MATH 232: ALGEBRAIC GEOMETRY I

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CONTENTS

1. References 2
   1.1. Books 2
   1.2. Lecture Notes 2
   1.3. Online textbooks 2
2. Lecture 1: What is algebraic geometry? 2
   2.1. Algebraic geometry beyond algebraic geometry 4
   2.2. Exercises 1: categorical preliminaries 5
3. Lecture 2: Prestacks 7
   3.1. Operation on prestacks I: fibered products 9
   3.2. Closed immersions 9
   3.3. Exercises 2 10
4. Lecture 3: Descent 12
   4.1. Unpacking the descent condition and Serre’s lemma 13
   4.2. Exercises 3 15
5. Lecture 4: Onto schemes 16
   5.1. Diversion: multiplicative groups and graded rings 16
   5.2. Complementation and open subfunctors 19
   5.3. Exercises 4 20
6. Lecture 5: Schemes, actually 21
   6.1. Quasi-affine prestacks are Zariski stacks 21
   6.2. General open covers 21
   6.3. Quasicompactness of affine schemes 22
   6.4. The definition of a scheme 24
   6.5. Exercises 5 24
7. Lecture 6: Universality of descent and closure of schemes under pullbacks 26
   7.1. Universality of descent and dèvissage 27
   7.2. Exercises 6 29
8. Lecture 7: Relative algebraic geometry and quasicoherent sheaves 30
   8.1. Relative algebraic geometry 30
   8.2. Linear algebra over schemes 32
   8.3. A word on why quasicoherent sheaves? 35
   8.4. Exercises 35
9. Lecture 8: vector bundles, affine morphisms 35
10. Lecture 9: projective space aka the most important scheme you should know 35
References 35
In our acquisition of knowledge of the Universe (whether mathematical or otherwise) that which renovates the quest is nothing more nor less than complete innocence.” - A. Grothendieck.

1. References

1.1. Books. In principle, this class is about Grothendieck’s [EGA1] which signals the birth of modern algebraic geometry. It is an extremely technical document on its own and shows one of the many ways mathematics was developed organically. The French is not too hard and I recommend that you look through the book before the start of class — I might also assign readings from here occasionally with the promise that the French (and some Google translate) will not hinder your mathematical understanding.

Here are some textbooks in algebraic geometry.

([Sha13]) Shafarevich’s book is a little more old school than the others in this list, but is valuable in the examples it gives.

([Har77]) Hartshorne’s book has long been the “gold-standard” for algebraic geometry textbook. I learned the subject from this book first. It is terse and has plenty of good exercises and problems. However, the point of view that this book takes will be substantially different from one we will take in this class, though I will most definitely steal problems from here.

([GW10]) This is essentially a translation of Grothendieck’s EGA (plus more) and is closer to the point of view of this class. Just like the original text, it is relentlessly general and very lucid in its exposition.

([Vak]) Arguably the most inviting book in this list, and modern in its outlook.

([DG80]) As far as I know this is still the only textbook reference to the functor-of-points point of view to algebraic geometry.

1.2. Lecture Notes. There are also several class notes online in algebraic geometry. I will add on to this list as the class progresses.

([Ras]) This is the closest document to our approach to this class. In fact, I will often present directly from these notes.

([Gat]) This is a “varieties” class, so the approach is very different, but I find it very helpful for lots of examples.

1.3. Online textbooks. There has been an explosion of online textbooks for algebraic geometry recently, though they are perhaps they are more like ”encyclopedias."

([Stacks]) Johan de Jong at Columbia was the trailblazer in this industry and most, if not all, facts about algebraic geometry that will be taught will appear here, with proofs.

([cri]) Similar but for commutative algebra. Much more incomplete.

([fpp]) A translation project for EGA.

2. Lecture 1: What is algebraic geometry?

In its essence, algebraic geometry is the study of solutions to polynomial equations. What one means by “polynomial equations,” however, has changed drastically throughout the latter part of the 20th century. To meet the demands in making constructions, ideas and theorems in classical algebraic geometry rigorous has given birth to a slew of techniques and ideas which are applicable to a much, much broader range of mathematical situations.

To begin with, let us recall the famous Fermat problem:

**Theorem 2.0.1** (Taylor-Wiles). Let \( n \geq 3 \), then \( x^n + y^n = 1 \) has no solutions over \( \mathbb{Q} \) when \( x, y \neq 0 \).
This is a problem in algebraic geometry. In the language that we will learn in this class, we will be able to associate a smooth, projective scheme $\widetilde{\text{Fer}}_n$ which is, informally, given by a homogeneous polynomial equation $x^n + y^n = z^n$, equipped with a canonical morphism

$$\begin{array}{rcl}
\text{Fer}_n & \rightarrow & \text{Spec } \mathbb{Z},
\end{array}$$

such that its set of sections

$$\begin{array}{rcl}
\text{Fer}_n & \hookrightarrow & \text{Spec } \mathbb{Z},
\end{array}$$

correspond to potential solutions to the Fermat equation. It is in this language that the Fermat problem was eventually solved.

The point-of-view we wish to adopt in this class, however, is one that goes by functor-of-points. In this highly abstract, but more flexible, approach schemes appear as what they are supposed to be which is often easier to think about. For us, the basic definition is:

**Definition 2.0.2.** A presheaf is a functor from the category of commutative rings to sets:

$$\mathcal{F} : \text{CAlg} \rightarrow \text{Set}.$$ 

**Remark 2.0.3.** A note on terminology: this is non-standard. What should be called (and was called by Grothendieck) a prestack is a functor

$$\mathcal{F} : \text{CAlg} \rightarrow \text{Cat},$$

where Cat is the (large, $(2, 1)$-)category of small categories. If we think of a set as a category with no non-trivial morphisms between the objects, then the above definition is a special case of this Grothendieck definition of a prestack. We will not consider functors into categories in this class so we will reserve the term prestack for such a functor above (as opposed to something like “a presheaf in sets”). Perhaps it should be called a presheaf, but a presheaf should really just be an arbitrary functor

$$\mathcal{F} : \mathcal{D}^{op} \rightarrow \text{Set}.$$ 

To make this definition jibe with the Fermat scheme above, let us note that the equation $x^n + y^n = 1$ is defined for any ring. Therefore we can define

$$\widetilde{\text{Fer}}_n(R) = \{(a, b, c) : a^n + b^n = 1\} \subset R^{x^2}.$$ 

The theorem of Taylor and Wiles can then be restated as the fact that

$$\widetilde{\text{Fer}}_n(\mathbb{Z}) = \emptyset \quad n \geq 3;$$

However we caution that this is not the same as the scheme $\text{Fer}_n$ that we have alluded to above since it is not projective — something that we will address in the class.

Another key idea in algebraic geometry is the question of parametrizing solutions of polynomial equations in a reasonable way. Let us consider $\text{Fer}_2$, which is the set of solutions to $x^2 + y^2 = 1$. We have a canonical equality (the first one is more or less the same as the above):

$$\widetilde{\text{Fer}}_2(R) = S^1,$$

as we all know. Here are three other possible answers:

1. $\text{Fer}_2(R) = (\cos \theta, \sin \theta) \quad 0 \leq \theta < 2\pi,$
2. $\overset{\sim}{\text{Fer}}_2(R) = \left(1 - \frac{t^2}{1 + t^2}, \frac{2t}{1 + t^2}\right) \quad t \in \mathbb{R},$
3. $\overset{\sim}{\text{Fer}}_2(R)$ is the set of all triangles with hypotenuse 1 up to congruence.
The first answer does not belong to the realm of (conventional) algebraic geometry which, by its very nature, concerns only polynomial functions. In other words, we dismiss transcendentals like exp and cos, sin. However the language of this class is actually powerful enough to capture transcendentals and reconstruct the familiar theory of differential geometry. The third answer will turn out to belong to the realm of algebraic geometry as well, but that will be reserved for a second course. The second answer does belong to the realm of algebraic geometry that we will study in this class: we can use rational functions of one variable in order to describe \( \text{Fer}_2(\mathbb{R}) \). In fact, this parametrization proves

**Theorem 2.0.4.** A quadric hypersurface in \( \mathbb{P}^2 \) with a rational point is rational. In fact, any quadric hypersurface with a rational point is rational.

The proof of this result is “basically known” to pre-Grothendieck algebraic geometers: we pick the rational point and stereographically project into a hyperplane. Since a quadric means that it is cut out by a degree two polynomial, it must hit one other point. This defines a rational map — one that is defined “almost everywhere” which is evidently an “isomorphism.” One of the major thread of investigation in algebraic geometry and comes under the name of birational geometry and the above result belongs to this area. An example of a beautiful result that belongs to modern birational geometry is:

**Theorem 2.0.5** (Clemens and Griffiths). A nonsingular cubic threefold over \( \mathbb{C} \) is not rational.

Theorem 2.0.5 is a non-existence proof — it says that there is no way to “rationally parametrize” the cubic threefold. If you have been trained in algebraic topology, you will feel like some kind of cohomological methods would be needed. The words that you should look for are “intermediate Jacobians,” an object whose real birthplace is Hodge theory.

One of the major, open problems in the subject is:

**Question 2.0.6.** Is a generic cubic fourfold over \( \mathbb{C} \) rational?

Recently, Katzarkov, Kontsevich and Pantev claimed to have made substantial progress towards this problem, but a write-up is yet to appear. More generally, a central question in algebraic geometry is:

**Question 2.0.7.** How does one classify algebraic varieties up to birational equivalence?

In topology, recall that a topological (closed) surface can be classified by genus or, better, Euler characteristic:

1. If \( \chi(\Sigma) < 0 \), then \( \Sigma \) must be the Riemann sphere,
2. If \( \chi(\Sigma) = 0 \), then \( \Sigma \) must be a torus — in the terminology of this class it is an elliptic curve,
3. Most surfaces have \( \chi(\Sigma) > 0 \) and they are, in some sense, the “generic situation.”

This kind of trichotomy can be extended to higher dimensional varieties (topological surfaces being a 1-dimensional algebraic variety over \( \mathbb{C} \)). The minimal model program seeks to find “preferred” representatives in each class.

### 2.1. Algebraic geometry beyond algebraic geometry

The field of birational geometry is extremely large and remains an active area of research. But classifying algebraic varieties is not the only thing that algebraic geometry is good for. We have seen how it can be used to phrase the Fermat problem and eventually hosts its solution. There are other areas where algebraic geometry has proven to be the optimal “hosts” for problems. One of the most prominent areas is representation theory where the central definition is very simple a group homomorphism

\[ \rho : G \rightarrow \text{GL}(V). \]

If we are interested in representations valued in \( k \)-vector spaces, then the collection of all \( \rho \)-representations form a category called \( \text{Rep}_k(G) \). This category has an algebro-geometric incarnation: it is the category of quasicoherent sheaves over the an algebro-geometric gadget
called a algebraic stack (in this case, denoted by BG) which is a special, more manageable class of prestacks but are slightly more mysterious gadgets than just algebraic varieties. Quasi-coherent sheaves are fancy versions of vector bundles — they include gadgets whose fibers “can jump” although we will study restrictions on how exactly they jump. In any case, the field of geometric representation theory takes as starting point that representation theory is “just” the study of the geometric object BG and brings to bear the tools of algebraic geometry onto representation theory.

We have seen that algebraic geometry hosts number theory through the problem of the existence of rational points on a variety. Another deep problem of number theory that lives within modern algebraic geometry is the Riemann hypothesis. In algebro-geometric terms it can be viewed as a way to assemble solutions of an equation over fields of different characteristics.

Soon we will learn what it means for a morphism of schemes \( f : X \to \text{Spec } \mathbb{Z} \) to be proper and for X to be regular, geometrically connected and dimension d. To this set-up we can associate the Hasse-Weil zeta function:

\[
\zeta_X(s) := \prod_{x \in |X|} (1 - \#(\kappa(x))^{-s})^{-1}.
\]

where:

1. the set \(|X|\) is the set of closed points of X,
2. \( \kappa(x) \) is the residue field of \( x \) which is a finite extension of \( \mathbb{F}_p \) for some prime \( p > 0 \).

This function is expected to be extending to all of the complex numbers (as a meromorphic function). There is a version \( \zeta_X(s) \) which takes into account the “analytic part” of X as well:

**Conjecture 2.1.1** (Generalized Riemann hypothesis). If \( s \in \mathbb{C} \) is a zero of \( \zeta_X(s) \) then:

\[
2\text{Re}(s) = \nu,
\]

where \( \nu \in [0, 2d] \).

One of the more viable approaches to verifying the generalized Riemann hypothesis is via cohomological methods — one would like to find a cohomology theory for schemes to which one can “extract” in a natural way the Hasse-Weil zeta function. One reason why one might expect this is the (also conjectured) functional equation

\[
\zeta_X(s) \sim \zeta_X(\dim(X) - s)
\]

where \( \sim \) indicates “up to some constant.” This is a manifestation of a certain Poincaré duality in this cohomology theory which witnesses a certain symmetry between the cohomology groups and governed by the dimension of X. If X is concentrated at a single prime, then the Riemann hypothesis was proved by Deligne using étale cohomology. Recent work of Hesselholt, Bhatt, Morrow and Scholze have made some breakthrough towards setting up this cohomology theory but the Riemann hypothesis is, to the instructor’s knowledge, still out of reach.

2.2. Exercises 1: categorical preliminaries. Here is a standard definition. We assume that every category in sight is locally small so that \( \text{Hom}(x, y) \) is a set, while the set of objects, \( \text{Obj}(\mathcal{C}) \), is not necessarily a set (so only a proper class).

**Definition 2.2.1.** A functor \( F : \mathcal{C} \to \mathcal{D} \) is fully faithful if for all \( x, y \in \mathcal{C} \), the canonical map

\[
\text{Hom}(x, y) \to \text{Hom}(Fx, Fy)
\]

is an isomorphism. We say that it is conservative if it reflects isomorphisms: an arrow \( f : c \to c' \) in \( \mathcal{C} \) is an isomorphism if and only if \( F(f) : F(c) \to F(c') \) is.

**Exercise 2.2.2.** Let \( F : \mathcal{C} \rightleftarrows \mathcal{D} : G \) be an adjunction (in these notes we always write the left adjoint on the left). Prove

1. \( F \) preserves all colimits,
2. \( G \) preserves all limits,
(3) The functor $F$ is fully faithful if and only if the unit transformation
$$\text{id} \to G \circ F$$
is an isomorphism.
(4) $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if $F$ is fully faithful and $G$ is conservative.

**Exercise 2.2.3.** Prove that $\mathcal{C}$ admits all colimits if and only if it admits coproducts and co-equalizers. What kind of colimits do the following categories have (you do not have to justify your answer):

1. the category of finite sets,
2. the category of sets,
3. the category of finitely generated free abelian groups,
4. the category of abelian groups,
5. the category of finite dimensional vector spaces,
6. the category of all vector spaces,
7. the category of finitely generated free modules over a commutative ring $R$,
8. the category of finitely generated projective modules over a ring $R$,
9. the category of all projective modules over a ring $R$.

**Exercise 2.2.4.** Give a very short proof (no more than one line) of the dual assertion: $\mathcal{C}$ admits all limits if and only if it admits products and equalizers.

**Exercise 2.2.5.** Prove that the limit over the empty diagram gives terminal object, while the colimit over the empty diagram gives the initial object.

**Exercise 2.2.6.** For any small category $\mathcal{C}$, we can form the presheaf category
$$\text{PSh}(\mathcal{C}) := \text{Fun}(\mathcal{C}^{\text{op}}, \text{Set}).$$
Prove:

1. If
$$F : I \to \text{PSh}(\mathcal{C}) \quad i \mapsto F_i$$
is a functor and $I$ is a small diagram, then for any $c \in \mathcal{C}$ the canonical map
$$(\text{colim} F_i)(c) \to \text{colim}(F_i(c))$$
is an isomorphism.
2. Formulate and prove a similar statement for limits.
3. Conclude that $\text{PSh}(\mathcal{C})$ admits all limits and colimits.

**Exercise 2.2.7.** Prove the Yoneda lemma in the following form: the functor
$$y : \mathcal{C} \to \text{PSh}(\mathcal{C}) \quad c \mapsto y(c)(x) = \text{Hom}(x, c).$$
is fully faithful. Any functor in the image of $y$ is called **representable**

**Exercise 2.2.8.** Prove that any functor $F \in \text{PSh}(\mathcal{C})$ is a colimit of representable functors. This entails constructing a natural transformation
$$\text{colim} \ y(c) \to F$$
where the domain is a colimit of a diagram of functors where each functor is representable, and proving that this natural transformation is an isomorphism when evaluated at each object of $\mathcal{C}$.

**Exercise 2.2.9.** We say that a category $\mathcal{C}$ is **essentially small** if it is equivalent to a small category. Let $R$ be a commutative ring and consider $\text{CAlg}_R$ to be the category of commutative $R$-algebras. We say that an $R$-algebra $S$ is **finite type** if it admits an $R$-linear surjective ring homomorphism
$$R[x_1, \cdots, x_n] \to S.$$Consider the full subcategory $\text{CAlg}_R^{\text{ft}} \subset \text{CAlg}_R$ of finite type $R$-algebras. Prove that:
Exercise 2.2.10. We define the subcategory of left exact functors

\[ PSh_{\text{lex}}(C) \subset PSh(C) \]

to be the subcategory of those functors which preserves finite limits. These are functors \( F \) such that for any finite diagram\(^1\) \( \alpha : I \to C \), the canonical map

\[ F(\text{colim}_I \alpha) \to \text{lim}_I F(\alpha) \]

is an isomorphism. Prove:

(1) a category \( C \) admits all finite limits if and only if it admits final objects and pullbacks;
(2) for a functor \( F \) to be left exact, it is necessary and sufficient that \( F \) preserves final objects and pullbacks.
(3) Prove that the yoneda functor factors as \( y : C \to PSh_{\text{lex}}(C) \).
(4) If \( f : C \to D \) is a functor, we define

\[ f^* : PSh(D) \to PSh(C) \quad f^* F = F \circ f. \]

Prove that if \( f \) preserves finite colimits, then we have an induced functor

\[ f^* : PSh_{\text{lex}}(D) \to PSh_{\text{lex}}(C). \]

In the third problem set, we will use this to prove the adjoint functor theorem and construct the sheafification functor.

Exercise 2.2.11. We say that \( C \) is locally presentable if there exists a subcategory \( i : C^c \subset C \)
(called the category of compact objects) which is essentially small and is closed under finite colimits such that the functor

\[ C \to PSh_{\text{lex}}(C) \overset{i^*}{\to} PSh_{\text{lex}}(C^c) \]

is an equivalence of categories. Prove

(1) the category Sets is locally presentable with Sets\(^c\) being the subcategory of finite sets,
(2) the category Vect\(_k\) is locally presentable with Vect\(_{k}^c\) being the subcategory of finite-dimensional vector spaces.

3. Lecture 2: Prestacks

Throughout the course we will denote by CAlg the category of commutative rings.

Definition 3.0.1. A prestack is a functor

\[ X : \text{CAlg} \to \text{Set}. \]

This means that to each commutative ring \( R \), we assign the set \( X(R) \) and for each morphism of commutative rings \( f : R \to S \) we have a morphism of sets

\[ f^* : X(R) \to X(S). \]

Furthermore, these satisfy the obvious compatibilities to be a functor.

\(^1\)This just means that I is a category with finitely many objects and finitely many morphisms.
Definition 3.0.2. A morphism of prestacks is a natural transformation \( g : X \rightarrow Y \) of functors. This means that for each morphism of commutative rings \( f : R \rightarrow S \) we have a commuting diagram

\[
\begin{array}{c}
X(R) \xrightarrow{f^*} X(S) \\
g_R \downarrow \quad \downarrow g_S \\
Y(R) \xrightarrow{f^*} X(S)
\end{array}
\]

An \( R \)-point of a prestack is point \( x \in X(R) \); this is the same thing as a morphism of prestacks \( \text{Spec } R \rightarrow X \) by the next

Lemma 3.0.3 (Yoneda). For all prestack \( X \) and all \( R \in \text{CAlg} \), we have a canonical isomorphism

\[
\text{Hom}(\text{Spec } R, X) \cong X(R).
\]

In particular we have that

\[
\text{Hom}(\text{Spec } R, \text{Spec } S) \cong \text{Spec } S(R) = \text{Hom}(S, R).
\]

Note the reversal of directions.

We denote by \( \text{PStk} \) the category of prestacks. We already have a wealth of examples:

Definition 3.0.4. Let \( R \) be a commutative ring, We define

\[
\text{Spec } R : \text{CAlg} \rightarrow \text{Set } S \mapsto \text{Hom}_{\text{CAlg}}(R, S).
\]

An affine scheme is a prestack of this form.

Remark 3.0.5. If a prestack is representable, then the ring representing it is unique up to unique isomorphism. This is a consequence of the Yoneda lemma. In more detail, the Yoneda functor takes the form

\[
\text{Spec} : \text{CAlg}^{\text{op}} \rightarrow \text{PStk} = \text{Fun(\text{CAlg}, \text{Set})}.
\]

This functor is fully faithful so we may (somewhat abusively) identify \( \text{CAlg}^{\text{op}} \) with its image in \( \text{PStk} \). The category of affine schemes is then taken to be the opposite category of commutative rings.

Example 3.0.6. Let \( n \geq 0 \). Then we define the prestack of affine space of dimension \( n \) as

\[
\mathbb{A}_\mathbb{Z}^n : \text{CAlg} \rightarrow \text{Set } R \mapsto R^{\times n}.
\]

In the homeworks, you will be asked to prove that this prestack is an affine scheme, represented by \( \mathbb{Z}[x_1, \ldots, x_n] \).

Example 3.0.7. Suppose that \( f(x) \in \mathbb{Z}[x, y, z] \) is a polynomial in three variables; for a famous example this could be \( f(x, y, z) = x^n + y^n - z^n \). For each ring \( A \), we define

\[
V(f)(R) := \{(a, b, c) : f(a, b, c) = 0\} \subseteq R^{\times 3}.
\]

Note that this indeed defines a prestack: given a morphism of rings \( \varphi : R \rightarrow S \), we have a morphism of sets

\[
V(f)(R) \rightarrow V(f)(S)
\]

since \( f(\varphi(a), \varphi(b), \varphi(c)) = \varphi(f(a, b, c)) = \varphi(0) = 0 \). In fact, we have a morphism of prestacks (in the sense of the next definition)

\[
V(f) \rightarrow \mathbb{A}_\mathbb{Z}^{\times 3},
\]

where \( \mathbb{A}_\mathbb{Z}^{\times 3}(R) = R^{\times 3} \).
3.1. **Operation on prestacks I: fibered products.** One of the key ideas behind algebraic geometry is to restrict ourselves to objects which are defined by polynomial functions. More abstractly we want to restrict ourselves to objects which arise from other objects in a constructive manner. This is both a blessing and a curse — on the one hand it makes objects in algebraic geometry rather rigid but, on the other, it gives objects in algebraic geometry a "tame" structure.

**Example 3.1.1.** As a warm-up, consider $n$-dimensional complex space $\mathbb{C}^n$ and suppose that we have a polynomial function $\mathbb{C}^n \to \mathbb{C}$. Then the zero set of $f$ is defined via the pullback

\[
\begin{array}{ccc}
Z(f) & \longrightarrow & \mathbb{C}^n \\
\downarrow & & \downarrow \\
\{0\} & \longrightarrow & \mathbb{C}.
\end{array}
\]

We want to say that $Z(f)$ has the structure of a prestack or, later, a scheme. Of course the above diagram presents $Z(f)$ as a set but we can also take the pullback in, say, the category of $\mathbb{C}$-analytic spaces so that $Z(f)$ inherits such a structure (if a pullback exists! and it does).

**Example 3.1.2.** Another important construction in algebraic geometry is the notion of the graph. Suppose that $f : X \to Y$, then its graph is the set $\Gamma_f := \{(x, y) : f(x) = y\} \subset X \times Y$.

Suppose that $X, Y$ have the structure of a scheme, or an $\mathbb{C}$-analytic spaces or a manifold etc., then we want to say that $\Gamma_f$ does inherit this structure. To do so we note that we can present $\Gamma_f$ in the following manner:

\[
\begin{array}{ccc}
\Gamma_f & \longrightarrow & Y \\
\downarrow & & \downarrow \Delta \\
X \times Y & \xrightarrow{f \times \text{id}} & Y \times Y
\end{array}
\]

**Definition 3.1.3.** Suppose that $X \to Y \leftarrow Z$ is a cospan of prestacks, then the fiber product of $X$ and $Y$ over $Z$ is defined as

\[(X \times_Y Z)(R) := X(R) \times_{Y(R)} Z(R).\]

It will be an exercise to verify the universal property of this construction.

**Example 3.1.4.** Suppose that we have a span of rings $R \leftarrow S \to T$ so that we have a cospan of prestacks $\text{Spec } R \to \text{Spec } S \leftarrow \text{Spec } T$. Then (exercise) we have a natural isomorphism

\[\text{Spec } R \times_{\text{Spec } S} \text{Spec } T \cong \text{Spec } (R \otimes_S T).\]

**Example 3.1.5.** A regular function on a prestack is a morphism of prestacks $X \to \mathbb{A}^1$. If $X = \text{Spec } R$, then this classifies a map of commutative rings $\mathbb{Z}[x] \to R$ which is equivalent to picking out a single element $f \in R$. The zero locus of $f$ is the prestack

\[Z(f) := X \times_{\mathbb{A}^1} \{0\}\]

where $\{0\} \to \mathbb{A}^1$ is the map corresponding to $\mathbb{Z}[x] \to \mathbb{Z}, x \mapsto 0$.

3.2. **Closed immersions.** A closed immersion is a special case of a subprestack.

**Definition 3.2.1.** A subprestack of a prestack $\mathcal{F}$ is a prestack $\mathcal{G}$ equipped with a natural transformation $\mathcal{G} \to \mathcal{F}$ such that for any $R \in \text{CAlg}$, the map

\[\mathcal{G}(R) \to \mathcal{F}(R)\]

is an injection. We will often write $\mathcal{F} \subset \mathcal{G}$ for subprestacks.

**Remark 3.2.2.** This is equivalent to saying that $\mathcal{G} \to \mathcal{F}$ is a monomorphism in the category of prestacks.
The important thing about a subprestack is that for any morphism $R \to R'$, the requirement that $\mathcal{G} \to \mathcal{F}$ is a natural transformation enforces the commutativity of the following diagram

$$
\begin{array}{ccc}
\mathcal{G}(R) & \longrightarrow & \mathcal{F}(R) \\
\downarrow & & \downarrow \\
\mathcal{G}(R') & \longrightarrow & \mathcal{F}(R').
\end{array}
$$

which should be read as: “the map $\mathcal{G}(R) \to \mathcal{F}(R')$ factors through the subset $\mathcal{G}(R')$.”

**Definition 3.2.3.** A closed immersion of affine schemes is a morphism $\text{Spec} \ R \to \text{Spec} \ S$ such that the induced map of rings $S \to R$ is surjective. In this case we say that $\text{Spec} \ R$ is a closed subscheme of $\text{Spec} \ S$.

Let us try to understand what this means. The map $\varphi : S \to R$, if surjective, is equivalent to the data of an ideal $I = \ker(\varphi)$. As stated before, we should think of $S$ as the ring of functions on a “space” $\text{Spec} \ S$ and so an ideal of $S$ is a collection of functions which are closed under the action of $S$. Now, the “space” $\text{Spec} \ R$ should be thought of as the space on which the functions that belong to $I$ vanish. In other words a closed immersion is one of the form $\text{Spec} \ R \to \text{Spec} \ R/I$.

**Exercise 3.2.4.** Suppose that $f : \mathbb{A}^n \to \mathbb{A}^1$ is regular function. Prove that the $\mathbb{Z}(f) \to \mathbb{A}^n$ is a closed immersion corresponding to a map of rings $\mathbb{Z}[x] \to \mathbb{Z}[x]/(f)$.

Here is how one can globalize this definition:

**Definition 3.2.5.** A morphism of prestacks $X \to Y$ is a closed immersion or a closed subprestack if for any morphism $\text{Spec} \ R \to Y$ then:

1. the prestack $\text{Spec} \times_Y X$ is representable and,
2. the morphism $\text{Spec} \times_Y X \to \text{Spec} \ R$ is a closed immersion.

### 3.3. Exercises 2.

**Exercise 3.3.1.** What does $\text{Spec}(0)$ represent?

**Exercise 3.3.2.** Prove that the category of prestacks admit all limits and all colimits.

**Exercise 3.3.3.** Prove that the prestack $\mathbb{A}^n_\mathbb{Z}$ is representable by $\mathbb{Z}[x_1, \ldots, x_n]$.

**Exercise 3.3.4.** Consider the prestack

$$
\mathbb{G}_m : R \mapsto \mathbb{R}^\times.
$$

Here $\mathbb{R}^\times$ is the multiplicative group of unit elements in $\mathbb{R}$. Prove that $\mathbb{G}_m$ is representable. What ring is it representable by?

**Exercise 3.3.5.** Suppose that $f : \mathbb{A}^n \to \mathbb{A}^1$ is regular function. Prove that the $\mathbb{Z}(f) \to \mathbb{A}^n$ is a closed immersion corresponding to a map of rings $\mathbb{Z}[x] \to \mathbb{Z}[x]/(f)$.

**Exercise 3.3.6.** Consider the prestack

$$
\text{GL}_n : R \mapsto \text{GL}_n(R).
$$

Prove that it is representable. What ring is it representable by?

If $R$ is a ring we write $R_p$ to be the localization of $R$ at $p$. We write $m_p$ to be the maximal ideal of said local ring and write

$$
\kappa(p) := R_p/m_p.
$$
Exercise 3.3.7. Let $R \in \text{CAlg}$ and $K$ a field. Prove that there is a natural bijection between 
\[ \{ \text{Spec} K \to \text{Spec} R \} \]
with 
\[ \{ \text{prime ideals } p \subset R \text{ with an inclusion } \kappa(p) \hookrightarrow K \} \].

The Zariski tangent space of $\text{Spec} R$ at a prime ideal $p$ is the $\kappa(p)$-vector space given by 
\[ T_p \text{Spec} R := (m_p/m_p^2)^\vee. \]

Exercise 3.3.8. Let $R \in \text{CAlg}$ and $k$ a field and suppose that $R$ is a $k$-algebra. Prove that there is a bijection between 
\[ \{ k\text{-morphisms of rings } R \to \kappa[x]/(x^2) \} \]
with 
\[ \{ \text{prime ideals } p \text{ of } R \text{ with residue field } k \text{ and an element of the Zariski tangent space at } p \} \].

Exercise 3.3.9. Prove that the functor 
\[ \text{Spec} : \text{CAlg} \to \text{PStk} \]

(1) is fully faithful, 
(2) preserves all colimits in the sense that if $\{ R_\alpha \}_{\alpha \in A}$ is a diagram of commutative rings then for all $S \in \text{CAlg}$, the canonical map 
\[ \colim_{\alpha} (\text{Spec } R_\alpha)(S) \to \lim_{\alpha} (\text{Spec } R_\alpha)(S) \]

is an isomorphism. Deduce, in particular, that $\text{Spec}$ converts tensor products of commutative rings to pullback. 
(3) Show, by example, that $\text{Spec}$ does not preserve limits.

Exercise 3.3.10. Prove that a closed immersion is a subprestack.

For the next exercise, recall that if $\mathcal{C}$ is a category with products and $X \in \mathcal{C}$, then the identity morphism $\text{id} : X \to X$ induces the diagonal map 
\[ \Delta : X \to X \times X. \]

Exercise 3.3.11. Let $R \in \text{CAlg}$ and consider the multiplication map $R \otimes_R R \to R$. Prove: 
(1) the corresponding map $\text{Spec } R \to \text{Spec } R \times \text{Spec } R$ is given by the diagonal morphism, 
(2) prove that $\Delta : \text{Spec } R \to \text{Spec } R \times \text{Spec } R$ is a closed immersion of prestacks.

Exercise 3.3.12. Let $\mathcal{C}$ be a category with all limits and suppose that we have a diagram
\[ \begin{array}{ccc}
X & \to & S \\
\downarrow & & \downarrow \\
Y & \to & T \\
\end{array} \]

Prove that the diagram (be sure to write down carefully how the maps are induced!) 
\[ \begin{array}{ccc}
X \times_S Y & \to & X \times_T Y \\
\downarrow & & \downarrow \\
S & \overset{\Delta}{\to} & S \times_T S \\
\end{array} \]

is cartesian (hint: if you are unable to prove this result in full generality, feel free to assume that $\mathcal{C} = \text{PStk}$).
Exercise 3.3.13. A morphism of prestacks $G \to F$ is said to be representable if for any $\text{Spec} R \to F$, the prestack $G \times_F \text{Spec} R$ is representable. Prove that the following are equivalent:

1. the diagonal $\Delta : F \to F \times F$ is representable;
2. any map $\text{Spec} S \to F$ (in other words any map from an affine scheme to $F$) is representable.

4. Lecture 3: Descent

We are about to define schemes. But first we define stacks\(^2\). In order to define the notion of open immersions, which are complementary to closed immersions.

Remark 4.0.1. Thinking about closed immersions of schemes is easier than thinking about open immersions, at least to the instructor. Indeed, every closed immersion of $\text{Spec} R$ corresponds to the set of all ideals of $R$. We can think of a poset of ideals $\{I \subset J\}$ which corresponds to a poset of closed subschemes $\{\text{Spec} R/I \to \text{Spec} R/J\}$. Of course, we want to say that open subschemes of $\text{Spec} R$ should be those of the form $D(J) := \text{Spec} R \setminus \text{Spec} R/J$.

But now $D(J)$ is in fact not representable — we will soon be able to prove this. In fact these $D(J)$’s are the first examples of non-affine schemes. In particular $D(J)$ is not the form $\text{Spec}$ of a ring. In order to formulate descent in a more digestible manner, we will restrict ourselves to open subschemes of Spec $R$ which are actually affine.

Suppose that $I$ is an ideal of a ring $R$, let us recall that the radical of $I$, denoted usually by $\sqrt{I}$ is defined as $\sqrt{I} = \{x : x^N \in I \text{ for some } N \gg 0\}$.

Example 4.0.2. Let $I = (0)$ be the zero ideal. Then the nilradical is $\sqrt{(0)} = \{f : f^N = 0 \text{ for some } N \gg 0\}$. We say that a ring is reduced if $\sqrt{(0)} = 0$.

Definition 4.0.3. Let $R$ be a ring and $f \in R$. A basic Zariski cover of the ring $R_f$ consists of a set $I$ and a collection $\mathcal{U} := \{f_i : f_i \in R\}_{i \in I}$ such that

$$f \in \sqrt{\Sigma(f_i)}.$$ 

In particular, a basic Zariski cover of a ring $R$ consists of a set $I$ and a collection $\mathcal{U} := \{f_i : f_i \in R\}_{i \in I}$ subject to the following condition:

$$1 \in \sqrt{\Sigma(f_i)}.$$ 

We write

$$\{\text{Spec} A_{f_i} \to \text{Spec} A_f\}_{i \in I}$$

to denote an arbitrary basic Zariski cover.

Remark 4.0.4. If $f$ is a unit so that $R_f = R$, this is a very redundant definition. Indeed, any element $x \in \sqrt{\Sigma(f_i)}$ means that

$$x^N \in \Sigma(f_i)$$

But the sum of ideals means we have a sum of elements in each ideal where all except finitely many elements are zero so:

$$x^N = a_1 f_1 + \cdots + a_n f_n,$$

up to rearrangements. But now

$$1 = 1^N = a_1 f_1 + \cdots + a_n f_n.$$ 

\(^2\)This is where some heavy conflict with the literature will occur so be wary. In the literature, the notion of stacks differs from this one in two, crucial ways. First the descent condition is asked with respect to something called the étale topology (which we will cover later in class) and, secondly, the functor lies in the $(2, 1)$-category of groupoids. Functors landing in said version of categories are not really functors in the sense we are used to in class.
Therefore we can find a subcover of \( U \) such that
\[
1 \in \Sigma(f_i).
\]
of course this argument also does show that a basic Zariski cover of \( R_f \) can be refined by a finite subcover.

**Example 4.0.5.** Let \( p,q \) be distinct primes in \( \mathbb{Z} \). This means, by Bézout’s identity, we can write
\[
1 = kp + rq,
\]
for some \( k, r \in \mathbb{Z} \). In the language above we find that
\[
\{ \text{Spec } \mathbb{Z}[\frac{1}{p}], \text{Spec } \mathbb{Z}[\frac{1}{q}] \} \hookrightarrow \text{Spec } \mathbb{Z}
\]
defines a basic Zariski cover of \( \text{Spec } \mathbb{Z} \).

**Example 4.0.6.** Let \( k \) be a field and consider \( k[x] \). Suppose that \( p(x), q(x) \) are polynomials which are irreducible and are coprime. Then Bézout’s identity again works in this situation:
\[
1 = k(x)p(x) + r(x)q(x).
\]
In this language we find that
\[
\{ \text{Spec } k[x][p(x)], \text{Spec } k[x][q(x)] \hookrightarrow \text{Spec } k[x] \}
\]
defines a basic Zariski cover of \( \text{Spec } k[x] \).

**Definition 4.0.7.** A prestack \( \mathcal{F} : \text{CAlg} \to \text{Set} \) is a (Zariski) stack if for any \( A \in \text{CAlg} \) and for all basic Zariski cover \( \{ \text{Spec } A_{f_i} \to \text{Spec } A \}_{i \in I} \) the diagram
\[
\mathcal{F}(A) \to \mathcal{F}(\prod_{i} A_{f_i}) \rightleftharpoons \mathcal{F}(\prod_{i_{0}, i_{1}} A_{f_{i_{0}}} \otimes_A A_{f_{i_{1}}})
\]
is an equalizer diagram where the maps are induced by
\[
\prod_{i_{0}, i_{1}} A_{f_{i_{0}}} \to \prod_{i_{0}, i_{1}} A_{f_{i_{0}}} \otimes_A A_{f_{i_{1}}}(g_{i}) \mapsto (g_{i_{0}}|_{A_{f_{i_{0}}}} \cdot f_{i_{1}}).
\]
and
\[
\prod_{i_{0}, i_{1}} A_{f_{i_{0}}} \to \prod_{i_{0}, i_{1}} A_{f_{i_{0}}} \otimes_A A_{f_{i_{1}}}(g_{i}) \mapsto (g_{i_{1}}|_{A_{f_{i_{0}}}} \cdot f_{i_{1}}).
\]

4.1. Unpacking the descent condition and Serre’s lemma. Let us note a couple of easy properties about localizations

**Lemma 4.1.1.** Let \( f_1, f_2 \in R \) then
\[
(R_{f_{1}})_{f_2} \cong R_{f_{1} \cdot f_2} \cong R_{f_1} \otimes_R R_{f_2}.
\]
This will be homework. For the rest of this section, we will seek be taking a map \( f : R \to A \) and then postcomposing then along some localization of \( A \), say \( A' \); for this it is convenient to use the notation
\[
f|_{A'}
\]
and think about “restriction.”

Let us fix a ring \( R \) and suppose that \( A \) is a “test-ring” and we are interested in the set
\[
\text{Hom}(\text{Spec } A, \text{Spec } R),
\]
and we would like to recover this set in terms of a given basic Zariski cover of \( A \). As we had discussed, this latter object is given by the data of elements \( g_1, \ldots, g_n \in A \) such that \( 1 = \sum_{i=1}^{n} g_i \). Let us consider the following set
\[
\text{Glue}(R, A, \{ g_i \}) \subset \prod_{i=1}^{n} \text{Hom}(\text{Spec } A_{g_i}, \text{Spec } R),
\]
consisting of the \( n \)-tuples \( \{ f_i : R \to A_{g_i} \} \) subject to the following condition
(cocycle) \( f|_{A_{g_i, g_j}} = f_j|_{A_{g_i, g_j}} \), called the cocycle condition.

**Lemma 4.1.2.** As above we have an isomorphism

\[
\text{Glue}(R, A, \{g_i\}) \cong \text{Eq}(\prod_i \text{Hom}(R, A_{g_i}) \Rightarrow \prod_{i_0, i_1} \text{Hom}(R, A_{g_{i_0} \cdot g_{i_1}}))
\]

This is an exercise in unpacking definitions. Even though Glue is more explicit, in order to prove actual results, we will work with the equalizer formulation. Our main theorem is as follows:

**Theorem 4.1.3.** The map

\[
\text{Hom}(\text{Spec } A, \text{Spec } R) \to \prod_{i=1}^n \text{Hom}(\text{Spec } A_{g_i}, \text{Spec } R)
\]

factors as

\[
\text{Hom}(\text{Spec } A, \text{Spec } R) \to \text{Glue}(R, A, \{g_i\})
\]

and induces an isomorphism. Equivalently, Spec R is a Zariski stack.

We will prove Theorem 4.1.3 in the right level of generality. The next lemma is called “Serre’s lemma for modules.”

**Lemma 4.1.4.** Let \( A \) be a ring and \( f_1, \ldots, f_n \) elements such that \( \sum_{i=1}^n f_i = 1 \). Then

\[
M \to \prod_i M_{f_i} \Rightarrow \prod_i M_{f_i, f_j}
\]

is an equalizer diagram.

**Proof.** We first assume

(1) there exists an element \( f_i \), say \( f_1 \), which is invertible. So that \( M \cong M_{f_1} \).

Then we prove the result: indeed, denote the equalizer by Eq. Indeed, the map \( M \to \prod_i M_{f_i} \)
factors through the equalizer since this is just the map

\[
m \mapsto (m|_{A_{f_i}}),
\]

and the cocycle condition is satisfied. On the other hand, If \( \{m_i\} \in \text{Eq} \subset \prod_i M_{f_i} \), then since \( m_1 \in M \) (under the assumed isomorphism), we get that the map is surjective.

To prove injectivity: if \( m_1 = 0 \), then we claim that \( m_i = 0 \) for all \( i > 0 \) if \( (m_i) \) is in the equalizer. But now, we note that, by the equalizer condition, we get that for \( i > 0 \)

\[
M_{f_i} \to M_{f_i, f_1}
\]
is an isomorphism and the image of \( m_i \) under this isomorphism must be the image of \( m_1 \) under the map \( M_{f_1} \to M_{f_i, f_1} \) because of the equalizer condition. Therefore \( m_i \) must be zero.

Now, to prove the desired claim: take kernels and cokernels:

\[
0 \to K \to M \to \text{Eq} \to C \to 0
\]
The claim is that \( K, C = 0 \). They are 0 after inverting each \( f_i \) by the previous claim. The proof the finishes by the next claim.

**Lemma 4.1.5.** Let \( A \) be a ring and \( f_1, \ldots, f_n \) elements such that \( \sum_{i=1}^n f_i = 1 \). Suppose that \( M_{f_i} = 0 \) for all \( i = 1, \ldots, n \). Then \( M = 0 \).

**Proof.** The condition means we can find an \( N \) such that \( f_i^N m = 0 \). But then there is an even larger \( M \):

\[
m = 1^M, m = (\sum f_i)^M, m = 0.
\]

\[ \square \]
Corollary 4.1.6. Let $A$ be a ring and $f_1, \ldots, f_n$ elements such that $\sum_{i=1}^n f_i = 1$. Then
\[ A \to \prod A_{f_i} \Rightarrow \prod A_{f_i: f_j} \]
is an equalizer diagram of rings. In other words, we have proved that $A^1$ is a Zariski stack.

Proof. This follows from what we have proved. The “in other words” part follows from the fact that
\[ A^1(A) = A. \]
\[ \square \]

Now let us prove

Proof of Theorem 4.1.3. We have proved that $A^1$ is a Zariski stack. Any affine scheme can be written as a pullback

\[
\begin{array}{ccc}
\text{Spec } \mathbb{Z} & \xrightarrow{0} & A^1 \\
\downarrow & & \downarrow \\
\text{Spec } R & \xrightarrow{\pi} & \text{Spec } \mathbb{Z}
\end{array}
\]

where $I, J$ are sets. The result then follows from the next lemmas (one of which is homework):

\[ \square \]

Lemma 4.1.7. Zariski stacks are preserved under limits.

Lemma 4.1.8. $\text{Spec } \mathbb{Z}$ is a Zariski stack.

Proof. For any ring $R$, $\text{Spec } \mathbb{Z}(R) = \ast$. The claim then follows from the observation that the diagram

\[ \ast \to \ast \cong \ast \]

is an equalizer.

\[ \square \]

4.2. Exercises 3.

Lemma 4.2.1. Let $R$ be a ring and consider the functor from $R$-algebras to $R$-modules
\[ U : \text{CAlg}_R \to \text{Mod}_R. \]

Proof:

(1) this functor preserves final objects,

(2) this functor creates pullbacks: a diagram of rings

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
C & \to & D
\end{array}
\]
is a pullback diagram if and only if the corresponding diagram of modules is,

(3) if you are feeling up to it: prove that in fact $U$ creates all limits.

Exercise 4.2.2. Here is a formula for $R_f$. We will work in the generality of the category $\text{Mod}_R$.

(1) Let $f \in R$ and consider the following $\mathbb{N}$-indexed diagram in the category of $R$-modules
\[ M \xrightarrow{f} M \xrightarrow{f} M \xrightarrow{f} \cdots. \]
Define the colimit to be the $R$-module $M_f$. Prove that we have a natural isomorphism: for any $N \in \text{Mod}_R$ such that the map
\[ f : N \to N, \]
is an isomorphism then:
\[ \text{Hom}_{\text{Mod}_R}(M_f, N) \cong \text{Hom}_{\text{Mod}_R}(M, N). \]
Construct explicitly a multiplication on $R_f$ and a compatible ring homomorphism $R \to R_f$.

Consider the functor $j_* : \text{Mod}_{R_f} \to \text{Mod}_R$ given by restriction of scalars. Prove that this functor admits a left adjoint given by $j^* : M \mapsto M_f$; part of the task is to explain why $M_f$ is naturally an $R_f$-module.

Use the formula from 1 to prove that $j_*$ is fully faithful and show that the essential image identifies with the subcategory of $R$-modules where $f \cdot$ acts by an isomorphism.

**Lemma 4.2.3.** Let $f_1, f_2 \in R$ then

$$(R_{f_1})_{f_2} \cong R_{f_1} \cdot f_2 \cong R_{f_1} \otimes_R R_{f_2}.$$ 

**Exercise 4.2.4.** In this exercise, we will give a proof of a basic, but very clear formulation of descent. Let $A$ be a ring and suppose that $f, g \in A$ are elements

1. Consider the square

$$
\begin{array}{ccc}
A & \longrightarrow & Af \\
\downarrow & & \downarrow \\
Ag & \longrightarrow & Af_g.
\end{array}
$$

Prove that the top and bottom arrows are isomorphisms after inverting $f$; conclude that the resulting square is cartesian.

2. Prove that the left vertical and the right vertical arrows are isomorphisms after inverting $g$; conclude that the resulting square is cartesian.

3. Now assume:

$$1 \in (f) + (g).$$

Conclude that the square is cartesian.

**Exercise 4.2.5.** Prove that Zariski stacks are preserved under limits: suppose that we have a diagram $I \to \text{PStk}$, then the functor

$$R \mapsto \lim_i \mathcal{F}_i(R),$$

defines a Zariski stack.

**Exercise 4.2.6.** Prove that any affine scheme $\text{Spec} R$ can be written as a pullback

$$
\begin{array}{ccc}
\text{Spec} R & \longrightarrow & A^1 \\
\downarrow & & \downarrow \\
\text{Spec} Z & \longrightarrow & A^1.
\end{array}
$$

5. Lecture 4: Onto schemes

We have done a bunch of abstract stuff. I would like to tell you how to say something concrete using abstract stuff.

5.1. Diversion: multiplicative groups and graded rings. Let us, for this section, consider what structure what one can endow on $G_m = \text{Spec} Z[t, t^{-1}]$. One suggestive way to think about $Z[t, t^{-1}]$ is as

$$Z[t, t^{-1}] \cong Z[Z] \cong \bigoplus_{j \in Z} Z(j).$$

This is also called the group algebra on the (commutative) group $Z$; we will say why this is a interesting at all later on. We want to say that $G_m$ is a group object in the category of affine schemes. Unwinding definitions, we need to provide three pieces of data
(Mult.) The multiplication:
$$\mu : \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}] \quad t \mapsto t \otimes t,$$

(Id.) The identity
$$\epsilon : \mathbb{Z}[t, t^{-1}] \to \mathbb{Z} \quad t \mapsto 1$$

(Inv.) The inverse
$$\iota : \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[t, t^{-1}] \quad t \mapsto t^{-1}.$$  

These data (or, more precisely, the opposites thereof) are subject to the compatibilities that prescribe $\mathbf{G}_m$ as a group object in $\text{PStk}$.

**Definition 5.1.1.** An **affine group scheme** is an affine scheme $G = \text{Spec } R$ with maps
$$\mu : G \times G \to G, \epsilon : \text{Spec } \mathbb{Z} \to G, \iota : G \to G$$
which endows it with the structure of a group object in prestacks.

When we speak of groups, we always want to speak about group actions. If $G$ is an affine group scheme and $\mathcal{F}$ is a prestack then a (left) action is given by a morphism of prestacks
$$a : G \times \mathcal{F} \to \mathcal{F},$$
satisfying the obvious compatibilities:

$$
\begin{array}{c}
G \times G \times \mathcal{F} \\
\downarrow \mu \times \text{id} \\
G \times \mathcal{F}
\end{array}
\quad
\begin{array}{c}
\mathcal{F} \\
\downarrow a \\
\mathcal{F}
\end{array}
\quad
\begin{array}{c}
G \times \mathcal{F} \\
\downarrow \text{id} \\
\mathcal{F}
\end{array}
\quad
\begin{array}{c}
\mathcal{F} \\
\downarrow a \\
\mathcal{F}
\end{array}
\quad
\begin{array}{c}
G \times \mathcal{F} \\
\downarrow \text{id} \\
\mathcal{F}
\end{array}
\quad
\begin{array}{c}
\mathcal{F} \\
\downarrow a \\
\mathcal{F}
\end{array}

If we restrict ourselves to $\mathbf{G}_m$ acting on affine schemes, we actually obtain the next result whose standard reference is [DG70, Exposé 1, 4.7.3]. Let us denote by $\text{Aff}^{\mathbf{B} \mathbf{G}_m}$ the category of affine schemes equipped with a $\mathbf{G}_m$-action and $\mathbf{G}_m$-equivariant morphisms. This is not a subcategory of prestacks, but admits a forgetful functor
$$\text{Aff}^{\mathbf{B} \mathbf{G}_m} \to \text{PStk}.$$

On the other hand a **$\mathbb{Z}$-graded ring** is a ring $R$ equipped with a decomposition:
$$R = \bigoplus_{i \in \mathbb{Z}} R_i$$
such that:

1. each $R_j$ is an additive subgroup of $R$ (in other words, the direct sum above is taken in the category of abelian groups) and,
2. the multiplication induces $R_i R_j \subseteq R_{i+j}$.

We say that an element $f \in R$ is a **homogeneous element of degree** $n$ if $f \in R_n$. A **graded morphism** of graded rings is just a ring homomorphism $\varphi : R \to S$ such that $\varphi(R_j) \subseteq S_j$. We denote by $\text{grCAlg}$ the category of $\mathbb{Z}$-graded rings.

**Theorem 5.1.2.** There is an equivalence of categories
$$\text{Aff}^{\mathbf{B} \mathbf{G}_m} \simeq (\text{grCAlg})^{\text{op}}.$$  

**Remark 5.1.3.** One of the main points of Theorem 5.1.2 is that it is interesting to read from left to right and right to left. One the one hand one can use the geometric language of groups acting on a scheme/variety to encode a combinatorial/algebraic structure. On the other hand, it gives a purely combinatorial/algebraic description of a geometric idea.

**Proof.** First we construct a functor:
$$\text{Aff}^{\mathbf{B} \mathbf{G}_m} \to (\text{grCAlg})^{\text{op}}.$$
Recall that the tensor product of rings is computed as the tensor product of underlying modules. Therefore we can write isomorphisms of \( \mathbb{Z} \)-modules:

\[
R \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}] \cong R \otimes_{\mathbb{Z}} \mathbb{Z}[Z] \cong \bigoplus_{j \in \mathbb{Z}} R(j).
\]

Hence a \( G_m \)-action on Spec \( R \) is the same data as giving a map

\[
\varphi: R \to \bigoplus_{j \in \mathbb{Z}} R(j) \quad f \mapsto (\varphi_j(f) \in R(j)),
\]
satisfying certain compatibilities. We note that the direct sum indicates that the components of \( (\varphi_j(f)) \) is finitely supported.

Using the identity axiom we see that the composite

\[
R \to \bigoplus_{j \in \mathbb{Z}} R(j) \xrightarrow{t^{-1}} R
\]

must be the identity. Therefore, in coordinates, we get that for any \( f \in R \), we get that

\[
f = \sum_{j \in \mathbb{Z}} \varphi_j(f),
\]

so that any \( f \) can be uniquely written as a finite sum of the \( \varphi_j(f) \)'s. To conclude that this defines a grading on \( R \) we need to prove that each \( \varphi_j \) is an idempotent. If this was proved, then the grading would be such that \( f \in R \) is of homogeneous degree \( j \) whenever \( \varphi(f) = ft^j \).

However, this is the case by associativity of the action:

\[
\begin{array}{ccc}
R & \xrightarrow{\varphi} & \bigoplus_{j \in \mathbb{Z}} R(j) \\
\downarrow{\varphi} & & \downarrow{\varphi} \\
\bigoplus_{j \in \mathbb{Z}} R(j) & \xrightarrow{\mu} & \bigoplus_{j \in \mathbb{Z}} \bigoplus_{k \in \mathbb{Z}} R(jk).
\end{array}
\]

Therefore we conclude that \( R \) splits, as a \( \mathbb{Z} \)-module (aka abelian group) as \( R \cong \bigoplus_j \varphi_j R := R_j \) and one can check that this defines a graded ring structure on \( R \) where the compatibility of multiplication originates from the fact that \( \varphi \) is a ring map.

On the other hand, given a ring \( R \) equipped with the structure of a graded ring \( R = \bigoplus R_j \) we define a map

\[
\varphi: R \to \bigoplus_{j \in \mathbb{Z}} R(j),
\]
on the level of abelian groups as

\[
R \to \pi_j R \subset R(j) \cong R,
\]
where \( \pi_j \) is the projection map. This is checked easily to define a \( G_m \)-action and the functors are mutually equivalent.

\[\square\]

Example 5.1.4. There is an action of \( G_m \) on \( \mathbb{A}^1 \) that “absorbs everything to the origin”; in coordinates this is written as \( t \cdot x = tx \). An exercise in this week’s homework will require you to translate this to a grading.

Example 5.1.5. The best way to define new graded rings is to mod out by homogenous polynomial equations. Recall that a polynomial \( f(x_1, \ldots, x_n) \in R[x_1, \ldots, x_n] \) (over any ring \( R \)) is said to be homogenous of degree \( d \) if for any \( r \in R \), \( r^d f(x_1, \ldots, x_n) = f(rx_1, \ldots, rx_n) \).

The instructor never found this a useful definition; we can equivalently define this to be a linear combination of monomials of degree \( d \), i.e.,

\[
a x_1^{r_1} \cdots x_k^{r_k} \sum_{j=1}^{k} r_j = d.
\]
Here is a nice visual example: consider the **quadric cone**:

\[ \text{Spec} \mathbb{Z}[x,y,z]/(x^2 + y^2 - z^2). \]

Since \( \mathbb{Z}[x,y,z]/(x^2 + y^2 - z^2) \) is the quotient of a graded ring by a homogeneous equation, it inherits a natural grading. This defines a \( \mathbb{G}_m \)-action. If we replace \( \mathbb{Z} \) by a field, convince yourself that this is pictorially the “absorbing” action of the cone onto its cone point.

### 5.2. Complementation and open subfunctors.

In the last class we defined the descent condition and also proved that \( \text{Spec} \mathbb{R} \) satisfies this condition. This is like choosing a basis in a vector space — we could have two covers which are specified by \( \{ f_i \} \) or \( \{ g_j \} \) and we have to say something in order to prove that descend with respect to one cover implies descent for the other.

Let us try to characterize open immersions of affine schemes in terms of its functor of points. We know that open subschemes of \( \text{Spec} \mathbb{A} \) should be one which is the complement of a closed subscheme where the latter is of the form \( \text{Spec} \mathbb{A}/I \). Furthermore we know the following example:

**Remark 5.2.1.** If \( I = (f) \), then \( \text{Spec} \mathbb{A}/f \hookrightarrow \text{Spec} \mathbb{A} \) has a complement which is actually an affine scheme given by \( \text{Spec} \mathbb{A}_f \). Indeed, let us attempt to unpack this: suppose that \( \text{Spec} \mathbb{R} \to \text{Spec} \mathbb{A} \) is a morphism of affine schemes corresponding to a map of rings \( \mathbb{A} \to \mathbb{R} \). We want to say that \( \text{Spec} \mathbb{R} \) lands in the open complement of \( \text{Spec} \mathbb{A}/f \) which translate algebraically to the following cartesian diagram

\[
\begin{array}{ccc}
\text{A} & \longrightarrow & \text{R} \\
\downarrow & & \downarrow \\
\text{A}/f & \longrightarrow & 0.
\end{array}
\]

This means that the map \( \varphi : \mathbb{A} \to \mathbb{R} \) must satisfy: \( \mathbb{R}/f\mathbb{R} = 0 \) and so \( f\mathbb{R} = \mathbb{R} \) which exactly means that \( f \) acts invertibly on \( \mathbb{R} \) and hence (by homework) defines uniquely a ring map

\( \mathbb{A}_f \to \mathbb{R} \).

To summarize our discussion:

1. intuitively (and actually!) the closed subscheme \( \text{Spec} \mathbb{A}/f \) is one which is cut out by \( f \) or, in other words, the locus where \( f \) vanishes. Its complement, which is an open subscheme (if you believe in topological spaces) is the locus where \( f \) is invertible so we should take something like \( \text{Spec} \mathbb{A}_f \).
2. we need a new definition to make sense of complementation of prestacks.

Let us also note the following:

**Lemma 5.2.2.** For any ring \( f \), the following diagram is cartesian

\[
\begin{array}{ccc}
\text{Spec} \mathbb{R}_f & \longrightarrow & \text{Spec} \mathbb{R} \\
\downarrow & & \downarrow \\
\text{Spec} \mathbb{Z}[t,t^{-1}] & \longrightarrow & \mathbb{A}^1 = \text{Spec} \mathbb{Z}.
\end{array}
\]

Let us recall that a morphism of prestacks \( X \to Y \) is a closed immersion if for any morphism \( \text{Spec} \mathbb{R} \to Y \) then:

1. the prestack \( \text{Spec} \mathbb{R} \times_Y X \) is representable and,
2. the morphism

\( \text{Spec} \mathbb{R} \times_Y X \to \text{Spec} \mathbb{R} \)

is a closed immersion.
Definition 5.2.3. Let $\mathcal{G} \subset \mathcal{F}$ be a closed immersion of prestacks. The **complement of** $\mathcal{G}$, defined by $\mathcal{F} \setminus \mathcal{G}$ is the prestack given in the following manner: a morphism $x : \text{Spec } R \to \mathcal{F}$ is in $\mathcal{F} \setminus \mathcal{G}$ if and only if the following diagram is cartesian

\[
\begin{array}{ccc}
\emptyset & \to & G \\
\downarrow & & \downarrow \\
\text{Spec } R & \xrightarrow{x} & \mathcal{F}
\end{array}
\]

We note that $\mathcal{F} \setminus \mathcal{G}$ is indeed a prestack because the empty scheme pulls back.

Lemma 5.2.4. Let $R \in \text{CAlg}$ and $I \subset R$ an ideal. Then there is a natural bijection between

1. maps $R \to A$ such $IA = A$ and,
2. morphisms $\text{Spec } A \to \text{Spec } R$ such that

\[
\begin{array}{ccc}
\emptyset & \to & \text{Spec } R/I \\
\downarrow & & \downarrow \\
\text{Spec } A & \xrightarrow{x} & \text{Spec } R.
\end{array}
\]

Proof. The condition of the second item says that the following diagram is cocartesian:

\[
\begin{array}{ccc}
R & \to & A \\
\downarrow & & \downarrow \\
R/I & \to & 0.
\end{array}
\]

which means that $A \otimes_R R/I = A/IA = 0$ which exactly means that $A = IA$.

Definition 5.2.5. A subfunctor $\mathcal{F} \to \text{Spec } R$ of the form in Lemma 5.2.4 is called an **open subscheme**, while $\mathcal{F}$ is called a **quasi-affine** prestack. In this case we write

$\mathcal{F} = D(I)$.

If $\mathcal{F} = \text{Spec } R_f$, we write $D(f)$.

We will soon learn how to prove that not all quasi-affine prestacks are affine.

5.3. Exercises 4.

Exercise 5.3.1. Consider the action of $\mathbb{G}_m$ on $\mathbb{A}^n$ given by

\[
\mathbb{Z}[x_1, \cdots, x_n] \to \mathbb{Z}[t, t^{-1}, x_1, \cdots, x_n] \quad x_j \mapsto t^{r_j} x_j.
\]

Calculate the induced grading on $\mathbb{Z}[x_1, \cdots, x_n]$.

Exercise 5.3.2. Let $R_*$ be a graded ring which is concentrated in $\mathbb{Z}_{\geq 0}$, i.e., $R_{<0} = 0$. Note that:

1. each $R_j$ is then canonically an $R_0$-module and, in fact, $R_*$ is an $R_0$-algebra;
2. the subset $R_+ := \bigoplus_{i \geq 1} R_i \subset R_*$ is an ideal

Prove that the following are equivalent:

1. the ideal $R_+$ is finitely generated as an $R_*$-ideal;
2. $R_*$ is generated as an $R_0$-algebra by finitely many homogeneous elements of positive degree.

In the above situation we say that $R_*$ is a **finitely generated** graded ring.

Exercise 5.3.3. Prove the following locality properties for open immersions:

1. Prove that the composite of open immersions (of prestacks) is an open immersion.
(2) Suppose that \( U \to \mathcal{F} \) is a morphism of prestacks and suppose that \( \mathcal{V} \) is an Zariski cover of \( U \) such that for each \( U \to U \) in \( \mathcal{V} \) the map \( U \to \mathcal{F} \) is an open immersion. Prove that \( U \to \mathcal{F} \) is an open immersion.

6. Lecture 5: Schemes, actually

6.1. Quasi-affine prestacks are Zariski stacks.

Proposition 6.1.1. Any quasi-affine scheme is a prestack.

Proof. Suppose that \( U = D(I) \subset \text{Spec } A \) is quasi-affine. We claim that it is a Zariski stack. Let \( R \) be a test ring and let \( \{f_i \in R\} \) determine a basic Zariski open cover. We first prove the following claim:

- a morphism \( \varphi : A \to R \) satisfies \( I \cdot R = R \) if and only if \( I \cdot R_{f_i} = R_{f_i} \).

Indeed, we have a short exact sequence of modules:

\[
0 \to I \to A \to A/I \to 0.
\]

Applying \( R \otimes_A - \), we get an exact sequence

\[
I \otimes_A R \to R \to R \otimes_A A/I.
\]

Since \( I \cdot R \) is the image of the left-most map, we need only prove that \( R \otimes_A A/I \) is zero as an \( R \)-module.

By Lemma 4.1.5, we need only check that for all \( f_i \), tensoring the above further with \( \otimes_R R_{f_i} \) is zero, But the map \( R \to R_{f_i} \) is flat. Therefore the claim follows from:

\[
R_{f_i} \otimes_R R \otimes_A A/I = R_{f_i} \otimes_A A/I = 0,
\]

as was assumed.

I claim that we are now done. Indeed, shorthanding the relevant equalizers as \( \text{Eq} (\mathcal{F}) (R) \) we get a diagram:

\[
\begin{array}{ccc}
D(I)(R) & \longrightarrow & \text{Eq}(D(I))(R) \\
\downarrow & & \downarrow \\
\text{Spec } A(R) & \xrightarrow{=} & \text{Eq}(\text{Spec } A)(R).
\end{array}
\]

Since the left vertical map is injective and the bottom horizontal map is an isomorphism, the top vertical map is injective. Now, the top vertical map is also surjective by what we have proved.

6.2. General open covers. We formulated a coordinate-dependent way of phrasing descent because we have used the notion of a basic Zariski cover. This is like choosing a basis for a topology which is, in turn, like choosing a basis for a vector space in linear algebra. One should not do this ever or, at least, whenever possible. We will get rid of these choices now.

Recall from last class that if \( \text{Spec } R = X \) is an affine scheme, then an open immersion is a morphism of prestacks where the domain is of the form \( D(I) \) where \( I \) is an ideal of \( R \). Recall also that closed immersions of prestacks are defined by appeal to the affine case: it is a morphism \( \mathcal{G} \to \mathcal{F} \) such that for every morphism \( \text{Spec } R \to \mathcal{F} \), the pullback \( \text{Spec } R \times_{\mathcal{F}} \mathcal{G} \) is representable by a scheme and the morphism to \( \text{Spec } R \) is indeed a closed immersion which is specified by an ideal.

Definition 6.2.1. Let \( \mathcal{F} \) be a prestack. Then an open subprestack of \( \mathcal{F} \) or an open immersion of prestacks is a morphism \( \mathcal{G} \to \mathcal{F} \) such that for any morphism \( \text{Spec } R \to \mathcal{F} \), the morphism \( \text{Spec } R \times_{\mathcal{F}} \mathcal{G} \) is representable by a scheme. In other words, \( \text{Spec } R \times_{\mathcal{F}} X \to \text{Spec } R \) is an open subscheme. In other words, \( \text{Spec } R \times_{\mathcal{F}} X \to \text{Spec } R \) is of the form \( D(I) \) where \( I \) is an ideal of \( R \).
Since we have plenty of examples of open immersions of affine schemes which are not themselves affine, we do not want the representability condition which we saw was imposed in the closed immersion case. The next lemma is left as an exercise.

**Lemma 6.2.2.** Let $Z \hookrightarrow \mathcal{F}$ be an closed immersion, then the complement $\mathcal{F} \setminus Z$ is canonically an open immersion.

**Definition 6.2.3.** A Zariski cover of an affine scheme $X = \text{Spec} A$ is a collection of open embeddings of prestacks $\mathcal{U} = \{U \hookrightarrow X\}$, such that for any nonzero ring $R$, $S = \text{Spec} R$ with a map $S \to X$ there exists a $U \hookrightarrow \text{Spec} A \in \mathcal{U}$ such that $U \times_X S \neq \emptyset$.

At this point nothing then stops us from defining Zariski covers of any prestack:

**Definition 6.2.4.** A Zariski cover of an prestack $\mathcal{F}$ is a collection of open embeddings of prestacks $\mathcal{U} = \{U \hookrightarrow \mathcal{F}\}$, such that for any nonzero ring $R$, $S = \text{Spec} R$ with a map $S \to \mathcal{F}$ there exists a $U \hookrightarrow \mathcal{F} \in \mathcal{U}$ such that $U \times_\mathcal{F} S \neq \emptyset$.

**Definition 6.2.5.** A prestack $\mathcal{F} : \text{CAlg} \to \text{Set}$ is a (Zariski) stack (resp. (Zariski) stack for the Zariski cover $\mathcal{U}$) if for all $A \in \text{CAlg}$ and all (resp. the) Zariski cover $\mathcal{U} : = \{U_i \to \text{Spec} A\}_{i \in I}$ the diagram

$$
\mathcal{F}(\text{Spec} A) \to \prod_i \text{Hom}(U_i, \mathcal{F}) \Rightarrow \prod_{i_0, i_1} \text{Hom}(U_{i_0} \times_{\text{Spec} A} U_{i_1}, \mathcal{F})
$$

is an equalizer diagram where the maps are induced are the obvious ones.

**Remark 6.2.6.** It is a bit dangerous to write $\mathcal{F}(U_i)$ since $U_i$ is not known (and will be known not to be) an affine scheme. So we will stick with the notation $\text{Hom}(U_i, \mathcal{F})$ in these notes, thought the instructor does lapse to writing $\mathcal{F}(U_i)$. This notation will be justified later on.

From this definition, our previous definition of a scheme asks that $\mathcal{F}$ is a Zariski stack for covers $\mathcal{U}$ which are made of Zariski open covers. Another exercise in unpacking definitions:

**Lemma 6.2.7.** A basic Zariski cover is a Zariski cover. In particular if $\mathcal{F}$ is a Zariski stack, then it is a Zariski stack with respect to basic open covers.

This tells us that Definition 6.2.4 is stronger than Definition 4.0.7. In some sense Definition 6.2.4 is preferable — it affords the flexibility of working with covers where the opens are not necessarily affine. We will work towards proving the equivalence of these two definitions shortly. More precisely:

**Theorem 6.2.8.** Suppose that $\mathcal{F} : \text{CAlg} \to \text{Set}$ is a functor. Then the following are equivalent:

1. $\mathcal{F}$ is a Zariski stack in the sense of Definition 6.2.4.
2. $\mathcal{F}$ is a Zariski stack in the sense of Definition 4.0.7.

### 6.3. Quasicompactness of affine schemes.

First let us consider the different ways we can think about Zariski covers.

**Remark 6.3.1.** One of the weird things about algebraic geometry is how large open sets are. For example, consider $\mathbb{R}$ with the usual topology. Then there exists many open covers of $\mathbb{R}$ with no finite subcovers. Now $\mathbb{R} = \text{Spec} \mathbb{A}^1(\mathbb{R})$ but, as an affine scheme $\text{Spec} \mathbb{A}^1$ is, in a precise way, compact.

We call the next lemma quasicompactness of affine schemes.
Lemma 6.3.2. Let $X = \text{Spec } A$, and $\mathcal{U} = \{U \hookrightarrow \text{Spec } A\}$ is a collection of open immersions. then the following are equivalent:

(1) $\mathcal{U}$ is a Zariski cover.
(2) $\mathcal{U}$ has a finite subset $\mathcal{V}$ which is also an open cover.
(3) for any field $k$ and any map $x : \text{Spec } k \to X$ there exists an $U \in \mathcal{U}$ such that $x$ factors through $U$.

Proof. The implication $(1) \Rightarrow (2)$ comes under the term “affine schemes are quasicompact” which is one way in which open subsets look very different in algebraic geometry than what you have experienced before. To prove this, we note that giving a Zariski open cover of an affine scheme is to give a collection of ideals $\{I_j\}$ such that $1 \in \sqrt{\sum I_j}$. But this means, from the definition of sums of ideals, there exists a finite subcollection $i_0, \ldots, i_k$ such that $1 \in \sqrt{\sum_{0 \leq s \leq k} I_{i_s}}$. This in turn means that we can refine the above open cover by $\{D(I_{i_s}) \to \text{Spec } A\}_{0 \leq s \leq k}$.

Let us prove $(2) \Rightarrow (3)$. Given a morphism into a field $\phi : A \to k$, we want to find $U \hookrightarrow X \in \mathcal{V}$ such that $x$ factors through $U$. Suppose that there is none, then $\phi(\Sigma I_j) = 0$, which means that $\phi(1) = 0$ but this is not possible.

Let us prove $(3) \Rightarrow (1)$. Suppose that $R$ is a nonzero ring with a map $\text{Spec } R \to \text{Spec } A$ so that we have a morphism $\varphi : A \to k$. Since $R$ is nonzero, it has a maximal ideal $m$ so that $R/m = \kappa$ a field. By hypothesis, there exists a $U \in \mathcal{U}$ such that $U \times_X \text{Spec } \kappa$ is nonempty, but this also means that $U \times_X \text{Spec } R$ is nonempty since there is a morphism $U \times_X \text{Spec } \kappa \to U \times_X \text{Spec } R$. □

We make this a definition:

Definition 6.3.3. A prestack $\mathcal{F}$ is said to be quasicompact if there exists an Zariski cover $\mathcal{U}$ of $\mathcal{F}$ such that $\mathcal{U}$ consists of a finite collection of affine schemes.

This provides the first mechanism by which a scheme can be non-affine.

Definition 6.3.4. Suppose that $R, S$ are rings, we define the scheme-theoretic coproduct or the coproduct in schemes as $\text{Spec } R \sqcup \text{Spec } S := \text{Spec } R \times S$.

This affine scheme equipped with maps $\text{Spec } R \hookrightarrow \text{Spec } R \sqcup \text{Spec } S \hookrightarrow \text{Spec } S$.

There is an obvious guess for what a coproduct of schemes could be. However, this latter notion is bad as it does not have Zariski descent — you will be asked to address this in the exercises.

Lemma 6.3.5. Suppose that we have a diagram $N \to \text{PStk}, i \mapsto \mathcal{F}_i$ such that for each $i$ the prestack $\mathcal{F}_i$ is a Zariski stack. Then $\text{colim}_i \mathcal{F}_i$ is a Zariski stack.

Proof. This follows from the fact that colimits in $\text{PStk}$ are computed pointwise (as was proved in the previous exercise) and $N$-indexed colimits commutes with finite limits (or, more generally, filtered colimits commute with finite limits) which is in this week’s problem set. □

Lemma 6.3.6. Let $(A_i)_{i \in N}$ be a collection of nonzero rings and consider a countable product $\coprod_N \text{Spec } A_i$. This prestack (which in fact a Zariski stack) is not an affine scheme.
Proof. We first note that $X := \prod \text{Spec } A_i = \varinjlim_{n \to \infty} \prod_{i=1}^{n} \text{Spec } A_i$ so the previous lemma asserts that $\prod \text{Spec } A_i$ is a Zariski stack. Consider the collection of maps

$$\{\text{Spec } A_j \to \prod X\}$$

I claim that $\text{Spec } A_j \to X$ is an open immersion for each fixed $j$. Indeed, suppose that we have a map $\text{Spec } R \to X$. In this case we note that:

$$X(R) = (\varinjlim_{n \to \infty} \prod_{i=1}^{n} \text{Spec } A_i)(R) = \varprojlim_{n \to \infty} (\prod_{i=1}^{n} \text{Spec } A_i)(R) = \varprojlim_{n \to \infty} \text{Hom}(\prod_{i=1}^{n} A_i, R),$$

which means that the map $\text{Spec } R \to X$ corresponds to a map $\prod_{i=1}^{n} A_i \to R$ or, equivalently, a map $\text{Spec } R \to \prod_{i=1}^{n} \text{Spec } A_i$, or, in other words, the map $\text{Spec } R \to X$ factors through a finite stage of the colimit. Therefore the pullback is computed as

$$\text{Spec } A_j \times_X \text{Spec } R = \text{Spec } A_j \times \varprojlim_{n \to \infty} \text{Spec } A_i \text{Spec } R.$$ 

But, by this week’s exercise we know that (1) $\text{Spec } A_j \to \prod_{i=1}^{n} \text{Spec } A_i \text{Spec } A_j$ is an open immersion (since summand inclusions are open immersions) and (2) open immersions are stable under pullbacks.

To conclude that the desired collection of maps are covers, we just need to check on $k$-points and invoke Lemma 6.3.2. To conclude, note that there is no refinement of this subcover. □

6.4. The definition of a scheme. Finally:

**Definition 6.4.1.** A prestack $X$ is a scheme if:

1. it is a Zariski stack, and
2. there exists an Zariski cover $\mathcal{U}$ of $X$ such that $\mathcal{U}$ consists of affine schemes.

So far we have seen that affine schemes are schemes; they are furthermore quasicompact schemes. We have also seen that infinite coproducts of schemes are schemes which are not quasicompact. Let us prove that quasi-affine schemes are schemes.

6.5. Exercises 5. Let $\mathsf{Eq}$ be the category displayed as $\bullet \Rightarrow \bullet$.

**Exercise 6.5.1.** Suppose that we have a diagram $F : N \times \mathsf{Eq} \to \text{Set}$. On the one hand we can view this as

$$N \to \text{Fun}(\mathsf{Eq}, \text{Set})$$

and take

$$\text{colim}_{\mathsf{Eq}} \lim_{N} F$$

or we can view this as

$$\text{Eq} \to \text{Fun}(N, \text{Set})$$

and take

$$\text{lim}_{\mathsf{Eq}} \text{colim}_{N} F.$$

Construct a natural map between the two sets and prove that they are isomorphic. Conclude the same commutation property for $\mathsf{PStk}$ in place of $\text{Set}$. This is the first instance of an unexpected commutation of limits versus a colimit that you have seen in this class.

**Exercise 6.5.2.** Prove that open immersions are stable under pullbacks: suppose that $\mathcal{U} \to \mathcal{F}$ is an open immersion of prestacks, then for each map $\mathcal{G} \to \mathcal{F}$, the induced map of prestacks $\mathcal{G} \times_{\mathcal{F}} \mathcal{U} \to \mathcal{G}$ is an open immersion. Prove the same property for closed immersions.

**Exercise 6.5.3.** Here is a simple algebra fact that you should know: let $R$ be a ring and $e$ is an idempotent. Then we have a decomposition $eR \oplus (1-e)R \cong R$ in the category of abelian groups, while neither $eR$ nor $(1-e)R$ are subrings unless $e$ is zero or 1. However note that the projections $R \to eR$ and $R \to (1-e)R$ are ring homomorphisms.
Exercise 6.5.4. Let $R, S$ be rings and consider a test ring $A$, then consider the map
\[ \text{Hom}(\text{Spec } A, \text{Spec } R) \sqcup \text{Hom}(\text{Spec } A, \text{Spec } S) \to \text{Hom}(\text{Spec } A, \text{Spec } R \times S), \]
given on one factor of the coproduct by
\[ g : R \to A \mapsto R \times S \to R, \]
\[ f : S \to A \mapsto R \times S \to S. \]

Prove:
(1) if $A$ has no nontrivial idempotent, then the map is an isomorphism.
(2) if $A$ has nontrivial idempotent, then the map is not an isomorphism so long as both $R, S \neq 0$.
(3) Prove that, in this case, the functor
\[ X(S) = \begin{cases} \text{Hom}(\text{Spec } A, \text{Spec } R) \sqcup \text{Hom}(\text{Spec } A, \text{Spec } S) & A \neq 0 \\ \ast & \text{else} \end{cases} \]
is not a Zariski stack.

Definition 6.5.5. We say that a scheme $X$ is connected if it is not the empty scheme and for all 2-fold open covers $U = \{U, V \hookrightarrow X\}$ such that $U \times_X V = \emptyset$, then $U = X$ or $V = X$.

Exercise 6.5.6. Prove that any Zariski stack converts a product of rings to a product of sets, i.e., if $A \cong B \times C$ then the map
\[ \mathcal{F}(B \times C) \to \mathcal{F}(B) \times \mathcal{F}(C), \]
is an isomorphism whenever $\mathcal{F}$ is a Zariski stack. Conclude that Zariski stack converts finite coproducts of affine schemes to finite products.

Exercise 6.5.7. Prove that:
(1) an affine scheme $X = \text{Spec } R$ is connected if and only if $R$ has no nontrivial idempotent,
(2) if $e$ is a nontrivial idempotent and $R = eR \times (1 - e)R$ prove that the map $R \to eR$ induces an map $\text{Spec } eR \to \text{Spec } R$ which is both an open and a closed immersion.
(3) let $R$ be a noetherian ring. Prove that we can write
\[ R \cong \prod_i R_i \]
where each $R_i$ is nonzero and has no nontrivial idempotent and $I$ is a finite set.

Exercise 6.5.8. The following is a generalization of Exercise 6.5.4. First we note that in point-set topology if $U \hookrightarrow X$ is a clopen subset of a topological space $X$, then $U$ is an union of connected components. So we can, in topological spaces, write $X = U \sqcup \text{Spec } R \times U$ where $X \setminus U$ is again clopen, whence also a union of its connected components. But we have seen that the operation of coproduct in schemes is not taken set-wise. The following exercise, however, still proves that we have a coproduct decomposition for clopen subfunctors:

Let $R$ be a ring and suppose that
\[ U \hookrightarrow \text{Spec } R \]
is both an open and closed subfunctor (in other words clopen). Note that this means that we also know that $\text{Spec } R \setminus U$ is quasi-affine.

(1) suppose that $\{\text{Spec } R_{f_i} \hookrightarrow U\}, \{\text{Spec } R_{g_j} \hookrightarrow \text{Spec } R \setminus U\}$ are open covers, which we can choose to be finite since we already know that quasi-affines are quasicompact. Prove that each $f_i g_j$ is nilpotent.
(2) Let $I = (f_1, \cdots, f_n), J = (g_1, \cdots, g_m)$. Prove that $\text{Spec } R/I \cong U, \text{Spec } R/J \cong \text{Spec } R \setminus U$. Conclude that $\text{Spec } R = \text{Spec } R/I \sqcup \text{Spec } R/J$.
(3) Prove that there exists an $N$ such that $I^N + J^N = R$ so that we can write $1 = x + y$ where $x \in I^N, y \in J^N$ and prove that $x$ and $y$ are idempotents.
(4) Conclude that if \( U \hookrightarrow \text{Spec } R \) is an open and closed subfunctor then
\[
\text{Spec } R \cong U \sqcup V,
\]
where \( U, V \) are affine schemes (remember what we mean by coproducts!).

(5) Conclude the same for arbitrary schemes: if \( U \hookrightarrow X \) is a clopen subscheme then
\[
X \cong U \sqcup V.
\]

(6) On the other hand, prove the following: if \( e \) is idempotent, then \( \text{Spec } R/e \hookrightarrow \text{Spec } R \) is a clopen embedding.

**Exercise 6.5.9.** Let \( A^{\infty} := \text{Spec } \mathbb{Z}[x_1, x_2, \ldots, x_n, \ldots] \). This an affine scheme hence quasicompact. Is the quasi-affine scheme \( A^{\infty} \setminus 0 \) quasicompact?

7. Lecture 6: Universality of descent and closure of schemes under pullbacks

Last time, we wanted to make more examples of schemes so we want to claim that quasi-affine schemes are schemes. We have already proved that quasi-affine schemes are Zariski stacks. It remains to produce a Zariski cover.

**Lemma 7.0.1.** Let \( A \) be a ring and \( I \) and ideal of \( A \). Then, the collection \( \{ \text{Spec } A_{f_i} \to D(I) : f_i \in I \setminus 0 \} \) is a Zariski open cover of the quasi-affine scheme \( D(I) \). In particular \( D(I) \) is a scheme.

**Proof.** This gives me an opportunity to compute pullbacks and show you how one can make computations of how some pullbacks look like. We want to compute
\[
\text{Spec } A_{f_i} \times_{\text{Spec } A} D(I).
\]

Recall that pullbacks are computed pointwise and therefore we want to understand explicitly how this prestack looks like when one maps into a ring \( R \) or, in other words, if one maps \( \text{Spec } R \) in. So let \( R \) be such a ring and, by definition, the following diagram is cartesian (in sets)
\[
\begin{array}{ccc}
(Spec A_{f_i} \times_{Spec A} D(I))(R) & \longrightarrow & \{ A \rightarrow R : IR = R \} \\
\downarrow & & \downarrow \\
\{ A_{f_i} \rightarrow R \} & \longrightarrow & \{ A \rightarrow R \}.
\end{array}
\]

Now we see that the pullback is given by the subset of maps \( \varphi : A_{f_i} \to R \) such that if we precompose with \( A \to A_{f_i} \) we have that \( IR = R \). Now, assume that \( f_i \in I \setminus 0 \). I claim that this last condition is no condition at all. Indeed, suppose that \( \varphi : A_{f_i} \to R \) is such a map, then given any \( r \in R \) we can write
\[
r = \varphi(f_i)/\varphi(f_i)^{-1}r = \varphi(f_i)r',
\]
which exactly means that \( IR = R \). Therefore we conclude that
\[
\text{Spec } A_{f_i} \times_{\text{Spec } A} D(I) \cong \text{Spec } A_{f_i}.
\]

We also know that open immersions are stable under pullbacks and therefore the map
\[
\text{Spec } A_{f_i} \hookrightarrow D(I).
\]
is open since the map \( \text{Spec } A_{f_i} \to \text{Spec } A \) is.
To prove the covering condition. Let \( \text{Spec } R \to D(I) \) be nonzero. We consider the diagram where each square is cartesian.

\[
\begin{array}{ccc}
\text{Spec } R & \times_{D(I)} & \text{Spec } A_{f_i} \\
\downarrow & & \downarrow \\
\text{Spec } A_{f_i} & \longrightarrow & D(I) \\
\doteq & & \doteq \\
\text{Spec } A_{f_i} & \longrightarrow & \text{Spec } A.
\end{array}
\]

Our goal is to find an \( f_i \) for which the top left corner is nonempty. But now, we note that the resulting rectangle is cartesian whenever \( f_i \in I \setminus 0 \) since the bottom square is from our previous computation. But by assumption on \( U \), we can indeed find such an \( f_i \).

\( \square \)

7.1. **Universality of descent and dévissage.** This is a technical section which proves the following result.

**Theorem 7.1.1.** Suppose that \( \mathcal{F} : \text{CAlg} \to \text{Set} \) is a functor. Then the following are equivalent:

1. \( \mathcal{F} \) is a Zariski stack in the sense of Definition 6.2.4.
2. \( \mathcal{F} \) is a Zariski stack in the sense of Definition 4.0.7.

Of course the direction (1) \( \Rightarrow \) (2) is immediate. We note that Theorem 7.1.1 is akin to a maneuver in point-set topology where we say that everything we want to know about the topology of a space is basically determined by the basic opens. We will prove this in a rather fancy way, but the crux is the next lemma:

**Lemma 7.1.2.** Any open cover \( \mathcal{U} \) of an affine scheme \( \text{Spec } A \) admits a refinement by basic Zariski covers. More precisely: given \( \mathcal{U} \) a Zariski open cover of \( \text{Spec } A \), there exists a basic Zariski open cover \( \mathcal{V} := \{ A \to A_{f_i} \} \) with the property that for each \( U \in \mathcal{U} \), the set

\[
\mathcal{V}_U := \{ \text{Spec } A_{f_i} \in \mathcal{V} : \text{Spec } A_{f_i} \hookrightarrow U \}
\]

is a Zariski open cover of \( U \).

**Proof.** By Lemma 5.2.4, \( \mathcal{U} = \{ D(I_\alpha) \}_\alpha \). By the proof of Lemma 6.3.2 we have that

\[
A = \sqrt{\sum I_\alpha}.
\]

From this, extract the set

\[
\{ f : \exists \alpha, f \in I_\alpha \setminus 0 \}
\]

Then \( \{ \text{Spec } A_f \} \) is the desired refinement after Lemma 7.0.1. \( \square \)

We now begin the proof. Our first goal is to observe that if \( \mathcal{F} \) is a Zariski stack in the sense of Definition 4.0.7, then there is an extension of \( \mathcal{F} \) to the category of quasi-affine schemes:

**Definition 7.1.3.** Let \( \text{QAff} \) to be the full subcategory of \( \mathcal{PStk} \) spanned by affine schemes and quasi-affine schemes. The objects of this category are prestacks of the form \( D(I) \) for some ideal \( I \subset A \) is a ring \( A \).

To see, this we make the following observation. Suppose that \( D(I) \hookrightarrow \text{Spec } A \) is a quasi-affine. Consider open immersions

\[
\text{Spec } B, \text{Spec } C \hookrightarrow D(I).
\]

Then we have a map

\[
\text{Spec } B \times_{D(I)} \text{Spec } C \to \text{Spec } B \times_{\text{Spec } A} \text{Spec } C.
\]

**Lemma 7.1.4.** This is an isomorphism.
Proof. According to Exercise 3.3.12, we have the following cartesian diagram

\[
\begin{array}{ccc}
\text{Spec } B \times_{D(I)} \text{Spec } C & \longrightarrow & \text{Spec } B \\
\downarrow & & \downarrow \\
D(I) & \longrightarrow & D(I) \times_{\text{Spec } A} D(I).
\end{array}
\]

Now, $\Delta$ is an isomorphism since $D(I)$ is open (exercise!), hence the claim follows. $\square$

In particular: $\text{Spec } B \times_{D(I)} \text{Spec } C$ turns out to be affine. Using this observation, we will extend $\mathcal{F} : \text{CAAlg} \to \text{Set}$.

Construction 7.1.5. Suppose that $\mathcal{F}$ is a Zariski stack and $D(I)$ is quasi-affine. For any basic open cover of $U = \{\text{Spec } A_i \hookrightarrow D(I)\}$ define:

$$\tilde{\mathcal{F}}(D(I)) := \text{Eq}(\prod_i \mathcal{F}(\text{Spec } A_i) \rightrightarrows \prod_{i,j} \mathcal{F}(\text{Spec } A_i \times_{D(I)} \text{Spec } A_j))$$

Lemma 7.1.6. The above construction gives a well-defined functor

$$\tilde{\mathcal{F}} : \text{QAff}^{op} \to \text{Set}.$$ 

This is kind of tedious, but you can imagine its proof

Proof Sketch. Given two covers $U = \{\text{Spec } A_i \to D(I)\}, V = \{\text{Spec } B_j \to D(I)\}$ then

1. for each $i$ we have that $\{\text{Spec } B_j \times_{D(I)} \text{Spec } A_i \to \text{Spec } A_i\}$ is a cover of Spec $A_i$ and
2. for each $i$ we have that $\{\text{Spec } A_i \times_{D(I)} \text{Spec } A_i \to \text{Spec } B_j\}$ is a cover of Spec $B_j$.

With this we can rewrite

$$\text{Eq}(\prod_i \mathcal{F}(\text{Spec } A_i) \rightrightarrows \prod_{i,j} \mathcal{F}(\text{Spec } A_i \times_{D(I)} \text{Spec } A_j))$$

as products of equalizers involving $B_j$’s and vice versa. The proof follows from the fact that equalizers and products are both limits and hence they commute. $\square$

The next concept is the key to proving a result like Theorem 7.1.1

Definition 7.1.7. Fix a prestack $\mathcal{F} : \text{QAff}^{op} \to \text{Set}$.

We say that a morphism of $X \to Y$ in $\text{QAff}$ is of $\mathcal{F}$-descent if

$$\mathcal{F}(Y) \to \mathcal{F}(X) \rightrightarrows \mathcal{F}(X \times_Y X),$$

is an equalizer diagram. It is of universal $\mathcal{F}$-descent if for any map $T \to Y$,

$$\mathcal{F}(T) \to \mathcal{F}(T \times_Y X) \rightrightarrows \mathcal{F}(T \times_X (X \times_Y X)).$$

is an equalizer diagram, i.e., the map $X \times_Y T \to T$ is also of $\mathcal{F}$-descent.

The next lemma summarizes why this definition is a good one:

Lemma 7.1.8. Let $f : Y \to Z, g : X \to Y$ be a morphism in $\text{QAff}$. Then

1. morphisms of universal $\mathcal{F}$-descent are stable under base change.
2. if $f$ admits a section, then it is of universal $\mathcal{F}$-descent.
3. suppose that we have a cartesian diagram

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
W & \overset{h}{\longrightarrow} & Z.
\end{array}
$$

Suppose that (1) the base change of $h$ to $Y$ and to $Y \times_Z Y$ and (2) the base change of $f$ to $Y \times_X Y$ are of universal $\mathcal{F}$-descent. Then $f$ is of universal $\mathcal{F}$-descent.
(4) If $f, g$ are of universal $\mathcal{F}$-descent then so is their composite.
(5) If $f \circ g$ is universal $\mathcal{F}$-descent, then so is $f$.
(6) If $\mathcal{F}$ converts coproducts to products, morphisms of the form $\bigcup_{i=1}^n \text{Spec } B_i \to \text{Spec } A$ is of universal $\mathcal{F}$-descent.

Proof sketch. Let us prove (5), assuming (1-3). We have a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\pi_X} & \text{Spec } A \\
\downarrow & & \downarrow \\
Y & \xrightarrow{f} & Z
\end{array}
$$

This proves that $\pi_X$ is a retraction and hence is of universal $\mathcal{F}$-descent by (2). In particular, the base change of $f$ along $h := f \circ g$ is of universal $\mathcal{F}$-descent and so are its base changes along $X \times_Z X \Rightarrow X$ using (2). Now, by assumption $h := f \circ g$ is of universal $\mathcal{F}$-descent so that its base change along $f$ is again of universal $\mathcal{F}$-descent and hence its base changes along $Y \times_Z Y \Rightarrow Z$, again using (2). Using (3), we conclude that $f$ is universal $\mathcal{F}$-descent.

Remark 7.1.9. Warning: this is not meant to be a technical remark. A lot of statements in algebraic geometry are proved by proving closure properties for that statement and then demonstrating a base case. This kind of argument style is called dévissage (which is the French word for “unscrewing”). Often this kind of argument can be replaced by a more ad hoc one which one proves by hand but we have chosen to give a demonstration of how this general principle can work.

Here is a sharper formulation of Theorem 7.1.1

**Theorem 7.1.10.** Suppose that $\mathcal{F} : \text{QAff}^{\text{op}} \to \text{Set}$ is a prestack. Assume that:

1. $\mathcal{F}$ converts finite coproducts to products,
2. $\mathcal{F}|_{\text{Aff}}$ is a Zariski stack.

Then $\mathcal{F}$ is a Zariski stack in the sense of Definition 6.2.4.

Proof. Suppose that $\mathcal{U}$ is a Zariski cover of $\text{Spec } R$. By the quasicompactness of $\text{Spec } R$, we can assume that $\mathcal{U}$ consists of a finite collection of morphisms. In this case, consider the map in $\text{QAff}$:

$$
f : \prod_{i=1}^n U_i \to \text{Spec } R,
$$

noting that quasi-affine schemes admit finite coproducts. We claim that $f$ is of universal $\mathcal{F}$-descent if $\mathcal{F}$ satisfies (1) and (2). Now, by Lemma 7.1.2, we can find a (finite) refinement of the $\mathcal{U}$ consisting of affine schemes so that we have a sequence of composable morphisms

$$
\prod_{i,j} V_{ij} \to \prod_{i=1}^n U_i \to \text{Spec } R.
$$

By Lemma 7.1.8.5, the composite is of universal $\mathcal{F}$-descent, which means we do know that the original map is too using Lemma 7.1.8.3.

7.2. Exercises 6.

**Definition 7.2.1.** Recall that a morphism $f : X \to Y$ is **monic** if for any $g, h : Z \to X$ and $fg = fh$ implies that $g = h$.

**Exercise 7.2.2.** Let $\mathcal{C}$ be a category with pullbacks. Then prove that the following are equivalent:

1. $f : X \to Y$ is monic,
2. $\Delta : Y \to Y \times_X Y$ is an isomorphism.
Corollary 7.2.3. Prove that Quasi-affine schemes are also quasi-compact.

Definition 7.2.4. Let $R$ be a ring. We define the reduction of $R$ to be the ring $R_{\text{red}} := R / \sqrt{(0)}$, i.e., it is the ring $R$ modulo its nilpotent elements.

Exercise 7.2.5. Let $X$ be a scheme. Then its reduced locus $X_{\text{red}} \hookrightarrow X$ is the subprestack defined as follows: an $R$-point $x : \text{Spec} R \to X$ lies in $X_{\text{red}}$ if and only if there exists a filler in the following diagram

$$
\begin{array}{ccc}
\text{Spec} R_{\text{red}} & \longrightarrow & X_{\text{red}} \\
\downarrow & & \downarrow \\
\text{Spec} R & \longrightarrow & X.
\end{array}
$$

Equivalently, an $R$-point of $X$ which lifts to $Z$ is a morphism $\text{Spec} R_{\text{red}} \to \text{Spec} R \times_X X_{\text{red}}$. Prove that:

1. if $X = \text{Spec} R$, then $X_{\text{red}} = \text{Spec} R_{\text{red}}$
2. $X_{\text{red}}$ is a closed subscheme of $X$.

We say that a scheme $X$ is reduced if $X = X_{\text{red}}$. Conclude that $X_{\text{red}}$ satisfies the following universal property: suppose that $Y$ is a reduced scheme with a map $f : Y \to X$, then $f$ factors uniquely through $X_{\text{red}}$.

Exercise 7.2.6. Let $f : X \to Y$ be a morphism of schemes, prove that there exists a unique morphism $f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$ rendering the diagram

$$
\begin{array}{ccc}
X_{\text{red}} & \longrightarrow & Y_{\text{red}} \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
$$

commutative. Furthermore:

1. prove that $(X \times_Z Y)_{\text{red}} \cong (X_{\text{red}} \times_{Z_{\text{red}}} Y_{\text{red}})_{\text{red}}$.
2. but even if $X, Y$ are reduced then $X \times_Z Y$ need not be.

Exercise 7.2.7 (Optional). Prove Lemma 7.1.8 (Hint: you should only use formal properties of pullbacks).

Definition 7.2.8. A scheme $X$ is noetherian if it is (1) quasicompact, and (2) there exists a Zariski cover $\mathcal{U}$ of $X$ consisting of affines schemes where each member is $\text{Spec}$ of a noetherian ring.

Exercise 7.2.9. Prove that a scheme $X$ is noetherian if and only if any chain of closed subschemes

$$
\cdots Z^i \subset Z^{i-1} \subset \cdots Z^1 \subset Z^0 = X,
$$

terminates.

Exercise 7.2.10. Let $f : X \to Y$ be a morphism of schemes. Define the graph prestack as

$$
\Gamma_f : R \mapsto \{(x, y) : f(x) = y\} \subset (X \times Y)(R).
$$

Prove that $\Gamma_f$ is a scheme.

8. Lecture 7: Relative algebraic geometry and quasicoherent sheaves

8.1. Relative algebraic geometry. One of the key innovations of the Grothendieck school is the idea that one should be working with algebraic geometry over a base scheme; this is also called “relative algebraic geometry.” At heart, the following proposition is key:
Proposition 8.1.1. Let \( X, Y, Z \) be schemes and suppose that we have cospan of prestacks

\[
\begin{array}{ccc}
X & \\ & Z \leftarrow \downarrow \rightarrow & Y
\end{array}
\]

then the fiber product in prestacks \( X \times_Z Y \) is, in fact, a scheme.

Proof. It is a Zariski stack since limits commute. The interesting part is to furnish an open cover of \( X, Y, Z \).

The proof breaks down naturally in steps:

1. If \( X = \text{Spec} \, A, Y = \text{Spec} \, B, Z = \text{Spec} \, C \), then \( X \times_Y Z = \text{Spec} \, A \otimes_C B \) as we have seen before.
2. Assume that \( Y, Z \) are affine. Our goal is to furnish an open cover \( X \times_Y Z \). So pick an open cover of \( X \) consisting of affines \( U = \{ \text{Spec} \, A_i \hookrightarrow X \} \). From the previous result, we do know that \( \text{Spec} \, A_i \times_Y Z \) is an (affine) scheme so taking
   \[
   U_X := \{ \text{Spec} \, A_i \times_Y Z \hookrightarrow X \times_Y Z \}
   \]
   furnishes an open cover since we do know that open immersions are stable under base change. Similarly, we are okay if \( Z, Y \) are both affines.
3. Now, let us assume that \( X, Y \) are both affines and suppose that \( Z \) is not. Take an affine open cover of \( Z \) \( V := \{ \text{Spec} \, C_i \hookrightarrow Z \} \). Now, the collection
   \[
   V_X := \{ X \times_Z \text{Spec} \, C_i \hookrightarrow X \},
   \]
   is a collection of open immersions of \( X \). We note that, however, \( X \times_Z \text{Spec} \, C_i \) is not necessarily affine— they are only quasi-affine. Similarly, we also have
   \[
   V_Y := \{ Y \times_Z \text{Spec} \, C_i \hookrightarrow Y \}.
   \]
   Now consider
   \[
   V_X \times_Z V_Y := \{ (Y \times_Z \text{Spec} \, C_i) \times_{\text{Spec} \, C_i} (\text{Spec} \, C_i \times_Z X) \hookrightarrow X \times_Y Z \}
   \]
   This is an improvement of the situation as the terms in the cover are made out of taking fiber products over an affine scheme so that we can just arrange one of the other terms, say \( (\text{Spec} \, C_i \times_Z X) \) to be affine by taking a further cover (using Lemma 7.1.2, say). We then appeal to the previous case.
4. Lastly, if none of them are affine, then we take open covers
   \[
   U := \{ \text{Spec} \, A_i \hookrightarrow X \} \quad V := \{ \text{Spec} \, B_j \hookrightarrow Y \},
   \]
   and note that
   \[
   U \times_Z V := \{ \text{Spec} \, A_i \times_Z \text{Spec} \, B_j \hookrightarrow X \times_Y Z \}
   \]
   is an open cover by the previous situation.

\[ \square \]

Definition 8.1.2. Let \( B \) be a scheme, the category of \( B \)-schemes is the slice category \( \text{Sch}_B := \text{Sch}_{/B} \). In other words, its objects are given by morphisms of schemes

\[
X \to B,
\]
while morphisms are

\[
\begin{array}{ccc}
X & \\ & Z \leftarrow \downarrow \rightarrow & Y
\end{array}
\]
Example 8.1.3. Let $R$ be a fixed commutative ring. Then the category of $\text{Spec } R$-schemes (sometimes also called $R$-schemes) which are also affine (this is different from saying “affine $R$-schemes) is equivalent to the category $R$-algebras, i.e., a commutative ring admitting a ring morphism from $R$ and those ring maps under $R$ which renders the obvious diagram commutative.

Remark 8.1.4. Let $X \to Y$ be a morphism between schemes, then we have an adjunction

$$\text{forget} : \text{Sch}_X \rightleftarrows \text{Sch}_Y : \times_Y X,$$

where the left adjoint is given by sending an $X$-scheme $T \to X$ to $T \to X \to Y$. Indeed, convince yourself that this is an adjunction. The right adjoint is often called the “base change functor.”

Example 8.1.5. Here are some examples of difference with working with absolute algebraic geometry versus relative algebraic geometry.

(1) consider the map

$$\mathbb{C} \to \mathbb{C} \quad z = a + ib \mapsto \overline{z} = a - ib.$$

This is is a $\mathbb{Z}$-linear map. However, it is not a $\mathbb{C}$-linear map:

$$(x + iy)ib = (x + iy)(-ib) = -xib + yb,$$

but

$$\overline{(x + iy)(ib)} = \overline{xyb} = -xib - yb.$$

This means that the involution

$$\iota : \text{Spec } \mathbb{C} \to \text{Spec } \mathbb{C},$$

is not a morphism in $\mathbb{C}$.

(2) Let $R$ be a commutative ring which is also an algebra over $\mathbb{F}_p$. Then there is a map

$$F_R : R \to R \quad x \mapsto x^p,$$

called the Frobenius. The Frobenius exhibits the following functoriality if $f : \text{Spec } R \to \text{Spec } S$ is a morphism then:

$$\begin{array}{ccc}
\text{Spec } R & \xrightarrow{F_R} & \text{Spec } R \\
\downarrow f & & \downarrow f \\
\text{Spec } S & \xrightarrow{F_S} & \text{Spec } S.
\end{array}$$

In particular, unless $F_S = \text{id}$, then $F_R$ is not a morphism of $S$-schemes.

8.2. Linear algebra over schemes. There is a one-shot definition of quasicoherent sheaves. This is the best definition but maybe not the most workable since it involves 2-categories:

$$\text{QCoh}(X) := \text{holim}_{\text{Spec } R \to X} \text{Mod}_R.$$

The the holim indicates a more sophisticated but correct notion of a limit in the context of the 2-category of 1-categories but I will spare everyone this formulation. If you have, however, worked with categories like this, I encourage you to think that way. The theory of quasicoherent sheaves can be quite difficult to stomach on first try. Here are some signposts in the wilderness:

(1) if $X = \text{Spec } A$, an affine scheme, then

$$\text{QCoh}(X) = \text{Mod}_A.$$

In other words, the theory of quasicoherent sheaves over an affine scheme is “just linear algebra.”
(2) Insider $\text{QCoh}(X)$ there is a distinguished (in the royal sense, but not in any precise sense):

$$\text{Vect}(X) \subset \text{QCoh}(X)$$

which are much much more manageable and also interesting — they are called “vector bundles” and behave like objects under the same name that you might have encountered in differential geometry or other contexts. If $X = \text{Spec} A$ then

$$\text{Vect}(X) = \text{Mod}^{fgproj}_A,$$

the category of finitely generated projective $A$-modules. Here’s one way to think about it from this point of view (which is not necessarily good since this is equivalent to picking a basis): to give finitely generated projective module $M$ is equivalent to giving a free module $A^{\oplus n}$ and an idempotent $e : A^{\oplus n} \to A^{\oplus n}$. In other words, these are just (square) matrices.

(3) Other than vector bundles, quasicohernet sheaves which are interesting are those “coming from closed subschemes.” In the affine case: if $X = \text{Spec} A$, recall that a closed subscheme is given by a surjection $A \to B$. From this we can extract $I := \ker(A \to B)$ which is thus an $A$-module and is thus a quasicoherent sheaf on $X$.

(4) Some good categorical/homological properties: $\text{QCoh}(X)$ is a symmetric monoidal Grothendieck abelian category with enough injectives. This is not an easy theorem and is due to Gabber (a name you will hear many times if you are an algebraic geometer) but, in particular, you can:

- take kernels and cokernels;
- take tensor products.
- if $0 \to F_i \to G_i \to H_i \to 0$ is an exact sequence, then the sum

$$0 \to \bigoplus_i F_i \to \bigoplus_i G_i \to \bigoplus_i H_i \to 0,$$

remains exact;

- and, most importantly, you can take cohomology which, by the end of this class, you will be addicted to doing.

Here comes the definition:

**Definition 8.2.1.** Let $X$ be a scheme. A quasicoherent sheaf $\mathcal{F}$ on $X$ is the data:

1. for each map $f : \text{Spec} R \to X$ an $A$-module which we denote by $f^* \mathcal{F} \in \text{Mod}_A$,
2. for each map $\text{Spec} S \rightarrow \text{Spec} R$ we are given an isomorphism $\alpha_{f,g} : (f \circ g)^* \mathcal{F} \cong g^* f^* \mathcal{F}$, subject to the following condition: given

$$\text{Spec} T \xrightarrow{h} \text{Spec} S \xrightarrow{g} \text{Spec} R \xrightarrow{f} X$$

we have an equality of maps

$$\alpha_{f,gh} \circ (f \circ g \circ h)^* \mathcal{F} \cong h^* g^* f^* \mathcal{F}.$$

A morphism of quasicoherent sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is the data: for each morphism $f : \text{Spec} R \to X$ a morphism $f^* \varphi : f^* \mathcal{F} \to f^* \mathcal{G}$ such that all the induced diagrams commute. We denote by $\text{QCoh}(X)$ the category of quasicoherent sheaves on $X$.

**Example 8.2.2.** Let $X$ be a scheme. Then the structure sheaf $\mathcal{O}$ on $X$, which we denote by $\mathcal{O}_X$ is the object $\mathcal{O}_X \in \text{QCoh}(X)$ given by

$$f^* \mathcal{O}_X = R$$

for any $f : \text{Spec} R \to X$. 
Definition 8.2.3. Let $\mathcal{F}$ be a quasicoherent sheaf on $X$, then the **global sections** of $\mathcal{F}$ is defined to be

$$\Gamma(X, \mathcal{F}) := \text{Hom}_{\text{QCoh}(X)}(\mathcal{O}_X, \mathcal{F})$$

Let us attempt to unpack Definition 8.2.3. Suppose that we have a morphism $f : \text{Spec } R \to X$, then to an element of $\Gamma(X, \mathcal{F})$ gives us a map of $R$-modules:

$$f^* \mathcal{O}_X = R \to f^* \mathcal{F},$$

which just means that we are picking out an element of $f^* \mathcal{F}$ since this map is $R$-linear. So if we think of $f^* \mathcal{F}$ as a sheaf over the space $X$, this picks out a local section of $\mathcal{F}$. As $R$ varies, these maps should then be compatible in a suitable way.

Let us see what happens in the affine situation. So suppose that $X = \text{Spec } A$ is an affine scheme and say, $\mathcal{F}$ is a quasicoherent sheaf on $X$. Among other things, we have the identity morphism $\text{id} : \text{Spec } A \to \text{Spec } A$ so that we have an $A$-module

$$\text{id}^* \mathcal{F} =: M.$$

On the other hand, if $M$ is an $A$-module, and we are given maps $\text{Spec } S \xrightarrow{g} \text{Spec } R \xrightarrow{f} X$ which means maps of rings

$$A \to R \to S.$$

Then we have a canonical isomorphism:

$$M \otimes_A R \otimes_R S \cong M \otimes_A S,$$

which automatically satisfies the cocycle condition. Furthermore, we note that $\Gamma(X, \mathcal{O}) = A$ and that $\Gamma(X, \mathcal{F}) = \text{Hom}_{\text{Mod}_A}(A, M) = M$. We can thus conclude that:

**Proposition 8.2.4.** Let $X = \text{Spec } A$, then we have an adjunction:

$$(-) : \text{Mod}_A \rightleftarrows \text{QCoh}(X) : \Gamma$$

where $f^* (M) = M \otimes_A R$ for any morphism $f : \text{Spec } R \to \text{Spec } A$. This adjunction is furthermore an equivalence of categories.

Notice that, in particular, $\Gamma(X, \mathcal{O}) = A$ as an $A$-module. We want to say that this recovers the algebra structure of $A$. To do so we note that it makes sense to speak of algebras in quasicoherent sheaves:

**Proposition 8.2.5.** The category of quasicoherent sheaves is a symmetric monoidal category with unit $\mathcal{O}_X$.

**Proof.** For a pair $\mathcal{F}, \mathcal{G}$ of quasicoherent sheaves, we define

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \in \text{QCoh}(X)$$

to be the quasicoherent sheaf such that for each $f : \text{Spec } R \to X$, we get

$$f^*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) := f^* \mathcal{F} \otimes_R f^* \mathcal{G}.$$

With this note that

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{F},$$

since for all $f : \text{Spec } R \to X$ we have that

$$f^*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X) = f^* \mathcal{F} \otimes_R R = f^* \mathcal{F}.$$

Checking the axioms of being a symmetric monoidal category is a standard check. \qed

With this, here’s a definition we will need in the next class.

**Definition 8.2.6.** A **quasicoherent algebra** or a **quasicoherent sheaves of algebras** is a quasicoherent sheaf $\mathcal{F}$ together with maps

$$m : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F} \quad \epsilon : \mathcal{O}_X \to \mathcal{F},$$

which makes $\mathcal{F}$ as a commutative algebra object in the symmetric monoidal category $\text{QCoh}(X)$.
Proposition 8.2.7. Let $X = \text{Spec } A$. The equivalence of Proposition 8.2.4 descends to a compatible equivalence:

$$\text{CAlg}_A \rightleftarrows \text{CAlg}(\text{QCoh}(X)).$$

In particular:

$$\Gamma(O_X) = A$$

as rings.

Therefore, we have an isomorphism $X \cong \text{Spec } \Gamma(O_X)$.

8.3. **A word on why quasicoherent sheaves?** It is algebro-geometric propaganda that quasicoherent sheaves are important and that they should be the next thing one introduces after introducing schemes. One motivation is that it that to have a mechanism of showing that some scheme is not affine, we will need to examine the global sections of the structure sheaf. Hence, the data of “global functions” or, more precisely, the global sections of the structure sheaf captures the entire affine scheme. On the other hand, the birational classification of algebraic varieties relies on studying invariants that one can extract out of quasicoherent sheaves.

In other words, to study an affine scheme it is not enough to look at global sections of $\mathcal{O}$, we must look at the global sections of $\text{QCoh}(\_ \_ \_ )$. I hope this is enough motivation for us to spend sometime trying to study quasicoherent sheaves.

There is however a more primary motivation. In the 60’s Gabriel and Rosenberg proved the following remarkable “reconstruction theorem.”

**Theorem 8.3.1** (Gabriel, Rosenberg). Let $X, Y$ be schemes which are quasiseparated (to be defined later), then the following are equivalent:

1. there is an equivalence of categories $\text{QCoh}(X) \cong \text{QCoh}(Y)$;
2. $X$ and $Y$ are isomorphic as schemes.

8.4. **Exercises.**

**Exercise 8.4.1.** Let $X$ be a scheme and $R$ a ring. Prove that the set $X(R)$ is canonically isomorphic to sections of $X \times_{\text{Spec } \mathbb{Z}} \text{Spec } R$. Compute the following sets:

1. sections of $A^1_R \to \text{Spec } R$,
2. sections of $A^1_\mathbb{C} \to \text{Spec } \mathbb{C}$.

**Exercise 8.4.2.** Let $\mathcal{F} : \text{CAlg}_A \to \text{Set}$ be a functor where $A$ is a nonzero ring.

1. prove that in $\text{Fun}(\text{CAlg}_A, \text{Set})$, the Yoneda embedding exhibits $\text{Spec } A$ as the final object by constructing a canonical map $\mathcal{F} \to \text{Spec } A$.
2. Prove that “being a scheme” is a local property in the following manner by proving that the following are equivalent:
   (a) the prestack
   $$\mathcal{F} : \text{CAlg} = \text{CAlg}_\mathbb{Z} \overset{\text{forget}}{\longrightarrow} \text{CAlg}_A \overset{\mathcal{F}}{\longrightarrow} \text{Set}$$
   is a scheme,
   (b) for each $\text{Spec } R \to \text{Spec } A$, the pullback $\mathcal{F} \times_{\text{Spec } R} \text{Spec } A$ is a scheme.

9. **Lecture 8: Vector Bundles, Affine Morphisms**

10. **Lecture 9: Projective Space aka the Most Important Scheme You Should Know**

References


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