A DESCENT VIEW ON MITCHELL'S THEOREM

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Abstract. In this short note, we give a new proof of Mitchell's theorem that $\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}(\mathbf{Z}) \simeq 0$ for $n \geqslant 2$. Instead of reducing the problem to delicate representation theory, we use recently established hyperdescent technology for chromatically-localized algebraic K-theory.

1. Introduction and background

In this note, we give an alternate proof of the following result:

Theorem 1.0.1 (Mitchell). *For all primes p and $n \geqslant 2$,* $\mathrm{K}(n)_*\mathrm{K}(\mathbf{Z}) = 0$.

The proof Theorem 1.0.1 given in [Mit87] is relatively self-contained and depends on showing that the unit map $\mathbf{1} \to \mathrm{K}(\mathbf{Z})$ factors through the "image of J." This factoring relies on delicate representation theory by way of the Barratt-Priddy-Quillen theorem. Since the latter spectrum is known to be acyclic for the Morava's $\mathrm{K}(n)$ for $n \geqslant 2$, the result follows. The value of the present note is that it locates the proof in its natural environment — Rognes' redshift philosophy in algebraic K-theory.

The starting point of Theorem 1.0.1 is Thomason's seminal result that K(1)-localized algebraic K-theory satisfies étale descent [Tho85]. Combined with the rigidity theorems of Suslin [Sus83] and Gabber [Gab92], one concludes that K(1)-local algebraic K-theory is, more or less, topological K-theory; we also refer the reader to [DFST82] for further elaboration on this point of view.

One can view Thomason's theorem as a "Bott-asymptotic" version of a more refined statement — the Quillen-Lichtenbaum conjecture (now the Voevodsky-Rost theorem [Voe03, Voe11]) which asserts that algebraic and étale K-theory agrees in high enough degrees. By analogy with the Quillen-Lichtenbaum conjectures, Rognes has formulated the idea that taking algebraic K-theory increases "chromatic complexity" — demonstrating a "redshift"; we refer the reader to [Rog14] for a discussion.

In line with this ideology, Thomason's theorem is then viewed as saying that taking algebraic K-theory of a discrete commutative ring (which is K(1)-acyclic) yields an interesting answer K(1)-locally. At the next height, results of Ausoni-Rognes [Aus10, AR02] who confirmed that v_2 acts invertibly on $\mathrm{K}(\mathrm{K}(\mathbf{C})^\wedge_p) \otimes \mathrm{V}(1)$ where $\mathrm{V}(1)$ is a type 2 complex, hence is interesting K(2)-locally.

We can view Mitchell's result anachronistically as a demonstration of the strictness of redshift: while the 2-fold algebraic K-theory of a discrete ring is interesting K(2)-locally, the algebraic K-theory thereof itself is not.

The value of our proof, if there is one, is that it is born in the same spirit as Thomason's results: we confirm Mitchell's vanishing by way of étale hyperdescent. We believe that proving the result this way places it within its proper context, at the cost of using more technology.

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2. On Mitchell's theorem

2.1. **A p-adic version of Mitchell's theorem.** As a warm-up, we first give a very simple proof of the p-adic version of Mitchell's theorem. So fix a prime $p > 0$; here our $\mathrm{T}(n)$-localizations will be at this implicit prime.

Theorem 2.1.1. *For $n \geqslant 2$, we have that $\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}(\mathbf{Z}_p) \simeq 0$.*

To begin, let C be the completion of the algebraic closure of $\mathbf{Q}_p$ and $\mathcal{O}_{\mathrm{C}}$ be its ring of integers which is an integral perfectoid ring.

Lemma 2.1.2. *For $n \geqslant 2$, we have that $\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}(\mathcal{O}_{\mathrm{C}}) \simeq 0$.*

Proof. Consider the zig-zag of maps

$$ku \to \mathrm{K}(\mathrm{C}) \xleftarrow{j^*} \mathrm{K}(\mathcal{O}_{\mathrm{C}}).$$

The maps above are all p-adic equivalences:

(1) for j^* this is [HN19, Lemma 1.3.7],
(2) for the unlabeled arrow, we have Suslin rigidity [Sus84].

Hence we conclude that $\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}(\mathcal{O}_{\mathrm{C}}) \simeq 0$. □

Remark 2.1.3. Note that [HN19, Lemma 1.3.7] only uses very basic facts about perfectoid rings (that we can choose a pseudouniformizer π such that $\mathcal{O}_{\mathrm{C}}/\pi \simeq \mathcal{O}_{\mathrm{C}^\flat}/\pi^\flat$) and the fact that the positive homotopy groups of the K-theory of local perfect $\mathbf{F}_p$-algebras are all p-divisible by [Hil81, Kra80].

Proof of Theorem 2.1.1. Since K-theory is a finitary invariant[1], we can write

$$\mathrm{K}(\mathcal{O}_{\mathrm{C}}; \mathbf{Z}_p) \simeq \operatorname*{colim}_{\mathbf{Q}_p \subset \mathrm{E} \subset \mathrm{C}} \mathrm{K}(\mathcal{O}_{\mathrm{E}}; \mathbf{Z}_p),$$

a colimit of $\mathbf{E}_\infty$-rings, and the colimit ranges along finite extensions of $\mathbf{Q}_p$ contained in C. Now, the source vanishes after applying $\mathrm{L}_{\mathrm{T}(n)}$, whence so is the target. Since $\mathrm{L}_{\mathrm{T}(n)}$ commutes with filtered colimits, we may find a finite extension E of $\mathbf{Q}_p$ such that $\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}(\mathcal{O}_{\mathrm{E}}) \simeq 0$ since the colimit in sight is computed in $\mathbf{E}_\infty$-ring; indeed, vanishing is equivalent to the unit being nullhomotopic. Since the morphism of rings $\mathbf{Z}_p \to \mathcal{O}_{\mathrm{E}}$ is finite flat, by the descent results of [CMNN20] we get that

$$\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}(\mathbf{Z}_p) \simeq \mathrm{Tot}\left(\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}\left(\mathcal{O}_{\mathrm{E}}^{\otimes_{\mathbf{Z}_p} \bullet+1}\right)\right).$$

We are now done, since all terms of the limit on the right hand side are modules over $\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}(\mathcal{O}_{\mathrm{E}}) = 0$. □

To conclude note that if A is $\mathrm{T}(n)$-acyclic, then it is also $\mathrm{K}(n)$-acyclic. Though we will not need it, we can also reverse the implication if A is, furthermore, an $\mathbf{E}_\infty$-ring spectrum by [LMT20, Lemma 2.3].

Corollary 2.1.4. *If $n \geqslant 2$, we have that $\mathrm{L}_{\mathrm{K}(n)}\mathrm{K}(\mathbf{Z}_p) \simeq 0$. In particular $\mathrm{K}(n)_*\mathrm{K}(\mathbf{Z}_p) = 0$.*

[1] In more details: $\mathcal{O}_{\mathrm{C}}$ is p-adically isomorphic to the colimit of the $\mathcal{O}_{\mathrm{E}}$'s and K-theory preserves p-adic equivalences in this setting. This follows from, for example, the fact that given a morphism of rings $\mathrm{A} \to \mathrm{B}$, $\mathrm{fib}(\mathrm{GL}(\mathrm{A}) \to \mathrm{GL}(\mathrm{B})) \simeq \mathrm{fib}(\mathrm{M}(\mathrm{A}) \to \mathrm{M}(\mathrm{B}))$ and the formation of matrix rings evidently preserves local equivalences.

2.2. **Mitchell's theorem.** We now give a proof of Mitchell's theorem. We phrase this as.

Theorem 2.2.1. *For all primes* p *and* $n \geqslant 2$, $\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}(\mathbf{Z}) \simeq 0$. *Equivalently,* $\mathrm{L}_{\mathrm{K}(n)}\mathrm{K}(\mathbf{Z}) \simeq 0$ *and, in particular,* $\mathrm{K}(n)_*\mathrm{K}(\mathbf{Z}) = 0$.

Proof. The "equivalently" part follows from [LMT20, Lemma 2.3]. We first claim that that we can work with rings which are (p)-local. To see this, we claim that the map

$$\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}\left(\mathbf{Z}\right) \to \mathrm{L}_{\mathrm{T}(n)}\mathrm{K}\left(\mathbf{Z}\left[\frac{1}{p}\right]\right),$$

is an equivalence. Indeed, by localization and dévissage [Qui10] we have a cofiber sequence

$$\mathrm{K}\left(\mathbf{F}_p\right) \to \mathrm{K}\left(\mathbf{Z}\right) \to \mathrm{K}\left(\mathbf{Z}\left[\frac{1}{p}\right]\right),$$

and $\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}(\mathbf{F}_p) \simeq \mathrm{L}_{\mathrm{T}(n)}\mathrm{H}\mathbf{Z}_p \simeq 0$ where the first equivalence is [Qui72] and the second equivalence follows since $n \geqslant 2$. Therefore, it suffices to show that $\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}\left(\mathbf{Z}\left[\frac{1}{p}\right]\right) \simeq 0$ which follows from the next two claims.

□

Claim 2.2.2. *For any* $n \geqslant 1$*, the presheaf,*

$$\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}(-) : \mathrm{Et}^{\mathrm{op}}_{\mathbf{Z}[\frac{1}{p}]} \to \mathrm{Spt}$$

is a hypercomplete sheaf of spectra.

Proof. While this is an immediate application of [CM19, Theorem 7.14], we will give a more detailed proof here to highlight the ingredients. Since telescopic localization is invariant under taking connective covers ([LMT20, Lemma 2.3(iii)]) we obtain a map of presheaves of $\mathbf{E}_\infty$-rings:

$$\mathrm{K}^{\mathrm{cn}}\left(-;\mathbf{Z}_p\right) \to \mathrm{L}_{\mathrm{T}(n)}\mathrm{K}^{\mathrm{cn}}\left(-;\mathbf{Z}_p\right) \cong \mathrm{L}_{\mathrm{T}(n)}\mathrm{K}\left(-\right)$$

Moreover $\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}(-)$ is an étale sheaf, thus this map factors through a canonical $\mathbf{E}_\infty$-map

$$\mathrm{K}^{\mathrm{cn}}\left(-;\mathbf{Z}_p\right)^{\text{ét}} \to \mathrm{L}_{\mathrm{T}(n)}\mathrm{K}\left(-\right)$$

Since hypercompletion is smashing by [CM19, Corollary 4.39], it suffices to then prove that $\mathrm{K}^{\mathrm{cn}}\left(-;\mathbf{Z}_p\right)^{\text{ét}}$ is a hypersheaf on $\mathrm{Et}_{\mathbf{Z}[\frac{1}{p}]}$.

As is proved by Thomason in [Tho85, Theorem 4.1] for odd primes (which relies on the Suslin-Merkerjuev theorem [MS82]) and Rosenschon and Østvær [ROsr05] for the prime 2 (which does rely on the Milnor conjecture [Voe03]), $\mathrm{L}_{\mathrm{T}(1)}\mathrm{K}$ does satisfy étale hyperdescent. We consider the canonical map $\mathrm{K}^{\mathrm{cn}}\left(-;\mathbf{Z}_p\right)^{\text{ét}} \to \mathrm{L}_{\mathrm{T}(1)}\mathrm{K}$ and $\mathcal{F}$ be the fiber. The claim then follows if one can show that the fiber $\mathcal{F}$ has étale hyperdescent.

Let $\mathcal{G}$ denote the fiber of the map $\mathrm{K}^{\mathrm{cn}}(-;\mathbf{Z}_p) \to \mathrm{L}_{\mathrm{T}(1)}\mathrm{K}$. By the full strength of the Bloch-Kato conjectures [Voe11, Voe03] (see [CM19, Theorem 6.18], noting that $\operatorname{Spec}\mathbf{Z}$ admits the desired global bound by [Ser02, I.3.2]) we see that the fiber is truncated. Therefore the sheafification, $\mathcal{F} \simeq \mathcal{G}^{\text{ét}}$ is Postnikov complete, whence is indeed hypercomplete as desired.

□

Claim 2.2.3. *For* $n \geqslant 2$ *and for all strictly hensel local ring* R *with residue field* κ *of characteristic* $\ell > 0$ *and* $(p, \ell) = 1$

$$\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}(\mathrm{R}) \simeq 0$$

Indeed, since strictly henselian local rings are the points in the étale topology and vanishing of étale hypersheaves are detected on points [Lur09, Remark 6.5.4.7], the claim implies our result.

Proof of Claim 2.2.3. By Gabber rigidity [Gab92], the map $\mathrm{K}^{\mathrm{cn}}(\mathrm{R}) \to \mathrm{K}^{\mathrm{cn}}(\kappa)$ is a p-adic equivalence, while by Suslin rigidity [Sus84] we have a further p-adic equivalence $\mathrm{K}^{\mathrm{cn}}(\kappa) \leftarrow \mathrm{ku}$. Since telescopic localization for $n \geqslant 1$ is invariant under taking connective covers by [LMT20, Lemma 2.3(iii)] we conclude:

$$\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}\,(\mathrm{R}) \simeq \mathrm{L}_{\mathrm{T}(n)}\mathrm{K}^{\mathrm{cn}}\,(\mathrm{R}) \simeq \mathrm{L}_{\mathrm{T}(n)}\mathrm{ku} \simeq 0.$$

□

Remark 2.2.4. Because of the appeal to [CM19, Theorem 7.14], our proof is not "simple", in contrast to the p-adic situation. This is because the Clausen-Mathew theorem depends on the resolution of the Quillen-Lichtenbaum conjectures. Specifically, the version of [CM19, Theorem 7.14] that we need, requires [CM19, Theorem 6.13] which ultimately proves that the restriction of $\mathrm{K}^{\text{ét}}$ to $\mathrm{Et}_{\mathbf{Z}[1/p]}$ is a étale hypersheaf. This latter result, in turn, relies on Rost-Voevodsky's resolution of the Bloch-Kato conjectures. Note that, in contrast to the p-adic situation, Theorem 2.2.1 asserts something global which prompts us to argue via stalks, whence an appeal to hyperdescent.

Remark 2.2.5 (Personal communication by J. Rognes)**.** Morally speaking, our proof is a cleaner packaging of the following method to prove Mitchell's theorem using the Rost-Voevodsky results. For simplicity let p be an odd prime and let $\mathrm{V}(1) := \mathbf{1}/(p, v_1)$ be a finite complex which admits a v_2-self map. Our goal is to prove that $\mathrm{K}(\mathbf{Z}) \otimes \mathrm{V}(1)$ is a bounded complex which suffices to prove Mitchell's theorem since inverting a positive degree self-map on a bounded complex annihilates it. Using the localization sequence, we reduce to proving the following assertions:

(1) if $\ell \neq p$, then v_1 acts invertibly on $\mathrm{K}_*(\mathbf{F}_\ell)/p$ for $*$ large enough,
(2) v_1 acts invertibly on $\mathrm{K}_*(\mathbf{Q})/p$ for $*$ large enough, and
(3) if $\ell = p$, then $\mathrm{K}_*(\mathbf{F}_p)/p$ is bounded above.

The last statement follows from Quillen's computation [Qui72] and the first statement follows by a further application of Suslin rigidity [Sus83]. The second statements is where Rost-Voevodsky's resolution of the Bloch-Kato conjectures is needed [Voe03, Voe11] (this is where the "overkill happens"), the motivic spectral sequence [FS02] and the fact that v_1 is detected by a periodicity operator on étale cohomology.

As a consequence of the main theorem we can easily obtain two corollaries:

Corollary 2.2.6. *The functor* $\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}|_{\mathrm{Cat}^{\mathrm{perf}}_{\mathbf{Z}}}$ *is zero for* $n \geqslant 2$*.*

Corollary 2.2.7. *For* $n \geqslant 2$*,* $\mathrm{L}_{\mathrm{T}(n)}\mathrm{TC}\,(\mathbf{Z}) \simeq 0$*. Consequently,* $\mathrm{L}_{\mathrm{T}(n)}\mathrm{TC}|_{\mathrm{Cat}^{\mathrm{perf}}_{\mathbf{Z}}}$ *is zero for* $n \geqslant 2$*.*

Proof. Via the trace map, $\mathrm{L}_{\mathrm{T}(n)}\mathrm{TC}(\mathbf{Z})$ is an $\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}\,(\mathbf{Z})$-$\mathbf{E}_\infty$-algebra, whence the result follows from Theorem 2.2.1. □

References

[AR02] C. Ausoni and J. Rognes, *Algebraic* K*-theory of topological* K*-theory*, Acta Math. **188** (2002), no. 1, pp. 1–39, https://doi.org/10.1007/BF02392794

[Aus10] C. Ausoni, *On the algebraic* K*-theory of the complex* K*-theory spectrum*, Invent. Math. **180** (2010), no. 3, pp. 611–668, https://doi.org/10.1007/s00222-010-0239-x

[CM19] D. Clausen and A. Mathew, *Hyperdescent and étale K-theory*, 2019, arXiv:1905.06611

[CMNN20] D. Clausen, A. Mathew, N. Naumann, and J. Noel, *Descent in algebraic* K*-theory and a conjecture of Ausoni-Rognes*, J. Eur. Math. Soc. (JEMS) **22** (2020), no. 4, pp. 1149–1200, https://doi-org.ezp-prod1.hul.harvard.edu/10.4171/JEMS/942

[DFST82] W. G. Dwyer, E. M. Friedlander, V. P. Snaith, and R. W. Thomason, *Algebraic* K*-theory eventually surjects onto topological* K*-theory*, Invent. Math. **66** (1982), no. 3, pp. 481–491, http://dx.doi.org/10.1007/BF01389225

[FS02] E. M. Friedlander and A. Suslin, *The spectral sequence relating algebraic* K*-theory to motivic cohomology*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 6, pp. 773–875, https://doi.org/10.1016/S0012-9593(02)01109-6

[Gab92] O. Gabber, K-*theory of Henselian local rings and Henselian pairs*, Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989), Contemp. Math., vol. 126, Amer. Math. Soc., Providence, RI, 1992, pp. 59–70, https://doi-org.ezp-prod1.hul.harvard.edu/10.1090/conm/126/00509

[Hil81] H. L. Hiller, *λ-rings and algebraic* K*-theory*, J. Pure Appl. Algebra **20** (1981), no. 3, pp. 241–266, https://doi.org/10.1016/0022-4049(81)90062-1

[HN19] L. Hesselholt and T. Nikolaus, *Topological cyclic homology*, Handbook of homotopy theory (Haynes Miller, ed.), CRC Press/Chapman and Hall, 2019

[Kra80] C. Kratzer, *λ-structure en* K*-théorie algébrique*, Comment. Math. Helv. **55** (1980), no. 2, pp. 233–254, https://doi.org/10.1007/BF02566684

[LMT20] M. Land, L. Meier, and G. Tamme, *Vanishing results for chromatic localizations of algebraic* K*-theory*, 2020, arXiv:2001.10425

[Lur09] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009

[Mit87] S. A. Mitchell, *The Morava* K*-theory of algebraic* K*-theory spectra*, K-Theory **1** (1987), no. 2, pp. 197–205, https://doi.org/10.1007/BF00533419

[MS82] A. S. Merkurjev and A. A. Suslin, K*-cohomology of Severi-Brauer varieties and the norm residue homomorphism*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), no. 5, pp. 1011–1046, 1135–1136

[Qui72] D. Quillen, *On the cohomology and* K*-theory of the general linear groups over a finite field*, Ann. of Math. (2) **96** (1972), pp. 552–586, https://doi.org/10.2307/1970825

[Qui10] ———, *Higher algebraic* K*-theory: I*, pp. 413–478

[Rog14] J. Rognes, *Chromatic redshift*, 2014, arXiv:1403.4838

[ROsr05] A. Rosenschon and P. A. Ø stvær, *The homotopy limit problem for two-primary algebraic* K*-theory*, Topology **44** (2005), no. 6, pp. 1159–1179, https://doi.org/10.1016/j.top.2005.04.004

[Ser02] J.-P. Serre, *Galois cohomology*, Springer-Verlag, 2002

[Sus83] A. Suslin, *On the* K*-theory of algebraically closed fields*, Invent. Math. **73** (1983), no. 2, pp. 241–245, https://doi-org.ezp-prod1.hul.harvard.edu/10.1007/BF01394024

[Sus84] A. A. Suslin, *On the* K*-theory of local fields*, Proceedings of the Luminy conference on algebraic K-theory (Luminy, 1983), vol. 34, 1984, pp. 301–318, https://doi-org.ezp-prod1.hul.harvard.edu/10.1016/0022-4049(84)90043-4

[Tho85] R. W. Thomason, *Algebraic* K*-theory and étale cohomology*, Ann. Sci. École Norm. Sup. (4) **18** (1985), no. 3, pp. 437–552, http://www.numdam.org/item?id=ASENS_1985_4_18_3_437_0

[Voe03] V. Voevodsky, *Motivic cohomology with* $\mathbf{Z}/2$*-coefficients*, Publ. Math. I.H.É.S. **98** (2003), no. 1, pp. 59–104

[Voe11] ———, *On motivic cohomology with* $\mathbf{Z}/l$*-coefficients*, Ann. Math. **174** (2011), no. 1, pp. 401–438, preprint arXiv:0805.4430

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