On p-adic comparison theorems for analytic spaces

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Algebraic comparison theorem

Notation: $K/\mathbf{Q}_p$ - finite, $\mathscr{G}_K = \mathrm{Gal}(\overline{K}/K)$, $C = \widehat{\overline{K}}$, $K \supset \mathscr{O}_K \to k$, $F = W(k)$.

Theorem (Algebraic comparison theorem) X/K – algebraic variety. There exists a natural $\mathbf{B}_{\mathrm{st}}$-linear, $\mathscr{G}_K$-equivariant period isomorphism $(r \geq 0)$

$$\begin{aligned} \alpha_{pst}: &\quad H^r_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}} \simeq H^r_{\mathrm{HK}}(X_{\overline{K}}) \otimes_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}}, \quad (\varphi, N, \mathscr{G}_K), \\ \alpha_{\mathrm{dR}}: &\quad H^r_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}} \simeq H^r_{\mathrm{dR}}(X_{\overline{K}}) \otimes_{\overline{K}} \mathbf{B}_{\mathrm{dR}}, \quad Fil, \end{aligned}$$

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Here:
(1) $H^r_{\mathrm{dR}}(X_{\overline{K}})$ – Deligne de Rham cohomology (uses resolution of singularities)
(2) $H^r_{\mathrm{HK}}(X_{\overline{K}})$ – Beilinson Hyodo-Kato cohomology (uses de Jong's alterations)

Hyodo-Kato cohomology

(i) **locally**: in *h*-topology alterations allow

$$\begin{array}{ccc}
\mathscr{U} & \hookleftarrow & U \\
{\scriptstyle sstable}\downarrow & & \downarrow{\scriptstyle h-map} \\
\mathrm{Spec}\,\mathscr{O}_L & & X \\
{\scriptstyle finite}\downarrow & \swarrow & \\
\mathrm{Spec}\,\mathscr{O}_K & &
\end{array}$$

Then we have

$$(a)\quad \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{U}_0/\mathscr{O}^0_{F_L}),\qquad H^*\text{- finite rank}/F_L,\qquad (\varphi, N),$$

$$(b)\quad \iota_{\mathrm{HK}}: \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{U}_0/\mathscr{O}^0_{F_L})\otimes^L_{F_L} L \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(U).$$

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(ii) **globalization**: make (i) geometric and glue in *h*-topology. Get

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X_{\overline{K}}), \quad H^*\text{- finite rank}/F^{\mathrm{nr}}, \quad (\varphi, N, \mathscr{G}_K),$$

$$\iota_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X_{\overline{K}}) \otimes_{F^{\mathrm{nr}}} \overline{K} \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}})$$

Restated algebraic comparison theorem

(i) **de Rham-to-étale comparison**:

$$H^r_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \simeq (H^r_{\text{HK}}(X_{\overline{K}}) \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}})^{\varphi=1, N=0} \cap F^0(H^r_{\text{dR}}(X) \otimes_K \mathbf{B}_{\text{dR}}), \quad \mathscr{G}_K,$$

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or: we have a bicartesian diagram ($r \geq 0$)

$$\begin{array}{ccc} H^r_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r)) & \longrightarrow & (H^r_{\text{HK}}(X_{\overline{K}}) \otimes_{F^{\text{nr}}} \mathbf{B}^+_{\text{st}})^{\varphi=p^r, N=0} \\ \downarrow & & \downarrow \\ F^r(H^r_{\text{dR}}(X) \otimes_K \mathbf{B}^+_{\text{dR}}) & \longrightarrow & H^r_{\text{dR}}(X) \otimes_K \mathbf{B}^+_{\text{dR}} \end{array}$$

We will write it as (upper index refers to cohomology degree)

$$\begin{array}{ccc} H^r_{\text{ét},r} & \longrightarrow & \text{HK}^r_r \\ \downarrow & & \downarrow \\ H^r(F^r) & \longrightarrow & \text{DR}^r \end{array}$$

or: there exists an exact sequence

$$0 \to H^r_{\text{ét},r} \to H^r(F^r) \oplus \mathrm{HK}^r_r \to \mathrm{DR}^r \to 0$$

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(ii) **étale-to-de Rham comparison**:

$$\mathrm{Hom}(H^r_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p), \mathbf{B}_{\mathrm{st}})^{\mathscr{G}_K - sm} \simeq H^r_{\mathrm{HK}}(X_{\overline{K}})^*, \quad (\varphi, N, \mathscr{G}_K),$$
$$\mathrm{Hom}_{\mathscr{G}_K}(H^r_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p), \mathbf{B}_{\mathrm{dR}}) \simeq H^r_{\mathrm{dR}}(X_{\overline{K}})^*, \quad Fil$$

Analytic varieties

X/K - smooth rigid analytic variety
Case 1 : X proper,
(A) Scholze:
(i) $H^r_{\text{ét}}(X_C, \mathbf{Q}_p)$ is finite rank over $\mathbf{Q}_p$:

- Artin-Schreier to pass to coherent cohomology
- Cartier-Serre argument for finitness of coherent cohomology

(ii) Hodge-de Rham spectral sequence degenerates
$\Rightarrow$ get **de Rham comparison isomorphism**:

$$\alpha_{\text{dR}} : \quad H^r_{\text{ét}}(X_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{dR}} \simeq H^r_{\text{dR}}(X) \otimes_K \mathbf{B}_{\text{dR}}, \quad Fil,$$

(B) Colmez-Nizioł: **Algebraic comparison theorem** holds (HK-cohomology is defined using Hartl and Temkin alterations instead of de Jong's)

$$\alpha_{pst}: \quad H^r_{\text{ét}}(X_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{st}} \simeq H^r_{\text{HK}}(X_C) \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}, \quad (\varphi, N, \mathscr{G}_K),$$
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(i) Tsuji, Kato, CN: p-adic nearby cycles = syntomic cohomology $(\tau_{\leq r})\Rightarrow$

$$\mathrm{DR}^{r-1}\xrightarrow{f_{r-1}}H^r_{\text{ét},r}\to H^r(F^r)\oplus\mathrm{HK}^r_r\to\mathrm{DR}^r\xrightarrow{f_r}H^{r+1}_{\text{syn},r}$$

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(ii) Lift the sequence to the category of Banach-Colmez (BC) spaces
Suffices: $f_{r-1}, f_r = 0$. For $\mathbf{f_{r-1}}$, have

$$\mathrm{DR}^i/H^i(F^r) - Dim = (d,0), \quad H^i_{\text{ét}} - Dim = (0,h), h \geq 0.$$

But in BC category there is no map between such spaces. For $\mathbf{f_r}$: bring to this situation by twisting.

Digression: Banach-Colmez spaces

What structure can we put on

$$\mathrm{HK}_r^r = (H^n_{\mathrm{HK}}(X_C) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}})^{\mathbf{N}=0,\varphi=1} \simeq (H^n_{\mathrm{HK}}(X_C) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{cr}})^{\varphi=1} \quad ?$$

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Example

$$0 \to \mathbf{Q}_p t \to \mathbf{B}_{\mathrm{cr}}^{+,\varphi=p} \to \mathbf{C} \to 0$$

So $\mathbf{B}_{\mathrm{cr}}^{+,\varphi=p} \sim \mathbf{C} \oplus \mathbf{Q}_p$.

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More generally, we have
Fundamental exact sequence:

$$0 \to \mathbf{Q}_p t^m \to \mathbf{B}^{+,\varphi=p^m}_{\mathrm{cr}} \to \mathbf{B}^+_{\mathrm{dR}}/t^m\mathbf{B}^+_{\mathrm{dR}} \to 0$$

So: $\mathbf{B}^{+,\varphi=p^m}_{\mathrm{cr}} \sim \mathbf{C}^m \oplus \mathbf{Q}_p$. But **In which category ?**

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So: $\mathbf{B}^{+,\varphi=p^m}_{\mathrm{cr}} \sim \mathbf{C}^m \oplus \mathbf{Q}_p$. But **In which category ?**

Remark The category of topological vector spaces is not good: $\mathbf{C} \oplus \mathbf{Q}_p \simeq \mathbf{C}$!

Theorem (Colmez, Fontaine) There exists an abelian category of Banach-Colmez vector spaces $\mathbb{W}$ which are finite dimensional $\mathbf{C}$-vector spaces $\pm$ finite dimensional $\mathbf{Q}_p$-vector spaces. We have

1. $\mathrm{Dim}(\mathbb{W}) := (\dim_{\mathbf{C}} \mathbb{W}, \dim_{\mathbf{Q}_p} \mathbb{W})$; set $\mathrm{ht}\, \mathbb{W} := \dim_{\mathbf{Q}_p} \mathbb{W}$
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Example

1. $\mathbf{B}_{\mathrm{dR}}^{+}/t^m$ is $\mathbb{B}_m$ with $\mathrm{Dim}(\mathbb{B}_m) = (m, 0)$.
2. $\mathbf{B}_{\mathrm{cr}}^{+,\varphi^a = p^b}$ is $\mathbb{U}_{a,b}$ with $\mathrm{Dim}(\mathbb{U}_{a,b}) = (b, a)$.
3. $\mathbf{C}/\mathbf{Q}_p$ is $\mathbb{V}^1/\mathbf{Q}_p$ with $\mathrm{Dim} = (1, -1)$.

Case 2:

X/K Stein:

1. there exists an admissible covering by affinoids

$$\cdots \Subset U_n \Subset U_{n+1} \Subset \cdots$$

2. $H^i(X, \mathscr{F}) = 0$, $\mathscr{F}$-coherent, $i > 0$
3. $\mathrm{R}\Gamma_{\text{proét}}(X_C, \mathbf{Q}_p) \simeq \operatorname{holim}_n \mathrm{R}\Gamma_{\text{ét}}(U_{n,C}, \mathbf{Q}_p)$

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Examples

(1) $X=\mathbb{A}_K$, $r>0$:

$$H^r_{\text{proét}}(X_C,\mathbf{Q}_p(r))\simeq \Omega^{r-1}(\mathbb{A}_C)/\ker d,$$
$$H^1_{\text{proét}}(X_C,\mathbf{Q}_p(1))\simeq \mathscr{O}(\mathbb{A}_C)/C$$

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(2) $X = \mathbb{G}_{m,K}$, there exists an exact sequence

$$0 \to \mathscr{O}(\mathbb{G}_{m,C})/C \to H^1_{\text{proét}}(\mathbb{G}_{m,C}, \mathbf{Q}_p(1)) \to \mathbf{Q}_p < \operatorname{dlog} z > \to 0$$

trivial $\mathscr{G}_K$-action on $\mathbf{Q}_p < \operatorname{dlog} z >$

(3) $X = \mathbb{P}^1_K \setminus \mathbb{P}^1(K)$ Drinfeld half-plane

$$0 \to \mathscr{O}(X_C)/C \to H^1_{\text{proét}}(X_C, \mathbf{Q}_p(1)) \to \mathrm{Sp}(\mathbf{Q}_p)^* \to 0$$

$\mathrm{Sp}(\mathbf{Q}_p) = \mathscr{C}^\infty(\mathbb{P}(K), \mathbf{Q}_p)/\mathbf{Q}_p$ – (smooth) Steinberg representation of $\mathrm{GL}_2(K)$.

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Theorem (Colmez-Dospinescu-N) X/K Stein smooth rigid space (or a dagger affinoid). There exists a map of exact sequences (all cohomologies are of X_C)

$$\begin{array}{ccccccccc}
0 \to & \Omega^{r-1}/\ker d & \to & H^r_{\text{proét}}(\mathbf{Q}_p(r)) & \to & (H^r_{\text{HK}} \widehat{\otimes}^R_{F^{\text{nr}}} \mathbf{B}^+_{\text{st}})^{\varphi=p^r, N=0} & \to & 0 \\
& \| & & \downarrow \alpha & & \downarrow \iota_{\text{HK}} \otimes \theta & & \\
0 \to & \Omega^{r-1}/\ker d & \to & \Omega^{r, d=0} & \to & H^r_{\text{dR}} & \to & 0
\end{array}$$

Main theorem

Theorem (Colmez-N) X/K smooth dagger variety.
(i) **de Rham-to-étale**: there exists a bicartesian diagram

$$\begin{array}{ccc} H^r_{\text{proét}}(X_C, \mathbf{Q}_p(r)) & \longrightarrow & (H^r_{\text{HK}}(X_C)\widehat{\otimes}^R_{F^{\text{nr}}}\mathbf{B}^+_{\text{st}})^{\varphi=p^r, N=0} \\ \downarrow & & \downarrow \iota_{\text{HK}}\otimes\iota \\ H^r(F^r(\mathrm{R}\Gamma_{\text{dR}}(X)\widehat{\otimes}^R_K\mathbf{B}^+_{\text{dR}})) & \longrightarrow & H^r_{\text{dR}}(X)\widehat{\otimes}^R_K\mathbf{B}^+_{\text{dR}} \end{array}$$

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(ii) **étale-to-de Rham**:

$$\mathrm{Hom}(H^r_{\text{proét}}(X_C,\mathbf{Q}_p),\mathbf{B}_{\text{st}})^{\mathscr{G}_K-prosm}\simeq H^r_{\text{HK}}(X_C)^* \quad (\varphi,N,\mathscr{G}_K),$$

$$\mathrm{Hom}_{\mathscr{G}_K}(H^r_{\text{proét}}(X_C,\mathbf{Q}_p),\mathbf{B}_{\text{dR}})\simeq H^r_{\text{dR}}(X)^*, \quad Fil???$$

Remarks

(1) X is **proper** then (degeneration of Hodge-de Rham sp. seq.)

$$H^r(F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}_K^R \mathbf{B}_{\mathrm{dR}}^+)) \simeq F^r(H^r_{\mathrm{dR}}(X)\widehat{\otimes}_K^R \mathbf{B}_{\mathrm{dR}}^+)$$

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(2) X is **Stein or an affinoid** then the two horizontal arrows are surjective and their kernels are $\Omega^{r-1}(X_C)/\ker d$.
(3) **Topology**: We work in the category of locally convex spaces (quasi-abelian).

- Tensor products are projective (commute with limits) and (right) derived.
- Overconvergence implies "good properties":
  1. higher derived functors of tensor products vanish,
  2. cohomology is "classical".

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Step 2: reduce to X quasi-compact: write

$$X=\cup_n U_n, \quad U_n\subset U_{n+1}, \quad U_n\text{-quasi-compact}$$

$$C(X):\quad 0\to H^r_{\text{proét},r}(X_C)\to H^r(F^r)(X_C)\oplus \mathrm{HK}^r_r(X_C)\to \mathrm{DR}^r(X_C)\to 0$$

Have $C(X)=\varprojlim_n C(U_n)$: use Mittag-Leffler in BC category to control $\mathrm{R}^1\varprojlim_n$.

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Step 3: Assume X quasi-compact
Lemma Main Theorem is equivalent to the following:

1. The pair $(H^r_{\text{HK}}(X_C), H^r_{\text{dR}}(X_C)$, $r \geq 0$, is *acyclic*.
2. $H^r_{\text{proét}}(X_C, \mathbf{Q}_p)$ is effective, i.e., has signature ≥ 0, for all r.
3. For all r,

$$\text{ht}(H^r_{\text{proét}}(X_C, \mathbf{Q}_p)) = \dim_K H^r_{\text{dR}}(X).$$

Acyclicity and signature

An (M, M_K)- filtered (φ, N)-module is called **acyclic** if (equivalently):

- the associated vector bundle $\mathscr{E}$ on X_{FF} is acyclic, i.e., $H^1(X_{\mathrm{FF}}, \mathscr{E}) = 0$
- $\mathscr{E}$ has HN slopes ≥ 0
- $(M \otimes \mathbf{B}_{\mathrm{st}})^{\varphi=1, N=0} \to (M \otimes \mathbf{B}_{\mathrm{dR}})/F^0$ is surjective

Remark If (M, M_K) is a weakly admissible filtered (φ, N)-module then it is acyclic: all Harder-Narasimhan slopes of $\mathscr{E}$ are 0.

Acyclicity and signature

An (M, M_K)- filtered (φ, N)-module is called **acyclic** if (equivalently):

- the associated vector bundle $\mathscr{E}$ on X_{FF} is acyclic, i.e., $H^1(X_{\mathrm{FF}}, \mathscr{E}) = 0$
- $\mathscr{E}$ has HN slopes ≥ 0
- $(M \otimes \mathbf{B}_{\mathrm{st}})^{\varphi=1, N=0} \to (M \otimes \mathbf{B}_{\mathrm{dR}})/F^0$ is surjective

Remark If (M, M_K) is a weakly admissible filtered (φ, N)-module then it is acyclic: all Harder-Narasimhan slopes of $\mathscr{E}$ are 0.

(1) **Signature** BC $\mathbb{W}$ has signature

- < 0 if $\mathrm{Hom}(\mathbb{W}, \mathbb{V}^1) = 0$; $\Leftarrow H^1(X_{FF}, \mathscr{E})$, $\mathscr{E}$ a vector bundle
- $= 0$ if it is affine, i.e., it is a successive extension of $\mathbb{V}^1$; think $H^0(X_{FF}, \mathscr{F})$, $\mathscr{F}$ coherent sheaf, supported at ∞, torsion
- > 0 if it injects into $\mathbf{B}_{\mathrm{dR}}^d$; think $H^0(X_{FF}, \mathscr{E})$.

Remark (1) signature ≥ 0 if $\mathbb{W} \hookrightarrow \mathbf{B}_{\mathrm{dR}}^{+}$ $-$ *module*
(2) signature ≤ 0 if $\mathrm{Hom}(\mathbb{W}, \mathbf{B}_{\mathrm{dR}}^{+}) = 0$

Example (i) $\mathbb{V}^1$ signature 0 and height 0
(ii) $\mathbb{V}^1/\mathbf{Q}_p$ signature < 0 and height $-1 < 0$
(iii)

- $\mathbb{U} = (\mathbf{B}_{\mathrm{cr}}^{+})^{\varphi=p}$ signature > 0 and height 1;
- $\mathbb{U}/\mathbf{Q}_p t$ signature 0 and height 0
- if $x \in \mathbb{U}(C) \setminus \mathbf{Q}_p t$ then $\mathbb{U}/\mathbf{Q}_p x$ signature < 0 and height 0

Proof of the main theorem

We will prove claim (3) of the lemma: X quasi-compact over K. For all r,

$$\mathrm{ht}(H^r_{\text{proét}}(X_C, \mathbf{Q}_p)) = \dim_K H^r_{\mathrm{dR}}(X).$$

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(ii) We will show that it is true for a union of two affinoids (the general case is similar). So, assume that U_1, U_2 are affinoids, let

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Note that

$$\mathrm{ht}(\mathrm{HK}^r_r) = \dim_K H^r_{\mathrm{dR}}(X)$$

$\Rightarrow$ it suffices to show that

$$\mathrm{ht}(H^r_{\text{proét},r}) = \mathrm{ht}(\mathrm{HK}^r_r).$$

(iii) Consider the map

$$g : H^r_{\text{proét},r} \to \mathrm{HK}^r_r$$

and let us pretend that

$$\mathrm{ht} : BC \text{ spaces } \to \text{an abelian category}$$

that is exact.
Show that

$$\mathrm{ht}(g) : \mathrm{ht}(H^r_{\text{proét},r}) \to \mathrm{ht}(\mathrm{HK}^r_r)$$

is an isomorphism.
It is clear what to do: Mayer-Vietoris yields the following map of exact sequences

$$\begin{array}{ccccccccc}
\mathrm{ht}^{r-1}_{\text{ét}}(U_1 \oplus U_2) & \to & \mathrm{ht}^{r-1}_{\text{ét}}(U_{12}) & \to & \mathrm{ht}^{r}_{\text{ét}}(U) & \to & \mathrm{ht}^{r}_{\text{ét}}(U_1 \oplus U_2) & \to & \mathrm{ht}^{r}_{\text{ét}}(U_{12}) \\
\wr\downarrow g & & \wr\downarrow g & & \downarrow g & & \wr\downarrow g & & \wr\downarrow g \\
\mathrm{ht}^{r-1}_{\mathrm{HK}}(U_1 \oplus U_2) & \to & \mathrm{ht}^{r-1}_{\mathrm{HK}}(U_{12}) & \to & \mathrm{ht}^{r}_{\mathrm{HK}}(U) & \to & \mathrm{ht}^{r}_{\mathrm{HK}}(U_1 \oplus U_2) & \to & \mathrm{ht}^{r}_{\mathrm{HK}}(U_{12})
\end{array}$$

Use five lemma.

(iv) But ht does not have these properties so we consider a partial

Categorification of height

Consider $h : BC \to C(\mathbf{B}_{\mathrm{dR}} - \mathit{modules})$,

$$\mathbb{W} \mapsto \mathrm{Hom}(\mathbb{W}, \mathbf{B}_{\mathrm{dR}}).$$

Facts:

(1) if $\mathbb{W}$ is effective then

$$\mathrm{rk}(h(\mathbb{W})) = \mathrm{ht}(\mathbb{W});$$

in general

$$\mathrm{rk}(h(\mathbb{W})) = \mathrm{ht}(W) + \mathrm{rk}(\mathrm{Ext}(\mathbb{W}, \mathbf{B}_{\mathrm{dR}})).$$

(2) h is an exact functor on effective BC's.

(v) It suffices to show that everything in sight is effective:

- we know it for all the affinoids by the inductive hypothesis
- it is clear for $\mathrm{HK}_r^r(U)$
- for $H^r_{\text{proét}}(U_C)$ we argue by induction on r using the fact

$$\text{acyclicity of } (H^{r-1}_{\mathrm{HK}}(X), H^{r-1}_{\mathrm{dR}}(X_K)) \Rightarrow \text{ effectiveness of } H^r_{\text{proét}}(U_C)$$
$$\Rightarrow \text{acyclicity of } (H^r_{\mathrm{HK}}(X), H^r_{\mathrm{dR}}(X_K)).$$

Thank you !