conjecture 1.1 by replacing Levine's use of his "reverse cycle map" with Hilbert schemes argument; see Remark 4.2.

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Notation and conventions. We fix a field k. We make use of the categories and functors depicted in the following commutative diagram.

Here $\mathcal{S}\mathrm{pc}(k)_*$ is the pointed unstable motivic ∞-category (see e.g. [BH17, §2.2]), $\mathcal{SH}^{S^1}(k)$ is the category of S^1-spectra, i.e. the stabilization of $\mathcal{S}\mathrm{pc}(k)_*$, $\mathcal{SH}(k)$ is the category of motivic spectra (see e.g. [BH17, §4.1]), and $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)$ is the category of motivic spaces with framed transfers [EHK$^+$17, §3.2]. All parallel functors in opposite directions are adjoints, the functors $\Sigma^\infty, \Sigma^\infty_{S^1}, \sigma^\infty$ are the evident infinite suspension functors, and U is the evident forgetful functor.

We freely use the language of ∞-categories as set out in [Lur09, Lur16].

2. SOME BIRATIONAL GEOMETRY OF FRAMED HILBERT SCHEMES

For a scheme X and a point $x \in X$, we write
$$\mathrm{cod}_X(x) = \dim(\mathcal{O}_{X,x})$$
for the codimension of x in X (see also [Sta18, Tag 02IZ]). We will use several times the following well-known "codimension formula".

Theorem 2.1. *Let $f : X \to Y$ be a flat morphism of locally noetherian schemes and $x \in X$. Then*
$$\mathrm{cod}_X(x) = \mathrm{cod}_Y(f(x)) + \mathrm{cod}_{X_{f(x)}}(x).$$

Proof. Let $y = f(x)$. Note that $\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} k(y)$. The theorem is now a restatement of [GW10, Corollary 14.95]. □

For the rest of this section, we assume that all schemes are locally noetherian.

Definition 2.2. Let $d \in \mathbb{Z}$. An open immersion $U \hookrightarrow X$ is called *d-birational* if whenever $x \in X$ with $\mathrm{cod}_X(x) \leq d$, then $x \in U$.

Example 2.3. If $d < 0$, the condition is vacuous. If $d = 0$, this coincides with what is usually called birational (U contains all generic points of X).

Remark 2.4. It follows from [Sta18, Tag 02I4] that d-birational open immersions are stable under composition.

The codimension formula tells us that d-birational open immersions is fpqc local (on the base).

Corollary 2.5. *Let $\alpha : U \to S$ be arbitrary and $p : Y \to S$ flat.*
  (1) *If α is a d-birational open immersion, then so is the base change $\alpha_Y : U_Y \hookrightarrow Y$.*
  (2) *If p is surjective and α_U is a d-birational open immersion, then so is α.*

Proof. (1) Since open immersions are stable under base change, it is enough to show that for $y \in Y$ with $\mathrm{cod}_Y(y) \leq d$ we have $y \in U_Y$. By the dimension formula we have $\mathrm{cod}_Y(y) \geq \mathrm{cod}_S(p(y))$, whence $p(y) \in U$ by d-birationality of α. Thus $y \in U_Y$, as needed.

(2) α is an open immersion by faithfully flat descent [Sta18, Tag 02L3]. Let $s \in S$ with $\mathrm{cod}_S(s) \leq d$. Let $y \in Y_s$ be a generic point. Then by the codimension formula we have $\mathrm{cod}_y(Y) = \mathrm{cod}_s(S) + \mathrm{cod}_{Y_s}(y) \leq d$, so that $y \in U_Y$. This implies $s \in U$, as needed. $\qquad\square$

Corollary 2.6. *If $U \hookrightarrow X$ and $V \hookrightarrow Y$ are d-birational open immersions of flat S-schemes, then so is $U \times_S V \hookrightarrow X \times_S Y$.*

Proof. We have the factorization

$$U \times_S V \hookrightarrow U \times_S Y \hookrightarrow X \times_S Y$$

in which both arrows are d-birational open immersions by Corollary 2.5(1), and hence so is the composite by Remark 2.4. $\qquad\square$

The next lemmas furnishes a fiberwise criterion for being d-birational.

Lemma 2.7. *Let $\alpha : U \hookrightarrow X$ be an open immersion of flat S-schemes. Then α is d-birational if and only if for every $s \in S$, the restriction $\alpha_s : U_s \hookrightarrow X_s$ is $(d - \mathrm{cod}_S(s))$-birational.*

Proof. Suppose α is d-birational. Let $s \in S$ and $x \in X_s$ with $\mathrm{cod}_{X_s}(x) \leq d - \mathrm{cod}_S(s)$. By the codimension formula we deduce

$$\mathrm{cod}_X(x) = \mathrm{cod}_S(s) + \mathrm{cod}_{X_s}(x) \leq d,$$

and hence $x \in U \cap X_s = U_s$.

Conversely, suppose the fiberwise condition holds. Let $x \in X$ with $f(x) = s$, and suppose that $\mathrm{cod}_x(X) \leq d$. Then by the codimension formula again we have

$$\mathrm{cod}_{X_s}(x) = \mathrm{cod}_X(x) - \mathrm{cod}_S(s) \leq d - \mathrm{cod}_S(s).$$

It follows that $x \in U_s \subset U$.

This concludes the proof. $\qquad\square$

For a (finite locally free) morphism $p : S' \to S$, we write R_p for the Weil restriction functor [BLR12, Chapter 7].

Proposition 2.8. *Let $p : S' \to S$ be finite locally free and $X \to S'$ smooth and quasi-projective. Let $\alpha : U \hookrightarrow X$ be a d-birational open immersion. Then $R_p(\alpha) : R_p(U) \to R_p(X)$ is a d-birational open immersion.*

Proof. Open immersions and smooth schemes are preserved under Weil restriction [BLR12, Proposition 7.6.2(i), Proposition 7.6.5(h)]. In particular $R_p(X) \to S$ is flat. It is thus enough to show that for $s \in S$, the restriction $R_p(\alpha)_s$ is $(d - \mathrm{cod}_S(s))$-birational. Let $s' \in S_s$. Since $\dim S_s = 0$, the codimension formula implies that $\mathrm{cod}_S(s) = \mathrm{cod}_{S'}(s')$, and hence α_s is $(d - \mathrm{cod}_S(s))$-birational. Since Weil restriction commutes with base change [CGP15, Proposition A.5.2(1)], we may thus assume that $S = Spec(k)$ is the spectrum of a field. Applying Corollary 2.5(2), we may assume that k is algebraically closed.

Writing S' as a finite disjoint union of its connected components and using Corollary 2.6, we may assume that S' is a finite local k-scheme, and so in particular $S' \to S$ is a universal homeomorphism ([Sta18, Tag 00J8] implies that $S'_{\mathrm{red}} \to S$ is an isomorphism, and hence the claim follows from [Sta18, 04DF] and 2-out-of-3). We claim that the canonical square

$$
\begin{array}{ccc}
p^* R_p U & \xrightarrow{\ p^* R_p(\alpha)\ } & p^* R_p X \\
\downarrow & & \downarrow \\
U & \xrightarrow{\quad \alpha \quad} & X
\end{array}
$$

is cartesian. Indeed for a scheme T over S', maps into $p^* R_p X$ (over S') are the same as maps $T \times_k S' \to X$. We thus need to show that a map $T \times_k S \to X$ factors through U if and only if the composite $T \to T \times_k S' \to X$ does, which follows from the fact that the first map is a homeomorphism.

By [CGP15, Lemma A.5.11], the map $p^* R_p X \to X$ is faithfully flat, and hence $p^* R_p(\alpha)$ is d-birational by Corollary 2.5(1). Since $X \to S$ is faithfully flat so is $p^* R_p X \to S$, and hence $R_p(\alpha)$ is d-birational by Corollary 2.5(2). $\qquad\square$

With this preparation out of the way, we come to our main topic, Hilbert schemes. For $X \in \mathrm{Sm}_k$ (quasi-projective, say) we write $\mathrm{Hilb}^{\mathrm{fr},n}(X)$ for the (ind-smooth) ind-scheme representing the functor $h^{\mathrm{nfr},n}(X)$ of [EHK$^+$17, §5.1.4]. We have

$$\mathrm{Hilb}^{\mathrm{fr},n}(X) = \coprod_{d \geq 0} \mathrm{Hilb}^{\mathrm{fr},n}_d(X),$$

the decomposition by degree; each $\mathrm{Hilb}^{\mathrm{fr},n}_d(X)$ is smooth. Finally

$$\mathrm{Hilb}^{\mathrm{fr}}(X) = \operatorname*{colim}_n \mathrm{Hilb}^{\mathrm{fr},n}(X).$$

Lemma 2.9. *If $\alpha : U \to X \in \mathrm{Sm}_k$ is a d-birational open immersion of smooth quasi-projective k-schemes, then so is $\mathrm{Hilb}^{\mathrm{fr},n}_d(U) \to \mathrm{Hilb}^{\mathrm{fr},n}_d(X)$.*

Proof. There are maps $I \xrightarrow{q} Z \xrightarrow{p} \mathrm{Hilb}^{\mathrm{flci}}_d(\mathbb{A}^n)$ with q smooth, p finite locally free, such that $\mathrm{Hilb}^{\mathrm{fr},n}_d(X) \simeq R_p(I \times X)$ [EHK$^+$17, §5.1.4]. The result thus follows from Corollary 2.6 and Proposition 2.8. $\square$

Recall that a k-scheme X is rational if there exists a span of 0-birational open immersions $X \leftarrow U \to \mathbb{A}^n_k$ for some n.

Lemma 2.10. $\mathrm{Hilb}^{\mathrm{fr},n}_d(*)$ *is rational.*

Proof. We use the notation from the proof of Lemma 2.9. By construction, I is a GL_n-torsor over Z (for the Zariski topology), and hence birational to $\mathrm{GL}_n \times Z$, whence birational to $\mathbb{A}^{n^2} \times Z$. By construction of the Weil restriction [BLR12, Theorem 7.6.4], $R_p(\mathbb{A}^{n^2} \times Z)$ is given by a vector bundle on $\mathrm{Hilb}^{\mathrm{flci}}(\mathbb{A}^n)$. Using Proposition 2.8, it is thus enough to show that $\mathrm{Hilb}^{\mathrm{flci}}_d(\mathbb{A}^n)$ is rational. It is well-known to be birational to $\mathrm{Sym}^d(\mathbb{A}^n)$ (see e.g. [Jel17, Lemma 4.28 and Theorem 4.36]), which is rational since $\mathbb{A}^n$ is [Mat68].

This concludes the proof. $\square$

3. The birational localizations

Denote by $L^d_{\mathrm{bir}}\mathcal{S}\mathrm{pc}(k)$ the (Bousfield) localization obtained by inverting d-birational open immersions of smooth k-schemes. See [Lur09, §5.4.4] for one account on localization of presentable ∞-categories. Variants of these localizations have been considered previously, for example by Kahn-Sujata [KS16] and Pelaez [Pel14].

Since $f \times \mathrm{id}_X$ is a d-birational open immersion whenever f is (see e.g. Corollary 2.6), this is a symmetric monoidal localization (e.g. apply [BH17, Proposition 6.16] to S the spectrum of an algebraically closed field and $\mathcal{C} = \mathrm{FEt}_S \simeq \mathrm{Fin}$). By Zariski descent, the same localization is obtained by inverting d-birational open immersions between smooth quasi-projective (or affine) k-schemes.

Write $L^d_{\mathrm{bir}}\mathcal{SH}^{S^1}(k)$ for the localization obtained by inverting maps of the form $\Sigma^{\infty+n}_{S^1} f_+$, with $n \in \mathbb{Z}$ and f a d-birational open immersion. Similarly write $L^d_{\mathrm{bir}}\mathcal{S}\mathrm{pc}(k)_*$ for the localization at maps of the form f_+, and $L^d_{\mathrm{bir}}\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)$ for the localization at maps of the form Ff. These are also symmetric monoidal localizations.

Recall that $\mathcal{SH}^{S^1}(k)(d)$ is defined as the localizing subcategory generated by $\mathcal{SH}^{S^1}(k) \wedge \mathbb{G}^{\wedge d}_m$. The reflection into its right orthogonal is denoted by $s_{[0,d-1]}$.

Lemma 3.1. *Let k be any field.*

(1) The functors

$$\mathcal{S}\mathrm{pc}(k) \to \mathcal{S}\mathrm{pc}(k)_* \to \mathcal{SH}^{S^1}(k)$$

and $F : \mathcal{S}\mathrm{pc}(k)_* \to \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)$ preserve L^d_{bir}-equivalences.
(2) The forgetful functor $\mathcal{S}\mathrm{pc}(k)_* \to \mathcal{S}\mathrm{pc}(k)$ commutes with L^d_{bir}.
(3) L^d_{bir}-equivalences in $\mathcal{S}\mathrm{pc}(k)_*$ are stable under finite products.
(4) The forgetful functor $U : \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k) \to \mathcal{S}\mathrm{pc}(k)_*$ commutes with L^d_{bir}.

If k is perfect, then also the following hold.

(5) A morphism $\alpha : E \to F \in \mathcal{SH}^{S^1}(k)$ is an L^d_{bir}-equivalence if and only if $\mathrm{cof}(\alpha) \in \mathcal{SH}^{S^1}(k)(d+1)$.
In fact the localizing subcategory generated by objects of the form $\mathrm{cof}(\alpha)$ is $\mathcal{SH}^{S^1}(k)(d+1)$.
(6) For $E \in \mathcal{SH}^{S^1}(k)$ we have $L^d_{\mathrm{bir}} E \simeq s_{[0,d]} E$.

Proof. (1) is clear by construction, (3) is immediate from (2), and (6) from (5).

(2) By construction the functor detects L_{bir}^d-local objects. This implies that it is enough to show that it preserves the strongly saturated class of morphisms (in $\mathcal{S}\mathrm{pc}(k)_*$) generated by d-birational open immersions of smooth schemes. By [BH17, Lemma 2.10], for this it is enough to show that if f is such a map and $X \in \mathrm{Sm}_k$, then $f \coprod \mathrm{id}_X \coprod \mathrm{id}_*$ is also a d-birational open immersion. This is clear.

(4) Using (2), it suffices to show that the functor $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k) \to \mathcal{S}\mathrm{pc}(k)$ commutes with L_{bir}^d. Arguing as in (2), we apply Lemma [BH17, Lemma 2.10] to $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)$. Using that $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)$ is semiadditive [EHK+17, Proposition 3.2.10(iii)] and the localization L_{bir}^d on $\mathcal{S}\mathrm{pc}(k)$ is symmetric monoidal, it is enough to prove that if $f : X \to Y$ is a d-birational open immersion of smooth quasi-projective k-schemes, then $h^{\mathrm{fr}}(f) : h^{\mathrm{fr}}(X) \to h^{\mathrm{fr}}(Y)$ becomes an equivalence in $L_{\mathrm{bir}}^d \mathcal{S}\mathrm{pc}(k)$. By [EHK+17, Corollary 2.3.25 and Theorem 5.1.8], it is enough to show that $\mathrm{Hilb}^{\mathrm{fr}}(X) \to \mathrm{Hilb}^{\mathrm{fr}}(Y)$ is an L_{bir}-equivalence. This follows from Lemma 2.9.

(5) Using Lemma 3.2 below, it suffices to show the "in fact" part. Since $\alpha : X \times (\mathbb{A}^n \setminus 0) \to X \times \mathbb{A}^n$ is birational and $\mathrm{cof}(\alpha) \simeq X_+ \wedge T^n$, one inclusion is clear; for the other one it is enough to show that if $U \hookrightarrow X$ is a d-birational open immersion, then $X/U \in \mathcal{SH}^{S^1}(k)(d+1)$. This is well-known, we include the proof for the convenience of the reader. Let $Z = X \setminus U$; we shall prove the claim by induction on $\dim Z$. By generic smoothness [Sta18, Tag 0B8X], there exists a smooth dense open $U' \subset Z$; let $Z' = Z \setminus U'$. The cofiber sequence

$$X \setminus Z'/U \setminus Z' \to X/U \to X/X \setminus Z'$$

implies that it is enough to show that $X \setminus Z'/U \setminus Z', X/X \setminus Z' \in \mathcal{SH}^{S^1}(k)(d+1)$. For the former this follows from homotopy purity [MV99, Theorem 3.2.23], for the latter it holds by induction.

This concludes the proof. $\square$

We made use of the following technical result, which is surely well-known.

Lemma 3.2. *Let $\mathcal{C}$ be a stable presentable ∞-category, and S a small set of morphisms in $\mathcal{C}$ closed under desuspension. Write S^0 for the localizing subcategory of $\mathcal{C}$ generated by cofibers of morphisms in S. Then the strong saturation of S consists of those maps α with $\mathrm{cof}(\alpha) \in S^0$.*

Proof. Let A be a strongly saturated class of morphisms stable under desuspensions, and write A^1 for the set of objects in $X \in \mathcal{C}$ such that $0 \to X \in A$. Then A^1 is closed under colimits and desuspensions, whence a localizing subcategory. Moreover by stability, $\alpha : X \to Y$ is an A-equivalence (i.e. in A) if and only if $0 \to \mathrm{cof}(\alpha)$ is an A-equivalence. It follows that $A^0 = A^1$, and also that $A \leftrightarrow A^0$ is an inclusion-preserving bijection between strongly saturated classes stable under desuspension and localizing subcategories. The result now follows from the observation that if S is closed under desuspension, then so is its strong saturation. $\square$

As usual we denote by $\mathcal{SH}(k)^{\mathrm{eff}} \subset \mathcal{SH}(k)$ the localizing subcategory generated by $\sigma^\infty \mathcal{SH}^{S^1}(k)$ and put $\mathcal{SH}(k)^{\mathrm{eff}}(d) = \mathbb{G}_m^{\wedge d} \wedge \mathcal{SH}(k)^{\mathrm{eff}}$; this is equivalently the localizing subcategory generated by $\sigma^\infty \mathcal{SH}^{S^1}(k)(d)$. We put $L_{\mathrm{bir}} = L_{\mathrm{bir}}^0$.

4. Proof of Conjecture 1.1

Recall from [Hoy18, §4] the framed presheaf $\mathbb{Z} \in \mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(k))$, and from [EHK+17, EHK+19] the presheaf $\mathcal{FS}\mathrm{yn}^{\mathrm{fr}} \simeq h^{\mathrm{fr}}(*)$. There is an evident "degree" map of framed presheaves $\mathcal{FS}\mathrm{yn}^{\mathrm{fr}} \to \mathbb{Z}$, factoring in fact through an evident sub-presheaf $\mathbb{N}$.

Theorem 4.1. *Let k be any field.*

(1) The map $\mathcal{FS}\mathrm{yn}^{\mathrm{fr}} \to \mathbb{N}$ is an L_{bir}-equivalence in $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)$.

(2) The map

$$\Sigma_{\mathrm{fr}}^\infty \mathcal{FS}\mathrm{yn}^{\mathrm{fr}} \to \Sigma_{\mathrm{fr}}^\infty \mathbb{Z}$$

identifies with the canonical map

$$\mathbb{1} \to s_0(\mathbb{1}) \in \mathcal{SH}(k).$$

Proof. (1) By Lemma 3.1(2,4), it suffices to show that the underlying map of unpointed motivic spaces is an L_{bir}-equivalence. Since it is the coproduct of the maps $\mathcal{FS}\mathrm{yn}_d^{\mathrm{fr}} \to *$, and $\mathcal{FS}\mathrm{yn}_d^{\mathrm{fr}} \overset{L_{\mathrm{mot}}}{\simeq} \mathrm{Hilb}_d^{\mathrm{fr},\infty}(*)$ [EHK+17, Corollary 2.3.25], it suffices to show that each $\mathrm{Hilb}_d^{\mathrm{fr},n}(*)$ is rational. This is Lemma 2.10.

(2) By [BH17, Lemma B.1] and [Hoy18, Lemma 20], all terms are stable under essentially smooth base change, so we may assume that k is perfect. It follows from (1) and Lemma 3.1(6) that $\mathbb{1} \simeq \Sigma_{\mathrm{fr}}^\infty \mathcal{FS}\mathrm{yn}^{\mathrm{fr}} \to$

$\Sigma_{\mathrm{fr}}^{\infty}\mathbb{N}$ induces an equivalence on s_0. Moreover $\Sigma_{\mathrm{fr}}^{\infty}\mathbb{N} \simeq \Sigma_{\mathrm{fr}}^{\infty}\mathbb{Z}$, since $\mathbb{Z}$ is the group completion of $\mathbb{N}$. It thus remains to show that $\Sigma_{\mathrm{fr}}^{\infty}\mathbb{Z}$ is right orthogonal to $\mathcal{SH}(k)^{\mathrm{eff}}(1)$. But now, $\underline{\pi}_*(\Sigma_{\mathrm{fr}}^{\infty}\mathbb{Z})_0 = 0$ if $* \neq 0$ and $= \mathbb{Z}$ else, so $\underline{\pi}_*(\Sigma_{\mathrm{fr}}^{\infty}\mathbb{Z})_{-1} = 0$, which implies what we want since homotopy sheaves are unramified.

This concludes the proof. □

Proof of Conjecture 1.1. The unit map $u : \mathbb{1} \to \mathrm{KGL}$ induces a map $s_0(u) : s_0(\mathbb{1}) \to s_0(\mathrm{KGL})$. Theorem 4.1(2), Conjecture 1.2 (to be proved in the next section) and Voevodsky's arguments from [Voe02c] (see also Example 5.6) imply that $s_0(u)$ is an equivalence. The zero slice of KGL is identified with higher Chow groups in [Lev08, Theorem 6.4.1]; whence the result. □

Remark 4.2. Theorem 4.1 characterizes the spectrum $s_0(\mathbb{1})$, over a field, uniquely as an effective spectrum with

$$\underline{\pi}_j(s_0(\mathbb{1}))_0 = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise} \end{cases}$$

and a map $\mathbb{1} \to s_0(\mathbb{1})$ inducing the canonical map $\underline{GW} \to \mathbb{Z}$ on $\underline{\pi}_0(-)_0$. However, in order to identify $s_0(\mathbb{1})$ with higher Chow groups, at some point it needs to be proved that the spectrum E representing higher Chow groups (even granting its existence) is effective. Voevodsky proves this (in characteristic zero) by studying the birational geometry of motivic Eilenberg MacLane spaces, and Levine deduces this from his homotopy coniveau tower theory. We have no new arguments for this. In effect, our argument replaces the deduction of $s_0(\mathbb{1})$ from $s_0(\mathrm{KGL})$ in [Lev08, §10] involving the "reverse cycle map."

5. PROOF OF CONJECTURE 1.2

For $d < 0$, we define $\mathcal{SH}^{S^1}(k)(d) = \mathcal{SH}^{S^1}(k)$, and $f_d = \mathrm{id} : \mathcal{SH}^{S^1}(k) \to \mathcal{SH}^{S^1}(k)(d)$. The next result is Conjecture 1.2. In Levine's approach [Lev08], he directly proves Corollary 5.3 below, which immediately implies the theorem.

Theorem 5.1. *Let k be a perfect field. Then*

$$\omega^{\infty}(\mathcal{SH}(k)^{\mathrm{eff}}(d+1)) \subset \mathcal{SH}^{S^1}(k)(d+1).$$

Proof. If $d < 0$ there is nothing to show. Otherwise by Lemma 3.1(5), it suffices to show that if $f : X \to Y$ is a d-birational open immersion of smooth, quasi-projective k-schemes, then $\omega^{\infty}\Sigma_+^{\infty} f \in \mathcal{SH}^{S^1}(k)$ is an L_{bir}^d-equivalence. Using Lemma 5.2 below, this follows from Lemma 3.1(1,3,4). □

The following technical result is a variant of [Voe02c, Proposition 4.4].

Lemma 5.2. *Let $\mathcal{X} \in \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)$ (where k is a perfect field). Then $\omega^{\infty}\Sigma_{\mathrm{fr}}^{\infty}\mathcal{X} \in \mathcal{SH}^{S^1}(k)$ can be obtained as a colimit of S^1-desuspensions of suspension spectra of the form $\Sigma_{S^1}^{\infty} U\mathcal{X}^{\times n}$. Moreover this expression is natural in $\mathcal{X}$.*

Proof. We have

$$\omega^{\infty}\Sigma_{\mathrm{fr}}^{\infty}\mathcal{X} \simeq \operatorname*{colim}_n \Sigma^{-n}\Sigma_{S^1}^{\infty}\Omega^{\infty}\Sigma^n\Sigma_{\mathrm{fr}}^{\infty}\mathcal{X}.$$

It is thus enough to prove that the spaces

$$\Omega^{\infty}\Sigma_{\mathrm{fr}}^{\infty}\Sigma^n\mathcal{X} \in \mathcal{S}\mathrm{pc}(k)_*$$

are of the desired form, say for $n \geq 1$. We view $\mathcal{X} \in \mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(k))$. It follows from the motivic recognition principle [EHK+17, Theorem 3.5.14] that

$$\Omega^{\infty}\Sigma_{\mathrm{fr}}^{\infty}\Sigma^n\mathcal{X} \simeq L_{\mathrm{mot}}^{\mathrm{gp}}\Sigma^n\mathcal{X}.$$

Writing $\Sigma^n\mathcal{X}$ as an iterated sifted colimit and using semiadditivity, we find that $\Sigma^n\mathcal{X} \simeq B^n\mathcal{X}$, where the bar construction B^n is just applied sectionwise. In particular, $\Sigma^n\mathcal{X} \simeq B^n\mathcal{X}$ is of the desired form. Since the forgetful functor $\mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(k)) \to \mathcal{P}_\Sigma(\mathrm{Sm}_k)$ preserves motivic equivalences [EHK+17, Proposition 3.2.14], it suffices to show that $B^n\mathcal{X}$ is motivically group complete (i.e. $L_{\mathrm{mot}}B^n\mathcal{X}$ is group complete). Since $n \geq 1$, $B^n\mathcal{X}$ is sectionwise connected (see e.g. [Seg74, Proposition 1.5]), and hence $L_{\mathrm{mot}}B^n\mathcal{X}$ is connected by [MV99, Corollary 3.3.22]. This concludes the proof. □

We deduce the following structural result, also originally due to Levine [Lev08, Theorem 9.0.3]

Corollary 5.3. (1) *The functor $\omega^{\infty} : \mathcal{SH}(k) \to \mathcal{SH}^{S^1}(k)$ commutes with the functors f_d and s_d, for all $d \in \mathbb{Z}$.*
  (2) *If $E \in \mathcal{SH}(k)$ is represented by the $\mathbb{G}_m$-Ω-spectrum $(E_0, E_1, \ldots)$, then $f_d E$ is represented by the $\mathbb{G}_m$-Ω-spectrum $(f_d E_0, f_{d+1} E_1, \ldots)$. In particular E is effective if and only if $E_i \in \mathcal{SH}^{S^1}(k)(i)$.*

Proof. (1) The categories $\mathcal{SH}(k)^{\text{eff}}(d)$ and $\mathcal{SH}^{S^1}(k)(d)$ define non-negative parts of t-structures by [Lur16, Proposition 1.4.4.11]. By construction σ^∞ is right-t-exact, and hence ω^∞ is left-t-exact. Since it is also right-t-exact by Theorem 5.1, we deduce that ω^∞ is t-exact, i.e. commutes with f_d. The claim about s_d follows immediately.

(2) We have $E_i \simeq \omega^\infty(\mathbb{G}_m^{\wedge i} \wedge E)$. Hence we compute

$$f_d(E)_i \simeq \omega^\infty(\mathbb{G}_m^{\wedge i} \wedge f_d E) \simeq \omega^\infty(f_{d+i}(\mathbb{G}_m^{\wedge i} \wedge E)) \simeq f_{d+i}(\omega^\infty(\mathbb{G}_m^{\wedge i} \wedge E)) \simeq f_{d+i}(E_i).$$

Here we have used part (1) for the third equivalence (which is the only non-trivial one in the string above). This concludes the proof. $\square$

We also deduce the following principle, slightly generalizing an argument of Voevodsky. The analog in topology is the following fact: if E is a connective spectrum, then it is d-connective as soon as its infinite loop space is d-connective.

Proposition 5.4. *Let k be a perfect field and $E \in \mathcal{SH}(k)^{veff}$. Then $E \in \mathcal{SH}(k)^{eff}(d)$ as soon as $\Sigma^\infty \Omega^\infty E \in \mathcal{SH}(k)^{eff}(d)$.*

Proof. Using the recognition principle and Lemma 5.2, or alternatively [Voe02c, Proposition 4.4], we find that $\omega^\infty E$ is in the localizing subcategory generated by suspension spectra of products of $\Omega^\infty E$. Since [Mor03, Lemma 6.2.2 and Footnote 45]

$$\Sigma^\infty_{S^1}(X \times Y) \simeq \Sigma^\infty_{S^1} X \vee \Sigma^\infty_{S^1} Y \vee \Sigma^\infty_{S^1} X \wedge Y,$$

this is equivalently the localizing subcategory generated by smash powers of $\Sigma^\infty_{S^1}\Omega^\infty E$. It follows that $\sigma^\infty \omega^\infty E$ is in the localizing subcategory generated by smash powers of $\Sigma^\infty \Omega^\infty E$, and hence $\sigma^\infty \omega^\infty E \in \mathcal{SH}(k)^{\text{eff}}(d)$.

The fibration sequence

$$\sigma^\infty \omega^\infty f_d E \to \sigma^\infty \omega^\infty E \to \sigma^\infty \omega^\infty s_{[0,d-1]}E$$

together with Theorem 5.1 now implies that $\sigma^\infty \omega^\infty s_{[0,d-1]}E \in \mathcal{SH}(k)^{\text{eff}}(d)$, and consequently the counit map

$$c : \sigma^\infty \omega^\infty s_{[0,d-1]}E \to s_{[0,d-1]}E$$

is null. On the other hand c induces a surjection on $[\Sigma^i \Sigma^\infty_+ X, -]$ for all $i \in \mathbb{Z}$ and $X \in \text{Sm}_k$, whence $s_{[0,d-1]}E \simeq 0$, being in the localizing subcategory generated by the $\Sigma^\infty_+ X$.

This concludes the proof. $\square$

Example 5.5. We can use this to give a slightly different proof of Theorem 4.1 over perfect fields. Namely let F denote the fiber of the degree map $\mathcal{FS}\text{yn}^{\text{fr,gp}} \to \mathbb{Z}$. It is enough to show that $\Sigma^\infty F \in \mathcal{SH}(k)^{\text{eff}}(1)$. But by [BEH$^+$19] we have

$$F \overset{L_{\text{mot}}}{\simeq} \mathcal{FS}\text{yn}^{\text{fr}+}_\infty \overset{L_{\text{mot}}}{\simeq} \text{Hilb}^{\text{fr},\infty+}_\infty(*).$$

Since Σ^∞ inverts acyclic maps, the result thus follows again from Lemma 2.10.

Example 5.6. The above argument is modeled on Voevodsky's determination of $s_0(\text{KGL})$, which we can restate in our language as follows: Using that $\Omega^\infty \text{KGL} = \mathbb{Z} \times \text{Gr}$, where Gr is the infinite Grassmannian variety, arguing as above one is reduced to showing that Gr is rational. This is well-known.

Example 5.7. The converse of Proposition 5.4 is false. Let $E = H\mathbb{Z}/2 \wedge \mathbb{G}_m$. We claim that $\Sigma^\infty \Omega^\infty E \notin \mathcal{SH}(k)^{\text{eff}}(1)$. For this it suffices to construct a non-zero map $\Sigma^\infty \Omega^\infty E \to \Sigma H\mathbb{Z}/2$, or equivalently a non-zero map $\Omega^\infty H\mathbb{Z}/2 \wedge \mathbb{G}_m \to \Omega^\infty \Sigma H\mathbb{Z}/2$. Any grouplike monoid M is equivalent as a pointed space to $\pi_0(M) \times M_{\geq 1}$; applying this construction sectionwise and projecting to the $M_{\geq 1}$ part, we obtain the desired non-zero map

$$\Omega^\infty H\mathbb{Z}/2 \wedge \mathbb{G}_m \to \Omega^\infty(H\mathbb{Z}/2 \wedge \mathbb{G}_m)_{\geq 1} \simeq K(\mathbb{Z}/2, 1) \simeq \Omega^\infty \Sigma H\mathbb{Z}/2.$$

REFERENCES

[BEH$^+$19] Tom Bachmann, Elden Elmanto, Marc Hoyois, Adeel A. Khan, Vladimir Sosnilo, and Maria Yakerson. On the infinite loop spaces of algebraic cobordism and the motivic sphere. arXiv:1911.02262, 2019.

[BH17] Tom Bachmann and Marc Hoyois. Norms in motivic homotopy theory. 2017. arXiv:1711.03061.

[Blo86] Spencer Bloch. Algebraic cycles and higher K-theory. *Adv. in Math.*, 61(3):267–304, 1986.

[BLR12] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21. Springer Science & Business Media, 2012.

[CGP15] Brian Conrad, Ofer Gabber, and Gopal Prasad. *Pseudo-reductive groups*, volume 26. Cambridge University Press, 2015.

[EHK$^+$17] Elden Elmanto, Marc Hoyois, Adeel A. Khan, Vladimir Sosnilo, and Maria Yakerson. Motivic infinite loop spaces. 2017.

[EHK⁺19] Elden Elmanto, Marc Hoyois, Adeel A. Khan, Vladimir Sosnilo, and Maria Yakerson. Modules over algebraic cobordism. 2019.

[GW10]   Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden, 2010. Schemes with examples and exercises.

[Hoy18]   Marc Hoyois. The localization theorem for framed motivic spaces. *arXiv preprint arXiv:1807.04253*, 2018.

[Jel17]   Joachim Jelisiejew. *Hilbert schemes of points and their applications*. PhD thesis, Repozytorium Uniwersytetu Warszawskiego, 2017.

[KS16]   Bruno Kahn and Ramdorai Sujatha. Birational motives, ii: triangulated birational motives. *International Mathematics Research Notices*, 2017(22):6778–6831, 2016.

[Lev08]   Marc Levine. The homotopy coniveau tower. *Journal of Topology*, 1(1):217–267, 2008.

[Lev18]   Marc Levine. Vladimir Voevodsky—an appreciation. *Bull. Amer. Math. Soc. (N.S.)*, 55(4):405–425, 2018.

[Lur09]   Jacob Lurie. *Higher topos theory*. Number 170. Princeton University Press, 2009.

[Lur16]   Jacob Lurie. Higher algebra, May 2016.

[Mat68]   Arthur Mattuck. The field of multisymmetric functions. *Proceedings of the American Mathematical Society*, 19(3):764–765, 1968.

[Mor03]   Fabien Morel. An introduction to $\mathbb{A}^1$-homotopy theory. *ICTP Trieste Lecture Note Ser. 15*, pages 357–441, 2003.

[MV99]   Fabien Morel and Vladimir Voevodsky. $\mathbb{A}^1$-homotopy theory of schemes. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 90(1):45–143, 1999.

[Pel14]   Pablo Pelaez. The unstable slice filtration. *Transactions of the American Mathematical Society*, 366(11):5991–6025, 2014.

[Seg74]   Graeme Segal. Categories and cohomology theories. *Topology*, 13(3):293–312, 1974.

[Sta18]   The Stacks Project Authors. *Stacks Project*. `http://stacks.math.columbia.edu`, 2018.

[Voe02a] V. Voevodsky. Open problems in the motivic stable homotopy theory , i. In *International Press Conference on Motives, Polylogarithms and Hodge Theory*. International Press, 2002.

[Voe02b] Vladimir Voevodsky. Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic. *Int. Math. Res. Not.*, (7):351–355, 2002.

[Voe02c] Vladimir Voevodsky. A possible new approach to the motivic spectral sequence for algebraic k-theory. 293:371–379, 2002.

[Voe04]   Vladimir Voevodsky. On the zero slice of the sphere spectrum. *Proc. Steklov Inst. Math.*, 246:93–102, 2004.

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