MILNOR EXCISION FOR MOTIVIC SPECTRA

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Abstract. We prove that the ∞-category of motivic spectra satisfies Milnor excision: if $A \to B$ is a morphism of commutative rings sending an ideal $I \subset A$ isomorphically onto an ideal of B, then a motivic spectrum over A is equivalent to a pair of motivic spectra over B and A/I that are identified over B/IB. Consequently, any cohomology theory represented by a motivic spectrum satisfies Milnor excision. We also prove Milnor excision for Ayoub's étale motives over schemes of finite virtual cohomological dimension.

For S a scheme, let $\mathbf{SH}(S)$ be the ∞-category of motivic spectra over S.

Theorem 1. *The presheaf of ∞-categories $\mathbf{SH}(-)\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ satisfies Milnor excision.*

This means the following [EHIK20, Definition 3.2.3]: given a cartesian square of schemes

$$\begin{array}{ccc} W & \xrightarrow{k} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array}$$

where f is affine, i is a closed immersion, and the induced map $Y \sqcup_W Z \to X$ is an isomorphism, the square of ∞-categories

$$\begin{array}{ccc} \mathbf{SH}(X) & \xrightarrow{i^*} & \mathbf{SH}(Z) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ \mathbf{SH}(Y) & \xrightarrow{k^*} & \mathbf{SH}(W) \end{array}$$

is cartesian.

If $E \in \mathbf{SH}(S)$ is a motivic spectrum and X is an S-scheme, we denote by $E(X) \in \mathrm{Spt}$ the mapping spectrum from $\mathbf{1}_X$ to E_X in $\mathbf{SH}(X)$. An immediate consequence of Theorem 1 is the following:

Corollary 2. *Let* S *be a scheme and* $E \in \mathbf{SH}(S)$*. Then the presheaf of spectra* $E(-)\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Spt}$ *satisfies Milnor excision.*

This corollary vastly generalizes [EHIK20, Theorem D]. For $S = \operatorname{Spec} \mathbf{Z}$ and $E = \mathrm{KGL}$, it recovers Weibel's excision theorem for the homotopy K-theory of commutative rings [Wei89, Theorem 2.1]. If $S = \operatorname{Spec} R$, an equivalent formulation of Corollary 2 is that the canonical extension of $E(-)$ to nonunital commutative R-algebras sends short exact sequences to fiber sequences (cf. [EHIK20, Remark 3.2.6]). Combining Corollary 2 with [KM18, Lemma 3.5(ii)], we obtain:

Corollary 3. *Let k be a perfect field and $E \in \mathbf{SH}(k)$. For every valuation ring* V *over k, henselian along an ideal $I \subset V$, the map $\pi_* E(V) \to \pi_* E(V/I)$ is surjective.*

This corollary verifies the property (G2) from [Kel19, Theorem 1] for the homotopy presheaves of any motivic spectrum over a perfect field.

Date: April 22, 2020.

Here is a brief outline of the proof of Theorem 1. Using the main result of [EHIK20], we first reduce it to the statement that $\mathbf{SH}(-)$ satisfies v-excision, which is special case of Milnor excision involving valuation rings (Theorem 6). In turn, this is equivalent to a certain unexpected functorial property of $\mathbf{SH}(-)$ with respect to localizations of valuation rings (Equation (9)). To prove the latter, the main idea is to pass to the larger ∞-category $\mathbf{SH}_{\mathrm{cdh}}(-)$ built from the cdh site instead of the smooth Nisnevich site; the cdh descent property of motivic spectra proved by Cisinski [Cis13] implies that $\mathbf{SH}_{\mathrm{cdh}}(-)$ contains $\mathbf{SH}(-)$ as a full subcategory. This allows us to take advantage of the fact that pushforward along an open immersion preserves cdh-local equivalences (Lemma 12). This fact further reduces the question to the level of presheaves (Lemma 11), where it boils down to a simple geometric property of valuation rings (Lemma 10).

Since $\mathbf{SH}(-)$ is a Zariski sheaf, we need only prove Theorem 1 for qcqs schemes. We shall do this by applying [EHIK20, Theorem 3.3.4] to the presheaf

$$\mathbf{SH}(-)^{\omega}\colon \mathrm{Sch}^{\mathrm{qcqs,op}} \to \mathrm{Cat}_{\infty},$$

where $\mathbf{SH}(\mathrm{X})^{\omega} \subset \mathbf{SH}(\mathrm{X})$ is the full subcategory of compact objects. To explain how, we need a pair of technical lemmas.

Lemma 4. *Let* Q *be a commutative square of small ∞-categories:*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ \mathcal{C} & \xrightarrow{g} & \mathcal{D}. \end{array}$$

(i) *If $\mathcal{A}$ is idempotent complete and* Ind(Q) *is cartesian, then* Q *is cartesian.*

(ii) *Suppose that* Q *is a square of stable ∞-categories and exact functors. If g has a fully faithful right adjoint and* Q *is cartesian, then* Ind(Q) *is cartesian.*

Proof. (i) This follows from [Lur17b, Lemma 5.4.5.7(2)].

(ii) Form the cartesian square

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{f} & \mathrm{Ind}(\mathcal{B}) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ \mathrm{Ind}(\mathcal{C}) & \xrightarrow{g} & \mathrm{Ind}(\mathcal{D}). \end{array}$$

It follows from [Lur17b, Lemma 5.4.5.7(2)] that $\mathcal{A} \subset \mathcal{E}^{\omega}$, so it suffices to show that $\mathcal{A}$ generates $\mathcal{E}$. Let $e \in \mathcal{E}$ be such that $\mathrm{Maps}(a, e) = 0$ for all $a \in \mathcal{A}$; we must show that $e = 0$. Let r be the right adjoint g. If $a \in \mathcal{A}$ is in the kernel of f (equivalently, of g), $\mathrm{Maps}(a, e) \simeq \mathrm{Maps}(a, h(e))$. Hence, $h(e)$ is right orthogonal to the kernel of g, so $h(e) = rgh(e)$. On the other hand, if a is the image of $b \in \mathcal{B}$ by the functor $\mathcal{B} \to \mathcal{A}$ induced by $r \circ k$, then

$$\mathrm{Maps}(a, e) \simeq \mathrm{Maps}(b, f(e)) \times_{\mathrm{Maps}(k(b),kf(e))} \mathrm{Maps}(rk(b), h(e)) \simeq \mathrm{Maps}(b, f(e)),$$

since $h(e) = rkf(e)$ and r is fully faithful. This shows that $f(e) = 0$, hence also $h(e) = 0$, hence $e = 0$. □

Lemma 5. *Let $\mathcal{K}$ be a filtered ∞-category and $\mathrm{D}\colon \mathcal{K}^{\triangleright} \to \mathrm{Cat}_{\infty}$ a diagram of small ∞-categories with finite colimits and right exact functors. Let $\widehat{\mathrm{D}}\colon (\mathcal{K}^{\mathrm{op}})^{\triangleleft} \to \mathrm{Cat}_{\infty}$ be the diagram obtained from* D *by applying* Ind *and passing to right adjoints.*

(i) *If $\mathrm{D}(k)$ is idempotent complete for all $k \in \mathcal{K}$ and $\widehat{\mathrm{D}}$ is a limit diagram, then* D *is a colimit diagram.*

(ii) *If* D *is a colimit diagram, then $\widehat{\mathrm{D}}$ is a limit diagram.*

Proof. By [Lur17b, Proposition 5.5.7.11], D is a colimit diagram in Cat_∞ if and only if it is so in $\mathrm{Cat}_\infty^{\mathrm{rex}}$. By [Lur17b, Proposition 5.5.7.6], $\widehat{\mathrm{D}}$ is a limit diagram in Cat_∞ if and only if it is so in $\mathrm{Pr}^{\mathrm{R}}_\omega$. Passing to adjoints gives an equivalence $(\mathrm{Pr}^{\mathrm{R}}_\omega)^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{L}}_\omega$ [Lur17b, Notation 5.5.7.7]. Assertion (i) now follows from [Lur17b, Proposition 5.5.7.8] and [Lur17a, Lemma 7.3.5.10], while assertion (ii) follows from [Lur17b, Proposition 5.5.7.10]. □

Recall that $\mathbf{SH}(-)$ is a cdh sheaf [Hoy17, Proposition 6.24] and that $\mathbf{SH}(\mathrm{X})$ is compactly generated when X is qcqs [Hoy14, Proposition C.12(1,2)]. Since cdh descent on $\mathrm{Sch}^{\mathrm{qcqs}}$ is equivalent to certain squares being taken to cartesian squares [EHIK20, Proposition 2.1.5(2)], it follows from Lemma 4(i) that $\mathbf{SH}(-)^\omega$ is a cdh sheaf on $\mathrm{Sch}^{\mathrm{qcqs}}$. The reason for passing to compact objects is that $\mathbf{SH}(-)^\omega$ is also a finitary presheaf, i.e., it transforms limits of cofiltered diagrams of qcqs schemes with affine transition maps into colimits of ∞-categories: this follows from [Hoy14, Proposition C.12(4)] and Lemma 5(i). Since the ∞-category of small ∞-categories is compactly generated, the presheaf $\mathbf{SH}(-)^\omega$ (more precisely, its right Kan extension to Sch) satisfies the assumptions of [EHIK20, Theorem 3.3.4] over the base $\operatorname{Spec}\mathbf{Z}$. The conclusion is that $\mathbf{SH}(-)^\omega$ satisfies Milnor excision if and only if it satisfies henselian v-excision. If $i\colon \mathrm{Z} \hookrightarrow \mathrm{X}$ is a closed immersion of qcqs schemes, the functor $i^*\colon \mathbf{SH}(\mathrm{X})^\omega \to \mathbf{SH}(\mathrm{Z})^\omega$ has a fully faithful right adjoint, since i_* preserves compact objects (by localization [Hoy14, Proposition C.10]). Hence, by Lemma 4, $\mathbf{SH}(-)^\omega$ satisfies Milnor excision or henselian v-excision if and only if $\mathbf{SH}(-)$ does. Theorem 1 is therefore reduced to the following theorem:

Theorem 6. *Let* V *be a valuation ring and* $\mathfrak{p} \subset \mathrm{V}$ *a prime ideal. Then the following square of* ∞*-categories is cartesian:*

$$\begin{array}{ccc} \mathbf{SH}(\mathrm{V}) & \longrightarrow & \mathbf{SH}(\mathrm{V}_\mathfrak{p}) \\ \downarrow & & \downarrow \\ \mathbf{SH}(\mathrm{V}/\mathfrak{p}) & \longrightarrow & \mathbf{SH}(\kappa(\mathfrak{p})). \end{array}$$

Moreover, $\mathbf{SH}(-)^\omega$ being finitary, it is enough to prove Theorem 6 for V a valuation ring of finite rank [EHIK20, Remark 3.3.3]. In this case, $\operatorname{Spec}\mathrm{V}_\mathfrak{p} \to \operatorname{Spec}\mathrm{V}$ is an open immersion.

Let us examine more generally under which conditions $\mathbf{SH}(-)$ sends a square to a cartesian square. A commutative square of schemes

$$\begin{array}{ccc} \mathrm{W} & \xrightarrow{k} & \mathrm{Y} \\ {\scriptstyle g}\downarrow & \searrow^{l} & \downarrow{\scriptstyle f} \\ \mathrm{Z} & \xrightarrow{h} & \mathrm{X} \end{array}$$

induces an adjunction

$$\mathbf{SH}(\mathrm{X}) \rightleftarrows \mathbf{SH}(\mathrm{Y}) \times_{\mathbf{SH}(\mathrm{W})} \mathbf{SH}(\mathrm{Z}). \tag{7}$$

This adjunction is an equivalence if and only if the left adjoint is conservative and the right adjoint is fully faithful, in other words if and only if:

(i) the functor $(f^*, h^*)\colon \mathbf{SH}(\mathrm{X}) \to \mathbf{SH}(\mathrm{Y}) \times \mathbf{SH}(\mathrm{Z})$ is conservative;

(ii) given $\mathrm{E_Y} \in \mathbf{SH}(\mathrm{Y})$, $\mathrm{E_Z} \in \mathbf{SH}(\mathrm{Z})$, $\mathrm{E_W} \in \mathbf{SH}(\mathrm{W})$, $k^*\mathrm{E_Y} \simeq \mathrm{E_W}$, and $g^*\mathrm{E_Z} \simeq \mathrm{E_W}$, if $\mathrm{E} = f_*(\mathrm{E_Y}) \times_{l_*\mathrm{E_W}} h_*(\mathrm{E_Z})$, then the canonical maps

$$f^*(\mathrm{E}) \to \mathrm{E_Y} \quad \text{and} \quad h^*(\mathrm{E}) \to \mathrm{E_Z}$$

are equivalences.

If f is an immersion, then f_* is fully faithful and hence $f^*(\mathrm{E}) \simeq \mathrm{E_Y} \times_{k_*g^*\mathrm{E_Z}} f^*h_*\mathrm{E_Z}$. It follows that $f^*(\mathrm{E}) \to \mathrm{E_Y}$ is an equivalence if and only if the exchange morphism $f^*h_*(\mathrm{E_Z}) \to k_*g^*(\mathrm{E_Z})$

is an equivalence. Thus, if f, g, h, and k are all immersions (so that k^* and g^* are essentially surjective), then (ii) holds if and only if the following exchange transformations are equivalences:

$$f^*h_* \to k_*g^* \colon \mathbf{SH}(\mathrm{Z}) \to \mathbf{SH}(\mathrm{Y}),$$
$$h^*f_* \to g_*k^* \colon \mathbf{SH}(\mathrm{Y}) \to \mathbf{SH}(\mathrm{Z}).$$

Remark 8. In the adjunction (7), the left adjoint functor is fully faithful if and only if, for all $\mathrm{E} \in \mathbf{SH}(\mathrm{X})$, $\mathrm{E}(-)$ converts every smooth base change of the given square to a cartesian square. For Milnor squares (which are preserved by smooth base change [EHIK20, Lemma 3.2.9]), this is precisely the content of Corollary 2. For abstract blowup squares, this was first proved by Cisinski in [Cis13, Proposition 3.7]. The stronger statement that $\mathbf{SH}(-)$ itself sends abstract blowup squares to cartesian squares was proved in [Hoy17, Proposition 6.24] by verifying conditions (i) and (ii) above.

Now let V be a valuation ring of finite rank and $\mathfrak{p} \subset \mathrm{V}$ a prime ideal. Set $\mathrm{X} = \operatorname{Spec} \mathrm{V}$, $\mathrm{U} = \operatorname{Spec} \mathrm{V}_{\mathfrak{p}}$, $\mathrm{Z} = \operatorname{Spec} \mathrm{V}/\mathfrak{p}$, and $\mathrm{T} = \mathrm{X} - \mathrm{Z}$. We have a commutative diagram

$$\begin{array}{ccccc} & & \mathrm{U} \cap \mathrm{Z} & \xrightarrow{v} & \mathrm{Z} \\ & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle i} \\ \mathrm{T} & \xrightarrow{t} & \mathrm{U} & \xrightarrow{u} & \mathrm{X} \end{array}$$

where the horizontal maps are open immersions and the vertical maps are closed immersions. Since U and Z cover X, the functor $\mathbf{SH}(\mathrm{X}) \to \mathbf{SH}(\mathrm{U}) \times \mathbf{SH}(\mathrm{Z})$ is conservative (by localization). Specializing the above discussion to this situation, we see that Theorem 6 holds if and only if the base change transformation

$$i^*u_* \to v_*k^* \colon \mathbf{SH}(\mathrm{U}) \to \mathbf{SH}(\mathrm{Z})$$

is an equivalence (the other transformation $u^*i_* \to k_*v^*$ being an equivalence by proper base change [Hoy14, Proposition C.13(1)]). This transformation induces the rightmost morphism in the diagram of localization sequences

$$\begin{array}{ccccc} u_!t_!t^* \simeq (ut)_!(ut)^*u_* & \longrightarrow & u_* & \longrightarrow & i_*i^*u_* \\ \downarrow & & \| & & \downarrow \\ u_*t_!t^* & \longrightarrow & u_* & \longrightarrow & u_*k_*k^* \simeq i_*v_*k^*. \end{array}$$

Since i_* is fully faithful and t^* is surjective, we deduce that Theorem 6 holds if and only if the canonical transformation

$$u_!t_! \to u_*t_! \colon \mathbf{SH}(\mathrm{T}) \to \mathbf{SH}(\mathrm{X}) \tag{9}$$

is an equivalence.

Let $\mathbf{H}_{\mathrm{cdh}}(\mathrm{X})$ and $\mathbf{SH}_{\mathrm{cdh}}(\mathrm{X})$ be the analogues of $\mathbf{H}(\mathrm{X})$ and $\mathbf{SH}(\mathrm{X})$ constructed using the cdh site $\mathrm{Sch}_{\mathrm{X}}^{\mathrm{lfp}}$ instead of the Nisnevich site Sm_{X}. The inclusion $\mathrm{Sm}_{\mathrm{X}} \subset \mathrm{Sch}_{\mathrm{X}}^{\mathrm{lfp}}$ induces left adjoint functors $\mathbf{H}(\mathrm{X}) \to \mathbf{H}_{\mathrm{cdh}}(\mathrm{X})$ and $\mathbf{SH}(\mathrm{X}) \to \mathbf{SH}_{\mathrm{cdh}}(\mathrm{X})$, and the fact that $\mathbf{SH}(-)$ satisfies cdh descent implies that the latter is fully faithful [Kha19]. For $f \colon \mathrm{Y} \to \mathrm{X}$ any morphism, we have commutative squares

$$\begin{array}{ccc} \mathbf{SH}(\mathrm{X}) & \xrightarrow{f^*} & \mathbf{SH}(\mathrm{Y}) \\ \downarrow & & \downarrow \\ \mathbf{SH}_{\mathrm{cdh}}(\mathrm{X}) & \xrightarrow{f^*} & \mathbf{SH}_{\mathrm{cdh}}(\mathrm{Y}) \end{array} \qquad \begin{array}{ccc} \mathbf{SH}(\mathrm{X}) & \xleftarrow{f_*} & \mathbf{SH}(\mathrm{Y}) \\ \uparrow & & \uparrow \\ \mathbf{SH}_{\mathrm{cdh}}(\mathrm{X}) & \xleftarrow{f_*} & \mathbf{SH}_{\mathrm{cdh}}(\mathrm{Y}). \end{array}$$

If $f\colon \mathrm{Y} \to \mathrm{X}$ is smooth, we moreover have a commutative square

$$\begin{array}{ccc} \mathbf{SH}(\mathrm{Y}) & \xrightarrow{f_\sharp} & \mathbf{SH}(\mathrm{X}) \\ \downarrow & & \downarrow \\ \mathbf{SH}_{\mathrm{cdh}}(\mathrm{Y}) & \xrightarrow{f_\sharp} & \mathbf{SH}_{\mathrm{cdh}}(\mathrm{X}). \end{array}$$

Hence, for $u\colon \mathrm{U} \hookrightarrow \mathrm{X}$ an open immersion, we have factorizations

$$\begin{array}{ccc} \mathbf{SH}(\mathrm{U}) & \xrightarrow{u_!} & \mathbf{SH}(\mathrm{X}) \\ \downarrow & & \uparrow \\ \mathbf{SH}_{\mathrm{cdh}}(\mathrm{U}) & \xrightarrow{u_!} & \mathbf{SH}_{\mathrm{cdh}}(\mathrm{X}) \end{array} \qquad \begin{array}{ccc} \mathbf{SH}(\mathrm{U}) & \xrightarrow{u_*} & \mathbf{SH}(\mathrm{X}) \\ \downarrow & & \uparrow \\ \mathbf{SH}_{\mathrm{cdh}}(\mathrm{U}) & \xrightarrow{u_*} & \mathbf{SH}_{\mathrm{cdh}}(\mathrm{X}). \end{array}$$

Thus, to show that (9) is an equivalence, it suffices to show that the natural transformation

$$u_!t_! \to u_*t_!\colon \mathbf{SH}_{\mathrm{cdh}}(\mathrm{T}) \to \mathbf{SH}_{\mathrm{cdh}}(\mathrm{X})$$

is an equivalence, which we do in Proposition 15 below. The following three lemmas are the heart of the proof.

Lemma 10. *Let* V *be a valuation ring and* X *a connected* V*-scheme. Then the image of* $\mathrm{X} \to \operatorname{Spec} \mathrm{V}$ *is an interval in the specialization poset.*

Proof. This follows from [EHIK20, Lemma 3.2.9], which says that Milnor squares are preserved by pullback to a reduced scheme: if the fiber over $\mathfrak{p} \subset \mathrm{V}$ is empty, then $\mathrm{X}_{\mathrm{red}}$ is the sum of its restrictions to $\mathrm{V}_{\mathfrak{p}}$ and $\mathrm{V}/\mathfrak{p}$. □

In the following lemmas, $\mathrm{PSh}_\varnothing \subset \mathrm{PSh}$ denotes the full subcategory of presheaves that send the initial object to the terminal object, and $\mathrm{PSh}_\Sigma \subset \mathrm{PSh}_\varnothing$ is the full subcategory of presheaves that transform finite sums into finite products.

Lemma 11. *Let* V *be a valuation ring of finite rank, let* $\mathrm{X} = \operatorname{Spec} \mathrm{V}$*, and let* $\mathrm{T} \xrightarrow{t} \mathrm{U} \xrightarrow{u} \mathrm{X}$ *be open immersions with* $\mathrm{T} \neq \mathrm{U}$*. Let* $\mathcal{C}$ *be a pointed* ∞*-category with finite products. Then the natural transformation*

$$u_!t_! \to u_*t_!\colon \mathrm{PSh}_\Sigma(\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{T}}, \mathcal{C}) \to \mathrm{PSh}_\Sigma(\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{X}}, \mathcal{C})$$

is an equivalence.

Proof. Since V has finite rank, every scheme in $\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{X}}$ has finitely many generic points and in particular is a finite sum of connected schemes. It therefore suffices to show that

$$(u_!t_!\mathcal{F})(\mathrm{Y}) \simeq (u_*t_!\mathcal{F})(\mathrm{Y})$$

for every connected X-scheme Y. We have

$$(u_!t_!\mathcal{F})(\mathrm{Y}) = \begin{cases} * & \text{if } \mathrm{Y}_{\mathrm{T}} \neq \mathrm{Y}, \\ \mathcal{F}(\mathrm{Y}_{\mathrm{T}}) & \text{otherwise,} \end{cases}$$

$$(u_*t_!\mathcal{F})(\mathrm{Y}) = \begin{cases} * & \text{if } \mathrm{Y}_{\mathrm{T}} \neq \mathrm{Y}_{\mathrm{U}}, \\ \mathcal{F}(\mathrm{Y}_{\mathrm{T}}) & \text{otherwise.} \end{cases}$$

These obviously agree if $\mathrm{Y}_{\mathrm{T}} = \varnothing$, since $\mathcal{F}(\varnothing) = *$, so we may assume $\mathrm{Y}_{\mathrm{T}} \neq \varnothing$. In this case, since $\mathrm{T} \neq \mathrm{U}$ and the image of $\mathrm{Y} \to \mathrm{X}$ is an interval (Lemma 10), $\mathrm{Y}_{\mathrm{T}} \neq \mathrm{Y}$ if and only if $\mathrm{Y}_{\mathrm{T}} \neq \mathrm{Y}_{\mathrm{U}}$. □

Lemma 12. *Let* $u\colon \mathrm{U} \hookrightarrow \mathrm{X}$ *be an open immersion between qcqs schemes. Then the functors*

$$u_*\colon \mathrm{PSh}_\varnothing(\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{U}}) \to \mathrm{PSh}_\varnothing(\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{X}})$$

$$u_*\colon \mathrm{PSh}_\varnothing(\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{U}})_* \to \mathrm{PSh}_\varnothing(\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{X}})_*$$

preserve cdh-local equivalences and motivic equivalences (i.e., morphisms that become equivalences in $\mathbf{H}_{\mathrm{cdh}}$*).*

Proof. The functor $u_*\colon \mathrm{PSh}_\varnothing(\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{U}}) \to \mathrm{PSh}_\varnothing(\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{X}})$ preserves colimits indexed by weakly contractible ∞-categories, since the inclusion $\mathrm{PSh}_\varnothing \subset \mathrm{PSh}$ does. If $\mathrm{L}_\varnothing\colon \mathrm{PSh}(\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{U}}) \to \mathrm{PSh}_\varnothing(\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{U}})$ is the left adjoint to the inclusion, then

$$(\mathrm{L}_\varnothing \mathcal{F})(\mathrm{Y}) = \begin{cases} * & \text{if } \mathrm{Y} = \varnothing, \\ \mathcal{F}(\mathrm{Y}) & \text{otherwise.} \end{cases}$$

In particular, if $\mathrm{Y} \in \mathrm{Sch}^{\mathrm{fp}}_{\mathrm{U}}$ and $i\colon \mathrm{R} \hookrightarrow \mathrm{Y}$ is a sieve, then $\mathrm{L}_\varnothing(i) = i$ unless the sieve is empty, in which case $\mathrm{L}_\varnothing(i)$ is the sieve on Y generated by the empty scheme. The collection of cdh-local equivalences in $\mathrm{PSh}_\varnothing(\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{U}})$ is therefore generated under 2-out-of-3 and colimits by nonempty cdh sieves, and the collection of motivic equivalences is similarly generated by cdh-local equivalences and $\mathbf{A}^1$-homotopy equivalences. The same collections are generated using only 2-out-of-3 and weakly contractible colimits, because the initial object of $\mathrm{Fun}(\Delta^1, \mathrm{PSh}_\varnothing(\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{U}}))$ is a cdh sieve and the colimit of any diagram $\mathcal{K} \to \mathcal{C}$ is the same as the colimit of an extension $\mathcal{K}^{\triangleleft} \to \mathcal{C}$ sending the cone point to an initial object. Since u_* preserves $\mathbf{A}^1$-homotopic maps, it remains to show that for every nonempty cdh sieve $\mathrm{R} \hookrightarrow \mathrm{Y}$ in $\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{U}}$, $u_*(\mathrm{R}) \hookrightarrow u_*(\mathrm{Y})$ is a cdh-local equivalence. Since it is a monomorphism, it suffices to check that it is surjective on stalks. If A is a henselian valuation ring and $\mathrm{Spec}\,\mathrm{A} \to u_*(\mathrm{Y})$ is a morphism, then $(\mathrm{Spec}\,\mathrm{A})_{\mathrm{U}}$ is either empty or the spectrum of a henselian valuation ring [EHIK20, Lemma 3.3.5]. In both cases, the map $(\mathrm{Spec}\,\mathrm{A})_{\mathrm{U}} \to \mathrm{Y}$ factors through R.

The functor $u_*\colon \mathrm{PSh}_\varnothing(\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{U}})_* \to \mathrm{PSh}_\varnothing(\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{X}})_*$ preserves colimits, so as before it suffices to show that $u_*\mathrm{L}_\varnothing(\mathrm{R}_+) \hookrightarrow u_*\mathrm{L}_\varnothing(\mathrm{Y}_+)$ is a cdh-local equivalence for every cdh sieve $\mathrm{R} \hookrightarrow \mathrm{Y}$. Since it is a monomorphism, this can be checked on stalks as above. □

Remark 13. If $u\colon \mathrm{U} \to \mathrm{X}$ is an étale morphism between qcqs schemes, the conclusions of Lemma 12 hold if one replaces $\mathrm{PSh}_\varnothing$ with PSh_Σ. Indeed, if V is a henselian valuation ring and $\mathrm{X} \to \mathrm{Spec}\,\mathrm{V}$ is a quasi-compact étale morphism, then X is the spectrum of a finite product of henselian valuation rings.

Remark 14. Lemma 12 (but not Remark 13) also holds for the rh topology, whose points are valuation rings.

Proposition 15. *Let* V *be a valuation ring of finite rank, let* $\mathrm{X} = \mathrm{Spec}\,\mathrm{V}$*, and let* $\mathrm{T} \xrightarrow{t} \mathrm{U} \xrightarrow{u} \mathrm{X}$ *be open immersions with* $\mathrm{T} \neq \mathrm{U}$*. Then the natural transformations*

$$u_!t_! \to u_*t_!\colon \mathbf{H}_{\mathrm{cdh}}(\mathrm{T})_* \to \mathbf{H}_{\mathrm{cdh}}(\mathrm{X})_*$$
$$u_!t_! \to u_*t_!\colon \mathbf{SH}_{\mathrm{cdh}}(\mathrm{T}) \to \mathbf{SH}_{\mathrm{cdh}}(\mathrm{X})$$

are equivalences.

Proof. The first equivalence follows directly from Lemmas 11 and 12. The functors $u_!$ and u_* extend to functors between the ∞-categories of $\mathbf{P}^1$-prespectra, which are computed levelwise. The second equivalence follows from the first since both $u_!$ and u_* commute with spectrification (the former because u^* preserves $\mathbf{P}^1$-spectra among $\mathbf{P}^1$-prespectra, and the latter because u_* commutes with $\mathbf{P}^1$-loops and filtered colimits). □

This completes the proof of Theorem 6, hence of Theorem 1.

Remark 16. Let S be a scheme and $\mathrm{E} \in \mathrm{Alg}(\mathbf{SH}(\mathrm{S}))$ a motivic ring spectrum over S. It follows formally from Theorem 1 that the presheaf of ∞-categories $\mathrm{Mod}_{\mathrm{E}}(\mathbf{SH}(-))\colon \mathrm{Sch}^{\mathrm{op}}_{\mathrm{S}} \to \mathrm{Cat}_\infty$ satisfies Milnor excision. For example, the ∞-category of Beilinson motives [CD19, §14.2], the

∞-category of Spitzweck motives [Spi18, Chapter 9], and the ∞-category of motivic spectra with finite syntomic transfers [EHK+19, §4.1] satisfy Milnor excision.

We conclude this article with a proof of Milnor excision for Ayoub's étale motives. For X a scheme and Λ a commutative ring, let $\mathbf{DA}^{\text{ét}}(X, \Lambda)$ be the ∞-category of étale motives constructed in [Ayo14, §3], and let $\mathbf{DA}^{\text{ét}}_{\text{ct}}(X, \Lambda) \subset \mathbf{DA}^{\text{ét}}(X, \Lambda)$ be the subcategory of constructible objects (defined as in [Ayo14, Définition 8.1] for X qcqs and using Zariski descent in general). Let $\mathbf{D}^{\text{ét}}(X, \Lambda)$ be the derived ∞-category of the abelian category of étale sheaves of Λ-modules on X (equivalently, the ∞-category of étale hypersheaves of Λ-module spectra), and let $\mathbf{D}^{\text{ét}}_{\text{ct}}(X, \Lambda) \subset \mathbf{D}^{\text{ét}}(X, \Lambda)$ be the subcategory of constructible objects (in the sense of [BS15, Definition 6.3.1]). We denote by $H\Lambda \in \mathbf{SH}(X)$ Spitzweck's motivic cohomology spectrum with coefficients in Λ [Hoy18, §4].

We shall say that X is *Λ-finite* if it has finite Krull dimension and

$$\sup_{x \in X, p \notin \Lambda^\times} \operatorname{cd}_p(\kappa(x)) < \infty,$$

where $\operatorname{cd}_p(k)$ is the mod p Galois cohomological dimension of a field k. If X is quasi-compact and Λ-finite and $Y \to X$ is of finite type, then Y is also Λ-finite. We shall say that X is *étale-locally Λ-finite* if it admits an étale covering by Λ-finite schemes. A quasi-compact scheme X is étale-locally Λ-finite if and only if the schemes $X[\frac{1}{2}, \zeta_4]$ and $X[\frac{1}{3}, \zeta_6]$ are Λ-finite, and also if and only if X has finite Krull dimension and $\sup_{x \in X, p \notin \Lambda^\times} \operatorname{vcd}_p(\kappa(x)) < \infty$. Every scheme essentially of finite type over $\mathbf{Z}$ is étale-locally Λ-finite, and the collection of étale-locally Λ-finite schemes is closed under Milnor pushouts.

Lemma 17. *Let X be a scheme and Λ a commutative ring.*

(i) *If X is qcqs and Λ-finite, then*

$$\mathbf{D}^{\text{ét}}(X, \Lambda) \simeq \operatorname{Ind}(\mathbf{D}^{\text{ét}}_{\text{ct}}(X, \Lambda)) \quad \textit{and} \quad \mathbf{DA}^{\text{ét}}(X, \Lambda) \simeq \operatorname{Ind}(\mathbf{DA}^{\text{ét}}_{\text{ct}}(X, \Lambda)).$$

(ii) *If X is the limit of a cofiltered diagram of schemes X_i with affine transition morphisms and if X and X_i are étale-locally Λ-finite, then*

$$\mathbf{D}^{\text{ét}}(X, \Lambda) \simeq \lim_i \mathbf{D}^{\text{ét}}(X_i, \Lambda) \quad \textit{and} \quad \mathbf{DA}^{\text{ét}}(X, \Lambda) \simeq \lim_i \mathbf{DA}^{\text{ét}}(X_i, \Lambda).$$

(iii) *If $f \colon Y \to X$ is a qcqs morphism and X and Y are étale-locally Λ-finite, then*

$$f_* \colon \mathbf{D}^{\text{ét}}(Y, \Lambda) \to \mathbf{D}^{\text{ét}}(X, \Lambda) \quad \textit{and} \quad f_* \colon \mathbf{DA}^{\text{ét}}(Y, \Lambda) \to \mathbf{DA}^{\text{ét}}(X, \Lambda)$$

preserve colimits.

Proof. (i) Let $d = \dim(X) + \sup_{x \in X, p \notin \Lambda^\times} \operatorname{cd}_p(\kappa(x)) + 1$. For any qcqs étale X-scheme U and any étale sheaf of Λ-modules $\mathcal{F}$ on U, we have $H^n_{\text{ét}}(U, \mathcal{F}) = 0$ for $n > d$. Indeed, this follows from [CM19, Corollary 3.28], noting that the p-local Galois cohomological dimension of a field k is at most $\operatorname{cd}_p(k) + 1$. The result for $\mathbf{D}^{\text{ét}}$ is now [BS15, Proposition 6.4.8], and the analogue for $\mathbf{DA}^{\text{ét}}$ is an easy consequence (cf. [Ayo14, Proposition 3.19]).

(ii) By Zariski descent, we may assume that X_i and hence X are qcqs. Using descent along the étale covering $\{\operatorname{Spec} \mathbf{Z}[\frac{1}{2}, \zeta_4], \operatorname{Spec} \mathbf{Z}[\frac{1}{3}, \zeta_6]\}$ of $\operatorname{Spec} \mathbf{Z}$, we may further assume that X and X_i are Λ-finite. The result is then a formal consequence of (i), cf. [Ayo14, Proposition 3.20].

(iii) By étale descent, we can assume that X and Y are qcqs and Λ-finite. Then the result follows immediately from (i). □

Remark 18. For $\mathbf{D}^{\text{ét}}(X, \Lambda)$ to be compactly generated, it suffices that X be qcqs and étale-locally Λ-finite. However, this does not suffice for the conclusion of Lemma 17(i), as the case $X = \operatorname{Spec} \mathbf{R}$ and $\Lambda = \mathbf{Z}/2$ shows: there the unit object is constructible but not compact.

Lemma 19. *Let X be a scheme and Λ a commutative ring.*

(i) *If* Λ *is a* $\mathbf{Q}$*-algebra and* X *is locally of finite Krull dimension, then* $\mathbf{DA}^{\text{ét}}(\mathrm{X},\Lambda) \simeq \mathbf{Mod}_{\mathrm{H}\Lambda}(\mathbf{SH}(\mathrm{X}))$.

(ii) *If* Λ *is a* $\mathbf{Z}/n$*-algebra for some integer* n *invertible on* X *and if* $\mathrm{X}_x^{\mathrm{sh}}$ *is* Λ*-finite for every geometric point* x *of* X*, then* $\mathbf{DA}^{\text{ét}}(\mathrm{X},\Lambda) \simeq \mathbf{D}^{\text{ét}}(\mathrm{X},\Lambda)$.

(iii) *If* p *is a prime that is locally nilpotent on* X*, then* $\mathbf{DA}^{\text{ét}}(\mathrm{X},\Lambda) \simeq \mathbf{DA}^{\text{ét}}(\mathrm{X},\Lambda[\frac{1}{p}])$.

Proof. (i) We may assume $\Lambda = \mathbf{Q}$, as both sides are obtained from this case by taking Λ-modules. By construction, Spitzweck's $\mathrm{H}\mathbf{Q}$ is the Beilinson motivic cohomology spectrum of Cisinski and Déglise. Using Zariski descent and Lemma 17(ii), we can assume X noetherian of finite Krull dimension. In this case the result is precisely [CD19, Theorem 16.2.18].

(ii) If X is of finite type over $\mathbf{Z}$, this follows from [Ayo14, Théorème 4.1]. By Lemma 17(ii), the conclusion holds whenever X is qcqs and étale-locally Λ-finite. The general case (which we will not use) follows from this case as in the second half of the proof of [Ayo14, Théorème 4.1].

(iii) By nilinvariance, we can assume that X is an $\mathbf{F}_p$-scheme. Then the result follows from the Artin–Schreier exact sequence, see [Ayo14, Lemma 3.10]. □

Remark 20. By a theorem of Gabber [ILO13, Exposé $\mathrm{XVIII}_{\mathrm{A}}$, Corollaire 1.2], the assumption on X in Lemma 19(ii) holds whenever X is locally noetherian. It does not hold in general, however, as for example the fraction field of a strictly henselian valuation ring of rank 1 can have infinite cohomological dimension.

Lemma 21. *Let* $\mathcal{C}$ *be an additive compactly generated* ∞*-category and* $\mathrm{X} \in \mathcal{C}$*. If* $\mathrm{X} \otimes \mathbf{Q} = 0$ *and* $\mathrm{X}/p = 0$ *for every prime* p*, then* $\mathrm{X} = 0$.

Proof. Since $\mathcal{C}$ is additive and compactly generated, it is enough to show that $[\mathrm{K},\mathrm{X}] = 0$ for every compact object $\mathrm{K} \in \mathcal{C}$. Since K is compact, $[\mathrm{K},\mathrm{X}] \otimes \mathbf{Q} \simeq [\mathrm{K},\mathrm{X} \otimes \mathbf{Q}] = 0$, so $[\mathrm{K},\mathrm{X}]$ is torsion. Moreover, $\mathrm{Tor}([\mathrm{K},\mathrm{X}],\mathbf{Z}/p)$ is a quotient of $[\Sigma\mathrm{K},\mathrm{X}/p] = 0$, so $[\mathrm{K},\mathrm{X}]$ is torsionfree. Hence, $[\mathrm{K},\mathrm{X}] = 0$. □

Theorem 22. *Let* Λ *be a commutative ring. The presheaf of* ∞*-categories* $\mathbf{DA}^{\text{ét}}(-,\Lambda)$ *satisfies Milnor excision on the category of étale-locally* Λ*-finite schemes.*

Proof. For any morphism $\Lambda \to \Lambda'$, we have $\mathbf{DA}^{\text{ét}}(-,\Lambda') \simeq \mathbf{Mod}_{\Lambda'}(\mathbf{DA}^{\text{ét}}(-,\Lambda))$. We can thus assume that Λ is a localization of $\mathbf{Z}$. Moreover, by Zariski descent and descent along the étale covering $\{\operatorname{Spec}\mathbf{Z}[\frac{1}{2},\zeta_4], \operatorname{Spec}\mathbf{Z}[\frac{1}{3},\zeta_6]\}$ of $\operatorname{Spec}\mathbf{Z}$, it suffices to consider Λ-finite qcqs schemes. Consider a Milnor square of such schemes

$$\begin{array}{ccc} \mathrm{W} & \xrightarrow{k} & \mathrm{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Z} & \xrightarrow{i} & \mathrm{X}. \end{array}$$

By localization, the functor $(f^*, i^*)\colon \mathbf{DA}^{\text{ét}}(\mathrm{X},\Lambda) \to \mathbf{DA}^{\text{ét}}(\mathrm{Y},\Lambda) \times \mathbf{DA}^{\text{ét}}(\mathrm{Z},\Lambda)$ is conservative. As explained after Theorem 6, $\mathbf{DA}^{\text{ét}}(-,\Lambda)$ takes this square to a cartesian square if and only if certain morphisms in $\mathbf{DA}^{\text{ét}}(\mathrm{Y},\Lambda)$ and $\mathbf{DA}^{\text{ét}}(\mathrm{Z},\Lambda)$ are equivalences. Since these are compactly generated stable ∞-categories by Lemma 17(i), this can be checked rationally and modulo p for every prime p (Lemma 21). Rationally, we have $\mathbf{DA}^{\text{ét}}(-,\mathbf{Q}) \simeq \mathbf{Mod}_{\mathrm{H}\mathbf{Q}}(\mathbf{SH}(-))$ by Lemma 19(i), and the latter satisfies Milnor excision by Theorem 1. By Lemma 17(iii), the morphisms witnessing Milnor excision for $\mathbf{DA}^{\text{ét}}(-,\mathbf{Q})$ are the rationalizations of the ones for $\mathbf{DA}^{\text{ét}}(-,\Lambda)$, hence the latter are rational equivalences. Modulo p, we have $\mathbf{DA}^{\text{ét}}(-,\mathbf{Z}/p) \simeq \mathbf{D}^{\text{ét}}(-[\frac{1}{p}],\mathbf{Z}/p)$ by Lemma 19(ii,iii) and localization. By Lemma 17(i) and Lemma 4(ii), it remains to show that $\mathbf{D}^{\text{ét}}_{\mathrm{ct}}(-[\frac{1}{p}],\mathbf{Z}/p)$ satisfies Milnor excision. This is true (on all schemes) by [BM18, Theorem 5.14] and [EHIK20, Corollary 3.2.12]. □

Acknowledgments. We are grateful to Joseph Ayoub for asking about Theorem 22 and for discussions about the proof.

We thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the program "K-theory, algebraic cycles and motivic homotopy theory" where this paper was written. This work was partially supported by EPSRC grant number EP/R014604/1.

References

[Ayo14] J. Ayoub, *La réalisation étale et les opérations de Grothendieck*, Ann. Sci. Éc. Norm. Supér. **47** (2014), no. 1, pp. 1–145

[BM18] B. Bhatt and A. Mathew, *The arc topology*, 2018, arXiv:1807.04725

[BS15] B. Bhatt and P. Scholze, *The pro-étale topology for schemes*, Astérisque **369** (2015), pp. 99–201

[CD19] D.-C. Cisinski and F. Déglise, *Triangulated categories of mixed motives*, Springer Monographs in Mathematics, Springer, 2019, preprint arXiv:0912.2110

[Cis13] D.-C. Cisinski, *Descente par éclatements en* K*-théorie invariante par homotopie*, Ann. Math. **177** (2013), no. 2, pp. 425–448

[CM19] D. Clausen and A. Mathew, *Hyperdescent and étale K-theory*, 2019, arXiv:1905.06611

[EHIK20] E. Elmanto, M. Hoyois, R. Iwasa, and S. Kelly, *Cdh descent, cdarc descent, and Milnor excision*, 2020, arXiv:2002.11647

[EHK+19] E. Elmanto, M. Hoyois, A. A. Khan, V. Sosnilo, and M. Yakerson, *Modules over algebraic cobordism*, 2019, arXiv:1908.02162v1

[Hoy14] M. Hoyois, *A quadratic refinement of the Grothendieck–Lefschetz–Verdier trace formula*, Algebr. Geom. Topol. **14** (2014), no. 6, pp. 3603–3658

[Hoy17] M. Hoyois, *The six operations in equivariant motivic homotopy theory*, Adv. Math. **305** (2017), pp. 197–279

[Hoy18] M. Hoyois, *The localization theorem for framed motivic spaces*, 2018, arXiv:1807.04253v2

[ILO13] L. Illusie, Y. Laszlo, and F. Orgogozo, *Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents*, Astérisque **363** (2013)

[Kel19] S. Kelly, *A better comparison of cdh- and ldh-cohomologies*, Nagoya Math. J. **236** (2019), pp. 183–213, preprint arXiv:1807.00158

[Kha19] A. A. Khan, *The cdh-local motivic homotopy category (after Cisinski)*, 2019, https://www.preschema.com/documents/shcdh.pdf

[KM18] S. Kelly and M. Morrow, K*-theory of valuation rings*, 2018, arXiv:1810.12203v1

[Lur17a] J. Lurie, *Higher Algebra*, September 2017, http://www.math.harvard.edu/~lurie/papers/HA.pdf

[Lur17b] ———, *Higher Topos Theory*, April 2017, http://www.math.harvard.edu/~lurie/papers/HTT.pdf

[Spi18] M. Spitzweck, *A commutative* $\mathbf{P}^1$*-spectrum representing motivic cohomology over Dedekind domains*, Mém. Soc. Math. Fr. (N.S.) (2018), no. 157, p. 110

[Wei89] C. A. Weibel, *Homotopy algebraic* K*-theory*, Algebraic K-Theory and Number Theory, Contemp. Math., vol. 83, AMS, 1989, pp. 461–488

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