MATH 232: ALGEBRAIC GEOMETRY I

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Contents

"In our acquisition of knowledge of the Universe (whether mathematical or otherwise) that which renovates the quest is nothing more nor less than complete innocence." - A. Grothendieck.

1. References

1.1. **Books.** In principle, this class is about Grothendieck's [EGA1] which signals the birth of modern algebraic geometry. It is an extremely technical document on its own and shows one of the many ways mathematics was developed organically. The French is not too hard and I recommend that you look through the book before the start of class — I might also assign readings from here occasionally with the promise that the French (and some Google translate) will not hinder your mathematical understanding.

Here are some textbooks in algebraic geometry.

- ([Sha13]) Shafarevich's book is a little more old school than the others in this list, but is valuable in the **examples** it gives.
- ([Har77]) Hartshorne's book has long been the "gold-standard" for algebraic geometry textbook. I learned the subject from this book first. It is **terse** and has plenty of **good exercises and problems**. However, the point of view that this book takes will be substantially different from one we will take in this class, though I will most definitely steal problems from here.
- ([GW10]) This is essentially a translation of Grothendieck's EGA (plus more) and is closer to the point of view of this class. Just like the original text, it is **relentlessly general** and very **lucid** in its exposition.
- ([Vak]) Arguably the most **inviting** book in this list, and modern in its outlook.
- ([DG80]) As far as I know this is still the only textbook reference to the **functor-of-points** point of view to algebraic geometry.

1.2. **Lecture Notes.** There are also several class notes online in algebraic geometry. I will add on to this list as the class progresses.

- ([Ras]) This is the closest document to our approach to this class. In fact, I will often present directly from these notes.
- ([Gat]) This is a "varieties" class, so the approach is very different, but I find it very helpful for lots of examples.

1.3. **Online textbooks.** There has been an explosion of online textbooks for algebraic geometry recently, though they are perhaps they are more like "encyclopedias."

- ([Stacks]) Johan de Jong at Columbia was the trailblazer in this industry and most, if not all, facts about algebraic geometry that will be taught will appear here, with proofs.
- ([cri]) Similar but for commutative algebra. Much more incomplete.
- ([fpp]) A translation project for EGA.

2. Lecture 1: What is algebraic geometry?

In its essence, algebraic geometry is the study of solutions to polynomial equations. What one means by "polynomial equations," however, has changed drastically throughout the latter part of the 20th century. To meet the demands in making constructions, ideas and theorems in classical algebraic geometry rigorous has given birth to a slew of techniques and ideas which are applicable to a much, much broader range of mathematical situations.

To begin with, let us recall the famous Fermat problem:

Theorem 2.0.1 (Taylor-Wiles). *Let $n \geqslant 3$, then $x^n + y^n = 1$ has no solutions over $\mathbf{Q}$ when $x, y \neq 0$.*

This is a problem in algebraic geometry. In the language that we will learn in this class, we will be able to associate a **smooth, projective scheme** Fer_n which is, informally, given by a homogeneous polynomial equation $x^n + y^n = z^n$, equipped with a canonical **morphism**

$$\begin{array}{c} \mathrm{Fer}_n \\ \downarrow \\ \operatorname{Spec} \mathbf{Z}, \end{array}$$

such that its set of sections

$$\begin{array}{c} \mathrm{Fer}_n \\ \uparrow\!\downarrow \\ \operatorname{Spec} \mathbf{Z}, \end{array}$$

correspond to potential solutions to the Fermat equation. It is in this language that the Fermat problem was eventually solved.

The point-of-view we wish to adopt in this class, however, is one that goes by **functor-of-points**. In this highly abstract, but more flexible, approach schemes appear as what they are *supposed to be* which is often easier to think about. For us, the basic definition is:

Definition 2.0.2. A **prestack** is a functor from the category of commutative rings to sets:

$$\mathcal{F} : \mathrm{CAlg} \to \mathrm{Set}.$$

Remark 2.0.3. A note on terminology: this is non-standard. What should be called (and was called by Grothendieck) a prestack is a functor

$$\mathcal{F} : \mathrm{CAlg} \to \mathrm{Cat},$$

where Cat is the (large, $(2,1)$-)category of small categories. If we think of a set as a category with no non-trivial morphisms between the objects, then the above definition is a special case of this Grothendieck definition of a prestack. We will not consider functors into categories in this class so we will reserve the term prestack for such a functor above (as opposed to something like "a prestack in sets"). Perhaps it should be called a presheaf, but a presheaf should really just be an arbitrary functor

$$\mathcal{F} : \mathcal{D}^{\mathrm{op}} \to \mathrm{Set}.$$

To make this definition jibe with the Fermat scheme above, let us note that the equation $x^n + y^n = 1$ is defined for *any* ring. Therefore we can define

$$\widetilde{\mathrm{Fer}}_n(\mathrm{R}) = \{(a,b,c) : a^n + b^n = 1\} \subset \mathrm{R}^{\times 2}.$$

The theorem of Taylor and Wiles can then be restated as the fact that

$$\widetilde{\mathrm{Fer}}_n(\mathbf{Z}) = \emptyset \qquad n \geqslant 3;$$

However we caution that this is not the same as the scheme Fer_n that we have alluded to above since it is not projective — something that we will address in the class.

Another key idea in algebraic geometry is the question of parametrizing solutions of polynomial equations in a reasonable way. Let us consider $\widetilde{\mathrm{Fer}}_2$, which is the set of solutions to $x^2 + y^2 = 1$. We have a canonical equality (the first one is more or less the same as the above):

$$\widetilde{\mathrm{Fer}}_2(\mathbf{R}) = \mathrm{S}^1,$$

as we all know. Here are three other possible answers:

(1) $\widetilde{\mathrm{Fer}}_2(\mathbf{R}) = (\cos\theta, \sin\theta) \qquad 0 \leqslant \theta < 2\pi$,
(2) $\widetilde{\mathrm{Fer}}_2(\mathbf{R}) = (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}) \qquad t \in \mathbf{R}$,
(3) $\widetilde{\mathrm{Fer}}_2(\mathbf{R})$ is the set of all triangles with hypotenuse 1 up to congruence.

The first answer *does not* belong to the realm of (conventional) algebraic geometry which, by its very nature, concerns only polynomial functions. In other words, we dismiss transcendentals like exp and cos, sin. However the language of this class is actually powerful enough to capture transcendentals and reconstruct the familiar theory of differential geometry. The third answer will turn out to belong to the realm of algebraic geometry as well, but that will be reserved for a second course. The second answer does belong to the realm of algebraic geometry that we will study in this class: we can use **rational functions** of one variable in order to describe $\mathrm{Fer}_2(\mathbf{R})$. In fact, this parametrization proves

Theorem 2.0.4. *A quadric hypersurface in $\mathbf{P}^2$ with a rational point is rational. In fact, any quadric hypersurface with a rational point is rational.*

The proof of this result is "basically known" to pre-Grothendieck algebraic geometers: we pick the rational point and stereographcially project into a hyperplane. Since a quadric means that it is cut out by a degree two polynomial, it must hit one other point. This defines a **rational map** — one that is defined "almost everywhere" which is evidently an "isomorphism."

One of the major thread of investigation in algebraic geometry and comes under the name of **birational geometry** and the above result belongs to this area. An example of a beautiful result that belongs to modern birational geometry is:

Theorem 2.0.5 (Clemens and Griffiths)**.** *A nonsingular cubic threefold over* $\mathbf{C}$ *is* not *rational.*

Theorem 2.0.5 is a non-existence proof — it says that there is no way to "rationally parametrize" the cubic threefold. If you have been trained in algebraic topology, you will feel like some kind of cohomological methods would be needed. The words that you should look for are "intermediate Jacobians," an object whose real birthplace is Hodge theory.

One of the major, open problems in the subject is:

Question 2.0.6. Is a generic cubic fourfold over $\mathbf{C}$ rational?

Recently, Katzarkov, Kontesvich and Pantev claimed to have made substantial progress towards this problem, but a write-up is yet to appear. More generally, a central question in algebraic geometry is:

Question 2.0.7. How does one classify algebraic varieties up to birational equivalence?

In topology, recall that a topological (closed) surface can be classified by genus or, better, Euler charateristic:

(1) if $\chi(\Sigma) < 0$, then Σ must be the Riemann sphere,
(2) if $\chi(\Sigma) = 0$ then Σ must be a torus — in the terminology of this class it is an **elliptic curve**,
(3) most surfaces have $\chi(\Sigma) > 0$ and they are, in some sense, the "generic situation."

This kind of trichotomy can be extended to higher dimensional varieties (topological surfaces being a 1-dimensional algebraic variety over $\mathbf{C}$). The **minimal model program** seeks to find "preferred" representatives in each class.

2.1. **Algebraic geometry beyond algebraic geometry.** The field of birational geometry is extremely large and remains an active area of research. But classifying algebraic varieties is not the only thing that algebraic geometry is good for. We have seen how it can be used to phrase the Fermat problem and eventually hosts its solution. There are other areas where algebraic geometry has proven to be the optimal "hosts" for problems.

One of the most prominent areas is representation theory where the central definition is very simple a group homomorphism

$$\rho : \mathrm{G} \to \mathrm{GL(V)}.$$

If we are interested in representations valued in k-vector spaces, then the collection of all G-representations form a category called $\mathbf{Rep}_k(\mathrm{G})$. This category has an algebro-geometric incarnation: it is the category of **quasicoherent sheaves** over the an algebro-geometric gadget called a **algebraic stack** (in this case, denoted by BG) which is a special, more manageable class of prestacks but are slightly more mysterious gadgets than just algebraic varieties. Quasicoherent sheaves are fancy versions of vector bundles — they include gadgets whose fibers "can jump" although we will study restrictions on how exactly they jump. In any case, the field of **geometric representation theory** takes as starting point that representation theory is "just" the study of the geometric object BG and brings to bear the tools of algebraic geometry onto representation theory.

We have seen that algebraic geometry hosts number theory through the problem of the existence of rational points on a variety. Another deep problem of number theory that lives within modern algebraic geometry is the **Riemann hypothesis**. In algebro-geometric terms it can be viewed as a way to assemble solutions of an equation over fields of different characteristics.

Soon we will learn what it means for a morphism of schemes $f : \mathrm{X} \to \operatorname{Spec} \mathbf{Z}$ to be **proper** and for X to be **regular, geometrically connected** and **dimension d**. To this set-up we can associate the **Hasse-Weil zeta function**:

$$\zeta_{\mathrm{X}}(s) := \prod_{x \in |\mathrm{X}|} (1 - \sharp(\kappa(x))^{-s})^{-1}.$$

where:

(1) the set $|\mathrm{X}|$ is the set of **closed points** of X,
(2) $\kappa(x)$ is the **residue field** of x which is a finite extension of $\mathbf{F}_p$ for some prime $p > 0$.

This function is expected to be extending to all of the complex numbers (as a meromorphic function). There is a version $\zeta_{\overline{\mathrm{X}}}(s)$ which takes into account the "analytic part" of X as well:

Conjecture 2.1.1 (Generalized Riemann hypothesis)**.** *If $s \in \mathbf{C}$ is a zero of $\zeta_{\overline{\mathrm{X}}}(s)$ then:*

$$2\mathrm{Re}(s) = \nu,$$

where $\nu \in [0, 2d]$.

One of the more viable approaches to verifying the generalized Riemann hypothesis is via **cohomological methods** — one would like to find a cohomology theory for schemes to which one can "extract" in a natural way the Hasse-Weil zeta function. One reason why one might expect this is the (also conjectured) functional equation

$$\zeta_{\overline{\mathrm{X}}}(s) \sim \zeta_{\overline{\mathrm{X}}}(\dim(\mathrm{X}) - s)$$

where $\sim$ indicates "up to some constant." This is a manifestation of a certain Poincaré duality in this cohomology theory which witnesses a certain symmetry between the cohomology groups and governed by the dimension of X. If X is concentrated at a single prime, then the Riemann hypothesis was proved by Deligne using **étale cohomology**. Recent work of Hesselholt, Bhatt, Morrow and Scholze have made some breakthrough towards setting up this cohomology theory but the Riemann hypothesis is, to the instructor's knowledge, still out of reach.

2.1.2. *Problem Set 1: categorical preliminaries.* Here is a standard definition. We assume that every category in sight is **locally small** so that $\mathrm{Hom}(x, y)$ is a set, while the set of objects, $\mathrm{Obj}(\mathcal{C})$, is not necessarily a set (so only a proper class).

Definition 2.1.3. A functor $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ is **fully faithful** if for all $x, y \in \mathcal{C}$, the canonical map

$$\mathrm{Hom}(x, y) \to \mathrm{H}om(\mathrm{F}x, \mathrm{F}y)$$

is an isomorphism. We say that it is **conservative** if it reflects isomorphisms: an arrow $f : c \to c'$ in $\mathcal{C}$ is an isomorphism if and onnly if $\mathrm{F}(f) : \mathrm{F}(c) \to \mathrm{F}(c')$ is.

Exercise 2.1.4. *Let*

$$\mathrm{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathrm{G}$$

be an adjunction (in these notes we always write the left adjoint on the left). Prove

(1) F *preserves all colimits,*
(2) G *preserves all limits,*
(3) The functor F *is fully faithful if and only if the unit transformation*

$$\mathrm{id} \to \mathrm{G} \circ \mathrm{F}$$

is an isomorphism.

(4) $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ *is an equivalence of categories if and only if* F *is fully faithful and* G *is conservative.*

Exercise 2.1.5. *Prove that $\mathcal{C}$ admits all colimits if and only if it admits coproducts and coequalizers. What kind of colimits do the following categories have (you do not have to justify your answer):*

(1) the category of finite sets,
(2) the category of sets,
(3) the category of finitely generated free abelian groups,

(4) the category of abelian groups,
(5) the category of finite dimensional vector spaces,
(6) the category of all vector spaces,
(7) the category of finitely generated free modules over a commutative ring R*,*
(8) the category of finitely generated projective modules over a ring R*,*
(9) the category of all projective modules over a ring R*.*

Exercise 2.1.6. *Give a very short proof (no more than one line) of the dual assertion:* $\mathcal{C}$ *admits all limits if and only if it admits products and equalizers.*

Exercise 2.1.7. *Prove that the limit over the empty diagram gives terminal object, while the colimit over the empty diagram gives the initial object.*

Exercise 2.1.8. *For any small category* $\mathcal{C}$*, we can form the presheaf category*

$$\mathrm{PSh}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}).$$

Prove:

(1) If

$$\mathrm{F} : \mathrm{I} \to \mathrm{PSh}(\mathcal{C}) \qquad i \mapsto \mathrm{F}_i$$

is a functor and I *is a small diagram, then for any* $c \in \mathcal{C}$ *the canonical map*

$$(\operatorname*{colim}_{\mathrm{I}} \mathrm{F}_i)(c) \to \operatorname*{colim}_{\mathrm{I}}(\mathrm{F}_i(c))$$

is an isomorphism.

(2) Formulate and prove a similar statement for limits.
(3) Conclude that $\mathrm{PSh}(\mathcal{C})$ *admits all limits and colimits.*

Exercise 2.1.9. *Prove the Yoneda lemma in the following form: the functor*

$$y : \mathcal{C} \to \mathrm{PSh}(\mathcal{C}) \qquad c \mapsto y(c)(x) = \mathrm{Hom}(x, c).$$

is fully faithful. Any functor in the image of y *is called* **representable**

Exercise 2.1.10. *Prove that any functor* $\mathrm{F} \in \mathrm{PSh}(\mathcal{C})$ *is a colimit of representable functors. This entails constructing a natural transformation*

$$\operatorname{colim} y(c) \to \mathrm{F}$$

where the domain is a colimit of a diagram of functors where each functor is representable, and proving that this natural transformation is an isomorphism when evaluated at each object of $\mathcal{C}$*.*

Exercise 2.1.11. *We say that a category* $\mathcal{C}$ *is* **essentially small** *if it is equivalent to small category. Let* R *be a commutative ring and consider* $\mathrm{CAlg}_{\mathrm{R}}$ *to be the category of commutative* R*-algebras. We say that an* R*-algebra* S *is* **finite type** *if it admits an* R*-linear* surjective *ring homomorphism*

$$\mathrm{R}[x_1, \cdots x_n] \to \mathrm{S}.$$

Consider the full subcategory $\mathrm{CAlg}^{\mathrm{ft}}_{\mathrm{R}} \subset \mathrm{CAlg}_{\mathrm{R}}$ *of finite type* R*-algebras. Prove that:*

(1) The collection of R*-algebras of the form*

$$\{\mathrm{R}[x_1, \cdots, x_n]/\mathrm{I} : \mathrm{I} \textit{ is an ideal}\}$$

forms a set (this is not meant to be hard and does not require knowledge of "set theory").

(2) Prove that the category of finite type R*-algebras are equivalent to the subcategory of* R*-algebras of the form* $\mathrm{R}[x_1, \cdots, x_n]/\mathrm{I}$ *(this is not meant to be hard and does not require knowledge of "set theory").*
(3) Conclude from this that $\mathrm{CAlg}^{\mathrm{ft}}_{\mathrm{R}}$ *is an essentially small category.*

Exercise 2.1.12. *We define the subcategory of* **left exact functors**

$$\mathrm{PSh}_{\mathrm{lex}}(\mathcal{C}) \subset \mathrm{PSh}(\mathcal{C})$$

to be the subcategory of those functors which preserves finite limits. These are functors F *such that for any finite diagram*[1] $\alpha : \mathrm{I} \to \mathcal{C}$*, the canonical map*

$$\mathrm{F}(\operatorname*{colim}_{\mathrm{I}} \alpha) \to \lim_{\mathrm{I}} \mathrm{F}(\alpha)$$

is an isomorphism. Prove:

(1) a category $\mathcal{C}$ *admits all finite limits if and only if it admits final objects and pullbacks;*
(2) for a functor F *to be left exact, it is necessary and sufficient that* F *preserves final objects and pullbacks.*
(3) Prove that the yoneda functor factors as $y : \mathcal{C} \to \mathrm{PSh}_{\mathrm{lex}}(\mathcal{C})$*.*
(4) If $f : \mathcal{C} \to \mathcal{D}$ *is a functor, we define*

$$f^* : \mathrm{PSh}(\mathcal{D}) \to \mathrm{PSh}(\mathcal{C}) \qquad f^*\mathrm{F} = \mathrm{F} \circ f.$$

Prove that if f *preserves finite colimits, then we have an induced functor*

$$f^* : \mathrm{PSh}_{\mathrm{lex}}(\mathcal{D}) \to \mathrm{PSh}_{\mathrm{lex}}(\mathcal{C}).$$

In the third problem set, we will use this to prove the adjoint functor theorem and construct the sheafification functor.

Exercise 2.1.13. *We say that* $\mathcal{C}$ *is* **locally presentable** *if there exists a subcategory* $i : \mathcal{C}^c \subset \mathcal{C}$ *(called the category of* **compact objects***) which is essentially small and is closed under finite colimits such that the functor*

$$\mathcal{C} \to \mathrm{PSh}_{\mathrm{lex}}(\mathcal{C}) \xrightarrow{i^*} \mathrm{PSh}_{\mathrm{lex}}(\mathcal{C}^c)$$

is an equivalence of categories. Prove

(1) the category Sets *is locally presentable with* Sets^c *being the subcategory of finite sets,*
(2) the category Vect_k *is locally presentable with* Vect^c *being the subcategory of finite-dimensional vector spaces.*

3. Lecture 2: Prestacks

Throughout the course we will denote by CAlg the category of commutative rings.

Definition 3.0.1. A **prestack** is a functor

$$\mathrm{X} : \mathrm{CAlg} \to \mathrm{Set}.$$

This means that to each commutative ring R, we assign the set X(R) and for each morphism of commutative rings $f : \mathrm{R} \to \mathrm{S}$ we have a morphism of sets

$$f^* : \mathrm{X(R)} \to \mathrm{X(S)}.$$

Furthermore, these satisfy the obvious compatibilities to be a functor.

Definition 3.0.2. A **morphism of prestacks** is a natural transformation $g : \mathrm{X} \to \mathrm{Y}$ of functors. This means that for each morphism of commutative rings $f : \mathrm{R} \to \mathrm{S}$ we have a commuting diagram

$$\begin{array}{ccc} \mathrm{X(R)} & \xrightarrow{f^*} & \mathrm{X(S)} \\ {\scriptstyle g_{\mathrm{R}}}\downarrow & & \downarrow{\scriptstyle g_{\mathrm{S}}} \\ \mathrm{Y(R)} & \xrightarrow{f^*} & \mathrm{X(S)}. \end{array}$$

An R**-point** of a prestack is point $x \in \mathrm{X(R)}$; this is the same thing as a morphism of prestacks $\operatorname{Spec} \mathrm{R} \to \mathrm{X}$ by the next

[1] This just means that I is a category with finitely many objects and finitely many morphisms.

Lemma 3.0.3 (Yoneda)**.** *For all prestack* X *and all* $\mathrm{R} \in \mathrm{CAlg}$*, we have a canonical isomorphism*

$$\mathrm{Hom}(\mathrm{Spec}\,\mathrm{R}, \mathrm{X}) \cong \mathrm{X}(\mathrm{R}).$$

In particular we have that

$$\mathrm{Hom}(\mathrm{Spec}\,\mathrm{R}, \mathrm{Spec}\,\mathrm{S}) \cong \mathrm{Spec}\,\mathrm{S}(\mathrm{R}) = \mathrm{Hom}(\mathrm{S}, \mathrm{R}).$$

Note the reversal of directions.

We denote by **PStk** the category of prestacks. We already have a wealth of examples:

Definition 3.0.4. Let R be a commutative ring, We define

$$\mathrm{Spec}\,\mathrm{R} : \mathrm{CAlg} \to \mathrm{Set} \qquad \mathrm{S} \mapsto \mathrm{Hom}_{\mathrm{CAlg}}(\mathrm{R}, \mathrm{S}).$$

An **affine scheme** is a prestack of this form.

Remark 3.0.5. If a prestack is representable, then the ring **representing** it is unique up to unique isomorphism. This is a consequence of the Yoneda lemma. In more detail, the Yoneda functor takes the form

$$\mathrm{Spec} : \mathrm{CAlg}^{\mathrm{op}} \to \mathbf{PStk} = \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}).$$

This functor is fully faithful so we may (somewhat abusively) identify $\mathrm{CAlg}^{\mathrm{op}}$ with its image in **PStk**. The category of affine schemes is then taken to be the opposite category of commutative rings.

Example 3.0.6. Let $n \geqslant 0$. Then we define the prestack of **affine space of dimension** n as

$$\mathbf{A}^n_{\mathbf{Z}} : \mathrm{CAlg} \to \mathrm{Set} \qquad \mathrm{R} \mapsto \mathrm{R}^{\times n}.$$

In the homeworks, you will be asked to prove that this prestack is an affine scheme, represented by $\mathbf{Z}[x_1, \cdots, x_n]$.

Example 3.0.7. Suppose that $f(x) \in \mathbf{Z}[x, y, z]$ is a polynomial in three variables; for a famous example this could be $f(x, y, z) = x^n + y^n - z^n$. For each ring A, we define

$$\mathrm{V}(f)(\mathrm{R}) := \{(a, b, c) : f(a, b, c) = 0\} \subset \mathrm{R}^{\times 3}.$$

Note that this indeed defines a prestack: given a morphism of rings $\varphi : \mathrm{R} \to \mathrm{S}$, we have a morphism of sets

$$\mathrm{V}(f)(\mathrm{R}) \to \mathrm{V}(f)(\mathrm{S})$$

since $f(\varphi(a), \varphi(b), \varphi(c)) = \varphi(f(a, b, c)) = \varphi(0) = 0$. In fact, we have a morphism of prestacks (in the sense of the next definition)

$$\mathrm{V}(f) \to \mathbf{A}^{\times 3}_{\mathbf{Z}},$$

where $\mathbf{A}^{\times 3}_{\mathbf{Z}}(\mathrm{R}) = \mathrm{R}^{\times 3}$.

3.1. **Operation on prestacks I: fibered products.** One of the key ideas behind algebraic geometry is to restrict ourselves to objects which are defined by polynomial functions. More abstractly we want to restrict ourselves to objects which arise from other objects in a *constructive manner*. This is both a blessing and a curse — on the one hand it makes objects in algebraic geometry rather rigid but, on the other, it gives objects in algebraic geometry a "tame" structure.

Example 3.1.1. As a warm-up, consider n-dimensional complex space $\mathbf{C}^n$ and suppose that we have a polynomial function $\mathbf{C}^n \to \mathbf{C}$. Then the **zero set** of f is defined via the pullback

$$\begin{array}{ccc} \mathrm{Z}(f) & \longrightarrow & \mathbf{C}^n \\ \downarrow & & \downarrow f \\ \{0\} & \longrightarrow & \mathbf{C}. \end{array}$$

We want to say that $\mathrm{Z}(f)$ has the structure of a prestack or, later, a scheme. Of course the above diagram presents $\mathrm{Z}(f)$ as a set but we can also take the pullback in, say, the category of $\mathbf{C}$-analytic spaces so that $\mathrm{Z}(f)$ inherits such a structure (if a pullback exists! and it does).

Example 3.1.2. Another important construction in algebraic geometry is the notion of the **graph**. Suppose that $f : \mathrm{X} \to \mathrm{Y}$, then its graph is the set

$$\Gamma_f := \{(x, y) : f(x) = y\} \subset \mathrm{X} \times \mathrm{Y}.$$

Suppose that X, Y have the structure of a scheme, or an $\mathbf{C}$-analytic spaces or a manifold etc., then we want to say that Γ_f does inherit this structure. To do so we note that we can present Γ_f in the following manner:

$$\begin{array}{ccc} \Gamma_f & \longrightarrow & \mathrm{Y} \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ \mathrm{X} \times \mathrm{Y} & \xrightarrow{f \times \mathrm{id}} & \mathrm{Y} \times \mathrm{Y} \end{array}$$

Definition 3.1.3. Suppose that $\mathrm{X} \to \mathrm{Y} \leftarrow \mathrm{Z}$ is a **cospan** of prestacks, then the **fiber product of** X **and** Y **over** Z is defined as

$$(\mathrm{X} \times_{\mathrm{Y}} \mathrm{Z})(\mathrm{R}) := \mathrm{X}(\mathrm{R}) \times_{\mathrm{Y}(\mathrm{R})} \mathrm{Z}(\mathrm{R}).$$

It will be an exercise to verify the universal property of this construction.

Example 3.1.4. Suppose that we have a **span** of rings $\mathrm{R} \leftarrow \mathrm{S} \to \mathrm{T}$ so that we have a cospan of prestacks $\operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{S} \leftarrow \operatorname{Spec} \mathrm{T}$. Then (exercise) we have a natural isomorphism

$$\operatorname{Spec} \mathrm{R} \times_{\operatorname{Spec} \mathrm{S}} \operatorname{Spec} \mathrm{T} \cong \operatorname{Spec}(\mathrm{R} \otimes_{\mathrm{S}} \mathrm{T}).$$

Example 3.1.5. A **regular function on a prestack** is a morphism of prestacks $\mathrm{X} \to \mathbf{A}^1$. If $\mathrm{X} = \operatorname{Spec} \mathrm{R}$, then this classifies a map of commutative rings $\mathbf{Z}[x] \to \mathrm{R}$ which is equivalent to picking out a single element $f \in \mathrm{R}$. The **zero locus** of f is the prestack

$$\mathrm{Z}(f) := \mathrm{X} \times_{\mathbf{A}^1} \{0\}$$

where $\{0\} \to \mathbf{A}^1$ is the map corresponding to $\mathbf{Z}[x] \to \mathbf{Z}, x \mapsto 0$.

3.2. Closed immersions.

A closed immersion is a special case of a **subprestack**

Definition 3.2.1. A **subprestack** of a prestack $\mathcal{F}$ is a prestack $\mathcal{G}$ equipped with a natural transformation $\mathcal{G} \to \mathcal{F}$ such that for any $\mathrm{R} \in \mathrm{CAlg}$, the map

$$\mathcal{G}(\mathrm{R}) \to \mathcal{F}(\mathrm{R})$$

is an injection. We will often write $\mathcal{F} \subset \mathcal{G}$ for subprestacks.

Remark 3.2.2. This is equivalent to saying that $\mathcal{G} \to \mathcal{F}$ is a monomorphism in the category of prestacks.

The important thing about a subprestack is that for any morphism $\mathrm{R} \to \mathrm{R}'$, the requirement that $\mathcal{G} \to \mathcal{F}$ is a natural transformation enforces the commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{G}(\mathrm{R}) & \longrightarrow & \mathcal{F}(\mathrm{R}) \\ \downarrow & & \downarrow \\ \mathcal{G}(\mathrm{R}') & \longrightarrow & \mathcal{F}(\mathrm{R}'). \end{array}$$

which should be read as: "the map $\mathcal{G}(\mathrm{R}) \to \mathcal{F}(\mathrm{R}')$ factors through the subset $\mathcal{G}(\mathrm{R}')$."

Definition 3.2.3. A **closed immersion** of affine schemes is a morphism $\operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{S}$ such that the induced map of rings $\mathrm{S} \to \mathrm{R}$ is surjective. In this case we say that $\operatorname{Spec} \mathrm{R}$ is a **closed subscheme** of $\operatorname{Spec} \mathrm{S}$.

Let us try to understand what this means. the map $\varphi : \mathrm{S} \to \mathrm{R}$, if surjective, is equivalent to the data of an ideal $\mathrm{I} = \ker(\varphi)$. As stated before, we should think of S as the ring of functions on a "space" Spec S and so an ideal of S is a collection of functions which are closed under the action of S. Now, the "space" Spec R should be thought of as the space on which the functions that belong to I vanish. In other words a closed immersion is one of the form $\operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{R}/\mathrm{I}$.

Exercise 3.2.4. *Suppose that* $f : \mathbf{A}^n \to \mathbf{A}^1$ *is regular function. Prove that the* $\mathrm{Z}(f) \to \mathbf{A}^n$ *is a closed immersion corresponding to a map of rings* $\mathbf{Z}[x] \to \mathbf{Z}[x]/(f)$.

Here is how one can globalize this definition:

Definition 3.2.5. A morphism of prestacks $\mathrm{X} \to \mathrm{Y}$ is a **closed immersion** or a **closed subprestack** if for any morphism $\operatorname{Spec} \mathrm{R} \to \mathrm{Y}$ then:

(1) the prestack $\operatorname{Spec} \mathrm{R} \times_{\mathrm{Y}} \mathrm{X}$ is representable and,
(2) the morphism
$$\operatorname{Spec} \mathrm{R} \times_{\mathrm{Y}} \mathrm{X} \to \operatorname{Spec} \mathrm{R}$$
is a closed immersion.

3.2.6. *Problem Set 2.*

Exercise 3.2.7. *What does* Spec(0) *represent?*

Exercise 3.2.8. *Prove that the category of prestacks admit all limits and all colimits.*

Exercise 3.2.9. *Prove that the prestack* $\mathbf{A}^n_{\mathbf{Z}}$ *is representable by* $\mathbf{Z}[x_1, \cdots, x_n]$.

Exercise 3.2.10. *Consider the prestack*
$$\mathbf{G}_m : \mathrm{R} \mapsto \mathrm{R}^{\times}.$$
Here $\mathrm{R}^{\times}$ *is the multiplicative group of unit elements in* R. *Prove that* $\mathbf{G}_m$ *is representable. What ring is it representable by?*

Exercise 3.2.11. *Suppose that* $f : \mathbf{A}^n \to \mathbf{A}^1$ *is regular function. Prove that the* $\mathrm{Z}(f) \to \mathbf{A}^n$ *is a closed immersion corresponding to a map of rings* $\mathbf{Z}[x] \to \mathbf{Z}[x]/(f)$.

Exercise 3.2.12. *Consider the prestack*
$$\mathrm{GL}_n : \mathrm{R} \mapsto \mathrm{GL}_n(\mathrm{R}).$$
Prove that it is representable. What ring is it representable by?

If R is a ring we write $\mathrm{R}_{\mathfrak{p}}$ to be the localization of R at $\mathfrak{p}$. We write $\mathfrak{m}_{\mathfrak{p}}$ to be the maximal ideal of said local ring and write
$$\kappa(\mathfrak{p}) := \mathrm{R}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}.$$

Exercise 3.2.13. *Let* $\mathrm{R} \in \mathrm{CAlg}$ *and* K *a field. Prove that there is a natural bijection between*
$$\{\operatorname{Spec} \mathrm{K} \to \operatorname{Spec} \mathrm{R}\}$$
with
$$\{\textit{prime ideals } \mathfrak{p} \subset \mathrm{R} \textit{ with an inclusion } \kappa(\mathfrak{p}) \hookrightarrow \mathrm{K}\}.$$

The **Zariski tangent space** of Spec R **at a prime ideal** $\mathfrak{p}$ is the $\kappa(\mathfrak{p})$-vector space given by
$$\mathrm{T}_{\mathfrak{p}} \operatorname{Spec} \mathrm{R} := (\mathfrak{m}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}^2)^{\vee}.$$

Exercise 3.2.14. *Let* $\mathrm{R} \in \mathrm{CAlg}$ *and* k *a field and suppose that* R *is a* k*-algebra. Prove that there is a bijection between*
$$\{k\textit{-morphisms of rings}\, \mathrm{R} \to \kappa[x]/(x^2)\}$$
with
$$\{\textit{prime ideals } \mathfrak{p} \textit{ of } \mathrm{R} \textit{ with residue field } k \textit{ and an element of the Zariski tangent space at } \mathfrak{p}\}.$$

Exercise 3.2.15. *Prove that the functor*

$$\operatorname{Spec} : \mathrm{CAlg} \to \mathbf{PStk}$$

1. *is fully faithful,*
2. *preserves all colimits in the sense that if* $\{R_\alpha\}_{\alpha \in A}$ *is a diagram of commutative rings then for all* $S \in \mathrm{CAlg}$*, the canonical map*
$$\operatorname*{colim}_{\alpha}(\operatorname{Spec} R_\alpha)(S) \to \lim(\operatorname{Spec} R_\alpha)(S)$$
is an isomorphism. Deduce, in particular, that Spec *converts tensor products of commutative rings to pullback.*
3. *Show, by example, that* Spec *does not preserve limits.*

Exercise 3.2.16. *Prove that a closed immersion is a subprestack.*

For the next exercise, recall that if $\mathcal{C}$ is a category with products and $X \in \mathcal{C}$, then the identity morphism $\mathrm{id} : X \to X$ induces the **diagonal** map

$$\Delta : X \to X \times X.$$

Exercise 3.2.17. *Let* $R \in \mathrm{CAlg}$ *and consider the multiplication map* $R \otimes_{\mathbf{Z}} R \to R$*. Prove:*

1. *the corresponding map* $\operatorname{Spec} R \to \operatorname{Spec} R \times \operatorname{Spec} R$ *is given by the diagonal morphism,*
2. *prove that* Δ *is a closed immersion of prestacks.*

Exercise 3.2.18. *Let* $\mathcal{C}$ *be a category with all limits and suppose that we have a diagram*

$$\begin{array}{ccccc} X & & & & \\ & \searrow & & & \\ & & S & \longrightarrow & T \\ & \nearrow & & & \\ Y & & & & \end{array}$$

Prove that the diagram (be sure to write down carefully how the maps are induced!)

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & X \times_T Y \\ \downarrow & & \downarrow \\ S & \xrightarrow{\Delta} & S \times_T S \end{array}$$

is cartesian (hint: if you are unable to prove this result in full generality, feel free to assume that $\mathcal{C} = \mathbf{PStk}$*).*

Exercise 3.2.19. *A morphism of prestacks* $G \to F$ *is said to be* **representable** *if for any* $\operatorname{Spec} R \to F$*, the prestack* $G \times_F \operatorname{Spec} R$ *is representable. Prove that the following are equivalent:*

1. *the diagonal* $\Delta : F \to F \times F$ *is representable;*
2. *any map* $\operatorname{Spec} S \to F$ *(in other words any map from an affine scheme to* $\mathcal{F}$*) is representable.*

4. Lecture 3: Descent

We are about to define schemes. But first we define stacks[2]. In order to define the notion of stacks, we need the notion of open immersions, which are complementary to closed immersions.

[2] This is where some heavy conflict with the literature will occur so be wary. In the literature, the notion of stacks differs from this one in two, crucial ways. First the descent condition is asked with respect to something called the étale topology (which we will cover later in class) and, secondly, the functor lies in the $(2,1)$-category of groupoids. Functors landing in said version of categories are not really functors in the sense we are used to in class.

Remark 4.0.1. Thinking about closed immersions of schemes is easier than thinking about open immersions, at least to the instructor. Indeed, every closed immersion of $\operatorname{Spec} R$ corresponds to the set of all ideals of R. We can think of a poset of ideals $\{I \subset J\}$ which corresponds to a poset of closed subschemes $\{\operatorname{Spec} R/I \to \operatorname{Spec} R/J\}$. Of course, we want to say that open subschemes of $\operatorname{Spec} R$ should be those of the form

$$D(J) := \operatorname{Spec} R \setminus \operatorname{Spec} R/J.$$

But now $D(J)$ is in fact **not** representable — we will soon be able to prove this. In fact these $D(J)$'s are the first examples of non-affine schemes. In particular $D(J)$ is not the form Spec of a ring. In order to fomulate descent in a more digestible manner, we will restrict ourselves to open subschemes of $\operatorname{Spec} R$ which are actually affine.

Suppose that I is an ideal of a ring R, let us recall that the **radical of** I, denoted usually by $\sqrt{I}$ is defined as

$$\sqrt{I} = \{x : x^N \in I \text{ for some } N \gg 0\}.$$

Example 4.0.2. Let $I = (0)$ be the zero ideal. Then the **nilradical** is $\sqrt{(0)} = \{f : f^N = 0 \text{ for some } N \gg 0\}$. We say that a ring is **reduced** if $\sqrt{(0)} = 0$.

Definition 4.0.3. Let R be a ring and $f \in R$. A **basic Zariski cover of the ring** R_f consists of a set I and a collection $\mathcal{U} := \{f_i : f_i \in R\}_{i \in I}$ such that

$$f \in \sqrt{\Sigma(f_i)}.$$

In particular, a **basic Zariski cover** of a ring R consists of a set I and a collection $\mathcal{U} := \{f_i : f_i \in R\}_{i \in I}$ subject to the following condition:

$$1 \in \sqrt{\Sigma(f_i)}.$$

We write

$$\{\operatorname{Spec} A_{f_i} \to \operatorname{Spec} A_f\}_{i \in I}$$

to denote an arbitrary basic Zariski cover.

Remark 4.0.4. If f is a unit so that $R_f = R$, this is a very redundant definition. Indeed, any element $x \in \sqrt{\Sigma(f_i)}$ means that

$$x^N \in \Sigma(f_i)$$

But the sum of ideals means we have a sum of elements in each ideal where all except finitely many elements are zero so:

$$x^N = a_1 f_1 + \cdots + a_n f_n,$$

up to rearrangements. But now

$$1 = 1^N = a_1 f_1 + \cdots + a_n f_n.$$

Therefore we can find a subcover of $\mathcal{U}$ such that

$$1 \in \Sigma(f_i).$$

Of course this argument also does show that a basic Zariski cover of R_f can be refined by a finite subcover.

Example 4.0.5. Let p, q be distinct primes in $\mathbf{Z}$. This means, by Bézout's identity, we can write

$$1 = kp + rq,$$

for some $k, r \in \mathbf{Z}$. In the language above we find that

$$\{\operatorname{Spec} \mathbf{Z}[\tfrac{1}{p}], \operatorname{Spec} \mathbf{Z}[\tfrac{1}{q}] \hookrightarrow \operatorname{Spec} \mathbf{Z}\}$$

defines a basic Zariski cover of $\operatorname{Spec} \mathbf{Z}$.

Example 4.0.6. Let k be a field and consider $k[x]$. Suppose that $p(x), q(x)$ are polynomials which are irreducible and are coprime. Then Bézout's identity again works in this situation:

$$1 = k(x)p(x) + r(x)q(x).$$

In this language we find that

$$\{\operatorname{Spec} k[x]_{p(x)}, \operatorname{Spec} k[x]_{q(x)} \hookrightarrow \operatorname{Spec} k[x]\}$$

defines a basic Zariski cover of $\operatorname{Spec} k[x]$.

Definition 4.0.7. A prestack $\mathcal{F} : \mathrm{CAlg} \to \mathrm{Set}$ is a **(Zariski) stack** if for any $\mathrm{A} \in \mathrm{CAlg}$ and any basic Zariski cover $\{\operatorname{Spec} \mathrm{A}_{f_i} \to \operatorname{Spec} \mathrm{A}\}_{i \in \mathrm{I}}$ the diagram

$$\mathcal{F}(\mathrm{A}) \to \mathcal{F}(\prod_i \mathrm{A}_{f_i}) \rightrightarrows \mathcal{F}(\prod_{i_0, i_1} \mathrm{A}_{f_{i_0}} \otimes_{\mathrm{A}} \mathrm{A}_{f_{i_1}})$$

is an equalizer diagram where the maps are induced by

$$\prod \mathrm{A}_{f_i} \to \prod_{i_0, i_1} \mathrm{A}_{f_{i_0}} \otimes_{\mathrm{A}} \mathrm{A}_{f_{i_1}} \qquad (g_i) \mapsto (g_{i_0}|_{\mathrm{A}_{f_{i_0} \cdot f_{i_1}}}).$$

and

$$\prod \mathrm{A}_{f_i} \to \prod_{i_0, i_1} \mathrm{A}_{f_{i_0}} \otimes_{\mathrm{A}} \mathrm{A}_{f_{i_1}} \qquad (g_i) \mapsto (g_{i_1}|_{\mathrm{A}_{f_{i_0} \cdot f_{i_1}}}).$$

4.1. **Unpacking the descent condition and Serre's lemma.** Let us note a couple of easy properties about localizations

Lemma 4.1.1. *Let $f_1, f_2 \in \mathrm{R}$ then*

$$(\mathrm{R}_{f_1})_{f_2} \cong \mathrm{R}_{f_1 \cdot f_2} \cong \mathrm{R}_{f_1} \otimes_{\mathrm{R}} \mathrm{R}_{f_2}.$$

This will be homework. For the rest of this section, we will seek be taking a map $f : \mathrm{R} \to \mathrm{A}$ and then postcomposing then along some localization of A, say A'; for this it is convenient to use the notation

$$f|_{\mathrm{A}'}$$

and think about "restriction."

Let us fix a ring R and suppose that A is a "test-ring" and we are interested in the set

$$\operatorname{Hom}(\operatorname{Spec} \mathrm{A}, \operatorname{Spec} \mathrm{R}),$$

and we would like to recover this set in terms of a given basic Zariski cover of A. As we had discussed, this latter object is given by the data of elements $g_1, \cdots g_n \in \mathrm{A}$ such that $1 = \sum_{i=1}^n g_i$. Let us consider the following set

$$\operatorname{Glue}(\mathrm{R}, \mathrm{A}, \{g_i\}) \subset \prod_{i=1}^n \operatorname{Hom}(\operatorname{Spec} \mathrm{A}_{g_i}, \operatorname{Spec} \mathrm{R}),$$

consisting of the n-tuples $\{f_i : \mathrm{R} \to \mathrm{A}_{g_i}\}$ subject to the following condition

(cocycle) $f_i|_{\mathrm{A}_{g_i \cdot g_j}} = f_j|_{\mathrm{A}_{g_j \cdot g_i}}$,

called the **cocycle condition**.

Lemma 4.1.2. *As above we have an isomorphism*

$$\operatorname{Glue}(\mathrm{R}, \mathrm{A}, \{g_i\}) \cong \operatorname{Eq}(\prod_i \operatorname{Hom}(\mathrm{R}, \mathrm{A}_{g_i}) \rightrightarrows \prod_{i_0, i_1} \operatorname{Hom}(\mathrm{R}, \mathrm{A}_{g_{i_0} \cdot g_{i_1}}))$$

This is an exercise in unpacking definitions. Even though Glue is more explicit, in order to prove actual results, we will work with the equalizer formulation. Our main theorem is as follows:

Theorem 4.1.3. *The map*

$$\mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{Spec}\,\mathrm{R}) \to \prod_{i=1}^{n} \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}_{g_i}, \mathrm{Spec}\,\mathrm{R}) \qquad f : \mathrm{R} \to \mathrm{A} \mapsto (f|_{\mathrm{A}_{g_i}})_{i \in \mathrm{I}}$$

factors as

$$\mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{Spec}\,\mathrm{R}) \to \mathrm{Glue}(\mathrm{R}, \mathrm{A}, \{g_i\})$$

and induces an isomorphism. Equivalently, $\mathrm{Spec}\,\mathrm{R}$ *is a Zariski stack.*

We will prove Theorem 4.1.3 in the right level of generality. The next lemma is called "Serre's lemma for modules."

Lemma 4.1.4. *Let* A *be a ring and* $f_1, \cdots, f_n$ *elements such that* $\sum_{i=1}^{n} f_i = 1$. *Then*

$$\mathrm{M} \to \prod \mathrm{M}_{f_i} \rightrightarrows \prod \mathrm{M}_{f_i \cdot f_j}$$

is an equalizer diagram.

Proof. We first assume

(1) there exists an element f_i, say f_1, which is invertible. So that $\mathrm{M} \cong \mathrm{M}_{f_1}$.

Then we prove the result: indeed, denote the equalizer by Eq. Indeed, the map $\mathrm{M} \to \prod \mathrm{M}_{f_i}$ factors through the equalizer since this is just the map

$$m \mapsto (m|_{\mathrm{A}_{f_i}}),$$

and the cocycle condition is satisfied. On the other hand, If $\{m_i\} \in \mathrm{Eq} \subset \prod \mathrm{M}_{f_i}$, then since $m_1 \in \mathrm{M}$ (under the assumed isomorphism), we get that the map is surjective.

To prove injectivity: if $m_1 = 0$, then we claim that $m_i = 0$ for all $i > 0$ if (m_i) is in the equalizer. But now, we note that, by the equalizer condition, we get that for $i > 0$

$$\mathrm{M}_{f_i} \to \mathrm{M}_{f_i \cdot f_1}$$

is an isomorphism and the image of m_i under this isomorphism must be the image of m_1 under the map $\mathrm{M}_{f_1} \to \mathrm{M}_{f_i \cdot f_1}$ because of the equalizer condition. Therefore m_i must be zero.

Now, to prove the desired claim: take kernels and cokernels:

$$0 \to \mathrm{K} \to \mathrm{M} \to \mathrm{Eq} \to \mathrm{C} \to 0$$

The claim is that $\mathrm{K}, \mathrm{C} = 0$. They are 0 after inverting each f_i by the previous claim. The proof the finishes by the next claim.

□

Lemma 4.1.5. *Let* A *be a ring and* $f_1, \cdots, f_n$ *elements such that* $\sum_{i=1}^{n} f_i = 1$. *Suppose that* $\mathrm{M}_{f_i} = 0$ *for all* $i = 1, \cdots, n$. *Then* $\mathrm{M} = 0$.

Proof. The condition means we can find an N such that $f_i^{\mathrm{N}} m = 0$. But then there is an even larger M:

$$m = 1^{\mathrm{M}} \cdot m = (\sum f_i)^{\mathrm{M}} \cdot m = 0.$$

□

Corollary 4.1.6. *Let* A *be a ring and* $f_1, \cdots, f_n$ *elements such that* $\sum_{i=1}^{n} f_i = 1$. *Then*

$$\mathrm{A} \to \prod \mathrm{A}_{f_i} \rightrightarrows \prod \mathrm{A}_{f_i \cdot f_j}$$

is an equalizer diagram of rings. In other words, we have proved that $\mathbf{A}^1$ *is a Zariski stack.*

Proof. This follows from what we have proved. The "in other words" part follows from the fact that

$$\mathbf{A}^1(\mathrm{A}) = \mathrm{A}.$$

□

Now let us prove

Proof of Theorem 4.1.3. We have proved that $\mathbf{A}^1$ is a Zariski stack. Any affine scheme can be written as a pullback

$$\begin{array}{ccc} \operatorname{Spec} \mathrm{R} & \longrightarrow & \mathbf{A}^{\mathrm{I}} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbf{Z} & \xrightarrow{0} & \mathbf{A}^{\mathrm{J}}, \end{array}$$

where I, J are sets. The result then follows from the next lemmas (one of which is homework): □

Lemma 4.1.7. *Zariski stacks are preserved under limits.*

Lemma 4.1.8. $\operatorname{Spec} \mathbf{Z}$ *is a Zariski stack.*

Proof. For any ring R, $\operatorname{Spec} \mathbf{Z}(\mathrm{R}) = *$. The claim then follows from the observation that the diagram

$$* \to * \rightrightarrows *$$

is an equalizer. □

4.1.9. *Problem set 3.*

Lemma 4.1.10. *Let* R *be a ring and consider the functor from* R*-algebras to* R*-modules*

$$\mathrm{U} : \mathrm{CAlg}_{\mathrm{R}} \to \mathrm{Mod}_{\mathrm{R}}.$$

Prove:

(1) *this functor preserves final objects,*
(2) *this functor creates pullbacks: a diagram of rings*

$$\begin{array}{ccc} \mathrm{A} & \longrightarrow & \mathrm{B} \\ \downarrow & & \downarrow \\ \mathrm{C} & \longrightarrow & \mathrm{D}, \end{array}$$

is a pullback diagram if and only if the corresponding diagram of modules is,
(3) *if you are feeling up to it: prove that in fact* U *creates all limits.*

Exercise 4.1.11. *Here is a formula for* R_f*. We will work in the generality of the category* $\mathrm{Mod}_{\mathrm{R}}$*.*

(1) *Let* $f \in \mathrm{R}$ *and consider the following* $\mathbf{N}$*-indexed diagram in the category of* R*-modules*

$$\mathrm{M} \xrightarrow{f\cdot} \mathrm{M} \xrightarrow{f\cdot} \mathrm{M} \xrightarrow{f\cdot} \cdots.$$

Define the colimit to be the R*-module* M_f*. Prove that we have a natural isomorphism: for any* $\mathrm{N} \in \mathrm{Mod}_{\mathrm{R}}$ *such that the map*

$$f\cdot : \mathrm{N} \to \mathrm{N},$$

is an isomorphism then:

$$\mathrm{Hom}_{\mathrm{Mod}_{\mathrm{R}}}(\mathrm{M}_f, \mathrm{N}) \cong \mathrm{Hom}_{\mathrm{Mod}_{\mathrm{R}}}(\mathrm{M}, \mathrm{N}).$$

(2) *Construct explicitly a multiplication on* R_f *and a compatible ring homomorphism* $\mathrm{R} \to \mathrm{R}_f$*.*
(3) *Consider the functor*

$$j_* : \mathrm{Mod}_{\mathrm{R}_f} \to \mathrm{Mod}_{\mathrm{R}}$$

given by restriction of scalars. Prove that this functor admits a left adjoint given by $j^* : \mathrm{M} \mapsto \mathrm{M}_f$*; part of the task is to explain why* M_f *is naturally an* R_f*-module.*
(4) *Use the formula from* 1 *to prove that* j_* *is fully faithful and show that the essential image identifies with the subcategory of* R*-modules where* $f\cdot$ *acts by an isomorphism.*

Lemma 4.1.12. *Let $f_1, f_2 \in \mathrm{R}$ then*

$$(\mathrm{R}_{f_1})_{f_2} \cong \mathrm{R}_{f_1 \cdot f_2} \cong \mathrm{R}_{f_1} \otimes_{\mathrm{R}} \mathrm{R}_{f_2}.$$

Exercise 4.1.13. *In this exercise, we will give a proof of a basic, but very clear formulation of descent. Let A be a ring and suppose that $f, g \in \mathrm{A}$ are elements*

(1) Consider the square

$$\begin{array}{ccc} \mathrm{A} & \longrightarrow & \mathrm{A}_f \\ \downarrow & & \downarrow \\ \mathrm{A}_g & \longrightarrow & \mathrm{A}_{fg}. \end{array}$$

Prove that it the top and bottom arrows are isomorphisms after inverting f; conclude that the resulting square is cartesian.

(2) Prove that the left vertical and the right vertical arrows are isomorphisms after inverting g; conclude that the resulting square is cartesian.

(3) Now assume:

$$1 \in (f) + (g).$$

Conclude that the square is cartesian.

Exercise 4.1.14. *Prove that Zariski stacks are preserved under limits: suppose that we have a diagram $\mathrm{I} \to \mathbf{PStk}$, then the functor*

$$\mathrm{R} \mapsto \lim_{\mathrm{I}} \mathcal{F}_i(\mathrm{R}),$$

defines a Zariski stack.

Exercise 4.1.15. *Prove that any affine scheme $\operatorname{Spec} \mathrm{R}$ can be written as a pullback*

$$\begin{array}{ccc} \operatorname{Spec} \mathrm{R} & \longrightarrow & \mathbf{A}^{\mathrm{I}} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbf{Z} & \xrightarrow{0} & \mathbf{A}^{\mathrm{J}}. \end{array}$$

5. Lecture 4: Schemes Part 1

We have done a bunch of abstract stuff. I would like to tell you how to say something concrete using abstract stuff.

5.1. Diversion: multiplicative groups and graded rings.

Let us, for this section, consider what structure what one can endow on $\mathbf{G}_m = \operatorname{Spec} \mathbf{Z}[t, t^{-1}]$. One suggestive way to think about $\mathbf{Z}[t, t^{-1}]$ is as

$$\mathbf{Z}[t, t^{-1}] \cong \mathbf{Z}[\mathbf{Z}] \cong \bigoplus_{j \in \mathbf{Z}} \mathbf{Z}(j).$$

This is also called the **group algebra** on the (commutative) group $\mathbf{Z}$; we will say why this is a interesting at all later on. We want to say that $\mathbf{G}_m$ is a group object in the category of affine schemes. Unwinding definitions, we need to provide three pieces of data

(Mult.) The multiplication:

$$\mu : \mathbf{Z}[t, t^{-1}] \to \mathbf{Z}[t, t^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[t, t^{-1}] \qquad t \mapsto t \otimes t,$$

(Id.) The identity

$$\epsilon : \mathbf{Z}[t, t^{-1}] \to \mathbf{Z} \qquad t \mapsto 1$$

(Inv.) The inverse

$$\iota : \mathbf{Z}[t, t^{-1}] \to \mathbf{Z}[t, t^{-1}] \qquad t \mapsto t^{-1}.$$

These data (or, more precisely, the opposites thereof) are subject to the compatibilities that prescribe $\mathbf{G}_m$ as a group object in $\mathbf{PStk}$.

Definition 5.1.1. An **affine group scheme** is an affine scheme $\mathrm{G} = \operatorname{Spec}\mathrm{R}$ with maps $\mu : \mathrm{G}\times\mathrm{G} \to \mathrm{G}, \epsilon : \operatorname{Spec}\mathbf{Z} \to \mathrm{G}, \iota : \mathrm{G} \to \mathrm{G}$ which endows it with the structure of a group object in prestacks.

When we speak of groups, we always want to speak about group actions. If G is an affine group scheme and $\mathcal{F}$ is a prestack then a **(left) action** is given by a morphism of prestacks

$$a : \mathrm{G} \times \mathcal{F} \to \mathcal{F},$$

satisfying the obvious compatibilities:

$$\begin{array}{ccc} \mathrm{G}\times\mathrm{G}\times\mathcal{F} & \xrightarrow{\mu\times\mathrm{id}} & \mathrm{G}\times\mathcal{F} \\ \downarrow{\scriptstyle \mathrm{id}\times a} & & \downarrow{\scriptstyle a} \\ \mathrm{G}\times\mathcal{F} & \xrightarrow{a} & \mathcal{F}. \end{array} \qquad \begin{array}{ccc} \mathcal{F} & \xrightarrow{\epsilon} & \mathrm{G}\times\mathcal{F} \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle a} \\ & & \mathcal{F}. \end{array}$$

If we restrict ourselves to $\mathbf{G}_m$ acting on affine schemes, we actually obtain the next result whose standard reference is [DG70, Exposé 1, 4.7.3]. Let us denote by $\mathrm{Aff}^{\mathrm{B}\mathbf{G}_m}$ the category of affine schemes equipped with a $\mathbf{G}_m$-action and $\mathbf{G}_m$-equivariant morphisms. This is not a subcategory of prestacks, but admits a forgetful functor

$$\mathrm{Aff}^{\mathrm{B}\mathbf{G}_m} \to \mathbf{PStk}.$$

On the other hand a **Z-graded ring** is a ring R equipped with a decomposition:

$$\mathrm{R} = \bigoplus_{i\in\mathbf{Z}} \mathrm{R}_i$$

such that:

(1) each R_j is an additive subgroup of R (in other words, the direct sum above is taken in the category of abelian groups) and,
(2) the multiplication induces $\mathrm{R}_i\mathrm{R}_j \subset \mathrm{R}_{i+j}$.

We say that an element $f \in \mathrm{R}$ is a **homogeneous element of degree** n if $f \in \mathrm{R}_n$. A **graded morphism** of graded rings is just a ring homomorphism $\varphi : \mathrm{R} \to \mathrm{S}$ such that $\varphi(\mathrm{R}_j) \subset \mathrm{S}_j$. We denote by grCAlg the category of **Z**-graded rings.

Theorem 5.1.2. *There is an equivalence of categories*

$$\mathrm{Aff}^{\mathrm{B}\mathbf{G}_m} \simeq (\mathrm{grCAlg})^{\mathrm{op}}.$$

Remark 5.1.3. One of the main points of Theorem 5.1.2 is that it is interesting to read from left to right and right to left. One the one hand one can use the geometric language of groups acting on a scheme/variety to encode a combinatorial/algebraic structure. On the other hand, it gives a purely combinatorial/algebraic description of a geometric idea.

Proof. First we construct a functor:

$$\mathrm{Aff}^{\mathrm{B}\mathbf{G}_m} \to (\mathrm{grCAlg})^{\mathrm{op}}.$$

Recall that the tensor product of rings is computed as the tensor product of underlying modules. Therefore we can write isomorphisms of **Z**-modules:

$$\mathrm{R} \otimes_{\mathbf{Z}} \mathbf{Z}[t, t^{-1}] \cong \mathrm{R} \otimes_{\mathbf{Z}} \mathbf{Z}[\mathbf{Z}] \cong \bigoplus_{j\in\mathbf{Z}} \mathrm{R}(j).$$

Hence a $\mathbf{G}_m$-action on $\operatorname{Spec}\mathrm{R}$ is the same data as giving a map

$$\varphi : \mathrm{R} \to \bigoplus_{j\in\mathbf{Z}} \mathrm{R}(j) \qquad f \mapsto (\varphi_j(f) \in \mathrm{R}(j)),$$

satisfying certain compatibilities. We note that the direct sum indicates that the components of $(\varphi_j(f))$ is finitely supported.

Using the identity axiom we see that the composite

$$\mathrm{R} \to \bigoplus_{j\in\mathbf{Z}} \mathrm{R}(j) \xrightarrow{t\mapsto 1} \mathrm{R}$$

must be the identity. Therefore, in coordinates, we get that for any $f \in \mathrm{R}$, we get that

$$f = \sum_{j\in\mathbf{Z}} \varphi_j(f),$$

so that any f can be uniquely written as a finite sum of the $\varphi_j(f)$'s. To conclude that this defines a grading on R we need to prove that each φ_j is an idempotent. If this was proved, then the grading would be such that $f \in \mathrm{R}$ is of homogeneous degree j whenever $\varphi(f) = ft^j$.

However, this is the case by associativity of the action:

$$\begin{array}{ccc} \mathrm{R} & \xrightarrow{\varphi} & \bigoplus_{j\in\mathbf{Z}} \mathrm{R}(j) \\ \downarrow \varphi & & \downarrow \varphi \\ \bigoplus_{j\in\mathbf{Z}} \mathrm{R}(j) & \xrightarrow{\mu} & \bigoplus_{j\in\mathbf{Z}} \bigoplus_{k\in\mathbf{Z}} \mathrm{R}(jk). \end{array}$$

Therefore we conclude that R splits, as a $\mathbf{Z}$-module (aka abelian group) as $\mathrm{R} \cong \bigoplus_j \varphi_j \mathrm{R}(:= \mathrm{R}_j)$ and one can check that this defines a graded ring structure on R where the compatibility of multiplication originates from the fact that φ is a ring map.

On the other hand, given a ring R equipped with the structure of a graded ring $\mathrm{R} = \bigoplus \mathrm{R}_j$ we define a map

$$\varphi : \mathrm{R} \to \bigoplus_{j\in\mathbf{Z}} \mathrm{R}(j),$$

on the level of abelian groups as

$$\mathrm{R} \to \pi_j \mathrm{R} \subset \mathrm{R}(j) \cong \mathrm{R},$$

where π_j is the projection map. This is checked easily to define a $\mathbf{G}_m$-action and the functors are mutually equivalent.

□

Example 5.1.4. There is an action of $\mathbf{G}_m$ on $\mathbf{A}^1$ that "absorbs everything to the origin"; in coordinates this is written as $t \cdot x = tx$. An exercise in this week's homework will require you to translate this to a grading.

Example 5.1.5. The best way to define new graded rings is to mod out by homogenenous polynomial equations. Recall that a polynomial $f(x_1, \cdots, x_n) \in \mathrm{R}[x_1, \cdots, x_n]$ (over any ring R) is said to be **homogeneous of degree** d if for any $r \in \mathrm{R}$ $r^d f(x_1, \cdots, x_n) = f(rx_1, \cdots, rx_n)$. The instructor never found this a useful definition; we can equivalently define this to be a linear combination of monomials of degree d, i.e.,

$$ax_1^{r_1} \cdots x_k^{r_k} \qquad \sum_{j=1}^{k} r_j = d.$$

Here is a nice visual example: consider the **quadric cone**:

$$\operatorname{Spec} \mathbf{Z}[x, y, z]/(x^2 + y^2 - z^2).$$

Since $\mathbf{Z}[x, y, z]/(x^2 + y^2 - z^2)$ is the quotient of a graded ring by a homogeneous equation, it inherits a natural grading. This defines a $\mathbf{G}_m$-action. If we replace $\mathbf{Z}$ by a field, convince yourself that this is pictorially the "absorbing" action of the cone onto its cone point.

5.2. **Complementation and open subfunctors.** In the last class we defined the descent condition and also proved that Spec R satisfies this condition. This is like choosing a basis in a vector space — we could have two covers which are specified by $\{f_i\}$ or $\{g_j\}$ and we have to say something in order to prove that descend with respect to one cover implies descent for the other.

Let us try to characterize open immersions of affine schemes in terms of its functor of points. We know that open subschemes of Spec A should be one which is the complement of a closed subscheme where the latter is of the form Spec A/I. Furthermore we know the following example:

Remark 5.2.1. If $\mathrm{I} = (f)$, then $\operatorname{Spec} \mathrm{A}/f \hookrightarrow \operatorname{Spec} \mathrm{A}$ has a complement which is actually an affine scheme given by $\operatorname{Spec} \mathrm{A}_f$. Indeed, let us attempt to unpack this: suppose that $\operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{A}$ is a morphism of affine schemes corresponding to a map of rings $\mathrm{A} \to \mathrm{R}$. We want to say that Spec R lands in the open complement of $\operatorname{Spec} \mathrm{A}/f$ which translate algebraically to the following cartesian diagram

$$\begin{array}{ccc} \mathrm{A} & \longrightarrow & \mathrm{R} \\ \downarrow & & \downarrow \\ \mathrm{A}/f & \longrightarrow & 0. \end{array}$$

This means that the map $\varphi : \mathrm{A} \to \mathrm{R}$ must satisfy: $\mathrm{R}/f\mathrm{R} = 0$ and so $f\mathrm{R} = \mathrm{R}$ which exactly means that f acts invertibly on R and hence (by homework) defines uniquely a ring map

$$\mathrm{A}_f \to \mathrm{R}.$$

To summarize our discussion:

(1) intuitively (and actually!) the closed subscheme $\operatorname{Spec} \mathrm{A}/f$ is one which is cut out by f or, in other words, the locus where f vanishes. Its complement, which is an open subscheme (if you believe in topological spaces) is the locus where f is invertible so we should take something like $\operatorname{Spec} \mathrm{A}_f$.
(2) we need a new definition to make sense of complementation of prestacks.

Let us also note the following:

Lemma 5.2.2. *For any ring f, the following diagram is cartesian*

$$\begin{array}{ccc} \operatorname{Spec} \mathrm{R}_f & \longrightarrow & \operatorname{Spec} \mathrm{R} \\ \downarrow & & \downarrow f \\ \operatorname{Spec} \mathbf{Z}[t, t^{-1}] & \longrightarrow & \mathbf{A}^1 = \operatorname{Spec} \mathbf{Z}. \end{array}$$

Let us recall that a morphism of prestacks $\mathrm{X} \to \mathrm{Y}$ is a closed immersion if for any morphism $\operatorname{Spec} \mathrm{R} \to \mathrm{Y}$ then:

(1) the prestack $\operatorname{Spec} \mathrm{R} \times_{\mathrm{Y}} \mathrm{X}$ is representable and,
(2) the morphism
$$\operatorname{Spec} \mathrm{R} \times_{\mathrm{Y}} \mathrm{X} \to \operatorname{Spec} \mathrm{R}$$
is a closed immersion.

Definition 5.2.3. Let $\mathcal{G} \subset \mathcal{F}$ be a closed immersion of prestacks. The **complement of** $\mathcal{G}$, defined by $\mathcal{F} \smallsetminus \mathcal{G}$ is the prestack given in the following manner: a morphism $x : \operatorname{Spec} \mathrm{R} \to \mathcal{F}$ is in $\mathcal{F} \smallsetminus \mathcal{G}$ if and only if the following diagram is cartesian

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathcal{G} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathrm{R} & \xrightarrow{x} & \mathcal{F} \end{array}$$

We note that $\mathcal{F} \smallsetminus \mathcal{G}$ is indeed a prestack because the empty scheme pullsback

Lemma 5.2.4. *Let $\mathrm{R} \in \mathrm{CAlg}$ and $\mathrm{I} \subset \mathrm{R}$ an ideal. Then there is a natural bijection between*

(1) maps $\mathrm{R} \to \mathrm{A}$ *such* $\mathrm{IA} = \mathrm{A}$ *and,*
(2) morphisms $\operatorname{Spec} \mathrm{A} \to \operatorname{Spec} \mathrm{R}$ *such that*

$$\begin{array}{ccc} \emptyset & \longrightarrow & \operatorname{Spec} \mathrm{R}/\mathrm{I} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathrm{A} & \xrightarrow{x} & \operatorname{Spec} \mathrm{R}. \end{array}$$

Proof. The condition of the second item says that the following diagram is cocartesian:

$$\begin{array}{ccc} \mathrm{R} & \longrightarrow & \mathrm{A} \\ \downarrow & & \downarrow \\ \mathrm{R}/\mathrm{I} & \longrightarrow & 0. \end{array}$$

which means that $\mathrm{A} \otimes_{\mathrm{R}} \mathrm{R}/\mathrm{I} = \mathrm{A}/\mathrm{IA} = 0$ which exactly means that $\mathrm{A} = \mathrm{IA}$.

□

Definition 5.2.5. A subfunctor $\mathcal{F} \to \operatorname{Spec} \mathrm{R}$ of the form in Lemma 5.2.4 is called an **open subscheme**, while $\mathcal{F}$ is called a **quasi-affine** prestack. In this case we write

$$\mathcal{F} = \mathrm{D}(\mathrm{I}).$$

If $\mathcal{F} = \operatorname{Spec} \mathrm{R}_f$, we write $\mathrm{D}(f)$.

We will soon learn how to prove that not all quasi-affine prestacks are affine.

5.3. Exercises.

Exercise 5.3.1. *Consider the action of* $\mathbf{G}_m$ *on* $\mathbf{A}^n$ *given by*

$$\mathbf{Z}[x_1, \cdots, x_n] \to \mathbf{Z}[t, t^{-1}, x_1, \cdots, x_n] \qquad x_j \mapsto t^{-k_j} x_j.$$

Calculate the induced grading on $\mathbf{Z}[x_1, \cdots, x_n]$.

Exercise 5.3.2. *Let* $\mathrm{R}_\bullet$ *be a graded ring which is concentrated in* $\mathbf{Z}_{\geqslant 0}$*, i.e.,* $\mathrm{R}_{<0} = 0$*. Note that:*

(1) each R_j *is then canonically an* R_0*-module and, in fact,* $\mathrm{R}_\bullet$ *is an* R_0*-algebra;*
(2) the subset $\mathrm{R}_+ := \bigoplus_{i \geqslant 1} \mathrm{R}_i \subset \mathrm{R}_\bullet$ *is an ideal*

Prove that the following are equivalent:

(1) the ideal R_+ *is finitely generated as an* $\mathrm{R}_\bullet$*-ideal;*
(2) $\mathrm{R}_\bullet$ *is generated as an* R_0*-algebra by finitely many homogeneous elements of positive degree.*

In the above situation we say that $\mathrm{R}_\bullet$ *is a* **finitely generated graded ring***.*

Exercise 5.3.3. *Let* A *be a ring which can be written as* $\mathrm{A} \cong \mathrm{B} \times \mathrm{C}$*. Prove that the projection map* $\mathrm{A} \to \mathrm{B}$ *is both an open and a closed immersion.*

Exercise 5.3.4. *Here we introduce another perspective on descent. Fix a prestack*

$$\mathcal{F} : \mathrm{CAlg} \to \mathrm{Set}.$$

We say that a morphism of affine schemes $\operatorname{Spec} \mathrm{S} \to \operatorname{Spec} \mathrm{R}$ *is* **of** $\mathcal{F}$**-descent** *if*

$$\mathcal{F}(\mathrm{R}) \to \mathcal{F}(\mathrm{S}) \rightrightarrows \mathcal{F}(\mathrm{S} \otimes_{\mathrm{R}} \mathrm{S}).$$

is an equalizer diagram. It is **of universal** $\mathcal{F}$**-descent** *if for any map* $\operatorname{Spec} \mathrm{T} \to \operatorname{Spec} \mathrm{R}$,

$$\mathcal{F}(\mathrm{T}) \to \mathcal{F}(\mathrm{T} \otimes_{\mathrm{R}} \mathrm{S}) \rightrightarrows \mathcal{F}(\mathrm{S} \otimes_{\mathrm{R}} \mathrm{T} \otimes_{\mathrm{R}} \mathrm{S} \cong (\mathrm{S} \otimes_{\mathrm{R}} \mathrm{T}) \otimes_{\mathrm{T}} (\mathrm{T} \otimes_{\mathrm{R}} \mathrm{S})).$$

is an equalizer diagram.

Prove:

(1) any morphism $\operatorname{Spec} \mathrm{S} \to \operatorname{Spec} \mathrm{R}$ *with a section is of universal* $\mathcal{F}$*-descent,*
(2) if f, g *are of universal* $\mathcal{F}$*-descent then so is their composite,*

(3) *if $f \circ g$ is universal $\mathcal{F}$-descent, and f is too, then g is of universal $\mathcal{F}$-descent.*
(4) *the collection of maps of universal $\mathcal{F}$-descent is closed under base change: if* $\operatorname{Spec} S \to \operatorname{Spec} R$ *is of universal $\mathcal{F}$-descent, then for any* $\operatorname{Spec} T \to \operatorname{Spec} R$*, the map* $\operatorname{Spec} T \otimes_R S \to \operatorname{Spec} T$ *is.*

Now, assume that $\mathcal{F}$ is a Zariski stack for any basic Zariski cover and further suppose that $\mathcal{F}$ converts finite coproduts of affine schemes to finite products. Let us prove that $\mathcal{F}$ is a Zariski stack in the following way:

(1) *fix once and for all* $\operatorname{Spec} R = X$ *and consider a Zariski cover* $\mathcal{U} = \{U_i \hookrightarrow X\}$ *which we may already assume to be finite. We want to prove that*

$$\sqcup_i U_i \to X,$$

6. Lecture 5: Schemes, finally

6.1. Quasi-affine prestacks are Zariski stacks.

Proposition 6.1.1. *Any quasi-affine scheme is a prestack.*

Proof. Suppose that $U = D(I) \subset \operatorname{Spec} A$ is quasi-affine. We claim that it is a Zariski stack. Let R be a test ring and let $\{f_i \in R\}$ determine a basic Zariski open cover. We first prove the following claim:

- a morphism $\varphi : A \to R$ satisfies $I \cdot R = R$ if and only if $I \cdot R_{f_i} = R_{f_i}$.

Indeed, we have a short exact sequence of modules:

$$0 \to I \to A \to A/I \to 0.$$

Applying $R \otimes_A -$, we get:

$$\mathrm{Tor}_1(R, A/I) \to I \otimes_A R \to R \to R \otimes_A A/I,$$

so we need only prove that $R \otimes_A A/I, \mathrm{Tor}_1(R, A/I)$ are zero as R-modules.

By Lemma 4.1.5, we need only check that for all f_i, tensoring the above further with $\otimes_R R_{f_i}$ is zero, But the map $R \to R_{f_i}$ is flat. Therefore the claim follows from:

$$R_{f_i} \otimes_R R \otimes_A A/I = R_{f_i} \otimes_A A/I = 0 \qquad R_{f_i} \otimes_R \mathrm{Tor}_1(R, A/I) = \mathrm{Tor}_1(R_{f_i}, A/I) = 0.$$

I claim that we are now done. Indeed, shorthanding the relevant equalizers as $\mathrm{Eq}(\mathcal{F})(R)$ we get a diagram:

$$\begin{array}{ccc} D(I)(R) & \longrightarrow & \mathrm{Eq}(D(I)(R)) \\ \downarrow & & \downarrow \\ \operatorname{Spec} A(R) & \longrightarrow & \mathrm{Eq}(\operatorname{Spec} A)(R). \end{array}$$

Since the left vertical map is injective and the bottom horizontal map is an isomorphism, the top vertical map is injective. Now, the top vertical map is also surjective by what we have proved. □

6.2. General open covers.

We formulated a coordinate-dependent way of phrasing descent because we have used the notion of a basic Zariski cover. We will get rid of these choices now.

Definition 6.2.1. Let $\mathcal{F}$ be a prestack. Then an **open subprestack** of $\mathcal{F}$ or an **open immersion of prestacks** is a morphism $\mathcal{G} \to \mathcal{F}$ such that for any morphism $\operatorname{Spec} R \to \mathcal{F}$,

(1) the prestack $\operatorname{Spec} R \times_Y X$ is representable and
(2) the morphism $\operatorname{Spec} R \times_Y X \to \operatorname{Spec} R$ is an open subscheme.

Lemma 6.2.2. *Let $\mathcal{Z} \hookrightarrow \mathcal{F}$ be an closed immersion, then the complement $\mathcal{F} \smallsetminus \mathcal{Z}$ is canonically an open immersion.*

Definition 6.2.3. A **Zariski cover** of an affine scheme $\mathrm{X} = \operatorname{Spec} \mathrm{A}$ is a collection of open embeddings of prestacks

$$\mathcal{U} = \{\mathrm{U} \hookrightarrow \mathrm{X}\},$$

such that for any nonzero ring R, $\mathrm{S} = \operatorname{Spec} \mathrm{R}$ with a map $\mathrm{S} \to \mathrm{X}$ there exists a $\mathrm{U} \hookrightarrow \operatorname{Spec} \mathrm{A} \in \mathcal{U}$ such that

$$\mathrm{U} \times_{\mathrm{X}} \mathrm{S} \neq \emptyset.$$

Definition 6.2.4. A prestack $\mathcal{F} : \mathrm{CAlg} \to \mathrm{Set}$ is a **(Zariski) stack** (resp. **(Zariski) stack for the Zariski cover** $\mathcal{U}$) if for any $\mathrm{A} \in \mathrm{CAlg}$ and any (resp. the) Zariski cover $\mathcal{U} := \{\mathrm{U}_i \to \operatorname{Spec} \mathrm{A}\}_{i \in \mathrm{I}}$ the diagram

$$\mathcal{F}(\operatorname{Spec} \mathrm{A}) \to \prod \operatorname{Hom}(\mathrm{U}_i, \mathcal{F}) \rightrightarrows \prod_{i_0, i_1} \operatorname{Hom}(\mathrm{U}_{i_0} \times_{\operatorname{Spec} \mathrm{A}} \mathrm{U}_{i_1}, \mathcal{F})$$

is an equalizer diagram where the maps are induced are the obvious ones.

Another exercise in unpacking definitions:

Lemma 6.2.5. *A basic Zariski cover is a Zariski cover. In particular if $\mathcal{F}$ is a Zariski stack, then it is a Zariski stack with respect to basic open covers.*

This tells us that Definition 6.3.2 is stronger than Definition 4.0.7. In some sense Definition 6.3.2 is preferable — it affords the flexibility of working with covers where the opens are not necessarily affine. We will work towards proving the equivalence of these two definitions shortly. First let us consider the different ways we can think about Zariski covers:

Lemma 6.2.6. *Let $\mathrm{X} = \operatorname{Spec} \mathrm{A}$, and $\mathcal{U} = \{\mathrm{U} \hookrightarrow \operatorname{Spec} \mathrm{A}\}$ is a collection of open immersions. then the following are equivalent:*

(1) $\mathcal{U}$ is a Zariski cover.
(2) $\mathcal{U}$ has a finite subset $\mathcal{V}$ which is also an open cover.
(3) for any field k and any map $x : \operatorname{Spec} k \to \mathrm{X}$ there exists an $\mathrm{U} \in \mathcal{U}$ such that x factors through U.

Proof. The implication (1) ⇒ (2) comes under the term "affine schemes are **quasicompact**" which is one way in which open subsets look very different in algebraic geometry than what you have experienced before. To prove this, we note that giving a Zariski open cover of an affine scheme is to give a collection of ideals $\{\mathrm{I}_j\}$ such that

$$1 \in \sqrt{\sum \mathrm{I}_j}.$$

But this means, from the definition of sums of ideals, there exists a finite subcollection $i_0, \cdots, i_k$ such that

$$1 \in \sqrt{\sum_{0 \leqslant s \leqslant k} \mathrm{I}_{i_s}}.$$

This in turn means that we can refine the above open cover by

$$\{\mathrm{D}(\mathrm{I}_{i_s}) \to \operatorname{Spec} \mathrm{A}\}_{0 \leqslant s \leqslant k}.$$

Let us prove (2) ⇒ (3). Given a morphism into a field $\varphi : \mathrm{A} \to k$, we want to find $\mathrm{U} \hookrightarrow \mathrm{X} \in \mathcal{V}$ such that x factors through U. Suppose that there is none, then $\varphi(\Sigma \mathrm{I}_j) = 0$, which means that $\varphi(1) = 0$ but this is not possible.

Let us prove (3) ⇒ (1). Suppose that R is a nonzero ring with a map $\operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{A}$ so that we have a morphism $\varphi : \mathrm{A} \to \mathrm{R}$. Since R is nonzero, it has a maximal ideal $\mathfrak{m}$ so that $\mathrm{R}/\mathfrak{m} = \kappa$ a field. By hypothesis, there exists a $\mathrm{U} \in \mathcal{U}$ such that $\mathrm{U} \times_{\mathrm{X}} \operatorname{Spec} \kappa$ is nonempty, but this also means that $\mathrm{U} \times_{\mathrm{X}} \operatorname{Spec} \mathrm{R}$ is nonempty. □

This provides the first mechanism by which a scheme can be non-affine.

Lemma 6.2.7. *Let* $(\mathrm{A}_i)_{i\in\mathbf{N}}$ *be a collection of nonzero rings and consider a countable product* $\coprod_{\mathbf{N}} \operatorname{Spec} \mathrm{A}_i$. *This prestack (which in fact a Zariski stack) is not an affine scheme.*

Proof. We first note that $\sqcup \operatorname{Spec} \mathrm{A}_i = \operatorname{colim}_{n\to\infty} \sqcup_{i=1}^n \operatorname{Spec} \mathrm{A}_i$. Now, since finite limits, in particular, equalizers commutes with **N**-indexed colimits, we conclude that $\sqcup \operatorname{Spec} \mathrm{A}_i$ is a stack. However we claim that it is not an affine scheme. Suppose that $\sqcup \operatorname{Spec} \mathrm{A}_i$ was representable by a ring B so that for any ring R, we have a functorial isomorphisms

$$\coprod \operatorname{Hom}(\mathrm{A}_i, \mathrm{R}) \cong (\coprod_{\mathbf{N}} \operatorname{Spec} \mathrm{A}_i)(\mathrm{R}) \cong \operatorname{Hom}(\mathrm{B}, \mathrm{R}).$$

Now, consider the collection of maps $\{\operatorname{Spec} \mathrm{A}_i \to \operatorname{Spec} \mathrm{B}\}$. I claim that this is an open cover.

(1) first we prove that $\operatorname{Spec} \mathrm{A}_i \to \operatorname{Spec} \mathrm{B}$ is an open immersion. Indeed, for each A_j, under the map
$$\operatorname{Hom}(\mathrm{A}_j, \mathrm{A}_j) \xrightarrow{\iota_j} \coprod \operatorname{Hom}(\mathrm{A}_i, \mathrm{A}_j) \cong \operatorname{Hom}(\mathrm{B}, \mathrm{A}_j)$$
the identity on the left hand side gets mapped to $\varphi_j : \mathrm{B} \to \mathrm{A}_j$. Take the ideal in B generated by $\ker(\varphi_j)_{j\neq i}$, determining a closed subset of Spec B. We can check that the complement is exactly $\operatorname{Spec} \mathrm{A}_i$, therefore it is open in Spec B.

(2) Next, we use criterion 3 of Lemma 6.2.6: given a map $\operatorname{Spec} k \to \operatorname{Spec} \mathrm{B}$, we get an element in a component of $\operatorname{Hom}(\mathrm{A}_i, k)$ so that, in particular, this collection is a cover.

But, there is no refinement of this subcover. □

6.3. Universality of descent.

We would like to prove the following result:

Theorem 6.3.1. *Suppose that* $\mathcal{F} : \mathrm{CAlg} \to \mathrm{Set}$ *is a functor. Then the following are equivalent:*

(1) $\mathcal{F}$ *is a Zariski stack in the sense of Definition 6.3.2.*
(2) $\mathcal{F}$ *is a Zariski stack in the sense of Definition 4.0.7.*

To do so, we define Zariski open covers of prestacks:

Definition 6.3.2. A **Zariski cover** of an prestack $\mathcal{F}$ is a collection of open embeddings of prestacks

$$\mathcal{U} = \{\mathrm{U} \hookrightarrow \mathcal{F}\},$$

such that for any nonzero ring R, $\mathrm{S} = \operatorname{Spec} \mathrm{R}$ with a map $\mathrm{S} \to \mathcal{F}$ there exists a $\mathrm{U} \hookrightarrow \mathcal{F} \in \mathcal{U}$ such that

$$\mathrm{U} \times_{\mathcal{F}} \mathrm{S} \neq \emptyset.$$

Lemma 6.3.3. *Let* A *be a ring and* I *and ideal of* A. *Suppose that* $\{\mathrm{A} \to \mathrm{A}_{f_i}\}$ *is a basic Zariski cover of* A. *Then* $\{\operatorname{Spec} \mathrm{A}_{f_i} \to \mathrm{D}(\mathrm{I}) : f_i \in \mathrm{I} \setminus 0\}$ *is a Zariski open cover of the quasi-affine scheme* D(I).

Proof. First, we may assume that none of the f_i's are nilpotent: if there was a nilpotent element then $\operatorname{Spec} \mathrm{A}_{f_i} = \operatorname{Spec} 0 = \emptyset$ and this makes no contribution to the Zariski cover.

Now it suffices to prove that for a morphism $\operatorname{Spec} \mathrm{B} \to \mathrm{D}(\mathrm{I})$ where B is nonzero, then $\operatorname{Spec} \mathrm{B} \times_{\mathrm{D}(\mathrm{I})} \operatorname{Spec} \mathrm{A}_{f_i} \neq 0$ whenever $f_i \in \mathrm{I} \setminus 0$. This classifies a morphism $\varphi : \mathrm{A} \to \mathrm{B}$ such that $\mathrm{IB} = \mathrm{B}$. It thus suffices to prove that:

$$\mathrm{A}_{f_i} \otimes_{\mathrm{A}}^{\varphi} \mathrm{B} \neq 0.$$

If this was zero then $1 = 0$. But $1 = \frac{1}{f_i} \otimes \varphi(f_i)$ and being zero means that there exist an N such that $f_i^{\mathrm{N}}(\frac{1}{f_i} \otimes \varphi(f_i)) = 0$. Since f_i was not nilpotent this means that $f_i^{\mathrm{N}}\varphi(f_i) = \varphi(f_i)^{\mathrm{N}} = 0$ so that f_i is nilpotent in B. But this cannot be since $\mathrm{B} = \mathrm{IB}$ was assumed. □

Lemma 6.3.4. *Any open cover* $\mathcal{U}$ *of an affine scheme* Spec A *admits a refinement by basic Zariski covers. More precisely: given* $\mathcal{U}$ *a Zariski open cover of* Spec A, *there exists a basic Zariski open cover* $\mathcal{V} := \{\mathrm{A} \to \mathrm{A}_{f_i}\}$ *with the property that for each* $\mathrm{U} \in \mathcal{U}$, *the set*

$$\mathcal{V}_{\mathrm{U}} := \{\operatorname{Spec} \mathrm{A}_{f_i} \in \mathcal{V} : \operatorname{Spec} \mathrm{A}_{f_i} \hookrightarrow \mathrm{U}\}$$

is a Zariski open cover of U.

Proof. By Lemma 5.2.4, $\mathcal{U} = \{D(I_\alpha)\}_\alpha$. By the proof of Lemma 6.2.6 we have that

$$A = \sqrt{\sum I_\alpha}.$$

From this, extract the set

$$\{f : \exists \alpha, f \in I_\alpha \setminus 0\}$$

Then $\{\operatorname{Spec} A_f\}$ is the desired refinement after Lemma 6.3.3. □

Here's an addendum to a previous definition. Let us first expand our minds and define QAff to be the full subcategory of **PStk** spanned by affine schemes and quasi-affine schemes. The objects of this category are prestacks of the form D(I) for some ideal $I \subset A$ is a ring A.

Definition 6.3.5. A **Zariski cover** of an quasi-affine scheme is a collection of open embeddings of prestacks

$$\mathcal{U} = \{U \hookrightarrow D(I)\},$$

such that for any nonzero ring R, Spec R with a map $\operatorname{Spec} R \to D(I)$ there exists a $U \hookrightarrow D(I) \in \mathcal{U}$ such that

$$U \times_X S \neq \emptyset.$$

Definition 6.3.6. A prestack $\mathcal{F} : \mathrm{QAff}^{\mathrm{op}} \to \mathrm{Set}$ is a **(Zariski) stack** (resp. **(Zariski) stack for the Zariski cover** $\mathcal{U}$) if for any $U \in \mathrm{QAff}$ and any (resp. the) Zariski cover $\mathcal{U} := \{U_i \to D(I)\}_{i \in I}$ the diagram

$$\mathcal{F}(D(I))) \to \prod \mathrm{Hom}(U_i, \mathcal{F}) \rightrightarrows \prod_{i_0, i_1} \mathrm{Hom}(U_{i_0} \times_{\operatorname{Spec} A} U_{i_1}, \mathcal{F})$$

is an equalizer diagram where the maps are induced are the obvious ones.

Theorem 6.3.7. *Suppose that* $\mathcal{F} : \mathrm{QAff}^{\mathrm{op}} \to \mathrm{Set}$ *is a prestack. The following are equivalent:*

(1) $\mathcal{F}$ *is a Zariski stack.*
(2) $\mathcal{F}$ *is a Zariski stack for basic open covers of affine schemes.*

6.4. **Refining covers.** We want to relate the notion of a Zariski stack defined in this class with the last. To do so, we need the language of refining covers.

Definition 6.4.1. Suppose that we have two families of morphism with fixed targets

$$\mathcal{U} = \{U_\alpha \hookrightarrow \mathcal{F}\}_{\alpha \in A}, \mathcal{V} = \{V_\beta \hookrightarrow \mathcal{G}\}_{\beta \in B}.$$

Then a **morphism of families with fixed targets**, written as

$$\mathcal{U} \to \mathcal{V},$$

is the data of (1) a morphism $\mathcal{F} \to \mathcal{G}$, (2) a map $t : A \to B$ and (3) a map $U_\alpha \to V_{t(\alpha)}$ such that the diagram

$$\begin{array}{ccc} U_\alpha & \longrightarrow & V_{t(\alpha)} \\ \downarrow & & \downarrow \\ \mathcal{F} & \longrightarrow & \mathcal{G} \end{array}$$

commutes. We say that a morphism of families with fixed targets is a **refinement** if $\mathcal{F} = \mathcal{G}$.

For a morphism of families with fixed targets $\mathcal{U} \to \mathcal{V}$, we write for each $V_\beta \in \mathcal{V}$:

$$\mathcal{U}_{V_\beta} := \{U_\alpha : U_\alpha \to V_\beta\}.$$

Definition 6.4.2. Let $\mathcal{F}$ be a prestack and $\mathcal{V}$ an open covering of $\mathcal{F}$. We say that a refinement

$$\mathcal{U} \to \mathcal{V}$$

is a **refinement of covers** if (1) $\mathcal{U}$ is also a Zariski cover, and (2) for each $V_\beta \in \mathcal{V}$, the family $\mathcal{U}_{V_\beta} := \{U_\alpha : U_\alpha \to V_\beta\}$ is a Zariski cover.

This is left to the reader.

Lemma 6.4.3. *Suppose that prestack $\mathcal{F}$ is a prestack and $\mathcal{V}$ is a Zariski cover. Suppose that we have a refinement of covers $\mathcal{U} \to \mathcal{V}$, $\mathcal{F}$ such that*

(1) $\mathcal{F}$ is a Zariski stack for $\mathcal{U}$ and,
(2) for any $\mathcal{U}_{\mathrm{V}_\beta}$, $\mathcal{F}$ is a Zariski stack for $\mathcal{U}_{\mathrm{V}_\beta}$ as well.

Then $\mathcal{F}$ is a Zariski stack for $\mathcal{V}$.

Proposition 6.4.4. *Let $\mathcal{F}$ be a prestack. Then the following are equivalent:*

(1) $\mathcal{F}$ is a Zariski stack,
(2) $\mathcal{F}$ is a Zariski stack for all basic Zariski covers of affine schemes.

6.5. **The definition of schemes.**

Definition 6.5.1. A **Zariski cover** of a prestack is a collection of open embeddings of prestacks

$$\mathcal{U} = \{\mathrm{U} \hookrightarrow \mathcal{F}\},$$

such that for any nonzero ring R, Spec R with a map Spec R $\to$ Spec A there exists a $\mathrm{U} \hookrightarrow \operatorname{Spec} \mathrm{A} \in \mathcal{U}$ such that

$$\mathrm{U} \times_{\mathrm{X}} \mathrm{S} \neq \emptyset.$$

Definition 6.5.2. A **scheme** is a Zariski stack $\mathcal{X}$ with an open cover $\mathcal{U} \to \mathcal{X}$ such that each $\mathrm{U} \in \mathcal{U}$ is an affine scheme.

Proposition 6.5.3. *Any affine scheme is a scheme.*

One of the most powerful aspects of algebraic geometry is the fact that we can work relative to a base scheme. Let us illustrate how this works with some examples.

(1) consider the map

$$\mathbf{C} \to \mathbf{C} \qquad z = a + ib \mapsto \overline{z} = a - ib.$$

This is is a **Z**-linear map. However, it is *not* a **C**-linear map:

$$(x+iy)\overline{ib} = (x+iy)(-ib) = -xib + yb,$$

but

$$\overline{(x+iy)(ib)} = \overline{(xib - yb)} = -xib - yb.$$

7. Lecture 6: Relative algebraic geometry

8. Lecture 7: Quasicoherent sheaves

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