MATH 232: ALGEBRAIC GEOMETRY I

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Contents

"In our acquisition of knowledge of the Universe (whether mathematical or otherwise) that which renovates the quest is nothing more nor less than complete innocence." - A. Grothendieck.

1. References

1.1. **Books.** In principle, this class is about Grothendieck's [EGA1] which signals the birth of modern algebraic geometry. It is an extremely technical document on its own and shows one of the many ways mathematics was developed organically. The French is not too hard and I recommend that you look through the book before the start of class — I might also assign readings from here occasionally with the promise that the French (and some Google translate) will not hinder your mathematical understanding.

Here are some textbooks in algebraic geometry.

([Sha13]) Shafarevich's book is a little more old school than the others in this list, but is valuable in the **examples** it gives.

([Har77]) Hartshorne's book has long been the "gold-standard" for algebraic geometry textbook. I learned the subject from this book first. It is **terse** and has plenty of **good exercises and problems**. However, the point of view that this book takes will be substantially different from one we will take in this class, though I will most definitely steal problems from here.

([GW10]) This is essentially a translation of Grothendieck's EGA (plus more) and is closer to the point of view of this class. Just like the original text, it is **relentlessly general** and very **lucid** in its exposition.

([Vak]) Arguably the most **inviting** book in this list, and modern in its outlook.

([DG80]) As far as I know this is still the only textbook reference to the **functor-of-points** point of view to algebraic geometry.

1.2. **Lecture Notes.** There are also several class notes online in algebraic geometry. I will add on to this list as the class progresses.

([Ras]) This is the closest document to our approach to this class. In fact, I will often present directly from these notes.

([Gat]) This is a "varieties" class, so the approach is very different, but I find it very helpful for lots of examples.

1.3. **Online textbooks.** There has been an explosion of online textbooks for algebraic geometry recently, though they are perhaps they are more like "encyclopedias."

([Stacks]) Johan de Jong at Columbia was the trailblazer in this industry and most, if not all, facts about algebraic geometry that will be taught will appear here, with proofs.

([cri]) Similar but for commutative algebra. Much more incomplete.

([fpp]) A translation project for EGA.

2. Lecture 1: What is algebraic geometry?

In its essence, algebraic geometry is the study of solutions to polynomial equations. What one means by "polynomial equations," however, has changed drastically throughout the latter part of the 20th century. To meet the demands in making constructions, ideas and theorems in classical algebraic geometry rigorous has given birth to a slew of techniques and ideas which are applicable to a much, much broader range of mathematical situations.

To begin with, let us recall the famous Fermat problem:

Theorem 2.0.1 (Taylor-Wiles)**.** *Let $n \geqslant 3$, then $x^n + y^n = 1$ has no solutions over $\mathbf{Q}$ when $x, y \neq 0$.*

This is a problem in algebraic geometry. In the language that we will learn in this class, we will be able to associate a **smooth, projective scheme** Fer_n which is, informally, given by a homogeneous polynomial equation $x^n + y^n = z^n$, equipped with a canonical **morphism**

$$\begin{tikzcd} \mathrm{Fer}_n \arrow[d] \\ \operatorname{Spec} \mathbf{Z}, \end{tikzcd}$$

such that its set of sections

$$\begin{tikzcd} \mathrm{Fer}_n \arrow[d] \\ \operatorname{Spec} \mathbf{Z}, \arrow[u, bend left] \end{tikzcd}$$

correspond to potential solutions to the Fermat equation. It is in this language that the Fermat problem was eventually solved.

The point-of-view we wish to adopt in this class, however, is one that goes by **functor-of-points**. In this highly abstract, but more flexible, approach schemes appear as what they are *supposed to be* which is often easier to think about. For us, the basic definition is:

Definition 2.0.2. A **prestack** is a functor from the category of commutative rings to sets:

$$\mathcal{F} : \mathrm{CAlg} \to \mathrm{Set}.$$

Remark 2.0.3. A note on terminology: this is non-standard. What should be called (and was called by Grothendieck) a prestack is a functor

$$\mathcal{F} : \mathrm{CAlg} \to \mathrm{Cat},$$

where Cat is the (large, $(2,1)$-)category of small categories. If we think of a set as a category with no non-trivial morphisms between the objects, then the above definition is a special case of this Grothendieck definition of a prestack. We will not consider functors into categories in this class so we will reserve the term prestack for such a functor above (as opposed to something like "a prestack in sets"). Perhaps it should be called a presheaf, but a presheaf should really just be an arbitrary functor

$$\mathcal{F} : \mathcal{D}^{\mathrm{op}} \to \mathrm{Set}.$$

To make this definition jibe with the Fermat scheme above, let us note that the equation $x^n + y^n = 1$ is defined for *any* ring. Therefore we can define

$$\widetilde{\mathrm{Fer}}_n(\mathrm{R}) = \{(a, b, c) : a^n + b^n = 1\} \subset \mathrm{R}^{\times 2}.$$

The theorem of Taylor and Wiles can then be restated as the fact that

$$\widetilde{\mathrm{Fer}}_n(\mathbf{Z}) = \emptyset \qquad n \geqslant 3;$$

However we caution that this is not the same as the scheme Fer_n that we have alluded to above since it is not projective — something that we will address in the class.

Another key idea in algebraic geometry is the question of parametrizing solutions of polynomial equations in a reasonable way. Let us consider $\widetilde{\mathrm{Fer}}_2$, which is the set of solutions to $x^2 + y^2 = 1$. We have a canonical equality (the first one is more or less the same as the above):

$$\widetilde{\mathrm{Fer}}_2(\mathbf{R}) = \mathrm{S}^1,$$

as we all know. Here are three other possible answers:

(1) $\widetilde{\mathrm{Fer}}_2(\mathbf{R}) = (\cos\theta, \sin\theta) \qquad 0 \leqslant \theta < 2\pi$,
(2) $\widetilde{\mathrm{Fer}}_2(\mathbf{R}) = (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}) \qquad t \in \mathbf{R}$,
(3) $\widetilde{\mathrm{Fer}}_2(\mathbf{R})$ is the set of all triangles with hypotenuse 1 up to congruence.

The first answer *does not* belong to the realm of (conventional) algebraic geometry which, by its very nature, concerns only polynomial functions. In other words, we dismiss transcendentals like exp and cos, sin. However the language of this class is actually powerful enough to capture transcendentals and reconstruct the familiar theory of differential geometry. The third answer will turn out to belong to the realm of algebraic geometry as well, but that will be reserved for a second course. The second answer does belong to the realm of algebraic geometry that we will study in this class: we can use **rational functions** of one variable in order to describe $\mathrm{Fer}_2(\mathbf{R})$. In fact, this parametrization proves

Theorem 2.0.4. *A quadric hypersurface in* $\mathbf{P}^2$ *with a rational point is rational. In fact, any quadric hypersurface with a rational point is rational.*

The proof of this result is "basically known" to pre-Grothendieck algebraic geometers: we pick the rational point and stereographcially project into a hyperplane. Since a quadric means that it is cut out by a degree two polynomial, it must hit one other point. This defines a **rational map** — one that is defined "almost everywhere" which is evidently an "isomorphism." One of the major thread of investigation in algebraic geometry and comes under the name of **birational geometry** and the above result belongs to this area. An example of a beautiful result that belongs to modern birational geometry is:

Theorem 2.0.5 (Clemens and Griffiths)**.** *A nonsingular cubic threefold over* $\mathbf{C}$ *is* not *rational.*

Theorem 2.0.5 is a non-existence proof — it says that there is no way to "rationally parametrize" the cubic threefold. If you have been trained in algebraic topology, you will feel like some kind of cohomological methods would be needed. The words that you should look for are "intermediate Jacobians," an object whose real birthplace is Hodge theory.

One of the major, open problems in the subject is:

Question 2.0.6. Is a generic cubic fourfold over $\mathbf{C}$ rational?

Recently, Katzarkov, Kontesvich and Pantev claimed to have made substantial progress towards this problem, but a write-up is yet to appear. More generally, a central question in algebraic geometry is:

Question 2.0.7. How does one classify algebraic varieties up to birational equivalence?

In topology, recall that a topological (closed) surface can be classified by genus or, better, Euler charateristic:

(1) if $\chi(\Sigma) < 0$, then Σ must be the Riemann sphere,
(2) if $\chi(\Sigma) = 0$ then Σ must be a torus — in the terminology of this class it is an **elliptic curve**,
(3) most surfaces have $\chi(\Sigma) > 0$ and they are, in some sense, the "generic situation."

This kind of trichotomy can be extended to higher dimensional varieties (topological surfaces being a 1-dimensional algebraic variety over $\mathbf{C}$). The **minimal model program** seeks to find "preferred" representatives in each class.

2.1. **Algebraic geometry beyond algebraic geometry.** The field of birational geometry is extremely large and remains an active area of research. But classifying algebraic varieties is not the only thing that algebraic geometry is good for. We have seen how it can be used to phrase the Fermat problem and eventually hosts its solution. There are other areas where algebraic geometry has proven to be the optimal "hosts" for problems.

One of the most prominent areas is representation theory where the central definition is very simple a group homomorphism

$$\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}).$$

If we are interested in representations valued in k-vector spaces, then the collection of all G-representations form a category called $\mathbf{Rep}_k(\mathrm{G})$. This category has an algebro-geometric incarnation: it is the category of **quasicoherent sheaves** over the an algebro-geometric gadget called a **algebraic stack** (in this case, denoted by BG) which is a special, more manageable class of prestacks but are slightly more mysterious gadgets than just algebraic varieties. Quasicoherent sheaves are fancy versions of vector bundles — they include gadgets whose fibers "can jump" although we will study restrictions on how exactly they jump. In any case, the field of **geometric representation theory** takes as starting point that representation theory is "just" the study of the geometric object BG and brings to bear the tools of algebraic geometry onto representation theory.

We have seen that algebraic geometry hosts number theory through the problem of the existence of rational points on a variety. Another deep problem of number theory that lives within modern algebraic geometry is the **Riemann hypothesis**. In algebro-geometric terms it can be viewed as a way to assemble solutions of an equation over fields of different characteristics.

Soon we will learn what it means for a morphism of schemes $f : \mathrm{X} \to \operatorname{Spec} \mathbf{Z}$ to be **proper** and for X to be **regular, geometrically connected** and **dimension d**. To this set-up we can associate the **Hasse-Weil zeta function**:

$$\zeta_{\mathrm{X}}(s) := \prod_{x \in |\mathrm{X}|} (1 - \sharp(\kappa(x))^{-s})^{-1}.$$

where:

(1) the set $|\mathrm{X}|$ is the set of **closed points** of X,
(2) $\kappa(x)$ is the **residue field** of x which is a finite extension of $\mathbf{F}_p$ for some prime $p > 0$.

This function is expected to be extending to all of the complex numbers (as a meromorphic function). There is a version $\zeta_{\overline{\mathrm{X}}}(s)$ which takes into account the "analytic part" of X as well:

Conjecture 2.1.1 (Generalized Riemann hypothesis). *If $s \in \mathbf{C}$ is a zero of $\zeta_{\overline{\mathrm{X}}}(s)$ then:*

$$2\mathrm{Re}(s) = \nu,$$

where $\nu \in [0, 2d]$.

One of the more viable approaches to verifying the generalized Riemann hypothesis is via **cohomological methods** — one would like to find a cohomology theory for schemes to which one can "extract" in a natural way the Hasse-Weil zeta function. One reason why one might expect this is the (also conjectured) functional equation

$$\zeta_{\overline{\mathrm{X}}}(s) \sim \zeta_{\overline{\mathrm{X}}}(\dim(\mathrm{X}) - s)$$

where $\sim$ indicates "up to some constant." This is a manifestation of a certain Poincaré duality in this cohomology theory which witnesses a certain symmetry between the cohomology groups and governed by the dimension of X. If X is concentrated at a single prime, then the Riemann hypothesis was proved by Deligne using **étale cohomology**. Recent work of Hesselholt, Bhatt, Morrow and Scholze have made some breakthrough towards setting up this cohomology theory but the Riemann hypothesis is, to the instructor's knowledge, still out of reach.

2.2. **Exercises 1: categorical preliminaries.** Here is a standard definition. We assume that every category in sight is **locally small** so that $\mathrm{Hom}(x, y)$ is a set, while the set of objects, $\mathrm{Obj}(\mathcal{C})$, is not necessarily a set (so only a proper class).

Definition 2.2.1. A functor $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ is **fully faithful** if for all $x, y \in \mathcal{C}$, the canonical map

$$\mathrm{Hom}(x, y) \to \mathrm{H}om(\mathrm{F}x, \mathrm{F}y)$$

is an isomorphism. We say that it is **conservative** if it reflects isomorphisms: an arrow $f : c \to c'$ in $\mathcal{C}$ is an isomorphism if and onnly if $\mathrm{F}(f) : \mathrm{F}(c) \to \mathrm{F}(c')$ is.

Exercise 2.2.2. *Let*

$$\mathrm{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathrm{G}$$

be an adjunction (in these notes we always write the left adjoint on the left). Prove

(1) F *preserves all colimits,*
(2) G *preserves all limits,*
(3) *The functor* F *is fully faithful if and only if the unit transformation*

$$\mathrm{id} \to \mathrm{G} \circ \mathrm{F}$$

is an isomorphism.
(4) $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ *is an equivalence of categories if and only if* F *is fully faithful and* G *is conservative.*

Exercise 2.2.3. *Prove that* $\mathcal{C}$ *admits all colimits if and only if it admits coproducts and co-equalizers. What kind of colimits do the following categories have (you do not have to justify your answer):*

(1) *the category of finite sets,*
(2) *the category of sets,*
(3) *the category of finitely generated free abelian groups,*
(4) *the category of abelian groups,*
(5) *the category of finite dimensional vector spaces,*
(6) *the category of all vector spaces,*
(7) *the category of finitely generated free modules over a commutative ring* R*,*
(8) *the category of finitely generated projective modules over a ring* R*,*
(9) *the category of all projective modules over a ring* R*.*

Exercise 2.2.4. *Give a very short proof (no more than one line) of the dual assertion:* $\mathcal{C}$ *admits all limits if and only if it admits products and equalizers.*

Exercise 2.2.5. *Prove that the limit over the empty diagram gives terminal object, while the colimit over the empty diagram gives the initial object.*

Exercise 2.2.6. *For any small category* $\mathcal{C}$*, we can form the presheaf category*

$$\mathrm{PSh}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}).$$

Prove:

(1) *If*

$$\mathrm{F} : \mathrm{I} \to \mathrm{PSh}(\mathcal{C}) \qquad i \mapsto \mathrm{F}_i$$

is a functor and I *is a small diagram, then for any* $c \in \mathcal{C}$ *the canonical map*

$$(\operatorname*{colim}_{\mathrm{I}} \mathrm{F}_i)(c) \to \operatorname*{colim}_{\mathrm{I}}(\mathrm{F}_i(c))$$

is an isomorphism.
(2) *Formulate and prove a similar statement for limits.*
(3) *Conclude that* $\mathrm{PSh}(\mathcal{C})$ *admits all limits and colimits.*

Exercise 2.2.7. *Prove the Yoneda lemma in the following form: the functor*

$$y : \mathcal{C} \to \mathrm{PSh}(\mathcal{C}) \qquad c \mapsto y(c)(x) = \mathrm{Hom}(x, c).$$

is fully faithful. Any functor in the image of y *is called* **representable**

Exercise 2.2.8. *Prove that any functor* $\mathrm{F} \in \mathrm{PSh}(\mathcal{C})$ *is a colimit of representable functors. This entails constructing a natural transformation*

$$\operatorname{colim} y(c) \to \mathrm{F}$$

where the domain is a colimit of a diagram of functors where each functor is representable, and proving that this natural transformation is an isomorphism when evaluated at each object of $\mathcal{C}$.

Exercise 2.2.9. *We say that a category* $\mathcal{C}$ *is* **essentially small** *if it is equivalent to small category. Let* R *be a commutative ring and consider* $\mathrm{CAlg}_{\mathrm{R}}$ *to be the category of commutative* R*-algebras. We say that an* R*-algebra* S *is* **finite type** *if it admits an* R*-linear* surjective *ring homomorphism*

$$\mathrm{R}[x_1, \cdots x_n] \to \mathrm{S}.$$

Consider the full subcategory $\mathrm{CAlg}^{\mathrm{ft}}_{\mathrm{R}} \subset \mathrm{CAlg}_{\mathrm{R}}$ *of finite type* R*-algebras. Prove that:*

(1) The collection of R*-algebras of the form*

$$\{\mathrm{R}[x_1, \cdots, x_n]/\mathrm{I} : \mathrm{I} \textit{ is an ideal}\}$$

forms a set (this is not meant to be hard and does not require knowledge of "set theory").

(2) Prove that the category of finite type R*-algebras are equivalent to the subcategory of* R*-algebras of the form* $\mathrm{R}[x_1, \cdots, x_n]/\mathrm{I}$ *(this is not meant to be hard and does not require knowledge of "set theory").*

(3) Conclude from this that $\mathrm{CAlg}^{\mathrm{ft}}_{\mathrm{R}}$ *is an essentially small category.*

Exercise 2.2.10. *We define the subcategory of* **left exact functors**

$$\mathrm{PSh}_{\mathrm{lex}}(\mathcal{C}) \subset \mathrm{PSh}(\mathcal{C})$$

to be the subcategory of those functors which preserves finite limits. These are functors F *such that for any finite diagram*[1] $\alpha : \mathrm{I} \to \mathcal{C}$*, the canonical map*

$$\mathrm{F}(\operatorname*{colim}_{\mathrm{I}} \alpha) \to \lim_{\mathrm{I}} \mathrm{F}(\alpha)$$

is an isomorphism. Prove:

(1) a category $\mathcal{C}$ *admits all finite limits if and only if it admits final objects and pullbacks;*

(2) for a functor F *to be left exact, it is necessary and sufficient that* F *preserves final objects and pullbacks.*

(3) Prove that the yoneda functor factors as $y : \mathcal{C} \to \mathrm{PSh}_{\mathrm{lex}}(\mathcal{C})$.

(4) If $f : \mathcal{C} \to \mathcal{D}$ *is a functor, we define*

$$f^* : \mathrm{PSh}(\mathcal{D}) \to \mathrm{PSh}(\mathcal{C}) \qquad f^*\mathrm{F} = \mathrm{F} \circ f.$$

Prove that if f *preserves finite colimits, then we have an induced functor*

$$f^* : \mathrm{PSh}_{\mathrm{lex}}(\mathcal{D}) \to \mathrm{PSh}_{\mathrm{lex}}(\mathcal{C}).$$

In the third problem set, we will use this to prove the adjoint functor theorem and construct the sheafification functor.

Exercise 2.2.11. *We say that* $\mathcal{C}$ *is* **locally presentable** *if there exists a subcategory* $i : \mathcal{C}^c \subset \mathcal{C}$ *(called the category of* **compact objects***) which is essentially small and is closed under finite colimits such that the functor*

$$\mathcal{C} \to \mathrm{PSh}_{\mathrm{lex}}(\mathcal{C}) \xrightarrow{i^*} \mathrm{PSh}_{\mathrm{lex}}(\mathcal{C}^c)$$

is an equivalence of categories. Prove

[1] This just means that I is a category with finitely many objects and finitely many morphisms.

(1) the category Sets *is locally presentable with* Sets^c *being the subcategory of finite sets,*
(2) the category Vect_k *is locally presentable with* Vect^c *being the subcategory of finite-dimensional vector spaces.*

3. Lecture 2: Prestacks

Throughout the course we will denote by CAlg the category of commutative rings.

Definition 3.0.1. A **prestack** is a functor

$$\mathrm{X} : \mathrm{CAlg} \to \mathrm{Set}.$$

This means that to each commutative ring R, we assign the set X(R) and for each morphism of commutative rings $f : \mathrm{R} \to \mathrm{S}$ we have a morphism of sets

$$f^* : \mathrm{X}(\mathrm{R}) \to \mathrm{X}(\mathrm{S}).$$

Furthermore, these satisfy the obvious compatibilities to be a functor.

Definition 3.0.2. A **morphism of prestacks** is a natural transformation $g : \mathrm{X} \to \mathrm{Y}$ of functors. This means that for each morphism of commutative rings $f : \mathrm{R} \to \mathrm{S}$ we have a commuting diagram

$$\begin{tikzcd} \mathrm{X}(\mathrm{R}) \arrow[r, "f^*"] \arrow[d, "g_{\mathrm{R}}"'] & \mathrm{X}(\mathrm{S}) \arrow[d, "g_{\mathrm{S}}"] \\ \mathrm{Y}(\mathrm{R}) \arrow[r, "f^*"] & \mathrm{X}(\mathrm{S}). \end{tikzcd}$$

An R-**point** of a prestack is point $x \in \mathrm{X}(\mathrm{R})$; this is the same thing as a morphism of prestacks $\mathrm{Spec}\,\mathrm{R} \to \mathrm{X}$ by the next

Lemma 3.0.3 (Yoneda)**.** *For all prestack* X *and all* $\mathrm{R} \in \mathrm{CAlg}$*, we have a canonical isomorphism*

$$\mathrm{Hom}(\mathrm{Spec}\,\mathrm{R}, \mathrm{X}) \cong \mathrm{X}(\mathrm{R}).$$

In particular we have that

$$\mathrm{Hom}(\mathrm{Spec}\,\mathrm{R}, \mathrm{Spec}\,\mathrm{S}) \cong \mathrm{Spec}\,\mathrm{S}(\mathrm{R}) = \mathrm{Hom}(\mathrm{S}, \mathrm{R}).$$

Note the reversal of directions.

We denote by **PStk** the category of prestacks. We already have a wealth of examples:

Definition 3.0.4. Let R be a commutative ring, We define

$$\mathrm{Spec}\,\mathrm{R} : \mathrm{CAlg} \to \mathrm{Set} \qquad \mathrm{S} \mapsto \mathrm{Hom}_{\mathrm{CAlg}}(\mathrm{R}, \mathrm{S}).$$

An **affine scheme** is a prestack of this form.

Remark 3.0.5. If a prestack is representable, then the ring **representing** it is unique up to unique isomorphism. This is a consequence of the Yoneda lemma. In more detail, the Yoneda functor takes the form

$$\mathrm{Spec} : \mathrm{CAlg}^{\mathrm{op}} \to \mathbf{PStk} = \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}).$$

This functor is fully faithful so we may (somewhat abusively) identify $\mathrm{CAlg}^{\mathrm{op}}$ with its image in **PStk**. The category of affine schemes is then taken to be the opposite category of commutative rings.

Example 3.0.6. Let $n \geqslant 0$. Then we define the prestack of **affine space of dimension** n as

$$\mathbf{A}^n_{\mathbf{Z}} : \mathrm{CAlg} \to \mathrm{Set} \qquad \mathrm{R} \mapsto \mathrm{R}^{\times n}.$$

In the homeworks, you will be asked to prove that this prestack is an affine scheme, represented by $\mathbf{Z}[x_1, \cdots, x_n]$.

Example 3.0.7. Suppose that $f(x) \in \mathbf{Z}[x,y,z]$ is a polynomial in three variables; for a famous example this could be $f(x,y,z) = x^n + y^n - z^n$. For each ring A, we define

$$\mathrm{V}(f)(\mathrm{R}) := \{(a,b,c) : f(a,b,c) = 0\} \subset \mathrm{R}^{\times 3}.$$

Note that this indeed defines a prestack: given a morphism of rings $\varphi : \mathrm{R} \to \mathrm{S}$, we have a morphism of sets

$$\mathrm{V}(f)(\mathrm{R}) \to \mathrm{V}(f)(\mathrm{S})$$

since $f(\varphi(a), \varphi(b), \varphi(c)) = \varphi(f(a,b,c)) = \varphi(0) = 0$. In fact, we have a morphism of prestacks (in the sense of the next definition)

$$\mathrm{V}(f) \to \mathbf{A}^{\times 3}_{\mathbf{Z}},$$

where $\mathbf{A}^{\times 3}_{\mathbf{Z}}(\mathrm{R}) = \mathrm{R}^{\times 3}$.

3.1. **Operation on prestacks I: fibered products.** One of the key ideas behind algebraic geometry is to restrict ourselves to objects which are defined by polynomial functions. More abstractly we want to restrict ourselves to objects which arise from other objects in a *constructive manner*. This is both a blessing and a curse — on the one hand it makes objects in algebraic geometry rather rigid but, on the other, it gives objects in algebraic geometry a "tame" structure.

Example 3.1.1. As a warm-up, consider n-dimensional complex space $\mathbf{C}^n$ and suppose that we have a polynomial function $\mathbf{C}^n \to \mathbf{C}$. Then the **zero set** of f is defined via the pullback

$$\begin{tikzcd} \mathrm{Z}(f) \arrow[r] \arrow[d] & \mathbf{C}^n \arrow[d, "f"] \\ \{0\} \arrow[r] & \mathbf{C}. \end{tikzcd}$$

We want to say that $\mathrm{Z}(f)$ has the structure of a prestack or, later, a scheme. Of course the above diagram presents $\mathrm{Z}(f)$ as a set but we can also take the pullback in, say, the category of $\mathbf{C}$-analytic spaces so that $\mathrm{Z}(f)$ inherits such a structure (if a pullback exists! and it does).

Example 3.1.2. Another important construction in algebraic geometry is the notion of the **graph**. Suppose that $f : \mathrm{X} \to \mathrm{Y}$, then its graph is the set

$$\Gamma_f := \{(x,y) : f(x) = y\} \subset \mathrm{X} \times \mathrm{Y}.$$

Suppose that X, Y have the structure of a scheme, or an $\mathbf{C}$-analytic spaces or a manifold etc., then we want to say that Γ_f does inherit this structure. To do so we note that we can present Γ_f in the following manner:

$$\begin{tikzcd} \Gamma_f \arrow[r] \arrow[d] & \mathrm{Y} \arrow[d, "\Delta"] \\ \mathrm{X} \times \mathrm{Y} \arrow[r, "f \times \mathrm{id}"] & \mathrm{Y} \times \mathrm{Y} \end{tikzcd}$$

Definition 3.1.3. Suppose that $\mathrm{X} \to \mathrm{Y} \leftarrow \mathrm{Z}$ is a **cospan** of prestacks, then the **fiber product of** X **and** Y **over** Z is defined as

$$(\mathrm{X} \times_{\mathrm{Y}} \mathrm{Z})(\mathrm{R}) := \mathrm{X}(\mathrm{R}) \times_{\mathrm{Y}(\mathrm{R})} \mathrm{Z}(\mathrm{R}).$$

It will be an exercise to verify the universal property of this construction.

Example 3.1.4. Suppose that we have a **span** of rings $\mathrm{R} \leftarrow \mathrm{S} \to \mathrm{T}$ so that we have a cospan of prestacks $\operatorname{Spec}\mathrm{R} \to \operatorname{Spec}\mathrm{S} \leftarrow \operatorname{Spec}\mathrm{T}$. Then (exercise) we have a natural isomorphism

$$\operatorname{Spec}\mathrm{R} \times_{\operatorname{Spec}\mathrm{S}} \operatorname{Spec}\mathrm{T} \cong \operatorname{Spec}(\mathrm{R} \otimes_{\mathrm{S}} \mathrm{T}).$$

Example 3.1.5. A **regular function on a prestack** is a morphism of prestacks $\mathrm{X} \to \mathbf{A}^1$. If $\mathrm{X} = \operatorname{Spec} \mathrm{R}$, then this classifies a map of commutative rings $\mathbf{Z}[x] \to \mathrm{R}$ which is equivalent to picking out a single element $f \in \mathrm{R}$. The **zero locus** of f is the prestack

$$\mathrm{Z}(f) := \mathrm{X} \times_{\mathbf{A}^1} \{0\}$$

where $\{0\} \to \mathbf{A}^1$ is the map corresponding to $\mathbf{Z}[x] \to \mathbf{Z}, x \mapsto 0$.

3.2. **Closed immersions.** A closed immersion is a special case of a **subprestack**

Definition 3.2.1. A **subprestack** of a prestack $\mathcal{F}$ is a prestack $\mathcal{G}$ equipped with a natural transformation $\mathcal{G} \to \mathcal{F}$ such that for any $\mathrm{R} \in \mathrm{CAlg}$, the map

$$\mathcal{G}(\mathrm{R}) \to \mathcal{F}(\mathrm{R})$$

is an injection. We will often write $\mathcal{F} \subset \mathcal{G}$ for subprestacks.

Remark 3.2.2. This is equivalent to saying that $\mathcal{G} \to \mathcal{F}$ is a monomorphism in the category of prestacks.

The important thing about a subprestack is that for any morphism $\mathrm{R} \to \mathrm{R}'$, the requirement that $\mathcal{G} \to \mathcal{F}$ is a natural transformation enforces the commutativity of the following diagram

$$\begin{tikzcd}
\mathcal{G}(\mathrm{R}) \arrow[r] \arrow[d] & \mathcal{F}(\mathrm{R}) \arrow[d] \\
\mathcal{G}(\mathrm{R}') \arrow[r] & \mathcal{F}(\mathrm{R}').
\end{tikzcd}$$

which should be read as: "the map $\mathcal{G}(\mathrm{R}) \to \mathcal{F}(\mathrm{R}')$ factors through the subset $\mathcal{G}(\mathrm{R}')$."

Definition 3.2.3. A **closed immersion** of affine schemes is a morphism $\operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{S}$ such that the induced map of rings $\mathrm{S} \to \mathrm{R}$ is surjective. In this case we say that $\operatorname{Spec} \mathrm{R}$ is a **closed subscheme** of $\operatorname{Spec} \mathrm{S}$.

Let us try to understand what this means. the map $\varphi : \mathrm{S} \to \mathrm{R}$, if surjective, is equivalent to the data of an ideal $\mathrm{I} = \ker(\varphi)$. As stated before, we should think of S as the ring of functions on a "space" $\operatorname{Spec} \mathrm{S}$ and so an ideal of S is a collection of functions which are closed under the action of S. Now, the "space" $\operatorname{Spec} \mathrm{R}$ should be thought of as the space on which the functions that belong to I vanish. In other words a closed immersion is one of the form $\operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{R}/\mathrm{I}$.

Exercise 3.2.4. *Suppose that $f : \mathbf{A}^n \to \mathbf{A}^1$ is regular function. Prove that the $\mathrm{Z}(f) \to \mathbf{A}^n$ is a closed immersion corresponding to a map of rings $\mathbf{Z}[x] \to \mathbf{Z}[x]/(f)$.*

Here is how one can globalize this definition:

Definition 3.2.5. A morphism of prestacks $\mathrm{X} \to \mathrm{Y}$ is a **closed immersion** or a **closed subprestack** if for any morphism $\operatorname{Spec} \mathrm{R} \to \mathrm{Y}$ then:

(1) the prestack $\operatorname{Spec} \mathrm{R} \times_{\mathrm{Y}} \mathrm{X}$ is representable and,
(2) the morphism
$$\operatorname{Spec} \mathrm{R} \times_{\mathrm{Y}} \mathrm{X} \to \operatorname{Spec} \mathrm{R}$$
is a closed immersion.

3.3. **Exercises 2.**

Exercise 3.3.1. *What does* $\operatorname{Spec}(0)$ *represent?*

Exercise 3.3.2. *Prove that the category of prestacks admit all limits and all colimits.*

Exercise 3.3.3. *Prove that the prestack $\mathbf{A}^n_{\mathbf{Z}}$ is representable by $\mathbf{Z}[x_1, \cdots, x_n]$.*

Exercise 3.3.4. *Consider the prestack*

$$\mathbf{G}_m : \mathrm{R} \mapsto \mathrm{R}^\times.$$

Here $\mathrm{R}^\times$ *is the multiplicative group of unit elements in* R*. Prove that* $\mathbf{G}_m$ *is representable. What ring is it representable by?*

Exercise 3.3.5. *Suppose that* $f : \mathbf{A}^n \to \mathbf{A}^1$ *is regular function. Prove that the* $\mathrm{Z}(f) \to \mathbf{A}^n$ *is a closed immersion corresponding to a map of rings* $\mathbf{Z}[x] \to \mathbf{Z}[x]/(f)$*.*

Exercise 3.3.6. *Consider the prestack*

$$\mathrm{GL}_n : \mathrm{R} \mapsto \mathrm{GL}_n(\mathrm{R}).$$

Prove that it is representable. What ring is it representable by?

If R is a ring we write $\mathrm{R}_\mathfrak{p}$ to be the localization of R at $\mathfrak{p}$. We write $\mathfrak{m}_\mathfrak{p}$ to be the maximal ideal of said local ring and write

$$\kappa(\mathfrak{p}) := \mathrm{R}_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}.$$

Exercise 3.3.7. *Let* $\mathrm{R} \in \mathrm{CAlg}$ *and* K *a field. Prove that there is a natural bijection between*

$$\{\operatorname{Spec} \mathrm{K} \to \operatorname{Spec} \mathrm{R}\}$$

with

$$\{\textit{prime ideals } \mathfrak{p} \subset \mathrm{R} \textit{ with an inclusion } \kappa(\mathfrak{p}) \hookrightarrow \mathrm{K}\}.$$

The **Zariski tangent space** of $\operatorname{Spec} \mathrm{R}$ **at a prime ideal** $\mathfrak{p}$ is the $\kappa(\mathfrak{p})$-vector space given by

$$\mathrm{T}_\mathfrak{p} \operatorname{Spec} \mathrm{R} := (\mathfrak{m}_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}^2)^\vee.$$

Exercise 3.3.8. *Let* $\mathrm{R} \in \mathrm{CAlg}$ *and* k *a field and suppose that* R *is a* k*-algebra. Prove that there is a bijection between*

$$\{k\textit{-morphisms of rings}\, \mathrm{R} \to \kappa[x]/(x^2)\}$$

with

$$\{\textit{prime ideals } \mathfrak{p} \textit{ of } \mathrm{R} \textit{ with residue field } k \textit{ and an element of the Zariski tangent space at } \mathfrak{p}\}.$$

Exercise 3.3.9. *Prove that the functor*

$$\operatorname{Spec} : \mathrm{CAlg} \to \mathbf{PStk}$$

1. *is fully faithful,*
2. *preserves all colimits in the sense that if* $\{\mathrm{R}_\alpha\}_{\alpha \in \mathrm{A}}$ *is a diagram of commutative rings then for all* $\mathrm{S} \in \mathrm{CAlg}$*, the canonical map*

$$\operatorname*{colim}_\alpha(\operatorname{Spec} \mathrm{R}_\alpha)(\mathrm{S}) \to \lim(\operatorname{Spec} \mathrm{R}_\alpha)(\mathrm{S})$$

   is an isomorphism. Deduce, in particular, that Spec *converts tensor products of commutative rings to pullback.*
3. *Show, by example, that* Spec *does not preserve limits.*

Exercise 3.3.10. *Prove that a closed immersion of schemes is a subprestack.*

For the next exercise, recall that if $\mathcal{C}$ is a category with products and $\mathrm{X} \in \mathcal{C}$, then the identity morphism $\mathrm{id} : \mathrm{X} \to \mathrm{X}$ induces the **diagonal** map

$$\Delta : \mathrm{X} \to \mathrm{X} \times \mathrm{X}.$$

Exercise 3.3.11. *Let* $\mathrm{R} \in \mathrm{CAlg}$ *and consider the multiplication map* $\mathrm{R} \otimes_\mathbf{Z} \mathrm{R} \to \mathrm{R}$*. Prove:*

1. *the corresponding map* $\operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{R} \times \operatorname{Spec} \mathrm{R}$ *is given by the diagonal morphism,*
2. *prove that* $\Delta : \operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{R} \times \operatorname{Spec} \mathrm{R}$ *is a closed immersion of prestacks.*

Exercise 3.3.12. *Let $\mathcal{C}$ be a category with all limits and suppose that we have a diagram*

$$\begin{tikzcd} \mathrm{X} \arrow[dr] & & \\ & \mathrm{S} \arrow[r] & \mathrm{T} \\ \mathrm{Y} \arrow[ur] & & \end{tikzcd}$$

Prove that the diagram (be sure to write down carefully how the maps are induced!)

$$\begin{tikzcd} \mathrm{X} \times_{\mathrm{S}} \mathrm{Y} \arrow[r] \arrow[d] & \mathrm{X} \times_{\mathrm{T}} \mathrm{Y} \arrow[d] \\ \mathrm{S} \arrow[r, "\Delta"] & \mathrm{S} \times_{\mathrm{T}} \mathrm{S} \end{tikzcd}$$

is cartesian (hint: if you are unable to prove this result in full generality, feel free to assume that $\mathcal{C} = \mathbf{PStk}$).

Exercise 3.3.13. *A morphism of prestacks* $\mathrm{G} \to \mathrm{F}$ *is said to be* **representable** *if for any* $\operatorname{Spec} \mathrm{R} \to \mathrm{F}$*, the prestack* $\mathrm{G} \times_{\mathrm{F}} \operatorname{Spec} \mathrm{R}$ *is representable. Prove that the following are equivalent:*

(1) the diagonal $\Delta : \mathrm{F} \to \mathrm{F} \times \mathrm{F}$ *is representable;*
(2) any map $\operatorname{Spec} \mathrm{S} \to \mathrm{F}$ *(in other words any map from an affine scheme to $\mathcal{F}$) is representable.*

4. Lecture 3: Descent

We are about to define schemes. But first we define stacks[2]. In order to define the notion of stacks, we need the notion of open immersions, which are complementary to closed immersions.

Remark 4.0.1. Thinking about closed immersions of schemes is easier than thinking about open immersions, at least to the instructor. Indeed, every closed immersion of $\operatorname{Spec} \mathrm{R}$ corresponds to the set of all ideals of R. We can think of a poset of ideals $\{\mathrm{I} \subset \mathrm{J}\}$ which corresponds to a poset of closed subschemes $\{\operatorname{Spec} \mathrm{R}/\mathrm{I} \to \operatorname{Spec} \mathrm{R}/\mathrm{J}\}$. Of course, we want to say that open subschemes of $\operatorname{Spec} \mathrm{R}$ should be those of the form

$$\mathrm{D}(\mathrm{J}) := \operatorname{Spec} \mathrm{R} \smallsetminus \operatorname{Spec} \mathrm{R}/\mathrm{J}.$$

But now $\mathrm{D}(\mathrm{J})$ is in fact **not** representable — we will soon be able to prove this. In fact these $\mathrm{D}(\mathrm{J})$'s are the first examples of non-affine schemes. In particular $\mathrm{D}(\mathrm{J})$ is not the form Spec of a ring. In order to fomulate descent in a more digestible manner, we will restrict ourselves to open subschemes of $\operatorname{Spec} \mathrm{R}$ which are actually affine.

Suppose that I is an ideal of a ring R, let us recall that the **radical of** I, denoted usually by $\sqrt{\mathrm{I}}$ is defined as

$$\sqrt{\mathrm{I}} = \{x : x^{\mathrm{N}} \in \mathrm{I} \text{ for some } \mathrm{N} \gg 0\}.$$

Example 4.0.2. Let $\mathrm{I} = (0)$ be the zero ideal. Then the **nilradical** is $\sqrt{(0)} = \{f : f^{\mathrm{N}} = 0 \text{ for some } \mathrm{N} \gg 0\}$. We say that a ring is **reduced** if $\sqrt{(0)} = 0$.

[2]This is where some heavy conflict with the literature will occur so be wary. In the literature, the notion of stacks differs from this one in two, crucial ways. First the descent condition is asked with respect to something called the étale topology (which we will cover later in class) and, secondly, the functor lies in the $(2,1)$-category of groupoids. Functors landing in said version of categories are not really functors in the sense we are used to in class.

Definition 4.0.3. Let R be a ring and $f \in \mathrm{R}$. A **basic Zariski cover of the ring** R_f consists of a set I and a collection $\mathcal{U} := \{f_i : f_i \in \mathrm{R}\}_{i \in \mathrm{I}}$ such that

$$f \in \sqrt{\Sigma(f_i)}.$$

In particular, a **basic Zariski cover** of a ring R consists of a set I and a collection $\mathcal{U} := \{f_i : f_i \in \mathrm{R}\}_{i \in \mathrm{I}}$ subject to the following condition:

$$1 \in \sqrt{\Sigma(f_i)}.$$

We write

$$\{\operatorname{Spec} \mathrm{A}_{f_i} \to \operatorname{Spec} \mathrm{A}_f\}_{i \in \mathrm{I}}$$

to denote an arbitrary basic Zariski cover.

Remark 4.0.4. If f is a unit so that $\mathrm{R}_f = \mathrm{R}$, this is a very redundant definition. Indeed, any element $x \in \sqrt{\Sigma(f_i)}$ means that

$$x^{\mathrm{N}} \in \Sigma(f_i)$$

But the sum of ideals means we have a sum of elements in each ideal where all except finitely many elements are zero so:

$$x^{\mathrm{N}} = a_1 f_1 + \cdots + a_n f_n,$$

up to rearrangements. But now

$$1 = 1^{\mathrm{N}} = a_1 f_1 + \cdots + a_n f_n.$$

Therefore we can find a subcover of $\mathcal{U}$ such that

$$1 \in \Sigma(f_i).$$

Of course this argument also does show that a basic Zariski cover of R_f can be refined by a finite subcover.

Example 4.0.5. Let p, q be distinct primes in $\mathbf{Z}$. This means, by Bézout's identity, we can write

$$1 = kp + rq,$$

for some $k, r \in \mathbf{Z}$. In the language above we find that

$$\{\operatorname{Spec} \mathbf{Z}[\tfrac{1}{p}], \operatorname{Spec} \mathbf{Z}[\tfrac{1}{q}] \hookrightarrow \operatorname{Spec} \mathbf{Z}\}$$

defines a basic Zariski cover of $\operatorname{Spec} \mathbf{Z}$.

Example 4.0.6. Let k be a field and consider $k[x]$. Suppose that $p(x), q(x)$ are polynomials which are irreducible and are coprime. Then Bézout's identity again works in this situation:

$$1 = k(x)p(x) + r(x)q(x).$$

In this language we find that

$$\{\operatorname{Spec} k[x]_{p(x)}, \operatorname{Spec} k[x]_{q(x)} \hookrightarrow \operatorname{Spec} k[x]\}$$

defines a basic Zariski cover of $\operatorname{Spec} k[x]$.

Definition 4.0.7. A prestack $\mathcal{F} : \mathrm{CAlg} \to \mathrm{Set}$ is a **(Zariski) stack** if for any $\mathrm{A} \in \mathrm{CAlg}$ and for all basic Zariski cover $\{\operatorname{Spec} \mathrm{A}_{f_i} \to \operatorname{Spec} \mathrm{A}\}_{i \in \mathrm{I}}$ the diagram

$$\mathcal{F}(\mathrm{A}) \to \prod_i \mathcal{F}(\mathrm{A}_{f_i}) \rightrightarrows \prod_{i_0, i_1} \mathcal{F}(\mathrm{A}_{f_{i_0}} \otimes_{\mathrm{A}} \mathrm{A}_{f_{i_1}})$$

is an equalizer diagram where the maps are induced by

$$\prod \mathrm{A}_{f_i} \to \prod_{i_0, i_1} \mathrm{A}_{f_{i_0}} \otimes_{\mathrm{A}} \mathrm{A}_{f_{i_1}} \qquad (g_i) \mapsto (g_{i_0}|_{\mathrm{A}_{f_{i_0} \cdot f_{i_1}}}).$$

and

$$\prod \mathrm{A}_{f_i} \to \prod_{i_0, i_1} \mathrm{A}_{f_{i_0}} \otimes_{\mathrm{A}} \mathrm{A}_{f_{i_1}} \qquad (g_i) \mapsto (g_{i_1}|_{\mathrm{A}_{f_{i_0} \cdot f_{i_1}}}).$$

4.1. **Unpacking the descent condition and Serre's lemma.** Let us note a couple of easy properties about localizations

Lemma 4.1.1. *Let* $f_1, f_2 \in \mathrm{R}$ *then*

$$(\mathrm{R}_{f_1})_{f_2} \cong \mathrm{R}_{f_1 \cdot f_2} \cong \mathrm{R}_{f_1} \otimes_{\mathrm{R}} \mathrm{R}_{f_2}.$$

This will be homework. For the rest of this section, we will seek be taking a map $f : \mathrm{R} \to \mathrm{A}$ and then postcomposing then along some localization of A, say A'; for this it is convenient to use the notation

$$f|_{\mathrm{A}'}$$

and think about "restriction."

Let us fix a ring R and suppose that A is a "test-ring" and we are interested in the set

$$\mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{Spec}\,\mathrm{R}),$$

and we would like to recover this set in terms of a given basic Zariski cover of A. As we had discussed, this latter object is given by the data of elements $g_1, \cdots g_n \in \mathrm{A}$ such that $1 = \sum_{i=1}^n g_i$. Let us consider the following set

$$\mathrm{Glue}(\mathrm{R}, \mathrm{A}, \{g_i\}) \subset \prod_{i=1}^n \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}_{g_i}, \mathrm{Spec}\,\mathrm{R}),$$

consisting of the n-tuples $\{f_i : \mathrm{R} \to \mathrm{A}_{g_i}\}$ subject to the following condition

$$f_i|_{\mathrm{A}_{g_i \cdot g_j}} = f_j|_{\mathrm{A}_{g_j \cdot g_i}}, \tag{cocycle}$$

called the **cocycle condition**.

Lemma 4.1.2. *As above we have an isomorphism*

$$\mathrm{Glue}(\mathrm{R}, \mathrm{A}, \{g_i\}) \cong \mathrm{Eq}(\prod_i \mathrm{Hom}(\mathrm{R}, \mathrm{A}_{g_i}) \rightrightarrows \prod_{i_0, i_1} \mathrm{Hom}(\mathrm{R}, \mathrm{A}_{g_{i_0} \cdot g_{i_1}}))$$

This is an exercise in unpacking definitions. Even though Glue is more explicit, in order to prove actual results, we will work with the equalizer formulation. Our main theorem is as follows:

Theorem 4.1.3. *The map*

$$\mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{Spec}\,\mathrm{R}) \to \prod_{i=1}^n \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}_{g_i}, \mathrm{Spec}\,\mathrm{R}) \qquad f : \mathrm{R} \to \mathrm{A} \mapsto (f|_{\mathrm{A}_{g_i}})_{i \in \mathrm{I}}$$

factors as

$$\mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{Spec}\,\mathrm{R}) \to \mathrm{Glue}(\mathrm{R}, \mathrm{A}, \{g_i\})$$

and induces an isomorphism. Equivalently, Spec R *is a Zariski stack.*

We will prove Theorem 4.1.3 in the right level of generality. The next lemma is called "Serre's lemma for modules."

Lemma 4.1.4. *Let* A *be a ring and* $f_1, \cdots, f_n$ *elements such that* $\sum_{i=1}^n f_i = 1$. *Then*

$$\mathrm{M} \to \prod \mathrm{M}_{f_i} \rightrightarrows \prod \mathrm{M}_{f_i \cdot f_j}$$

is an equalizer diagram.

Proof. We first assume

(1) there exists an element f_i, say f_1, which is invertible. So that $\mathrm{M} \cong \mathrm{M}_{f_1}$, via a map φ_1.

Then we prove the result: indeed, denote the equalizer by Eq. Indeed, the map $\mathrm{M} \to \prod \mathrm{M}_{f_i}$ factors through the equalizer since this is just the map

$$m \mapsto (m|_{\mathrm{A}_{f_i}}),$$

and the cocycle condition is satisfied. We construct a map

$$\mathrm{Eq} \to \mathrm{M}$$

given by

$$(m_i)_{i\in\mathrm{I}} \mapsto \varphi_1^{-1}(m_1).$$

From this, we check the two composites. First, consider:

$$\mathrm{Eq} \to \mathrm{M} \to \mathrm{Eq} \qquad (m_i) \mapsto \varphi^{-1}(m_1) \mapsto (m_1, m_1|_{\mathrm{A}_{f_2}}, \cdots, m_1|_{\mathrm{A}_{f_n}}).$$

But now, for $j > 1$ we have that

$$m_1|_{\mathrm{A}_{f_j}} = \varphi^{-1}(m_1)|_{\mathrm{A}_{f_1\cdot f_j}} = m_j|_{\mathrm{A}_{f_j\cdot f_1}} = m_j|_{\mathrm{A}_{f_1\cdot f_j}} = m_j,$$

where the last equality comes from the assumption that $\mathrm{M} = \mathrm{M}_{f_1}$. This proves that the composite is the identity.

Second, note that:

$$\mathrm{M} \to \mathrm{Eq} \to \mathrm{M} \qquad m \mapsto (m|_{\mathrm{A}_{f_j}}) \mapsto \varphi^{-1}(m_{\mathrm{A}_{f_j}}).$$

is easily seen to be the identity.

Now, to prove the desired claim: take kernels and cokernels:

$$0 \to \mathrm{K} \to \mathrm{M} \to \mathrm{Eq} \to \mathrm{C} \to 0$$

The claim is that $\mathrm{K}, \mathrm{C} = 0$. They are 0 after inverting each f_i by the previous claim. The proof the finishes by the next claim.

□

Lemma 4.1.5. *Let* A *be a ring and* $f_1, \cdots, f_n$ *elements such that* $\sum_{i=1}^n f_i = 1$. *Suppose that* $\mathrm{M}_{f_i} = 0$ *for all* $i = 1, \cdots, n$. *Then* $\mathrm{M} = 0$.

Proof. The condition means we can find an N such that $f_i^{\mathrm{N}} m = 0$. But then there is an even larger M:

$$m = 1^{\mathrm{M}} \cdot m = (\sum f_i)^{\mathrm{M}} \cdot m = 0.$$

□

Corollary 4.1.6. *Let* A *be a ring and* $f_1, \cdots, f_n$ *elements such that* $\sum_{i=1}^n f_i = 1$. *Then*

$$\mathrm{A} \to \prod \mathrm{A}_{f_i} \rightrightarrows \prod \mathrm{A}_{f_i\cdot f_j}$$

is an equalizer diagram of rings. In other words, we have proved that $\mathbf{A}^1$ *is a Zariski stack.*

Proof. This follows from what we have proved. The "in other words" part follows from the fact that

$$\mathbf{A}^1(\mathrm{A}) = \mathrm{A}.$$

□

Now let us prove

Proof of Theorem 4.1.3. We have proved that $\mathbf{A}^1$ is a Zariski stack. Any affine scheme can be written as a pullback

$$\begin{array}{ccc} \operatorname{Spec}\mathrm{R} & \longrightarrow & \mathbf{A}^{\mathrm{I}} \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbf{Z} & \xrightarrow{0} & \mathbf{A}^{\mathrm{J}}, \end{array}$$

where I, J are sets. The result then follows from the next lemmas (one of which is homework):

□

Lemma 4.1.7. *Zariski stacks are preserved under limits.*

Lemma 4.1.8. $\mathrm{Spec}\,\mathbf{Z}$ *is a Zariski stack.*

Proof. For any ring R, $\mathrm{Spec}\,\mathbf{Z}(\mathrm{R}) = *$. The claim then follows from the observation that the diagram

$$* \to * \rightrightarrows *$$

is an equalizer. □

4.2. **Exercises 3.**

Lemma 4.2.1. *Let* R *be a ring and consider the functor from* R*-algebras to* R*-modules*

$$\mathrm{U} : \mathrm{CAlg}_{\mathrm{R}} \to \mathrm{Mod}_{\mathrm{R}}.$$

Prove:

(1) *this functor preserves final objects,*
(2) *this functor creates pullbacks: a diagram of rings*

$$\begin{array}{ccc} \mathrm{A} & \longrightarrow & \mathrm{B} \\ \downarrow & & \downarrow \\ \mathrm{C} & \longrightarrow & \mathrm{D}, \end{array}$$

is a pullback diagram if and only if the corresponding diagram of modules is,
(3) *if you are feeling up to it: prove that in fact* U *creates all limits.*

Exercise 4.2.2. *Here is a formula for* R_f*. We will work in the generality of the category* $\mathrm{Mod}_{\mathrm{R}}$*.*

(1) *Let* $f \in \mathrm{R}$ *and consider the following* $\mathbf{N}$*-indexed diagram in the category of* R*-modules*

$$\mathrm{M} \xrightarrow{f\cdot} \mathrm{M} \xrightarrow{f\cdot} \mathrm{M} \xrightarrow{f\cdot} \cdots.$$

Define the colimit to be the R*-module* M_f*. Prove that we have a natural isomorphism: for any* $\mathrm{N} \in \mathrm{Mod}_{\mathrm{R}}$ *such that the map*

$$f\cdot : \mathrm{N} \to \mathrm{N},$$

is an isomorphism then:

$$\mathrm{Hom}_{\mathrm{Mod}_{\mathrm{R}}}(\mathrm{M}_f, \mathrm{N}) \cong \mathrm{Hom}_{\mathrm{Mod}_{\mathrm{R}}}(\mathrm{M}, \mathrm{N}).$$

(2) *Construct explicitly a multiplication on* R_f *and a compatible ring homomorphism* $\mathrm{R} \to \mathrm{R}_f$*.*
(3) *Consider the functor*

$$j_* : \mathrm{Mod}_{\mathrm{R}_f} \to \mathrm{Mod}_{\mathrm{R}}$$

given by restriction of scalars. Prove that this functor admits a left adjoint given by $j^* : \mathrm{M} \mapsto \mathrm{M}_f$*; part of the task is to explain why* M_f *is naturally an* R_f*-module.*
(4) *Use the formula from* 1 *to prove that* j_* *is fully faithful and show that the essential image identifies with the subcategory of* R*-modules where* $f\cdot$ *acts by an isomorphism.*

Lemma 4.2.3. *Let* $f_1, f_2 \in \mathrm{R}$ *then*

$$(\mathrm{R}_{f_1})_{f_2} \cong \mathrm{R}_{f_1 \cdot f_2} \cong \mathrm{R}_{f_1} \otimes_{\mathrm{R}} \mathrm{R}_{f_2}.$$

Exercise 4.2.4. *In this exercise, we will give a proof of a basic, but very clear formulation of descent. Let* A *be a ring and suppose that* $f, g \in \mathrm{A}$ *are elements*

(1) Consider the square

$$\begin{tikzcd} \mathrm{A} \arrow[r] \arrow[d] & \mathrm{A}_f \arrow[d] \\ \mathrm{A}_g \arrow[r] & \mathrm{A}_{fg}. \end{tikzcd}$$

Prove that it the top and bottom arrows are isomorphisms after inverting f; conclude that the resulting square is cartesian.

(2) Prove that the left vertical and the right vertical arrows are isomorphisms after inverting g; conclude that the resulting square is cartesian.

(3) Now assume:

$$1 \in (f) + (g).$$

Conclude that the square is cartesian.

Exercise 4.2.5. *Prove that Zariski stacks are preserved under limits: suppose that we have a diagram* $\mathrm{I} \to \mathbf{PStk}$*, then the functor*

$$\mathrm{R} \mapsto \lim_{\mathrm{I}} \mathcal{F}_i(\mathrm{R}),$$

defines a Zariski stack.

Exercise 4.2.6. *Prove that any affine scheme* $\operatorname{Spec}\mathrm{R}$ *can be written as a pullback*

$$\begin{tikzcd} \operatorname{Spec}\mathrm{R} \arrow[r] \arrow[d] & \mathbf{A}^{\mathrm{I}} \arrow[d] \\ \operatorname{Spec}\mathbf{Z} \arrow[r, "0"] & \mathbf{A}^{\mathrm{J}}. \end{tikzcd}$$

5. Lecture 4: Onto schemes

We have done a bunch of abstract stuff. I would like to tell you how to say something concrete using abstract stuff.

5.1. Diversion: multiplicative groups and graded rings.

Let us, for this section, consider what structure what one can endow on $\mathbf{G}_m = \operatorname{Spec}\mathbf{Z}[t, t^{-1}]$. One suggestive way to think about $\mathbf{Z}[t, t^{-1}]$ is as

$$\mathbf{Z}[t, t^{-1}] \cong \mathbf{Z}[\mathbf{Z}] \cong \bigoplus_{j \in \mathbf{Z}} \mathbf{Z}(j).$$

This is also called the **group algebra** on the (commutative) group $\mathbf{Z}$; we will say why this is a interesting at all later on. We want to say that $\mathbf{G}_m$ is a group object in the category of affine schemes. Unwinding definitions, we need to provide three pieces of data

(Mult.) The multiplication:

$$\mu : \mathbf{Z}[t, t^{-1}] \to \mathbf{Z}[t, t^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[t, t^{-1}] \qquad t \mapsto t \otimes t,$$

(Id.) The identity

$$\epsilon : \mathbf{Z}[t, t^{-1}] \to \mathbf{Z} \qquad t \mapsto 1$$

(Inv.) The inverse

$$\iota : \mathbf{Z}[t, t^{-1}] \to \mathbf{Z}[t, t^{-1}] \qquad t \mapsto t^{-1}.$$

These data (or, more precisely, the opposites thereof) are subject to the compatibilities that prescribe $\mathbf{G}_m$ as a group object in $\mathbf{PStk}$.

Definition 5.1.1. An **affine group scheme** is an affine scheme $\mathrm{G} = \operatorname{Spec}\mathrm{R}$ with maps $\mu : \mathrm{G} \times \mathrm{G} \to \mathrm{G}, \epsilon : \operatorname{Spec}\mathbf{Z} \to \mathrm{G}, \iota : \mathrm{G} \to \mathrm{G}$ which endows it with the structure of a group object in prestacks.

When we speak of groups, we always want to speak about group actions. If G is an affine group scheme and $\mathcal{F}$ is a prestack then a **(left) action** is given by a morphism of prestacks

$$a : \mathrm{G} \times \mathcal{F} \to \mathcal{F},$$

satisfying the obvious compatibilities:

$$\begin{tikzcd} \mathrm{G}\times\mathrm{G}\times\mathcal{F} \arrow[r, "\mu\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times a"'] & \mathrm{G}\times\mathcal{F} \arrow[d, "a"] \\ \mathrm{G}\times\mathcal{F} \arrow[r, "a"] & \mathcal{F}. \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F} \arrow[r, "\epsilon"] \arrow[dr, "\mathrm{id}"'] & \mathrm{G}\times\mathcal{F} \arrow[d, "a"] \\ & \mathcal{F}. \end{tikzcd}$$

If we restrict ourselves to $\mathbf{G}_m$ acting on affine schemes, we actually obtain the next result whose standard reference is [DG70, Exposé 1, 4.7.3]. Let us denote by $\mathrm{Aff}^{\mathrm{B}\mathbf{G}_m}$ the category of affine schemes equipped with a $\mathbf{G}_m$-action and $\mathbf{G}_m$-equivariant morphisms. This is not a subcategory of prestacks, but admits a forgetful functor

$$\mathrm{Aff}^{\mathrm{B}\mathbf{G}_m} \to \mathbf{PStk}.$$

On the other hand a **Z-graded ring** is a ring R equipped with a decomposition:

$$\mathrm{R} = \bigoplus_{i\in\mathbf{Z}} \mathrm{R}_i$$

such that:

(1) each R_j is an additive subgroup of R (in other words, the direct sum above is taken in the category of abelian groups) and,
(2) the multiplication induces $\mathrm{R}_i\mathrm{R}_j \subset \mathrm{R}_{i+j}$.

We say that an element $f \in \mathrm{R}$ is a **homogeneous element of degree** n if $f \in \mathrm{R}_n$. A **graded morphism** of graded rings is just a ring homomorphism $\varphi : \mathrm{R} \to \mathrm{S}$ such that $\varphi(\mathrm{R}_j) \subset \mathrm{S}_j$. We denote by grCAlg the category of $\mathbf{Z}$-graded rings.

Theorem 5.1.2. *There is an equivalence of categories*

$$\mathrm{Aff}^{\mathrm{B}\mathbf{G}_m} \simeq (\mathrm{grCAlg})^{\mathrm{op}}.$$

Remark 5.1.3. One of the main points of Theorem 5.1.2 is that it is interesting to read from left to right and right to left. One the one hand one can use the geometric language of groups acting on a scheme/variety to encode a combinatorial/algebraic structure. On the other hand, it gives a purely combinatorial/algebraic description of a geometric idea.

Proof. First we construct a functor:

$$\mathrm{Aff}^{\mathrm{B}\mathbf{G}_m} \to (\mathrm{grCAlg})^{\mathrm{op}}.$$

Recall that the tensor product of rings is computed as the tensor product of underlying modules. Therefore we can write isomorphisms of $\mathbf{Z}$-modules:

$$\mathrm{R} \otimes_{\mathbf{Z}} \mathbf{Z}[t, t^{-1}] \cong \mathrm{R} \otimes_{\mathbf{Z}} \mathbf{Z}[\mathbf{Z}] \cong \bigoplus_{j\in\mathbf{Z}} \mathrm{R}(j).$$

Hence a $\mathbf{G}_m$-action on $\mathrm{Spec}\,\mathrm{R}$ is the same data as giving a map

$$\varphi : \mathrm{R} \to \bigoplus_{j\in\mathbf{Z}} \mathrm{R}(j) \qquad f \mapsto (\varphi_j(f) \in \mathrm{R}(j)),$$

satisfying certain compatibilities. We note that the direct sum indicates that the components of $(\varphi_j(f))$ is finitely supported.

Using the identity axiom we see that the composite

$$\mathrm{R} \to \bigoplus_{j\in\mathbf{Z}} \mathrm{R}(j) \xrightarrow{t\mapsto 1} \mathrm{R}$$

must be the identity. Therefore, in coordinates, we get that for any $f \in \mathrm{R}$, we get that

$$f = \sum_{j \in \mathbf{Z}} \varphi_j(f),$$

so that any f can be uniquely written as a finite sum of the $\varphi_j(f)$'s. To conclude that this defines a grading on R we need to prove that each φ_j is an idempotent. If this was proved, then the grading would be such that $f \in \mathrm{R}$ is of homogeneous degree j whenever $\varphi(f) = ft^j$.

However, this is the case by associativity of the action:

$$\begin{tikzcd}
\mathrm{R} \arrow[rr, "\varphi"] \arrow[d, "\varphi"] & & \bigoplus_{j\in\mathbf{Z}} \mathrm{R}(j) \arrow[d, "\varphi"] \\
\bigoplus_{j\in\mathbf{Z}} \mathrm{R}(j) \arrow[r, "\mu"] & \bigoplus_{j\in\mathbf{Z}} \bigoplus_{k\in\mathbf{Z}} \mathrm{R}(jk). &
\end{tikzcd}$$

Therefore we conclude that R splits, as a $\mathbf{Z}$-module (aka abelian group) as $\mathrm{R} \cong \bigoplus_j \varphi_j \mathrm{R}(:= \mathrm{R}_j)$ and one can check that this defines a graded ring structure on R where the compatibility of multiplication originates from the fact that φ is a ring map.

In more details. let $f \in \mathrm{R}$; generically we can write $\varphi(f) = \sum_i f_i t^i$ then going to the right and down gives:

$$\varphi(\varphi(f)) = \varphi(\sum_i f_i t^i) = \sum_i \varphi(f_i) t^i u^i,$$

while going down and then left gives

$$\mu(\varphi(f)) = \mu(\sum_i f_i t^i) = \sum_i f_i \mu(t^i) = \sum_i f_i t^i u^i;$$

so that $f_i = \varphi(f_i)$.

On the other hand, given a ring R equipped with the structure of a graded ring $\mathrm{R} = \bigoplus \mathrm{R}_j$ we define a map

$$\varphi : \mathrm{R} \to \bigoplus_{j \in \mathbf{Z}} \mathrm{R}(j),$$

on the level of abelian groups as

$$\mathrm{R} \to \pi_j \mathrm{R} \subset \mathrm{R}(j) \cong \mathrm{R},$$

where π_j is the projection map. This is checked easily to define a $\mathbf{G}_m$-action and the functors are mutually equivalent.

□

Example 5.1.4. There is an action of $\mathbf{G}_m$ on $\mathbf{A}^1$ that "absorbs everything to the origin"; in coordinates this is written as $t \cdot x = tx$. An exercise in this week's homework will require you to translate this to a grading.

Example 5.1.5. The best way to define new graded rings is to mod out by homogenenous polynomial equations. Recall that a polynomial $f(x_1, \cdots, x_n) \in \mathrm{R}[x_1, \cdots, x_n]$ (over any ring R) is said to be **homogeneous of degree** d if for any $r \in \mathrm{R}$ $r^d f(x_1, \cdots, x_n) = f(rx_1, \cdots, rx_n)$. The instructor never found this a useful definition; we can equivalently define this to be a linear combination of monomials of degree d, i.e.,

$$a x_1^{r_1} \cdots x_k^{r_k} \qquad \sum_{j=1}^{k} r_j = d.$$

Here is a nice visual example: consider the **quadric cone**:

$$\operatorname{Spec} \mathbf{Z}[x, y, z]/(x^2 + y^2 - z^2).$$

Since $\mathbf{Z}[x, y, z]/(x^2 + y^2 - z^2)$ is the quotient of a graded ring by a homogeneous equation, it inherits a natural grading. This defines a $\mathbf{G}_m$-action. If we replace $\mathbf{Z}$ by a field, convince yourself that this is pictorially the "absorbing" action of the cone onto its cone point.

5.2. **Complementation and open subfunctors.** In the last class we defined the descent condition and also proved that $\operatorname{Spec} \mathrm{R}$ satisfies this condition. This is like choosing a basis in a vector space — we could have two covers which are specified by $\{f_i\}$ or $\{g_j\}$ and we have to say something in order to prove that descend with respect to one cover implies descent for the other.

Let us try to characterize open immersions of affine schemes in terms of its functor of points. We know that open subschemes of $\operatorname{Spec} \mathrm{A}$ should be one which is the complement of a closed subscheme where the latter is of the form $\operatorname{Spec} \mathrm{A}/\mathrm{I}$. Furthermore we know the following example:

Remark 5.2.1. If $\mathrm{I} = (f)$, then $\operatorname{Spec} \mathrm{A}/f \hookrightarrow \operatorname{Spec} \mathrm{A}$ has a complement which is actually an affine scheme given by $\operatorname{Spec} \mathrm{A}_f$. Indeed, let us attempt to unpack this: suppose that $\operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{A}$ is a morphism of affine schemes corresponding to a map of rings $\mathrm{A} \to \mathrm{R}$. We want to say that $\operatorname{Spec} \mathrm{R}$ lands in the open complement of $\operatorname{Spec} \mathrm{A}/f$ which translate algebraically to the following cartesian diagram

$$\begin{tikzcd} \mathrm{A} \arrow[r] \arrow[d] & \mathrm{R} \arrow[d] \\ \mathrm{A}/f \arrow[r] & 0. \end{tikzcd}$$

This means that the map $\varphi : \mathrm{A} \to \mathrm{R}$ must satisfy: $\mathrm{R}/f\mathrm{R} = 0$ and so $f\mathrm{R} = \mathrm{R}$ which exactly means that f acts invertibly on R and hence (by homework) defines uniquely a ring map

$$\mathrm{A}_f \to \mathrm{R}.$$

To summarize our discussion:

(1) intuitively (and actually!) the closed subscheme $\operatorname{Spec} \mathrm{A}/f$ is one which is cut out by f or, in other words, the locus where f vanishes. Its complement, which is an open subscheme (if you believe in topological spaces) is the locus where f is invertible so we should take something like $\operatorname{Spec} \mathrm{A}_f$.
(2) we need a new definition to make sense of complementation of prestacks.

Let us also note the following:

Lemma 5.2.2. *For any ring f, the following diagram is cartesian*

$$\begin{tikzcd} \operatorname{Spec} \mathrm{R}_f \arrow[r] \arrow[d] & \operatorname{Spec} \mathrm{R} \arrow[d, "f"] \\ \operatorname{Spec} \mathbf{Z}[t, t^{-1}] \arrow[r] & \mathbf{A}^1 = \operatorname{Spec} \mathbf{Z}. \end{tikzcd}$$

Let us recall that a morphism of prestacks $\mathrm{X} \to \mathrm{Y}$ is a closed immersion if for any morphism $\operatorname{Spec} \mathrm{R} \to \mathrm{Y}$ then:

(1) the prestack $\operatorname{Spec} \mathrm{R} \times_{\mathrm{Y}} \mathrm{X}$ is representable and,
(2) the morphism
$$\operatorname{Spec} \mathrm{R} \times_{\mathrm{Y}} \mathrm{X} \to \operatorname{Spec} \mathrm{R}$$
is a closed immersion.

Definition 5.2.3. Let $\mathcal{G} \subset \mathcal{F}$ be a closed immersion of prestacks. The **complement of** $\mathcal{G}$, defined by $\mathcal{F} \smallsetminus \mathcal{G}$ is the prestack given in the following manner: a morphism $x : \operatorname{Spec} \mathrm{R} \to \mathcal{F}$ is in $(\mathcal{F} \smallsetminus \mathcal{G})(\mathrm{R})$ if and only if the following diagram is cartesian

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathcal{G} \arrow[d] \\ \operatorname{Spec} \mathrm{R} \arrow[r, "x"] & \mathcal{F} \end{tikzcd}$$

Definition 5.2.4. A morphism of prestacks $\mathcal{G} \to \mathcal{F}$ is an **open immersions** if for each $\operatorname{Spec} R \to \mathcal{F}$, the map $\operatorname{Spec} R \times_{\mathcal{F}} \mathcal{G} \to \operatorname{Spec} R$ is an open immersions. Equivalently, it is the complement of a closed immersion.

We note that $\mathcal{F} \setminus \mathcal{G}$ is indeed a prestack because the empty scheme pullsback

Lemma 5.2.5. *Let* $R \in \mathrm{CAlg}$ *and* $I \subset R$ *an ideal. Then there is a natural bijection between*

(1) maps $R \to A$ *such* $IA = A$ *and,*
(2) morphisms $\operatorname{Spec} A \to \operatorname{Spec} R$ *such that*

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & \operatorname{Spec} R/I \arrow[d] \\
\operatorname{Spec} A \arrow[r, "x"] & \operatorname{Spec} R.
\end{tikzcd}$$

Proof. The condition of the second item says that the following diagram is cocartesian:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & A \arrow[d] \\
R/I \arrow[r] & 0.
\end{tikzcd}$$

which means that $A \otimes_R R/I = A/IA = 0$ which exactly means that $A = IA$.

□

Definition 5.2.6. A subfunctor $\mathcal{F} \to \operatorname{Spec} R$ of the form in Lemma 5.2.5 is called an **open subscheme**, while $\mathcal{F}$ is called a **quasi-affine** prestack. In this case we write

$$\mathcal{F} = D(I).$$

If $\mathcal{F} = \operatorname{Spec} R_f$, we write $D(f)$.

We will soon learn how to prove that not all quasi-affine prestacks are affine.

5.3. **Exercises 4.**

Exercise 5.3.1. *Consider the action of* $\mathbf{G}_m$ *on* $\mathbf{A}^n$ *given by*

$$\mathbf{Z}[x_1, \cdots, x_n] \to \mathbf{Z}[t, t^{-1}, x_1, \cdots, x_n] \qquad x_j \mapsto t^{-k_j} x_j.$$

Calculate the induced grading on $\mathbf{Z}[x_1, \cdots, x_n]$.

Exercise 5.3.2. *Let* $R_\bullet$ *be a graded ring which is concentrated in* $\mathbf{Z}_{\geqslant 0}$, *i.e.,* $R_{<0} = 0$. *Note that:*

(1) each R_j *is then canonically an* R_0*-module and, in fact,* $R_\bullet$ *is an* R_0*-algebra;*
(2) the subset $R_+ := \bigoplus_{i \geqslant 1} R_i \subset R_\bullet$ *is an ideal*

Prove that the following are equivalent:

(1) the ideal R_+ *is finitely generated as an* $R_\bullet$*-ideal;*
(2) $R_\bullet$ *is generated as an* R_0*-algebra by finitely many homogeneous elements of positive degree.*

In the above situation we say that $R_\bullet$ *is a* **finitely generated graded ring**.

Exercise 5.3.3. *Prove the following locality properties for open immersions:*

(1) Prove that the composite of open immersions of schemes is an open immersion.
(2) Suppose that $X \to Y$ *is a morphism of schemes and suppose that* $\mathcal{V}$ *is an Zariski cover of* X *such that for each* $U \to X$ *in* $\mathcal{V}$ *the map*

$$U \to Y$$

is an open immersion. Prove that $X \to Y$ *is an open immersion.*

6. Lecture 5: Schemes, actually

6.1. **Quasi-affine prestacks are Zariski stacks.**

Proposition 6.1.1. *Any quasi-affine scheme is a prestack.*

Proof. Suppose that $\mathrm{U} = \mathrm{D}(\mathrm{I}) \subset \operatorname{Spec} \mathrm{A}$ is quasi-affine. We claim that it is a Zariski stack. Let R be a test ring and let $\{f_i \in \mathrm{R}\}$ determine a basic Zariski open cover. We first prove the following claim:

- a morphism $\varphi : \mathrm{A} \to \mathrm{R}$ satisfies $\mathrm{I} \cdot \mathrm{R} = \mathrm{R}$ if and only if $\mathrm{I} \cdot \mathrm{R}_{f_i} = \mathrm{R}_{f_i}$.

Indeed, we have a short exact sequence of modules:

$$0 \to \mathrm{I} \to \mathrm{A} \to \mathrm{A}/\mathrm{I} \to 0.$$

Applying $\mathrm{R} \otimes_{\mathrm{A}} -$, we get an exact sequence

$$\mathrm{I} \otimes_{\mathrm{A}} \mathrm{R} \to \mathrm{R} \to \mathrm{R} \otimes_{\mathrm{A}} \mathrm{A}/\mathrm{I}.$$

Since $\mathrm{I} \cdot \mathrm{R}$ is the image of the left-most map, we need only prove that $\mathrm{R} \otimes_{\mathrm{A}} \mathrm{A}/\mathrm{I}$ is zero as an R-module.

By Lemma 4.1.5, we need only check that for all f_i, tensoring the above further with $\otimes_{\mathrm{R}} \mathrm{R}_{f_i}$ is zero, But the map $\mathrm{R} \to \mathrm{R}_{f_i}$ is flat. Therefore the claim follows from:

$$\mathrm{R}_{f_i} \otimes_{\mathrm{R}} \mathrm{R} \otimes_{\mathrm{A}} \mathrm{A}/\mathrm{I} = \mathrm{R}_{f_i} \otimes_{\mathrm{A}} \mathrm{A}/\mathrm{I} = 0,$$

as was assumed.

I claim that we are now done. Indeed, shorthanding the relevant equalizers as $\mathrm{Eq}(\mathcal{F})(\mathrm{R})$ we get a diagram:

$$\begin{tikzcd} \mathrm{D}(\mathrm{I})(\mathrm{R}) \arrow[r] \arrow[d, hook] & \mathrm{Eq}(\mathrm{D}(\mathrm{I}))(\mathrm{R}) \arrow[d] \\ \operatorname{Spec} \mathrm{A}(\mathrm{R}) \arrow[r, "\cong"] & \mathrm{Eq}(\operatorname{Spec} \mathrm{A})(\mathrm{R}). \end{tikzcd}$$

Since the left vertical map is injective and the bottom horizontal map is an isomorphism, the top vertical map is injective. Now, the top vertical map is also surjective by what we have proved. ☐

6.2. **General open covers.**

We formulated a coordinate-dependent way of phrasing descent because we have used the notion of a basic Zariski cover. This is like choosing a basis for a topology which is, in turn, like choosing a basis for a vector space in linear algebra. One should not do this ever or, at least, whenever possible. We will get rid of these choices now.

Recall from last class that if $\operatorname{Spec} \mathrm{R} = \mathrm{X}$ is an affine scheme, then an open immersion is a morphism of prestacks where the domain is of the form $\mathrm{D}(\mathrm{I})$ where I is an ideal of R. Recall also that closed immersions of prestacks are defined by appeal to the affine case: it is a morphism $\mathcal{G} \to \mathcal{F}$ such that for every morphism $\operatorname{Spec} \mathrm{R} \to \mathcal{F}$, the pullback $\operatorname{Spec} \mathrm{R} \times_{\mathcal{F}} \mathcal{G}$ is representable by a scheme and the morphism to $\operatorname{Spec} \mathrm{R}$ is indeed a closed immersion which is specified by an ideal.

Definition 6.2.1. Let $\mathcal{F}$ be a prestack. Then an **open subprestack** of $\mathcal{F}$ or an **open immersion of prestacks** is a morphism $\mathcal{G} \to \mathcal{F}$ such that for any morphism $\operatorname{Spec} \mathrm{R} \to \mathcal{F}$, the morphism $\operatorname{Spec} \mathrm{R} \times_{\mathrm{Y}} \mathrm{X} \to \operatorname{Spec} \mathrm{R}$ is an open subscheme. In other words, $\operatorname{Spec} \mathrm{R} \times_{\mathrm{Y}} \mathrm{X}$ is of the form $\mathrm{D}(\mathrm{I})$ where I is an ideal of R.

Since we have plenty of examples of open immersions of affine schemes which are not themselves affine, we do not want the representability condition which we saw was imposed in the closed immersion case. The next lemma is left as an exercise.

Lemma 6.2.2. *Let $\mathcal{Z} \hookrightarrow \mathcal{F}$ be an closed immersion, then the complement $\mathcal{F} \smallsetminus \mathcal{Z}$ is canonically an open immersion.*

Definition 6.2.3. A **Zariski cover** of an affine scheme $\mathrm{X} = \operatorname{Spec}\mathrm{A}$ is a collection of open embeddings of prestacks
$$\mathcal{U} = \{\mathrm{U} \hookrightarrow \mathrm{X}\},$$
such that for any nonzero ring R, $\mathrm{S} = \operatorname{Spec}\mathrm{R}$ with a map $\mathrm{S} \to \mathrm{X}$ there exists a $\mathrm{U} \hookrightarrow \operatorname{Spec}\mathrm{A} \in \mathcal{U}$ such that
$$\mathrm{U} \times_{\mathrm{X}} \mathrm{S} \neq \emptyset.$$

At this point nothing then stops us from defining Zariski covers of any prestack:

Definition 6.2.4. A **Zariski cover** of an prestack $\mathcal{F}$ is a collection of open embeddings of prestacks
$$\mathcal{U} = \{\mathrm{U} \hookrightarrow \mathcal{F}\},$$
such that for any nonzero ring R, $\mathrm{S} = \operatorname{Spec}\mathrm{R}$ with a map $\mathrm{S} \to \mathcal{F}$ there exists a $\mathrm{U} \hookrightarrow \mathcal{F} \in \mathcal{U}$ such that
$$\mathrm{U} \times_{\mathcal{F}} \mathrm{S} \neq \emptyset.$$

Definition 6.2.5. A prestack $\mathcal{F} : \mathrm{CAlg} \to \mathrm{Set}$ is a **(Zariski) stack** (resp. **(Zariski) stack for the Zariski cover** $\mathcal{U}$) if for all $\mathrm{A} \in \mathrm{CAlg}$ and all (resp. the) Zariski cover $\mathcal{U} := \{\mathrm{U}_i \to \operatorname{Spec}\mathrm{A}\}_{i\in\mathrm{I}}$ the diagram
$$\mathcal{F}(\operatorname{Spec}\mathrm{A}) \to \prod \operatorname{Hom}(\mathrm{U}_i, \mathcal{F}) \rightrightarrows \prod_{i_0, i_1} \operatorname{Hom}(\mathrm{U}_{i_0} \times_{\operatorname{Spec}\mathrm{A}} \mathrm{U}_{i_1}, \mathcal{F})$$
is an equalizer diagram where the maps are induced are the obvious ones.

Remark 6.2.6. It is a bit dangerous to write $\mathcal{F}(\mathrm{U}_i)$ since U_i is not known (and will be known not to be) an affine scheme. So we will stick with the notation $\operatorname{Hom}(\mathrm{U}_i, \mathcal{F})$ in these notes, though the instructor does lapse to writing $\mathcal{F}(\mathrm{U}_i)$. This notation will be justified later on.

From this definition, our previous definition of a scheme asks that $\mathcal{F}$ is a Zariski stack for covers $\mathcal{U}$ which are made of Zariski open covers. Another exercise in unpacking definitions:

Lemma 6.2.7. *A basic Zariski cover is a Zariski cover. In particular if $\mathcal{F}$ is a Zariski stack, then it is a Zariski stack with respect to basic open covers.*

This tells us that Definition 6.2.4 is stronger than Definition 4.0.7. In some sense Definition 6.2.4 is preferable — it affords the flexibility of working with covers where the opens are not necessarily affine. We will work towards proving the equivalence of these two definitions shortly. More precisely:

Theorem 6.2.8. *Suppose that $\mathcal{F} : \mathrm{CAlg} \to \mathrm{Set}$ is a functor. Then the following are equivalent:*
(1) $\mathcal{F}$ is a Zariski stack in the sense of Definition 6.2.4.
(2) $\mathcal{F}$ is a Zariski stack in the sense of Definition 4.0.7.

6.3. **Quasicompactness of affine schemes.** First let us consider the different ways we can think about Zariski covers.

Remark 6.3.1. One of the weird things about algebraic geometry is how large open sets are. For example, consider $\mathbf{R}$ with the usual topology. Then there exists many open covers of $\mathbf{R}$ with no finite subcovers. Now $\mathbf{R} = \operatorname{Spec}\mathbf{A}^1(\mathbf{R})$ but, as an affine scheme $\operatorname{Spec}\mathbf{A}^1$ is, in a precise way, compact.

We call the next lemma *quasicompactness of affine schemes.*

Lemma 6.3.2. *Let $\mathrm{X} = \operatorname{Spec}\mathrm{A}$, and $\mathcal{U} = \{\mathrm{U} \hookrightarrow \operatorname{Spec}\mathrm{A}\}$ is a collection of open immersions. then the following are equivalent:*
(1) $\mathcal{U}$ is a Zariski cover.
(2) $\mathcal{U}$ has a finite subset $\mathcal{V}$ which is also an open cover.
(3) for any field k and any map $x : \operatorname{Spec} k \to \mathrm{X}$ there exists an $\mathrm{U} \in \mathcal{U}$ such that x factors through U.

Proof. The implication (1) $\Rightarrow$ (2) comes under the term "affine schemes are **quasicompact**" which is one way in which open subsets look very different in algebraic geometry than what you have experienced before. To prove this, we note that giving a Zariski open cover of an affine scheme is to give a collection of ideals $\{\mathrm{I}_j\}$ such that

$$1 \in \sqrt{\sum \mathrm{I}_j}. \tag{6.3.3}$$

Indeed, we know that each element of $\mathcal{U}$ is of the form $\mathrm{D}(\mathrm{I}_j) \hookrightarrow \operatorname{Spec} \mathrm{A}$. Suppose that there exists a cover $\mathcal{U}$ for which (6.3.3) is not satisfied. Then, consider the ring:

$$\mathrm{B} := \mathrm{A}/(\sqrt{\Sigma \mathrm{I}_j}),$$

which induces a morphism $\operatorname{Spec} \mathrm{B} \to \operatorname{Spec} \mathrm{A}$. By assumption $\mathrm{B} \neq 0$, however for no j do we have $\operatorname{Spec} \mathrm{B} \times_{\operatorname{Spec} \mathrm{A}} \mathrm{D}(\mathrm{I}_j) \neq \emptyset$, contradicting the condition to be an open cover.

Continuing with the proof, from the definition of sums of ideals, there exists a finite subcollection $i_0, \cdots, i_k$ such that

$$1 \in \sqrt{\sum_{0 \leqslant s \leqslant k} \mathrm{I}_{i_s}}.$$

This in turn means that we can refine the above open cover by

$$\{\mathrm{D}(\mathrm{I}_{i_s}) \to \operatorname{Spec} \mathrm{A}\}_{0 \leqslant s \leqslant k}.$$

Let us prove (2) $\Rightarrow$ (3). Given a morphism into a field $\varphi : \mathrm{A} \to k$, we want to find $\mathrm{U} \hookrightarrow \mathrm{X} \in \mathcal{V}$ such that x factors through U. Suppose that there is none, then $\varphi(\Sigma \mathrm{I}_j) = 0$, which means that $\varphi(1) = 0$ but this is not possible.

Let us prove (3) $\Rightarrow$ (1). Suppose that R is a nonzero ring with a map $\operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{A}$ so that we have a morphism $\varphi : \mathrm{A} \to \mathrm{R}$. Since R is nonzero, it has a maximal ideal $\mathfrak{m}$ so that $\mathrm{R}/\mathfrak{m} = \kappa$ a field. By hypothesis, there exists a $\mathrm{U} \in \mathcal{U}$ such that $\mathrm{U} \times_{\mathrm{X}} \operatorname{Spec} \kappa$ is nonempty, but this also means that $\mathrm{U} \times_{\mathrm{X}} \operatorname{Spec} \mathrm{R}$ is nonempty since there is a morphism $\mathrm{U} \times_{\mathrm{X}} \operatorname{Spec} \kappa \to \mathrm{U} \times_{\mathrm{X}} \operatorname{Spec} \mathrm{R}$. □

We make this a definition:

Definition 6.3.4. A prestack $\mathcal{F}$ is said to be **quasicompact** if there exists an Zariski cover $\mathcal{U}$ of $\mathcal{F}$ such that $\mathcal{U}$ consists of a *finite* collection of affine schemes.

This provides the first mechanism by which a scheme can be non-affine.

Definition 6.3.5. Suppose that R, S are rings, we define the **scheme-theoretic coproduct** or the **coproduct in schemes** as

$$\operatorname{Spec} \mathrm{R} \sqcup \operatorname{Spec} \mathrm{S} := \operatorname{Spec} \mathrm{R} \times \mathrm{S}.$$

This affine scheme equipped with maps

$$\operatorname{Spec} \mathrm{R} \hookrightarrow \operatorname{Spec} \mathrm{R} \sqcup \operatorname{Spec} \mathrm{S} \hookleftarrow \operatorname{Spec} \mathrm{S}.$$

There is an obvious guess for what a coproduct of schemes could be. However, this latter notion is bad as it does not have Zariski descent — you will be asked to address this in the exercises.

Lemma 6.3.6. *Suppose that we have a diagram* $\mathbf{N} \to \mathbf{PStk}, i \mapsto \mathcal{F}_i$ *such that for each* i *the prestack* $\mathcal{F}_i$ *is a Zariski stack. Then* $\operatorname{colim}_i \mathcal{F}_i$ *is a Zariski stack.*

Proof. This follows from the fact that colimits in **PStk** are computed pointwise (as was proved in the previous exercise) and **N**-indexed colimits commutes with finite limits (or, more generally, filtered colimits commute with finite limits) which is in this week's problem set. □

Lemma 6.3.7. *Let* $(\mathrm{A}_i)_{i \in \mathbf{N}}$ *be a collection of nonzero rings and consider a countable product* $\coprod_{\mathbf{N}} \operatorname{Spec} \mathrm{A}_i$. *This prestack (which in fact a Zariski stack) is not an affine scheme.*

Proof. We first note that $\mathrm{X} := \coprod \operatorname{Spec} \mathrm{A}_i = \operatorname{colim}_{n\to\infty} \coprod_{i=1}^n \operatorname{Spec} \mathrm{A}_i$ so the previous lemma asserts that $\coprod \operatorname{Spec} \mathrm{A}_i$ is a Zariski stack. Consider the collection of maps

$$\{\operatorname{Spec} \mathrm{A}_i \to \coprod \mathrm{X}\}$$

I claim that $\operatorname{Spec} \mathrm{A}_j \to \mathrm{X}$ is an open immersion for each fixed j. Indeed, suppose that we have a map $\operatorname{Spec} \mathrm{R} \to \mathrm{X}$. In this case we note that:

$$\mathrm{X}(\mathrm{R}) = (\operatorname*{colim}_{n\to\infty} \coprod_{i=1}^{n} \operatorname{Spec} \mathrm{A}_i)(\mathrm{R}) = \operatorname*{colim}_{n\to\infty} (\coprod_{i=1}^{n} \operatorname{Spec} \mathrm{A}_i)(\mathrm{R}) = \operatorname*{colim}_{n\to\infty} \operatorname{Hom}(\prod_{i=1}^{n} \mathrm{A}_i, \mathrm{R}),$$

which means that the map $\operatorname{Spec} \mathrm{R} \to \mathrm{X}$ corresponds to a map $\prod_{i=1}^n \mathrm{A}_i \to \mathrm{R}$ or, equivalently, a map $\operatorname{Spec} \mathrm{R} \to \coprod_{i=1}^n \operatorname{Spec} \mathrm{A}_i$ or, in other words, the map $\operatorname{Spec} \mathrm{R} \to \mathrm{X}$ factors through a finite stage of the colimit. Therefore the pullback is computed as

$$\operatorname{Spec} \mathrm{A}_j \times_{\mathrm{X}} \operatorname{Spec} \mathrm{R} = \operatorname{Spec} \mathrm{A}_j \times_{\coprod_{i=1}^n \operatorname{Spec} \mathrm{A}_i} \operatorname{Spec} \mathrm{R}.$$

But, by this week's exercise we know that (1) $\operatorname{Spec} \mathrm{A}_j \to \coprod_{i=1}^n \operatorname{Spec} \mathrm{A}_i$ is an open immersion (since summand inclusions are open immersions) and (2) open immersions are stable under pullbacks and therefore, $\operatorname{Spec} \mathrm{A}_j \times_{\coprod_{i=1}^n \operatorname{Spec} \mathrm{A}_i} \operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{R}$ is an open immersion.

To conclude that the desired collection of maps are covers, we just need to check on k-points and invoke Lemma 6.3.2. To conclude, note that there is no refinement of this subcover.

□

6.4. **The definition of a scheme.** Finally:

Definition 6.4.1. A prestack X is a **scheme** if:

(1) it is a Zariski stack, and
(2) there exists an Zariski cover $\mathcal{U}$ of X such that $\mathcal{U}$ consists of affine schemes.

So far we have seen that affine schemes are schemes; they are furthermore quasicompact schemes. We have also seen that infinite coproducts of schemes are schemes which are not quasicompact. We will prove that quasi-affine schemes are schemes in the next lecture.

6.5. **Exercises 5.** Let Eq be the category displayed as $\bullet \rightrightarrows \bullet$.

Exercise 6.5.1. *Suppose that we have a diagram* $\mathrm{F} : \mathbf{N} \times \mathrm{Eq} \to \mathrm{Set}$. *On the one hand we can view this as*

$$\mathbf{N} \to \operatorname{Fun}(\mathrm{Eq}, \mathrm{Set})$$

and take

$$\operatorname*{colim}_{\mathbf{N}} \lim_{\mathrm{Eq}} \mathrm{F}$$

or we can view this as

$$\mathrm{Eq} \to \operatorname{Fun}(\mathbf{N}, \mathrm{Set})$$

and take

$$\lim_{\mathrm{Eq}} \operatorname*{colim}_{\mathbf{N}} \mathrm{F}.$$

Construct a natural map between the two sets and prove that they are isomorphic. Conclude the same commutation property for **PStk** *in place of* Set. *This is the first instance of an unexpected commutation of limits versus a colimit that you have seen in this class.*

Exercise 6.5.2. *Prove that open immersions are stable under pullbacks: suppose that* $\mathcal{U} \to \mathcal{F}$ *is an open immersion of prestacks, then for each map* $\mathcal{G} \to \mathcal{F}$, *the induced map of prestacks* $\mathcal{G} \times_{\mathcal{F}} \mathcal{U} \to \mathcal{G}$ *is an open immersion. Prove the same property for closed immersions.*

Exercise 6.5.3. *Here is a simple algebra fact that you should know: let* R *be a ring and* e *is an idempotent. Then prove that we have a decomposition* $e\mathrm{R} \oplus (1-e)\mathrm{R} \cong \mathrm{R}$ *in the category of abelian groups, while neither* $e\mathrm{R}$ *nor* $(1-e)\mathrm{R}$ *are subrings unless* e *is zero or* 1. *However note that the projections* $\mathrm{R} \to e\mathrm{R}$ *and* $\mathrm{R} \to (1-e)\mathrm{R}$ *are ring homomorphisms.*

Exercise 6.5.4. *Let* R, S *be rings and consider a test ring* A*, then consider the map*

$$\mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{Spec}\,\mathrm{R}) \sqcup \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{Spec}\,\mathrm{S}) \to \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{Spec}\,\mathrm{R} \times \mathrm{S}),$$

given on one factor of the coproduct by

$$g : \mathrm{R} \to \mathrm{A} \mapsto \mathrm{R} \times \mathrm{S} \to \mathrm{R} \xrightarrow{g} \mathrm{A},$$

$$f : \mathrm{S} \to \mathrm{A} \mapsto \mathrm{R} \times \mathrm{S} \to \mathrm{S} \xrightarrow{f} \mathrm{A}.$$

Prove:

(1) *if* A *has no nontrivial idempotent, then the map is an isomorphism.*
(2) *if* A *has nontrivial idempotent, then the map is not an isomorphism in general.*
(3) *Prove that, in this case, the functor*

$$\mathrm{X}(\mathrm{A}) = \begin{cases} \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{Spec}\,\mathrm{R}) \sqcup \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{Spec}\,\mathrm{S}) & \mathrm{A} \neq 0 \\ * & \textit{else}, \end{cases}$$

is not *a Zariski stack.*

Definition 6.5.5. We say that a scheme X is **connected** if it is not the empty scheme and for all 2-fold open covers $\mathcal{U} = \{\mathrm{U}, \mathrm{V} \hookrightarrow \mathrm{X}\}$ such that $\mathrm{U} \times_{\mathrm{X}} \mathrm{V} = \emptyset$, then $\mathrm{U} = \mathrm{X}$ or $\mathrm{V} = \mathrm{X}$.

Exercise 6.5.6. *Prove that any Zariski stack converts a product of rings to a product of sets, i.e., if* $\mathrm{A} \cong \mathrm{B} \times \mathrm{C}$ *then the map*

$$\mathcal{F}(\mathrm{B} \times \mathrm{C}) \to \mathcal{F}(\mathrm{B}) \times \mathcal{F}(\mathrm{C}),$$

is an isomorphism whenever $\mathcal{F}$ *is a Zariski stack. Conclude that Zariski stack converts finite coproducts of affine schemes to finite products.*

Exercise 6.5.7. *Prove that:*

(1) *an affine scheme* $\mathrm{X} = \mathrm{Spec}\,\mathrm{R}$ *is connected if and only if* R *has no nontrivial idempotent,*
(2) *if* e *is a nontrivial idempotent and* $\mathrm{R} = e\mathrm{R} \times (1-e)\mathrm{R}$ *prove that the map* $\mathrm{R} \to e\mathrm{R}$ *induces an map* $\mathrm{Spec}\,e\mathrm{R} \to \mathrm{Spec}\,\mathrm{R}$ *which is both an open and a closed immersion.*
(3) *let* R *be a noetherian ring. Prove that we can write*

$$\mathrm{R} \simeq \prod_{\mathrm{I}} \mathrm{R}_i$$

where each R_i *is nonzero and has no nontrivial idempotent and* I *is a finite set.*

Exercise 6.5.8. *The following is a generalization of Exercise 6.5.4. First we note that in point-set topology if* $\mathrm{U} \hookrightarrow \mathrm{X}$ *is a clopen subset of a topological space* X*, then* U *is a union of connected components. So we can, in topological spaces, write* $\mathrm{X} = \mathrm{U} \sqcup \mathrm{X} \smallsetminus \mathrm{U}$ *where* $\mathrm{X} \smallsetminus \mathrm{U}$ *is again clopen, whence also a union of its connected components. But we have seen that the operation of coproduct in schemes is not taken set-wise. The following exercise, however, still proves that we have a coproduct decomposition for clopen subfunctors:*

Let R *be a ring and suppose that*

$$\mathrm{U} \hookrightarrow \mathrm{Spec}\,\mathrm{R}$$

is both an open and closed subfunctor (in other words **clopen***). Note that this means that we also know that* $\mathrm{Spec}\,\mathrm{R} \smallsetminus \mathrm{U}$ *is quasi-affine.*

(1) *suppose that* $\{\mathrm{Spec}\,\mathrm{R}_{f_i} \hookrightarrow \mathrm{U}\}, \{\mathrm{Spec}\,\mathrm{R}_{g_j} \hookrightarrow \mathrm{Spec}\,\mathrm{R} \smallsetminus \mathrm{U}\}$ *are open covers, which we can choose to be finite since we already know that quasi-affines are quasicompact. Prove that each* $f_i g_j$ *is nilpotent.*
(2) *Prove that there exists an* N *such that* $\mathrm{I}^{\mathrm{N}} + \mathrm{J}^{\mathrm{N}} = \mathrm{R}$ *so that we can write* $1 = x + y$ *where* $x \in \mathrm{I}^{\mathrm{N}}, y = \mathrm{J}^{\mathrm{N}}$ *and prove that* x *and* y *are idempotents.*
(3) *Let* $\mathrm{J} = (f_1, \cdots, f_n), \mathrm{I} = (g_1, \cdots, g_m)$*. Prove that* $\mathrm{Spec}\,\mathrm{R}/\mathrm{I}^{\mathrm{N}} \cong \mathrm{U}, \mathrm{Spec}\,\mathrm{R}/\mathrm{J}^{\mathrm{M}} \cong \mathrm{Spec}\,\mathrm{R} \smallsetminus \mathrm{U}$ *for* M, N *large enough. Conclude that* $\mathrm{Spec}\,\mathrm{R} = \mathrm{Spec}\,\mathrm{R}/\mathrm{I} \sqcup \mathrm{Spec}\,\mathrm{R}/\mathrm{J}$.

(4) *Conclude that if* $\mathrm{U} \hookrightarrow \operatorname{Spec} \mathrm{R}$ *is an open and closed subfunctor then*

$$\operatorname{Spec} \mathrm{R} \cong \mathrm{U} \sqcup \mathrm{V},$$

where U, V *are affine schemes (remember what we mean by coproducts!).*

(5) *Conclude the same for arbitrary schemes: if* $\mathrm{U} \hookrightarrow \mathrm{X}$ *is a clopen subscheme then*

$$\mathrm{X} \cong \mathrm{U} \sqcup \mathrm{V}.$$

(6) *On the other hand, prove the following: if* e *is idempotent, then* $\operatorname{Spec} \mathrm{R}/e \hookrightarrow \operatorname{Spec} \mathrm{R}$ *is a clopen embedding.*

Exercise 6.5.9. *Let* $\mathbf{A}^{\infty} := \operatorname{Spec} \mathbf{Z}[x_1, x_2, \cdots, x_n, \cdots]$. *This an affine scheme hence quasicompact. Is the quasi-affine scheme* $\mathbf{A}^{\infty} \setminus 0$ *quasicompact?*

7. Lecture 6: Quasi-affine schemes, a dévissage in action

Last time, we wanted to make more examples of schemes so we want to claim that quasi-affine schemes are schemes. We have already proved that quasi-affine schemes are Zariski stacks. It remains to produce a Zariski cover.

Lemma 7.0.1. *Let* A *be a ring and* I *and ideal of* A. *Then, the collection* $\{\operatorname{Spec} \mathrm{A}_{f_i} \to \mathrm{D}(\mathrm{I}) : f_i \in \mathrm{I} \setminus 0\}$ *is a Zariski open cover of the quasi-affine scheme* $\mathrm{D}(\mathrm{I})$. *In particular* $\mathrm{D}(\mathrm{I})$ *is a scheme.*

Proof. This gives me an opportunity to compute pullbacks and show you how one can make computations of how some pullbacks look like. We want to compute

$$\operatorname{Spec} \mathrm{A}_{f_i} \times_{\operatorname{Spec} \mathrm{A}} \mathrm{D}(\mathrm{I}).$$

Recall that pullbacks are computed pointwise and therefore we want to understand explicitly how this prestack looks like when one maps into a ring R or, in other words, if one maps Spec R in. So let R be such a ring and, by definition, the following diagram is cartesian (in sets)

$$\begin{tikzcd}
(\operatorname{Spec} \mathrm{A}_{f_i} \times_{\operatorname{Spec} \mathrm{A}} \mathrm{D}(\mathrm{I}))(\mathrm{R}) \arrow[r] \arrow[d] & \{\mathrm{A} \to \mathrm{R} : \mathrm{IR} = \mathrm{R}\} \arrow[d] \\
\{\mathrm{A}_{f_i} \to \mathrm{R}\} \arrow[r] & \{\mathrm{A} \to \mathrm{R}\}.
\end{tikzcd}$$

Now we see that the pullback is given by the subset of maps $\varphi : \mathrm{A}_{f_i} \to \mathrm{R}$ such that if we precompose with $\mathrm{A} \to \mathrm{A}_{f_i}$ we have that $\mathrm{IR} = \mathrm{R}$. Now, assume that $f_i \in \mathrm{I} \setminus 0$. I claim that this last condition is no condition at all. Indeed, suppose that $\varphi : \mathrm{A}_{f_i} \to \mathrm{R}$ is such a map, then given any $r \in \mathrm{R}$ we can write

$$r = \varphi(f_i)/\varphi(f_i)^{-1} r = \varphi(f_i) r'$$

which exactly means that $\mathrm{IR} = \mathrm{R}$. Therefore we conclude that

$$\operatorname{Spec} \mathrm{A}_{f_i} \times_{\operatorname{Spec} \mathrm{A}} \mathrm{D}(\mathrm{I}) \cong \operatorname{Spec}_{\mathrm{A}_{f_i}}.$$

We also know that open immersions are stable under pullbacks and therefore the map

$$\operatorname{Spec}_{\mathrm{A}_{f_i}} \hookrightarrow \mathrm{D}(\mathrm{I}).$$

is open since the map $\operatorname{Spec} \mathrm{A}_{f_i} \to \operatorname{Spec} \mathrm{A}$ is.

To prove the covering condition. Let $\operatorname{Spec} \mathrm{R} \to \mathrm{D}(\mathrm{I})$ be nonzero. We consider the diagram where each square is cartesian.

$$\begin{tikzcd}
\operatorname{Spec} \mathrm{R} \times_{\mathrm{D}(\mathrm{I})} \operatorname{Spec} \mathrm{A}_{f_i} \arrow[r] \arrow[d] & \operatorname{Spec} \mathrm{R} \arrow[d] \\
\operatorname{Spec} \mathrm{A}_{f_i} \arrow[r] \arrow[d, "\cong"] & \mathrm{D}(\mathrm{I}) \arrow[d] \\
\operatorname{Spec} \mathrm{A}_{f_i} \arrow[r] & \operatorname{Spec} \mathrm{A}.
\end{tikzcd}$$

Our goal is to find an f_i for which the top left corner is nonempty. But now, we note that the resulting rectangle is cartesian whenever $f_i \in \mathrm{I} \setminus 0$ since the bottom square is from our previous computation. But by assumption on $\mathcal{U}$, we can indeed find such an f_i.

□

7.1. **Universality of descent and dévissage.** This is a technical section which proves the following result.

Theorem 7.1.1. *Suppose that $\mathcal{F} : \mathrm{CAlg} \to \mathrm{Set}$ is a functor. Then the following are equivalent:*

(1) $\mathcal{F}$ is a Zariski stack in the sense of Definition 6.2.4.
(2) $\mathcal{F}$ is a Zariski stack in the sense of Definition 4.0.7.

Of course the direction (1)⇒(2) is immediate. We note that Theorem 7.1.1 is akin to a maneuver in point-set topology where we say that everything we want to know about the topology of a space is basically determined by the basic opens. We will prove this in a rather fancy way, but the crux is the next lemma:

Lemma 7.1.2. *Any open cover $\mathcal{U}$ of an affine scheme $\operatorname{Spec} \mathrm{A}$ admits a refinement by basic Zariski covers. More precisely: given $\mathcal{U}$ a Zariski open cover of $\operatorname{Spec} \mathrm{A}$, there exists a basic Zariski open cover $\mathcal{V} := \{\mathrm{A} \to \mathrm{A}_{f_i}\}$ with the property that for each $\mathrm{U} \in \mathcal{U}$, the set*

$$\mathcal{V}_{\mathrm{U}} := \{\operatorname{Spec} \mathrm{A}_{f_i} \in \mathcal{V} : \operatorname{Spec} \mathrm{A}_{f_i} \hookrightarrow \mathrm{U}\}$$

is a Zariski open cover of U.

Proof. By Lemma 5.2.5, $\mathcal{U} = \{\mathrm{D}(\mathrm{I}_\alpha)\}_\alpha$. By the proof of Lemma 6.3.2 we have that

$$\mathrm{A} = \sqrt{\sum \mathrm{I}_\alpha}.$$

From this, extract the set

$$\{f : \exists \alpha, f \in \mathrm{I}_\alpha \setminus 0\}$$

Then $\{\operatorname{Spec} \mathrm{A}_f\}$ is the desired refinement after Lemma 7.0.1. □

We now begin the proof. Our first goal is to observe that if $\mathcal{F}$ is a Zariski stack in the sense of Definition 4.0.7, then there is an extension of $\mathcal{F}$ to the category of quasi-affine schemes:

Definition 7.1.3. Let QAff to be the full subcategory of **PStk** spanned by affine schemes and quasi-affine schemes. The objects of this category are prestacks of the form $\mathrm{D}(\mathrm{I})$ for some ideal $\mathrm{I} \subset \mathrm{A}$ is a ring A.

To see, this we make the following observation. Suppose that $\mathrm{D}(\mathrm{I}) \hookrightarrow \operatorname{Spec} \mathrm{A}$ is a quasi-affine. Consider open immersions

$$\operatorname{Spec} \mathrm{B}, \operatorname{Spec} \mathrm{C} \hookrightarrow \mathrm{D}(\mathrm{I}).$$

Then we have a map

$$\operatorname{Spec} \mathrm{B} \times_{\mathrm{D}(\mathrm{I})} \operatorname{Spec} \mathrm{C} \to \operatorname{Spec} \mathrm{B} \times_{\operatorname{Spec} \mathrm{A}} \operatorname{Spec} \mathrm{C}.$$

Lemma 7.1.4. *This is an isomorphism.*

Proof. According to Exercise 3.3.12, we have the following cartesian diagram

$$\begin{tikzcd}
\operatorname{Spec} B \times_{D(I)} \operatorname{Spec} C \arrow[r] \arrow[d] & \operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} C \arrow[d] \\
D(I) \arrow[r, "\Delta"] & D(I) \times_{\operatorname{Spec} A} D(I).
\end{tikzcd}$$

Now, Δ is an isomorphism since $D(I)$ is open (exercise!), hence the claim follows. $\square$

In particular: $\operatorname{Spec} B \times_{D(I)} \operatorname{Spec} C$ turns out to be affine. Using this observation, we will extend $\mathcal{F} : \mathrm{CAlg} \to \mathrm{Set}$.

Construction 7.1.5. Suppose that $\mathcal{F}$ is a Zariski stack and $D(I)$ is quasi-affine. For any basic open cover of $\mathcal{U} = \{\operatorname{Spec} A_i \hookrightarrow D(I)\}$ define:

$$\widetilde{\mathcal{F}}(D(I)) := \mathrm{Eq}(\prod_i \mathcal{F}(\operatorname{Spec} A_i) \rightrightarrows \prod_{i,j} \mathcal{F}(\operatorname{Spec} A_i \times_{D(I)} \operatorname{Spec} A_j)$$

Lemma 7.1.6. *The above construction gives a well-defined functor*

$$\widetilde{\mathcal{F}} : \mathrm{QAff}^{\mathrm{op}} \to \mathrm{Set}.$$

This is kind of tedious, but you can imagine its proof

Proof Sketch. Given two covers $\mathcal{U} = \{\operatorname{Spec} A_i \to D(I)\}, \mathcal{V} = \{\operatorname{Spec} B_j \to D(I)\}$ then

(1) for each i we have that $\{\operatorname{Spec} B_j \times_{D(I)} \operatorname{Spec} A_i \to \operatorname{Spec} A_i\}$ is a cover of $\operatorname{Spec} A_i$ and
(2) for each i we have that $\{\operatorname{Spec} A_i \times_{D(I)} \operatorname{Spec} A_i \to \operatorname{Spec} B_j\}$ is a cover of $\operatorname{Spec} B_j$.

With this we can rewrite

$$\mathrm{Eq}(\prod_i \mathcal{F}(\operatorname{Spec} A_i) \rightrightarrows \prod_{i,j} \mathcal{F}(\operatorname{Spec} A_i \times_{D(I)} \operatorname{Spec} A_j))$$

as products of equalizers involving B_j's and vice versa. The proof follows from the fact that equalizers and products are both limits and hence they commute. $\square$

The next concept is the key to proving a result like Theorem 7.1.1

Definition 7.1.7. Fix a prestack

$$\mathcal{F} : \mathrm{QAff}^{\mathrm{op}} \to \mathrm{Set}.$$

We say that a morphism of $X \to Y$ in QAff is of **$\mathcal{F}$-descent** if

$$\mathcal{F}(Y) \to \mathcal{F}(X) \rightrightarrows \mathcal{F}(X \times_Y X),$$

is an equalizer diagram. It is **of universal $\mathcal{F}$-descent** if for any map $T \to Y$,

$$\mathcal{F}(T) \to \mathcal{F}(T \times_Y X) \rightrightarrows \mathcal{F}(T \times_Y (X \times_Y X)).$$

is an equalizer diagram, i.e., the map $X \times_Y T \to T$ is also of **$\mathcal{F}$-descent**.

The next lemma summarizes why this definition is a good one:

Lemma 7.1.8. *Let $f : Y \to Z, g : X \to Y$ be a morphism in* QAff*. Then*

(1) morphisms of universal $\mathcal{F}$-descent are stable under base change.
(2) it f admits a section, then it is of universal $\mathcal{F}$-descent.
(3) suppose that we have a cartesian diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
W \arrow[r, "h"] & Z.
\end{tikzcd}$$

Suppose that (1) the base change of h to Y and to $Y \times_Z Y$ and (2) the base change of f to $Y \times_X Y$ are of universal $\mathcal{F}$-descent. Then f is of universal $\mathcal{F}$-descent.

(4) If f, g are of universal $\mathcal{F}$-descent then so is their composite.
(5) If $f \circ g$ is universal $\mathcal{F}$-descent, then so is f.
(6) If $\mathcal{F}$ converts coproducts to products, morphisms of the form $\sqcup_{i=1}^{n} \operatorname{Spec} \mathrm{B}_i \to \operatorname{Spec} \mathrm{A}$ is of universal $\mathcal{F}$-descent.

Proof sketch. Let us prove (5), assuming (1-3). We have a diagram

$$\begin{tikzcd}
\mathrm{X} \arrow[dr] \arrow[drr, "\mathrm{id}"] \arrow[ddr, "g"'] & & \\
& \mathrm{X} \times_{\mathrm{Y}} \mathrm{Z} \arrow[r, "\pi_{\mathrm{X}}"] \arrow[d, "\pi_{\mathrm{Y}}"] & \mathrm{X} \arrow[d, "f \circ g"] \\
& \mathrm{Y} \arrow[r, "f"] & \mathrm{Z}.
\end{tikzcd}$$

This proves that π_{X} is a retraction and hence is of universal $\mathcal{F}$-descent by (2). In particular, the base change of f along $h := f \circ g$ is of universal $\mathcal{F}$-descent and so are its base changes along $\mathrm{X} \times_{\mathrm{Z}} \mathrm{X} \rightrightarrows \mathrm{X}$ using (2). Now, by assumption $h := f \circ g$ is of universal $\mathcal{F}$-descent so that its base change along f is again of universal $\mathcal{F}$-descent and hence its base changes along $\mathrm{Y} \times_{\mathrm{Z}} \mathrm{Y} \rightrightarrows \mathrm{Z}$, again using (2). Using (3), we conclude that f is universal $\mathcal{F}$-descent. □

Remark 7.1.9. Warning: this is not meant to be a technical remark. A lot of statements in algebraic geometry are proved by proving closure properties for that statement and then demonstrating a base case. This kind of argument style is called *dévissage* (which is the French word for "unscrewing"). Often this kind of argument can be replaced by a more ad hoc one which one proves by hand but we have chosen to give a demonstration of how this general principle can work.

Here is a sharper formulation of Theorem 7.1.1

Theorem 7.1.10. *Suppose that $\mathcal{F} : \mathrm{QAff}^{\mathrm{op}} \to \mathrm{Set}$ is a prestack. Assume that:*
(1) $\mathcal{F}$ converts finite coproducts to products,
(2) $\mathcal{F}|_{\mathrm{Aff}}$ is a Zariski stack.
Then $\mathcal{F}$ is a Zariski stack in the sense of Definition 6.2.4.

Proof. Suppose that $\mathcal{U}$ is a Zariski cover of $\operatorname{Spec} \mathrm{R}$. By the quasicompactness of $\operatorname{Spec} \mathrm{R}$, we can assume that $\mathcal{U}$ consists of a finite collection of morphisms. In this case, consider the map in QAff:

$$f : \coprod_{i=1}^{n} \mathrm{U}_i \to \operatorname{Spec} \mathrm{R},$$

noting that quasi-affine schemes admit finite coproducts. We claim that f is of universal $\mathcal{F}$-descent if $\mathcal{F}$ satisfies (1) and (2). Now, by Lemma 7.1.2, we can find a (finite) refinement of the $\mathcal{U}$ consisting of affine schemes so that we have a sequence of composable morphisms

$$\coprod_{i,j} \mathrm{V}_{ij} \to \coprod_{i=1}^{n} \mathrm{U}_i \to \operatorname{Spec} \mathrm{R}.$$

By Lemma 7.1.8.5, the composite is of universal $\mathcal{F}$-descent, which means we do know that the original map is too using Lemma 7.1.8.3. □

7.2. **Exercises 6.**

Definition 7.2.1. Recall that a morphism $f : \mathrm{X} \to \mathrm{Y}$ is **monic** if for any $g, h : \mathrm{Z} \to \mathrm{X}$ and $fg = fh$ implies that $g = h$.

Exercise 7.2.2. *Let $\mathcal{C}$ be a category with pullbacks. Then prove that the following are equivalent:*
(1) $f : \mathrm{X} \to \mathrm{Y}$ is monic,
(2) $\Delta : \mathrm{Y} \to \mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}$ is an isomorphism.

Corollary 7.2.3. *Prove that Quasi-affine schemes are also quasi-compact.*

Definition 7.2.4. Let R be a ring. We define the **reduction of** R to be the ring $\mathrm{R}_{\mathrm{red}} := \mathrm{R}/\sqrt{(0)}$, i.e., it is the ring R modulo its nilpotent elements.

Exercise 7.2.5. *Let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism of schemes, prove that there exists a unique morphism $f_{\mathrm{red}} : \mathrm{X}_{\mathrm{red}} \to \mathrm{Y}_{\mathrm{red}}$ rendering the diagram*

$$\begin{tikzcd} \mathrm{X}_{\mathrm{red}} \arrow[r, "f_{\mathrm{red}}"] \arrow[d] & \mathrm{Y}_{\mathrm{red}} \arrow[d] \\ \mathrm{X} \arrow[r, "f"] & \mathrm{Y} \end{tikzcd}$$

commutative. Furthermore:

(1) prove that $(\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y})_{\mathrm{red}} \cong (\mathrm{X}_{\mathrm{red}} \times_{\mathrm{Z}_{\mathrm{red}}} \mathrm{Y}_{\mathrm{red}})_{\mathrm{red}}$,
(2) but even if X, Y are reduced then $\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}$ need not be.

Exercise 7.2.6 (Optional)**.** *Prove Lemma 7.1.8 (Hint: you should only use formal properties of pullbacks).*

Definition 7.2.7. A scheme X is **noetherian** if it is (1) quasicompact, and (2) there exists a Zariski cover $\mathcal{U}$ of X consisting of affines schemes where each member is Spec of a noetherian ring.

Exercise 7.2.8. *Prove that a scheme X is noetherian if and only if any chain of closed subschemes*

$$\cdots \mathrm{Z}^i \subset \mathrm{Z}^{i-1} \subset \cdots \mathrm{Z}^1 \subset \mathrm{Z}^0 = \mathrm{X},$$

terminates.

Exercise 7.2.9. *Let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism of schemes. Define the graph prestack as*

$$\Gamma_f : \mathrm{R} \mapsto \{(x, y) : f(x) = y\} \subset (\mathrm{X} \times \mathrm{Y})(\mathrm{R}).$$

Prove that Γ_f is a scheme.

8. Lecture 7: Relative algebraic geometry and quasicoherent sheaves

We have defined schemes and proved an independence result on the condition of being a Zariski stack. Next, we will discuss relative algebraic geometry and the theory of quasicoherent sheaves.

8.1. **Relative algebraic geometry.**

One of the key innovations of the Grothendieck school is the idea that one should be working with algebraic geometry over a base scheme; this is also called "*relative algebraic geometry.*" At heart, the following proposition is key:

Proposition 8.1.1. *Let $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ be schemes and suppose that we have cospan of prestacks*

$$\begin{tikzcd} \mathrm{X} \arrow[dr] & & \mathrm{Y} \arrow[dl] \\ & \mathrm{Z} & \end{tikzcd},$$

then the fiber product in prestacks $\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}$ is, in fact, a scheme.

Proof. It is a Zariski stack since limits commute with each other. The interesting part is to furnish an open cover of $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$.

The proof breaks down naturally in steps:

(1) If $\mathrm{X} = \operatorname{Spec} \mathrm{A}, \mathrm{Y} = \operatorname{Spec} \mathrm{B}, \mathrm{Z} = \operatorname{Spec} \mathrm{C}$, then $\mathrm{X} \times_{\mathrm{Y}} \mathrm{Z} = \operatorname{Spec} \mathrm{A} \otimes_{\mathrm{C}} \mathrm{B}$ as we have seen before.

(2) Assume that Y, Z are affine. Our goal is to furnish an open cover $X \times_Y Z$. So pick an open cover of X consisting of affines $\mathcal{U} = \{\operatorname{Spec} A_i \hookrightarrow X\}$. From the previous result, we do know that $\operatorname{Spec} A_i \times_Y Z$ is an (affine) scheme so taking

$$\mathcal{U}_X := \{\operatorname{Spec} A_i \times_Y Z \to X \times_Y Z\}$$

furnishes an open cover since we do know that open immersions are stable under base change. Similarly, we are okay if Z, Y are both affines.

(3) Now, let us assume that X, Y are both affines and suppose that Z is not. Take an affine open cover of Z $\mathcal{V} := \{\operatorname{Spec} C_i \hookrightarrow Z\}$. Now, the collection

$$\mathcal{V}_X := \{X \times_Z \operatorname{Spec} C_i \hookrightarrow X\},$$

is a collection of open immersions of X. We note that, however, $X \times_Z \operatorname{Spec} C_i$ is not necessarily affine — they are only quasi-affine. Similarly, we also have

$$\mathcal{V}_Y := \{Y \times_Z \operatorname{Spec} C_i \hookrightarrow Y\}.$$

Now consider

$$\mathcal{V}_X \times_Z \mathcal{V}_Y := \{(Y \times_Z \operatorname{Spec} C_i) \times_{\operatorname{Spec} C_i} (\operatorname{Spec} C_i \times_Z X) \hookrightarrow X \times_Z Y\}$$

This is an improvement of the situation as the terms in the cover are made out of *taking fiber products over an affine scheme* so that we can just arrange one of the other terms, say $(\operatorname{Spec} C_i \times_Z X)$ to be affine by taking a further cover (using Lemma 7.0.1, say). We then appeal to the previous case.

(4) Lastly, if none of them are affine, then we take open covers

$$\mathcal{U} := \{\operatorname{Spec} A_i \hookrightarrow X\} \qquad \mathcal{V} := \{\operatorname{Spec} B_j \hookrightarrow Y\},$$

and note that

$$\mathcal{U} \times_Z \mathcal{V} := \{\operatorname{Spec} A_i \times_Z \operatorname{Spec} B_i \hookrightarrow X \times_Z Y\}$$

is an open cover by the previous situation.

□

Here is a way to phrase what the above says:

Corollary 8.1.2. *The inclusion* Sch $\subset$ **PStk** *creates fiber products. In fact, it creates finite limits.*

Equipped with the above result, we define:

Definition 8.1.3. Let B be a scheme, the category of **B-schemes** is the slice category $\mathrm{Sch}_B := \mathrm{Sch}_{/B}$. In other words, its objects are given by morphisms of schemes

$$X \to B,$$

while morphisms are

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd} .$$

The following lemma is a consequence of the existence of fiber products:

Lemma 8.1.4. *The category* Sch_B *admits finite limits.*

We note that products in Sch_B are given by fiber products.

Example 8.1.5. Let R be a fixed commutative ring. Then the category of Spec R-schemes (sometimes also called R-schemes) which are *also affine* (this is different from saying "affine R-schemes) is equivalent to the category R-algebras, i.e., a commutative ring admitting a ring morphism from R and those ring maps under R which renders the obvious diagram commutative.

Remark 8.1.6. Let $X \to Y$ be a morphism between schemes, then we have an adjunction

$$\text{forget} : \mathrm{Sch}_X \rightleftarrows \mathrm{Sch}_Y : \times_Y X,$$

where the left adjoint is given by sending an X-scheme $T \to X$ to $T \to X \to Y$. Indeed, convince yourself that this is an adjunction. The right adjoint is often called the "base change functor."

Example 8.1.7. Here are some examples of difference with working with *absolute* algebraic geometry versus *relative* algebraic geometry.

(1) consider the map

$$\mathbf{C} \to \mathbf{C} \qquad z = a + ib \mapsto \overline{z} = a - ib.$$

This is is a $\mathbf{Z}$-linear map. However, it is *not* a $\mathbf{C}$-linear map:

$$(x+iy)\overline{ib} = (x+iy)(-ib) = -xib + yb,$$

but

$$\overline{(x+iy)(ib)} = \overline{(xib - yb)} = -xib - yb.$$

This means that the involution

$$\iota : \operatorname{Spec} \mathbf{C} \to \operatorname{Spec} \mathbf{C},$$

is *not* a morphism in $\mathbf{C}$.

(2) Let R be a commutative ring which is also an algebra over $\mathbf{F}_p$. Then there is a map

$$\mathrm{F_R} : \mathrm{R} \to \mathrm{R} \qquad x \mapsto x^p,$$

called the Frobenius. The Frobenius exhbits the following functoriality if $f : \operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{S}$ is a morphism then:

$$\begin{tikzcd} \operatorname{Spec} \mathrm{R} \arrow[r, "\mathrm{F_R}"] \arrow[d, "f"] & \operatorname{Spec} \mathrm{R} \arrow[d, "f"] \\ \operatorname{Spec} \mathrm{S} \arrow[r, "\mathrm{F_S}"] & \operatorname{Spec} \mathrm{S}. \end{tikzcd}$$

In particular, unless $\mathrm{F_S} = \mathrm{id}$, then $\mathrm{F_R}$ is *not* a morphism of S-schemes.

Relative algebraic geometry prompts us to ask the following question:

Question 8.1.8. Is there a good theory of "algebra" over a base scheme B?

It turns out, as you will prove in the exercises, that repeating the theory of prestacks as functors

$$\mathcal{F} : \mathrm{CAlg_A} \to \mathrm{Set}$$

does recover relative algebraic geometry over $\operatorname{Spec} \mathrm{A}$.

8.2. **Linear algebra over schemes.** There is a one-shot definition of quasicoherent sheaves. This is the best definition but maybe not the most workable since it involves 2-categories:

$$\mathbf{QCoh}(\mathrm{X}) := \operatorname{holim}_{\operatorname{Spec} \mathrm{R} \to \mathrm{X}} \mathbf{Mod}_\mathrm{R}.$$

The the holim indicates a more sophisticated but correct notion of a limit in the context of the 2-category of 1-categories but I will spare everyone this formulation. If you have, however, worked with categories like this, I encourage you to think that way. The theory of quasicoherent sheaves can be quite difficult to stomach on first try. Here are some signposts in the wilderness:

(1) if $\mathrm{X} = \operatorname{Spec} \mathrm{A}$, an affine scheme, then

$$\mathbf{QCoh}(\mathrm{X}) = \mathbf{Mod}_\mathrm{A}.$$

In other words, the theory of quasicoherent sheaves over an affine scheme is "just linear algebra."

(2) Insider **QCoh**(X) there is a distinguished (in the royal sense, but not in any precise sense):

$$\mathbf{Vect}(\mathrm{X}) \subset \mathbf{QCoh}(\mathrm{X})$$

which are much much more manageable and also interesting — they are called "vector bundles" and behave like objects under the same name that you might have encountered in differential geometry or other contexts. If $\mathrm{X} = \operatorname{Spec} \mathrm{A}$ then

$$\mathbf{Vect}(\mathrm{X}) = \mathbf{Mod}_{\mathrm{A}}^{\mathrm{fgproj}},$$

the category of finitely generated projective A-modules. Here's one way to think about it from this point of view (which is not necessarily good since this is equivalent to picking a basis): to give finitely generated projective module M is equivalent to giving a free module $\mathrm{A}^{\oplus n}$ and an idempotent $e : \mathrm{A}^{\oplus n} \to \mathrm{A}^{\oplus n}$. In other words, these are just *(square) matrices.*

(3) Other than vector bundles, quasicohernet sheaves which are interesting are those "coming from closed subschemes." In the affine case: if $\mathrm{X} = \operatorname{Spec} \mathrm{A}$, recall that a closed subscheme is given by a surjection $\mathrm{A} \to \mathrm{B}$. From this we can extract $\mathrm{I} := \ker(\mathrm{A} \to \mathrm{B})$ which is thus an A-module and is thus a quasicoherent sheaf on X.

(4) Some good categorical/homological properties: **QCoh**(X) is a symmetric monoidal Grothendieck abelian category with enough injectives. This is not an easy theorem and is due to Gabber (a name you will hear many times if you are an algebraic geometer) but, in particular, you can:
- take kernels and cokernels;
- take tensor products.
- if $0 \to \mathcal{F}_i \to \mathcal{G}_i \to \mathcal{H}_i \to 0$ is an exact sequence, then the sum
$$0 \to \bigoplus_i \mathcal{F}_i \to \bigoplus_i \mathcal{G}_i \to \bigoplus_i \mathcal{H}_i \to 0,$$
remains exact;
- and, most importantly, you can *take cohomology* which, by the end of this class, you will be addicted to doing.

Here comes the definition:

Definition 8.2.1. Let X be a scheme. A **quasicoherent sheaf** $\mathcal{F}$ on X is the data:

(1) for each map $x : \operatorname{Spec} \mathrm{R} \to \mathrm{X}$ an A-module which we denote by $x^*\mathcal{F} \in \mathbf{Mod}_{\mathrm{A}}$,
(2) for each map $\operatorname{Spec} \mathrm{S} \xrightarrow{g} \operatorname{Spec} \mathrm{R} \xrightarrow{x} \mathrm{X}$ whose composite we denote by
$$y = x \circ g : \operatorname{Spec} \mathrm{S} \to \mathrm{X}.$$
we are given an isomorphism
$$\alpha_{x,g} : (x \circ g)^*\mathcal{F} \xrightarrow{\cong} g^*x^*\mathcal{F},$$

subject to the following condition: given

$$\operatorname{Spec} \mathrm{T} \xrightarrow{h} \operatorname{Spec} \mathrm{S} \xrightarrow{g} \operatorname{Spec} \mathrm{R} \xrightarrow{x} \mathrm{X}$$

so that the composite is denoted as

$$z : \operatorname{Spec} \mathrm{T} \to \mathrm{X}$$

we have an equality of maps

$$\alpha_{x,g\circ h} = \alpha_{x\circ g,h} : z^*\mathcal{F} \xrightarrow{\cong} h^*g^*f^*\mathcal{F}.$$

A **morphism of quasicoherent sheaves** $q : \mathcal{F} \to \mathcal{G}$ is the data: for each morphism $x : \operatorname{Spec} \mathrm{R} \to \mathrm{X}$ a morphism

$$x^*q : f^*\mathcal{F} \to x^*\mathcal{G}$$

of R-modules such that all the induced diagrams commute. We denote by **QCoh**(X) the category of quasicoherent sheaves on X.

Example 8.2.2. Let X be a scheme. Then the **structure sheaf** on X, which we denote by $\mathcal{O}_X$ is the object $\mathcal{O}_X \in \mathbf{QCoh}(X)$ given by

$$f^*\mathcal{O}_X = R$$

for any $f : \operatorname{Spec} R \to X$. Let us unpack the compatibility condition. Suppose that we have a composite

$$\operatorname{Spec} T \xrightarrow{h} \operatorname{Spec} S \xrightarrow{g} \operatorname{Spec} R \xrightarrow{x} X,$$

so that, on the level of rings, we have morphisms

$$R \to S \to T.$$

We are two different isomorphisms of

$$z^*\mathcal{O}_X = T \xrightarrow{\cong} h^*g^*x^*\mathcal{O}_X = h^*g^*R$$

given by

$$T \cong R \otimes_S (S \otimes_S T)$$

and

$$T \cong (R \otimes_S S) \otimes_S T.$$

The claim that $\mathcal{O}_X$ is a quasicoherent sheaf follows from the associativity of the tensor product.

8.3. **A word on: why quasicoherent sheaves?** It is algebro-geometric propaganda that quasicoherent sheaves are important and that they should be the next thing one introduces after introducing schemes. One motivation is that it that to have a mechanism of showing that some scheme is not affine, we will need to examine the global sections of the structure sheaf. Hence, the data of "global functions" or, more precisely, the global sections of the structure sheaf captures the entire affine scheme. On the other hand, the birational classification of algebraic varieties relies on studying invariants that one can extract out of quasicoherent sheaves.

In other words, to study an affine scheme it is not enough to look at global sections of $\mathcal{O}$, we must look at the global sections of $\mathbf{QCoh}(-)$. I hope this is enough motivation for us to spend sometime trying to study quasicoherent sheaves.

There is however a more primary motivation. In the 60's Gabriel and Rosenberg proved the following remarkable "reconstruction theorem."

Theorem 8.3.1 (Gabriel, Rosenberg)**.** *Let* X, Y *be schemes which are quasiseparated (to be defined later), then the following are equivalent:*

(1) there is an equivalence of categories $\mathbf{QCoh}(X) \cong \mathbf{QCoh}(Y)$*;*

(2) X *and* Y *are isomorphic as schemes.*

8.4. **Exercises.**

Definition 8.4.1. A **closed point** of a scheme X is a closed immersion of the form

$$x : \operatorname{Spec} k \hookrightarrow X.$$

Exercise 8.4.2. *Prove that under the correspondence of Exercise 3.3.7 a closed point of* $X = \operatorname{Spec} A$, $\operatorname{Spec} k \hookrightarrow X$ *is the same thing as a maximal ideal of* A *with* $A/\mathfrak{m} = k$.

Definition 8.4.3. Let R be a ring and $R \to S$ an R-algebra. Then S is said to be **of finite type** if it is of the form (as an R-algebra)

$$S \cong R[x_1, \cdots, x_n]/I,$$

where I is an ideal. In this case, we say that $\operatorname{Spec} S \to \operatorname{Spec} R$ is of **finite type**.

For the next exercise, you may, and should, invoke Hilbert's Nullstellensatz.

Exercise 8.4.4. *Suppose that* $X = \operatorname{Spec} A \to \operatorname{Spec} k$ *is of finite type. Then the following are equivalent:*

(1) the k*-morphism* $\operatorname{Spec} K \to \operatorname{Spec} A$ *is a closed point,*

(2) K *is a finite field extension of* k*,*
(3) K *is algebraic as an extension over* k*.*

Exercise 8.4.5. *Let* $\mathcal{O}$ *be a discrete valuation ring with fraction field* K*. Prove that the map*

$$\operatorname{Spec} \mathrm{K} \to \operatorname{Spec} \mathcal{O},$$

is an open immersion. Conclude that $\operatorname{Spec} \mathrm{K}$ *is not a closed point of* $\operatorname{Spec} \mathcal{O}$ *but has an open neighborhood in which it is closed.*

Exercise 8.4.6. *Let* $\mathcal{C}$ *be a category and* $\mathrm{X} \in \mathcal{C}$ *a fixed object. Consider* $\mathrm{Z} \xrightarrow{f} \mathrm{X}, \mathrm{Y} \to \mathrm{X}$ *Then prove that the following is a pullback diagram in sets*

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}_{/\mathrm{X}}}(\mathrm{Z},\mathrm{Y}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(\mathrm{Z},\mathrm{Y}) \arrow[d] \\ \{f\} \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(\mathrm{Z},\mathrm{X}) \end{tikzcd}$$

where the right vertical map is postcomposition:

$$\mathrm{Z} \to \mathrm{Y} \mapsto \mathrm{Z} \to \mathrm{Y} \to \mathrm{X}.$$

Exercise 8.4.7. *Let* X *be a scheme and* R *a ring. Prove that the set* $\mathrm{X}(\mathrm{R})$ *is canonically isomorphic to sections of* $\mathrm{X} \times_{\operatorname{Spec} \mathbf{Z}} \operatorname{Spec} \mathrm{R}$*. Compute the following sets:*

(1) *sections of* $\mathbf{A}^1_{\mathbf{R}} \to \operatorname{Spec} \mathbf{R}$*,*
(2) *sections of* $\mathbf{A}^1_{\mathbf{C}} \to \operatorname{Spec} \mathbf{C}$*.*

This illustrates how rational points can differ over different base rings.

Exercise 8.4.8. *The following exercises corrects a mistake made in the formulation of a previous exercise. Let* B *be a base scheme and* K *a field. Consider the following commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} \mathrm{K} \arrow[rr, "\xi"] \arrow[dr] & & \mathrm{X} \arrow[dl] \\ & \mathrm{B} & \end{tikzcd} .$$

Define

$$\mathrm{T}_\xi(\mathrm{X}/\mathrm{B}) := \mathrm{Hom}_{\mathrm{S}}(\operatorname{Spec} \mathrm{K}[x]/(x^2), \mathrm{X}) \times_{\mathrm{Hom}_{\mathrm{S}}(\operatorname{Spec} \mathrm{K}, \mathrm{X})} \{x\}.$$

Prove:

(1) *we can endow* $\mathrm{T}_\xi(\mathrm{X}/\mathrm{B})$ *with the structure of a* K*-vector space;*
(2) *if* $\mathrm{S} = \operatorname{Spec} k$ *where* k *is a field and* $\mathrm{K} = k$*, prove that* $\mathrm{T}_\xi(\mathrm{X}/\mathrm{B})$ *recovers the Zariski tangent space from Exercise 3.3.7.*
(3) *Let* S *remain the same but let* K *be a finite extension of* k*. Let* $\mathrm{X} = \operatorname{Spec} \mathrm{K}$ *and let* $x : \operatorname{Spec} \mathrm{K} \to \mathrm{X}$ *classify the identity. Prove that*

$$\mathrm{T}_x \mathrm{X} = 0,$$

but

$$\mathrm{T}_\xi(\mathrm{X}/\operatorname{Spec} k) = 0$$

if and only if K/k *is a separable extension.*

Exercise 8.4.9. *Let* $\mathcal{F} : \mathrm{CAlg}_{\mathrm{A}} \to \mathrm{Set}$ *be a functor where* A *is a nonzero ring.*

(1) *prove that in* $\mathrm{Fun}(\mathrm{CAlg}_{\mathrm{A}}, \mathrm{Set})$*, the Yoneda embedding exhibits* $\operatorname{Spec} \mathrm{A}$ *as the final object by constructing a canonical map* $\mathcal{F} \to \operatorname{Spec} \mathrm{A}$*.*
(2) *Prove that "being a scheme" is a local property by proving that the following are equivalent:*

(a) *the prestack*

$$\mathcal{F} : \mathrm{CAlg} = \mathrm{CAlg}_{\mathbf{Z}} \xrightarrow{\text{forget}} \mathrm{CAlg}_{\mathrm{A}} \xrightarrow{\mathcal{F}} \mathrm{Set}$$

is a scheme,

(b) *for each* $\mathrm{Spec}\,\mathrm{R} \to \mathrm{Spec}\,\mathrm{A}$*, the pullback* $\mathcal{F} \times_{\mathrm{Spec}\,\mathrm{R}} \mathrm{Spec}\,\mathrm{A}$ *is a scheme.*

Exercise 8.4.10. *Let* X *be a scheme. We say that a quasicoherent sheaf* $\mathcal{F}$ *is* **finite type** *if for all* $f : \mathrm{Spec}\,\mathrm{R} \to \mathrm{X}$*, the pullback* $f^*\mathcal{M} \in \mathbf{Mod}_{\mathrm{R}}$ *is a finitely generated* R*-module. Define the subprestack*

$$\mathrm{Supp}(\mathcal{F}) \hookrightarrow \mathrm{X}$$

as follows: a morphism $g : \mathrm{Spec}\,\mathrm{R} \to \mathrm{X}$ *factors through* $\mathrm{Supp}(\mathcal{F})$ *if and only if* $g^*\mathcal{F} \neq 0$*. Prove that* $\mathrm{Supp}(\mathcal{F})$ *is a closed subscheme (Hint: you may take for granted the following fact: if* M *is a module over a ring* R*, then the support of a module* M *(in the sense of commutative algebra) which is finite type is given by the quotient ring* $\mathrm{R}/\mathrm{Ann}_{\mathrm{R}}(\mathrm{M})$*; but be sure to know that this* not *true when* M *is note finitely generated).*

Exercise 8.4.11. *Let* k *be an algebraically closed field. Consider the* $\mathbf{A}^1$*-scheme* $\mathrm{Spec}\,k[x,y,t]/(ty-x^2) \to \mathrm{Spec}\,k[t]$ *induced by the obvious morphism.*

(1) *Prove that this morphism is surjective on all* R*-points where* R *is a* k*-algebra.*
(2) *For each* $a \in k$ *which is not zero consider the map* $\mathrm{Spec}\,k \to \mathrm{Spec}\,k[t]$ *induced by the map* $k[t] \to k$ *sending* t *to* a*. Prove that the pullback*

$$\mathrm{Spec}\,k \times_{\mathrm{Spec}\,k[t]} \mathrm{Spec}\,k[x,y,t]/(ty-x^2),$$

is a reduced scheme.
(3) *If* $\mathrm{Spec}\,k \to \mathrm{Spec}\,k[t]$ *is induced by the map* $k[t] \to k$ *sending* t *to zero, then prove that*

$$\mathrm{Spec}\,k \times_{\mathrm{Spec}\,k[t]} \mathrm{Spec}\,k[x,y,t]/(ty-x^2),$$

is, however, not reduced.

9. Lecture 8: more quasicoherent sheaves

Last class, we defined the category $\mathbf{QCoh}(\mathrm{X})$. Let us see what happens in the affine situation. So suppose that $\mathrm{X} = \mathrm{Spec}\,\mathrm{A}$ is an affine scheme and say, $\mathcal{F}$ is a quasicoherent sheaf on X. Among other things, we have the identity morphism $\mathrm{id} : \mathrm{Spec}\,\mathrm{A} \to \mathrm{Spec}\,\mathrm{A}$ so that we have an A-module

$$\mathrm{id}^*\mathcal{F} =: \mathrm{M}.$$

Furthermore, we also note that $\mathrm{id}^*\mathcal{O}_{\mathrm{X}} = \mathrm{A}$.

On the other hand, if M is an A-module, and we are given maps $\mathrm{Spec}\,\mathrm{S} \xrightarrow{g} \mathrm{Spec}\,\mathrm{R} \xrightarrow{f} \mathrm{X}$ which means maps of rings

$$\mathrm{A} \to \mathrm{R} \to \mathrm{S},$$

then we have a canonical isomorphism:

$$\mathrm{M} \otimes_{\mathrm{A}} \mathrm{R} \otimes_{\mathrm{R}} \mathrm{S} \cong \mathrm{M} \otimes_{\mathrm{A}} \mathrm{S}.$$

The canoncity of this isomorphism implies that the cocycle condition is satisfied. We denote this quasicoherent sheaf by $\widetilde{\mathrm{M}}$ and this assembles into a functor

$$\mathbf{Mod}_{\mathrm{A}} \to \mathbf{QCoh}(\mathrm{X}) \qquad \mathrm{M} \mapsto \widetilde{\mathrm{M}}.$$

Proposition 9.0.1. *Let* $\mathrm{X} = \mathrm{Spec}\,\mathrm{A}$*. Consider the functor*

$$\Gamma : \mathbf{QCoh}(\mathrm{X}) \to \mathbf{Mod}_{\mathrm{A}},$$

given by

$$\mathcal{F} \mapsto \mathrm{id}^*\mathcal{F}.$$

This functor is an equivalence of categories.

Proof. We have a functor $\mathbf{Mod}_{\mathrm{A}} \to \mathbf{QCoh}(\mathrm{X})$ given by sending an A-module M to $\widetilde{\mathrm{M}}$. But now we note that

$$\Gamma(\widetilde{\mathrm{M}}) = \mathrm{id}^*\widetilde{\mathrm{M}} = \mathrm{M},$$

by construction. On the other hand, for any $x : \operatorname{Spec} \mathrm{R} \to \mathrm{X}$, we have that

$$x^*\widetilde{\Gamma(\mathcal{F})} = x^*\Gamma(\mathcal{F}) = x^*\mathrm{id}^*\mathcal{F} = x^*\mathcal{F}.$$

Hence we have that $\widetilde{\Gamma(\mathcal{F})} \cong \mathcal{F}$. □

More generally, we define:

Definition 9.0.2. Let $\mathcal{F}$ be a quasicoherent sheaf on X, then the set of **global sections** of $\mathcal{F}$ is defined to be

$$\Gamma(\mathrm{X}, \mathcal{F}) := \mathrm{Hom}_{\mathbf{QCoh}(\mathrm{X})}(\mathcal{O}_{\mathrm{X}}, \mathcal{F})$$

Let us attempt to unpack Definition 9.0.2. Suppose that we have a morphism $f : \operatorname{Spec} \mathrm{R} \to \mathrm{X}$, then to an element of $\Gamma(\mathrm{X}, \mathcal{F})$ gives us a map of R-modules:

$$f^*\mathcal{O}_{\mathrm{X}} = \mathrm{R} \to f^*\mathcal{F},$$

which just means that we are picking out an element of $f^*\mathcal{F}$ since this map is R-linear. So if we think of $f^*\mathcal{F}$ as a sheaf over the space X, this picks out a local section of $\mathcal{F}$. As R varies, these maps should then be compatible in a suitable way. In particular we note that if $\mathrm{X} = \operatorname{Spec} \mathrm{A}$ then

$$\Gamma(\mathrm{X}, \widetilde{\mathrm{M}}) = \mathrm{id}^*\widetilde{\mathrm{M}} = \mathrm{M},$$

and is thus naturally an A-module. We will later say what extract structure $\Gamma(\mathrm{X}, \mathcal{F})$ is.

9.1. A(nother) result of Serre's.

In the next topic, we will "calculate" what the category $\mathbf{QCoh}(\mathrm{X})$ looks like, i.e., produce a smaller amount of data that determines the whole category of quasicoherent sheaves. We have two motivations for this:

9.1.1. *Ideal sheaves.* Let X be a scheme. If X is affine, we note that an closed subscheme $\mathrm{Z} \hookrightarrow \mathrm{X}$ determines and is determined by an ideal I, i.e., an algebraic data. We wish to prove that there is a global version of this phenomenon. We would like to define the *ideal sheaf* of a closed immersion $\mathrm{Z} \hookrightarrow \mathrm{X}$ which is often denoted by $\mathcal{I}_{\mathrm{Z}}$. Here's one way to begin doing so: suppose that $x : \operatorname{Spec} \mathrm{R} \to \mathrm{X}$ is a morphism, then we know that

$$\operatorname{Spec} \mathrm{R} \times_{\mathrm{X}} \mathrm{Z} \cong \operatorname{Spec} \mathrm{R}/\mathrm{I}.$$

We would like to set

$$x^*\mathcal{I}_{\mathrm{Z}} := \mathrm{I} \in \mathbf{Mod}_{\mathrm{R}}.$$

However we note that if $\operatorname{Spec} \mathrm{S} \xrightarrow{f} \operatorname{Spec} \mathrm{R}$ and

$$\operatorname{Spec} \mathrm{S} \times_{\mathrm{X}} \mathrm{Z} \cong \operatorname{Spec} \mathrm{S}/\mathrm{J},$$

then the ideal J need not be equal to $f^*\mathrm{I}$. Indeed, we have an exact sequence of R-modules

$$0 \to \mathrm{I} \to \mathrm{R} \to \mathrm{R}/\mathrm{I} \to 0.$$

If we apply f^* we get a sequence which may not be left exact

$$f^*\mathrm{I} \to \mathrm{S} \to \mathrm{S}/\mathrm{J} = \mathrm{S} \otimes_{\mathrm{R}} \mathrm{R}/\mathrm{I} \to 0,$$

though there is a comparison map

$$f^*\mathrm{I} \to \mathrm{J}.$$

To fix this problem, we will have an excuse to discuss the first basic functoriality of quasicoherent sheaves.

Construction 9.1.2. Let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism of schemes. Then the **pullback functor**

$$f^* : \mathbf{QCoh}(\mathrm{Y}) \to \mathbf{QCoh}(\mathrm{X})$$

is constructed as follows: for $\mathcal{F} \in \mathbf{QCoh}(\mathrm{Y})$, then for any $x : \operatorname{Spec} \mathrm{R} \to \mathrm{X}$, $x^*(f^*\mathcal{F})$ is defined as

$$x^* f^* \mathcal{F}.$$

The compatibility condiitions for $f^*\mathcal{F}$ is inherited by those of $\mathcal{F}$'s.

We observe

Proposition 9.1.3. *Let* $\mathrm{X} = \operatorname{Spec} \mathrm{A}$ *be an affine scheme and suppose that* $j : \mathrm{U} \hookrightarrow \mathrm{X}$ *is an open immersions. Then given an injection of* A*-modules*

$$0 \to \mathrm{M}' \to \mathrm{M},$$

we have an injection

$$0 \to j^*\mathrm{M}' \to j^*\mathrm{M}$$

Proof. We will soon develop the full-blown theory of exact sequences in quasicoherent sheaves. For now, we only note that in order to check that the map $j^*\mathrm{M}' \to j^*\mathrm{M}$ is injective, it suffices to prove injectivity on a cover of U: for an open affine cover $\{j_i : \operatorname{Spec} \mathrm{R}_i \to \mathrm{U}\}$, the map

$$j_i^* j^* \mathrm{M}' \to j_i^* j^* \mathrm{M}$$

is injective. It suffices to then check for any open immersion of affine schemes $j : \operatorname{Spec} \mathrm{S} \to \operatorname{Spec} \mathrm{R}$, the functor $j^* = - \otimes_{\mathrm{R}} \mathrm{S}$ is flat (which is homework). □

9.1.4. *Non-affineness results.* Suppose that we want to prove the following result:

Theorem 9.1.5. *The scheme* $\mathbf{A}^2 \smallsetminus 0$ *is not an affine scheme.*

For this we need to find a property of affine schemes which makes them "affine." One of the things that makes a scheme is affine is the fact that $\mathbf{QCoh}(\mathrm{X}) \simeq \mathbf{Mod}_{\mathrm{A}}$. In fact,

$$\mathrm{A} \cong \Gamma(\mathcal{O}_{\mathrm{X}}),$$

as A-modules. But in fact, this equivalence is an equivalence of algebras. To make sense of this we need to talk about the symmetric monoidal structure on $\mathbf{QCoh}(\mathrm{X})$.

Proposition 9.1.6. *The category of quasicoherent sheaves is a symmetric monoidal category with unit* $\mathcal{O}_{\mathrm{X}}$. *Furthermore, if* $f : \mathrm{X} \to \mathrm{Y}$ *is a morphism of schemes, then the induced functor*

$$\mathbf{QCoh}(\mathrm{Y}) \xrightarrow{f^*} \mathbf{QCoh}(\mathrm{X})$$

is strongly symmetric monoidal.

Proof. For a pair $\mathcal{F}, \mathcal{G}$ of quasicoherent sheaves, we define

$$\mathcal{F} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{G} \in \mathbf{QCoh}(\mathrm{X})$$

to be the quasicoherent sheaf such that for each $f : \operatorname{Spec} \mathrm{R} \to \mathrm{X}$, we get

$$f^*(\mathcal{F} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{G}) := f^*\mathcal{F} \otimes_{\mathrm{R}} f^*\mathcal{G}.$$

With this note that

$$\mathcal{F} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{O}_{\mathrm{X}} = \mathcal{F},$$

since for all $f : \operatorname{Spec} \mathrm{R} \to \mathrm{X}$ we have that

$$f^*(\mathcal{F} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{O}_{\mathrm{X}}) = f^*\mathcal{F} \otimes_{\mathrm{R}} \mathrm{R} = f^*\mathcal{F}.$$

Checking the axioms of being a symmetric monoidal category is a standard check.

For the second claim, this follows from the fact that if $f : \operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{S}$ is a morphism of affine schemes, then

$$f^*(\mathrm{M} \otimes_{\mathrm{S}} \mathrm{N}) = \mathrm{R} \otimes_{\mathrm{S}} (\mathrm{M} \otimes_{\mathrm{S}} \mathrm{N}) \cong (\mathrm{M} \otimes_{\mathrm{S}} \mathrm{R}) \otimes_{\mathrm{R}} (\mathrm{R} \otimes_{\mathrm{S}} \mathrm{N}) = f^*(\mathrm{M}) \otimes_{\mathrm{S}} f^*(\mathrm{N}),$$

and

$$f^*(\mathrm{S}) = \mathrm{R}.$$

□

By construction, we see that if X is affine, then the equivalence $\mathbf{QCoh}(\mathrm{X}) \cong \mathbf{Mod}_{\mathrm{A}}$ respects symmetric monoidal structures. In particular we can recover A as a ring from this equivalence. To make this more precise, we introduce the main focus of next week's lectures:

Definition 9.1.7. A **quasicoherent algebra** or a **quasicoherent sheaves of algebras** is a quasicoherent sheaf $\mathcal{F}$ together with maps

$$m : \mathcal{F} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F} \to \mathcal{F} \qquad \epsilon : \mathcal{O}_{\mathrm{X}} \to \mathcal{F},$$

which makes $\mathcal{F}$ as a commutative algebra object in the symmetric monoidal category $\mathbf{QCoh}(\mathrm{X})$.

Corollary 9.1.8. *Let* $\mathrm{X} = \operatorname{Spec} \mathrm{A}$. *The equivalence of Proposition 9.0.1 descends to a compatible equivalence:*

$$\mathrm{CAlg}_{\mathrm{A}} \rightleftarrows \mathrm{CAlg}(\mathbf{QCoh}(\mathrm{X})).$$

In particular:

$$\Gamma(\mathcal{O}_{\mathrm{X}}) = \mathrm{A}$$

as ring and thus $\operatorname{Spec} \Gamma(\mathcal{O}_{\mathrm{X}}) \cong \mathrm{X}$.

Now, let X be a scheme in general; we do not expect $\Gamma(\mathcal{O}_{\mathrm{X}})$ to recover X in general. But we can still ask what kind of structure $\Gamma(\mathcal{F})$ has. This will sharpen our picture a little bit more:

(1) if $s, t : \mathcal{O}_{\mathrm{X}} \to \mathcal{F}$ are two sections, then we can add sections

$$s + t : \mathcal{O}_{\mathrm{X}} \to \mathcal{F},$$

which is locally given by

$$s + t : \mathrm{R} \to x^*\mathcal{F} \qquad (s + t)(r) = s(r) + t(r),$$

using the module addition. Therefore $\Gamma(\mathcal{F})$ is an abelian group.

(2) Furthermore, if $\mathcal{A}$ is a quasicoherent sheaf of algebras, then we can multiply sections:

$$s \cdot t : \mathcal{O}_{\mathrm{X}} \to \mathcal{A},$$

which is locally given by

$$s \cdot t : \mathrm{R} \to x^*\mathcal{A} \qquad s \cdot t(r) = s(t)r(t).$$

From these observations, we conclude that

$$\Gamma(\mathcal{O}_{\mathrm{X}})$$

is an algebra.

As a result, Γ assembles into a functor

$$\Gamma : \mathbf{QCoh}(\mathrm{X}) \to \mathbf{Mod}_{\mathbf{Z}} = \mathbf{QCoh}(\operatorname{Spec} \mathbf{Z}).$$

Soon, will recast the above discussion in a more general context later and discover Γ as the pushforward map

$$\pi_{\mathrm{X}*} : \mathbf{QCoh}(\mathrm{X}) \to \mathbf{QCoh}(\operatorname{Spec} \mathbf{Z}),$$

where $\pi_{\mathrm{X}} : \mathrm{X} \to \operatorname{Spec} \mathbf{Z}$ is the canonical map.

Anyway, we now have a clear strategy to prove that $\mathbf{A}^2 \setminus 0$ is not affine: we just prove that

$$\operatorname{Spec} \Gamma(\mathcal{O}_{\mathbf{A}^2 \setminus 0}) \ncong \mathbf{A}^2 \setminus 0.$$

In fact we know exactly that $\Gamma(\mathcal{O}_{\mathbf{A}^2 \setminus 0})$ is:

Lemma 9.1.9. *The ring* $\Gamma(\mathcal{O}_{\mathbf{A}^2 \setminus 0}) \cong \mathbf{Z}[x, y]$.

This begs the question:

Question 9.1.10. How does one compute $\Gamma(\mathcal{O}_{\mathrm{X}})$?

9.2. **Formulation of Serre's theorem.** Suppose that $j : \mathrm{U} \hookrightarrow \mathrm{V}$ be an open immersion of schemes and $\mathcal{F} \in \mathbf{QCoh}(\mathrm{V})$. We will sometimes abusively write

$$j^*\mathcal{F} = \mathcal{F}|_{\mathrm{V}}.$$

If $\mathrm{W} \hookrightarrow \mathrm{V}$ is another open immersion we also abusively write

$$\mathrm{W} \times_{\mathrm{V}} \mathrm{U} =: \mathrm{W} \cap \mathrm{U}$$

With this:

Definition 9.2.1. Let X be a scheme and $\mathcal{U}$ an open cover of X. We define

$$\mathbf{QCoh}(\mathrm{X}, \mathcal{U})$$

to be the category consisting of $\mathcal{F}_{\mathrm{U}} \in \mathbf{QCoh}(\mathrm{U})$ and isomorphisms

$$\alpha_{\mathrm{UV}} : \mathcal{F}_{\mathrm{U}}|_{\mathrm{U}\cap\mathrm{V}} \cong \mathcal{F}_{\mathrm{V}}|_{\mathrm{U}\cap\mathrm{V}},$$

satisfying the cocycle condition: for triple intersections $\mathrm{U}, \mathrm{V}, \mathrm{W}$ the composite of isomorphisms

$$\begin{aligned}
\mathcal{F}_{\mathrm{U}}|_{\mathrm{U}\cap\mathrm{V}\cap\mathrm{W}} &\simeq (\mathcal{F}_{\mathrm{U}}|_{\mathrm{U}\cap\mathrm{V}})|_{\mathrm{U}\cap\mathrm{V}\cap\mathrm{W}} \\
&\xrightarrow{\alpha_{\mathrm{UV}}} (\mathcal{F}_{\mathrm{V}}|_{\mathrm{U}\cap\mathrm{V}})|_{\mathrm{U}\cap\mathrm{V}\cap\mathrm{W}} \\
&\xrightarrow{\alpha_{\mathrm{VW}}} (\mathcal{F}_{\mathrm{W}}|_{\mathrm{V}\cap\mathrm{W}})|_{\mathrm{U}\cap\mathrm{V}\cap\mathrm{W}} \\
&\simeq \mathcal{F}_{\mathrm{W}}|_{\mathrm{U}\cap\mathrm{V}\cap\mathrm{W}}.
\end{aligned}$$

must be equal to

$$\mathcal{F}_{\mathrm{U}}|_{\mathrm{U}\cap\mathrm{V}\cap\mathrm{W}} \simeq (\mathcal{F}_{\mathrm{U}}|_{\mathrm{U}\cap\mathrm{W}})|_{\mathrm{U}\cap\mathrm{V}\cap\mathrm{W}} \xrightarrow{\alpha_{\mathrm{UW}}} \mathcal{F}_{\mathrm{W}}|_{\mathrm{U}\cap\mathrm{V}\cap\mathrm{W}}.$$

Remark 9.2.2. The cocycle condition appearing in the above definition makes it clearer how a quasicoherent sheaf is actually a "sheaf" in some sense.

Now, we note that we can indeed define the ideal sheaf $\mathcal{I}_{\mathrm{Z}}$ of a closed immersion as an object of $\mathbf{QCoh}(\mathrm{X}, \mathcal{U})$. We have a functor

$$\text{forget} : \mathbf{QCoh}(\mathrm{X}) \to \mathbf{QCoh}(\mathrm{X}, \mathcal{U})$$

which *a priori* seems to forget a lot of information. This is not the case.

9.3. **Exercises.** We begin to elucidate the structure of $\mathbf{QCoh}(\mathrm{X})$ in the exercises; fix a Zariski cover of X consisting of affine schemes $\mathcal{U}$. We say that a sequence of quasicoherent sheaves

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

is a **exact** if, under the identification of Serre's theorem

$$\mathbf{QCoh}(\mathrm{X}, \mathcal{U}) \simeq \mathbf{QCoh}(\mathrm{X})$$

the sequence

$$\mathcal{F}'|_{\mathrm{U}} \to \mathcal{F}|_{\mathrm{U}} \to \mathcal{F}''|_{\mathrm{U}}$$

is exact for all $\mathrm{U} \hookrightarrow \mathrm{X}$.

Exercise 9.3.1. *Prove that for any open immersion $j : \mathrm{U}' \hookrightarrow \mathrm{X}$, the functor*

$$j^* : \mathbf{QCoh}(\mathrm{X}, \mathcal{U}) \to \mathbf{QCoh}(\mathrm{U}')$$

is exact.

Exercise 9.3.2. *Prove that a sequence*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

is exact if and only if for all $\mathcal{G} \in \mathbf{QCoh}(\mathrm{U})$, the sequence of $\Gamma(\mathrm{U})$-modules

$$0 \to \mathrm{Hom}_{\mathbf{QCoh}(\mathrm{U})}(\mathcal{G}, \mathcal{F}'|_{\mathrm{U}}) \to \mathrm{Hom}_{\mathbf{QCoh}(\mathrm{U})}(\mathcal{G}, \mathcal{F}|_{\mathrm{U}}) \to \mathrm{Hom}_{\mathbf{QCoh}(\mathrm{U})}(\mathcal{G}, \mathcal{F}''|_{\mathrm{U}}).$$

Prove that

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is exact if and only if for all $\mathcal{G} \in \mathbf{QCoh}(\mathrm{U})$*, the sequence of* $\Gamma(\mathrm{U})$*-modules*

$$0 \to \mathrm{Hom}_{\mathbf{QCoh}(\mathrm{U})}(\mathcal{F}''|_{\mathrm{U}}, \mathcal{G}) \to \mathrm{H}om_{\mathbf{QCoh}(\mathrm{U})}(\mathcal{F}|_{\mathrm{U}}, \mathcal{G})) \to \mathrm{Hom}_{\mathbf{QCoh}(\mathrm{U})}(\mathcal{F}'|_{\mathrm{U}}, \mathcal{G}).$$

is exact.

Exercise 9.3.3. *Let* X *be a scheme and* $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathbf{QCoh}(\mathrm{X})$*. Prove that there exists*

$$\mathcal{H}\mathrm{om}(\mathcal{G}, \mathcal{H}) \in \mathbf{QCoh}(\mathrm{X})$$

such that we have a natural isomorphism of abelian groups

$$\mathrm{Hom}_{\mathbf{QCoh}(\mathrm{X})}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \cong \mathrm{Hom}_{\mathbf{QCoh}(\mathrm{X})}(\mathcal{F}, \mathcal{H}\mathrm{om}(\mathcal{G}, \mathcal{H})).$$

Exercise 9.3.4. *Consider the scheme* $\mathrm{X} = \mathbf{A}^2 \smallsetminus 0 \subset \operatorname{Spec} \mathbf{Z}[x_1, x_2]$*. Consider the quadric*

$$\mathrm{Q}_3 := \operatorname{Spec} \mathbf{Z}[x_1, x_2, y_1, y_2]/(x_1 y_1 + x_2 y_2 = 1) \subset \mathbf{A}^4.$$

Consider the map $p : \mathrm{Q}_3 \to \mathbf{A}^2$ *given by*

$$\mathbf{Z}[x_1, x_2] \to \mathbf{Z}[x_1, x_2, y_1, y_2]/(x_1 y_1 + x_2 y_2 = 1) \qquad x_1 \mapsto x_1, x_2 \mapsto x_2.$$

Prove that

(1) *the map* p *factors through* $\mathbf{A}^2 \smallsetminus 0$*,*
(2) *over any field point* $\operatorname{Spec} k \to \mathbf{A}^2 \smallsetminus 0$*, the pullback* $\operatorname{Spec} k \times_{\mathbf{A}^2 \smallsetminus 0} \mathrm{Q}_3 \cong \mathbf{A}^1_k$*.*
(3) *Furnish an open cover of* $\mathbf{A}^2 \smallsetminus 0$ *by affine schemes,* $\{\mathrm{U} \hookrightarrow \mathbf{A}^2 \smallsetminus 0\}$ *such that*

$$\mathrm{U} \times_{\mathbf{A}^2 \smallsetminus 0} \mathrm{Q}_3 \cong \mathbf{A}^1_{\mathrm{U}}.$$

The scheme Q_3 *is called the* **Jouanalou-Thomason device** *of* $\mathbf{A}^2 \smallsetminus 0$*: it is a "bundle" (which we have not yet defined in this class) of affine spaces over a non-affine scheme whose "total space" is an affine scheme. We will later see that most reasonable schemes admit Jouanalou devices.*

Exercise 9.3.5. *Generalize Exercise 9.3.4 to*

$$\mathrm{Q}_{2n} := \operatorname{Spec} \mathbf{Z}[x_i, \cdots, x_n, y_1, \cdots, y_n]/(x_1 y_1 + \cdots + x_n y_n = 1) \to \mathbf{A}^n \smallsetminus 0.$$

10. Lecture 9: vector and line bundles

Here is Serre's theorem:

Theorem 10.0.1. *The functor*

$$\text{forget} : \mathbf{QCoh}(\mathrm{X}) \to \mathbf{QCoh}(\mathrm{X}, \mathcal{U})$$

is an equivalence of categories.

Proof sketch. This proof is tedious, so let me indicate the main ideas:

(Reduce to affines) By definition, a quasicoherent sheaf is a compatible collection of $x^*\mathcal{F}$ for each $x : \operatorname{Spec} \mathrm{R} \to \mathrm{X}$ and isomorphisms as $\operatorname{Spec} \mathrm{R}$ varies. For each point $x : \operatorname{Spec} \mathrm{R} \to \mathrm{X}$, we can consider the cover of $\operatorname{Spec} \mathrm{R}$ given by $\mathcal{U}_{\mathrm{R}} := \{\operatorname{Spec} \mathrm{R} \times_{\mathrm{X}} \mathrm{U}_i \to \operatorname{Spec} \mathrm{R}\}$. Suppose that we have proved the claim for affine schemes, then an object of $\mathbf{QCoh}(\mathrm{X}, \mathcal{U})$ determines an object of $\mathbf{QCoh}(\operatorname{Spec} \mathrm{R}, \mathcal{U}_{\mathrm{R}})$ which then defines a quasicoherent sheaf on $\mathbf{QCoh}(\operatorname{Spec} \mathrm{R}) = \mathbf{Mod}_{\mathrm{R}}$. Doing this as x varies, we procure an R-module with the desired compatibilities and isomorphisms.

(Reduce to basic opens) This is left to the reader and formulated precisely in the exercises.

(Case of basic opens) We are now left with the case of $\mathrm{X} = \operatorname{Spec} \mathrm{R}$ and $\mathcal{U} = \{\mathrm{R} \to \mathrm{R}_{f_i}\}$ a Zariski cover. An object of $\mathbf{QCoh}(\mathrm{X}, \mathcal{U})$ is the data of R_{f_i}-modules M_i with isomorphisms

$$\alpha_{ij} : \mathrm{M}_i[f_j^{-1}] \cong \mathrm{M}_j[f_i^{-1}].$$

I claim that we can produce an R-module. There is only one possible way to do this of course. Form the diagram in $\mathbf{Mod}_{\mathrm{R}}$

$$\prod_i \mathrm{M}_i \rightrightarrows \prod_{i,j} \mathrm{M}_{ij},$$

where $\mathrm{M}_{ij} := \mathrm{M}_i[f_j^{-1}] \cong \mathrm{M}_j[f_i^{-1}]$. Then taking equalizers, we produce $\mathrm{M} \in \mathbf{Mod}_{\mathrm{R}}$.

(Assembling the proof) We we have done is to construct a functor

$$\mathrm{glue} : \mathbf{QCoh}(\mathrm{X}, \mathcal{U}) \to \mathbf{QCoh}(\mathrm{X}).$$

I claim that the composite

$$\mathbf{QCoh}(\mathrm{X}) \to \mathbf{QCoh}(\mathrm{X}, \mathcal{U}) \to \mathbf{QCoh}(\mathrm{X}).$$

Indeed this composite is equivalent to the identity from our old friend Lemma 4.1.4. In particular, this prove that the forgetful functor is fully faithful. Now, we note that forget and glue are adjoint (the counit is tautologically id = glue ∘ forget). To finish off, prove that the other composite is equivalent to the identity (another exercise, using an old trick).

□

We can now conclude:

Proof of Lemma 9.1.9. We have a morphism $\mathbf{A}^2 \setminus 0 \to \mathbf{A}^2$. I claim that the map

$$\Gamma(\mathcal{O}_{\mathbf{A}^2}) \cong \Gamma(\mathcal{O}_{\mathbf{A}^2 \setminus 0})$$

is an isomorphism. Consider the following commutative diagram

$$\begin{tikzcd}
\mathbf{A}^1 \setminus 0 \times \mathbf{A}^1 \setminus 0 \arrow[r] \arrow[d] & \mathbf{A}^2 \setminus \mathbf{A}^1_v \arrow[d] \\
\mathbf{A}^2 \setminus \mathbf{A}^1_h \arrow[r] & \mathbf{A}^2 \setminus 0.
\end{tikzcd}$$

Here $\mathbf{A}^1_h$ (resp. $\mathbf{A}^1_z$) is the horizontal $\mathbf{A}^1$ (resp. vertical $\mathbf{A}^1$). We note that

$$\{\mathbf{A}^2 \setminus \mathbf{A}^1_v, \mathbf{A}^2 \setminus \mathbf{A}^1_h \hookrightarrow \mathbf{A}^2 \setminus 0\}$$

is an open cover of $\mathbf{A}^2 \setminus 0$ and furthermore we have:

$$\mathbf{A}^2 \setminus \mathbf{A}^1_h = \operatorname{Spec} \mathbf{Z}[x, y, x^{-1}] \qquad \mathbf{A}^2 \setminus \mathbf{A}^1_v = \operatorname{Spec} \mathbf{Z}[x, y, y^{-1}].$$

Using Serre's theorem, a quasicoherent sheaf on $\mathbf{A}^2 \setminus 0$ is given by:

(1) a $\mathbf{Z}[x, y, x^{-1}]$-module M,
(2) a $\mathbf{Z}[x, y, y^{-1}]$-module N,
(3) an isomorphism $\alpha : \mathrm{M}[y^{-1}] \cong \mathrm{M}[x^{-1}]$.

And so $\Gamma(\mathcal{O}_{\mathrm{X}})$ is exactly (by an exercise in this week's homework):

$$f \in \mathbf{Z}[x, y, x^{-1}] \qquad g \in \mathbf{Z}[x, y, y^{-1}],$$

such that $f = g$ as elements in $\mathbf{Z}[x, y, x^{-1}, y^{-1}]$. But this means that $f = g \in \mathbf{Z}[x, y]$. □

Along the way we have used:

Lemma 10.0.2. *Let $\{j_i : \mathrm{U}_i \hookrightarrow \mathrm{X}\}$ be a cover of X by affine schemes. Under the identification of Theorem 10.0.1, the functor*

$$\Gamma : \mathbf{QCoh}(\mathrm{X}, \mathcal{U}) \to \mathbf{QCoh}(\mathbf{Z})$$

is given by

$$(\mathcal{F}_{\mathrm{U}_i}, \alpha_{ij}) \mapsto \{f_i \in \mathcal{F}(\mathrm{U}_i, \mathcal{F}_{\mathrm{U}_i}) \in \mathbf{QCoh}(\mathrm{U}_i) : \alpha_{ij}(f_i) = f_j|_{\mathrm{U}_j}\}.$$

10.1. **Vector bundles.** One of the reasons to develop the theory of quasicoherent sheaves over a scheme was to do algebra over a scheme. We are on our way to doing linear algebra. But to actually do linear algebra we need a good theory of projective modules.

Here's a motivation:

Lemma 10.1.1. *Let* R *be a ring and* M *an* R*-module. Then the following are equivalent:*

1. M *is in the smallest idempotent complete additive subcategory of* $\mathbf{Mod}_{\mathrm{R}}$ *containing* R.
2. M *is finitely generated and projective,*
3. M *is a direct summand of a finitely generated free module,*
4. M *is finite and locally free in the sense that for any prime ideal* $\mathfrak{p}$ *of* R*, the module*
$$\mathrm{M}_{\mathfrak{p}} \in \mathbf{Mod}_{\mathrm{R}_{\mathfrak{p}}}$$
is a free module of finite rank,
5. *for any open cover* $\{j_{\mathrm{U}} : \mathrm{U} \hookrightarrow \operatorname{Spec} \mathrm{R}\}$ *of affine schemes,* $j_{\mathrm{U}}^{*}\widetilde{\mathrm{M}}$ *is projective and finitely generated.*
6. *there exists an open cover* $\{j_{\mathrm{U}} : \mathrm{U} \hookrightarrow \operatorname{Spec} \mathrm{R}\}$ *of affine schemes such that* $j_{\mathrm{U}}^{*}\widetilde{\mathrm{M}}$ *is finitely generated and free.*

The first definition gives finitely generated projective modules a kind of universal property. The third definition gives it an "equational property" — it is basically the same thing as an idempotent square matrix with entries in R. The fourth definition gives it a "local description".

I would like to prove the equivalence between the last two definitions which are geometric in nature. Having this we define

Definition 10.1.2. Let X be a scheme and $\mathcal{E} \in \mathbf{QCoh}(\mathrm{X})$ is a quasicoherent sheaf. Then $\mathcal{E}$ is a **vector bundle on** X (also often called **locally free sheaf of finite rank** if it is finite type and locally projective, i.e., there exists an cover of X of affines $\{j : \mathrm{U}_{\alpha} = \operatorname{Spec} \mathrm{A}_{\alpha} \hookrightarrow \mathrm{X}\}$ such that $j^{*}\mathcal{F} \in \mathbf{Mod}_{\mathrm{A}_{\alpha}}$ is finitely generated and projective.

We have the subcategory
$$\mathbf{Vect}(\mathrm{X}) \subset \mathbf{QCoh}(\mathrm{X})$$
spanned by those quasicoherent sheaves which are vector bundles.

In order to prove the equivalence of parts (4) and (5) of Lemma 10.1.1, let us formulate a version of Nakayama's lemma which is geometric in nature

11. Exercises

Exercise 11.0.1. *Complete the proof of "reduction to basic opens" in Theorem 10.0.1.*

Exercise 11.0.2. *Prove Lemma 10.0.2.*

Exercise 11.0.3. *Construct a natural map in* $\mathbf{QCoh}(\mathrm{X})$
$$\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G}) \otimes \mathcal{H} \to \mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G} \otimes \mathcal{H})$$
such that if $\mathcal{F}$ *or* $\mathcal{H}$ *are vector bundles, then this map is an isomorphism.*

Exercise 11.0.4. *If* $\mathcal{E}$ *is a vector bundle, then denote its* **dual** *by*
$$\mathcal{E}^{\vee} := \mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{O}_{\mathrm{X}})$$
Prove:

1. $(\mathcal{E}^{\vee})^{\vee} \cong \mathcal{E}$,
2. *for any quasicoherent sheaf* $\mathcal{F}$*, then* $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^{\vee} \otimes \mathcal{F}$.

Exercise 11.0.5. *Let* X *be a scheme. Let* $\mathcal{F} \in \mathbf{QCoh}(\mathrm{X}) \cong \mathbf{QCoh}(\mathrm{X}, \mathcal{U})$ *under Serre's theorem. Then* $\mathcal{I} \to \mathcal{O}_{\mathrm{X}}$ *is a* **quasicoherent ideal** *or also called an* **ideal sheaf** *if for each* $\mathrm{U} = \operatorname{Spec} \mathrm{A} \in \mathcal{U}$
$$\mathcal{I}_{\mathrm{U}} \subset \mathcal{O}_{\mathrm{U}}$$

is the inclusion of an ideal in $\mathbf{Mod}_{\mathrm{A}}$*. Let* $\mathcal{I}$ *be a quasicoherent ideal and define the following data: on* $j : \mathrm{U} \hookrightarrow \mathrm{X}$ *where* $\mathrm{U} \in \mathcal{U}$ *consider the closed subscheme of* U *defined via* $\mathcal{I}_{\mathrm{U}}$*, i.e.,*

$$\mathrm{Z}_{\mathrm{U}} := \operatorname{Spec} \Gamma(\mathcal{O}_{\mathrm{U}})/\mathcal{I}_{\mathrm{U}}.$$

Show that this defines subprestack $\mathrm{Z} \hookrightarrow \mathrm{X}$ *which is a scheme and is, furthermore, a closed subscheme.*

Prove that there is a canonical bijection between quasicoherent ideals of $\mathcal{O}_{\mathrm{X}}$ *and closed subschemes of* X*.*

Exercise 11.0.6. *Let* $f : \mathrm{Y} \to \mathrm{X}$ *be a morphism of schemes and* $\mathcal{F}, \mathcal{G} \in \mathbf{QCoh}(\mathrm{X})$*.*

(1) *construct a natural morphism*

$$\alpha_{\mathcal{F},\mathcal{G}} : f^*\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}\mathrm{om}(f^*\mathcal{F}, f^*\mathcal{G}),$$

which, if $f : \operatorname{Spec} \mathrm{A} \to \operatorname{Spec} \mathrm{B}$ *is a morphism of affine schemes, then the morphism is given by a morphism of* A*-modules*

$$\mathrm{Hom}_{\mathrm{B}}(\mathrm{M}, \mathrm{N}) \otimes_{\mathrm{B}} \mathrm{A} \to \mathrm{Hom}_{\mathrm{A}}(\mathrm{M} \otimes_{\mathrm{B}} \mathrm{A}, \mathrm{N} \otimes_{\mathrm{B}} \mathrm{A}),$$

adjoint to a map of B*-modules*

$$\mathrm{Hom}_{\mathrm{B}}(\mathrm{M}, \mathrm{N}) \to \mathrm{Hom}_{\mathrm{A}}(\mathrm{M} \otimes_{\mathrm{B}} \mathrm{A}, \mathrm{N} \otimes_{\mathrm{B}} \mathrm{A}) \qquad f \mapsto f \otimes \mathrm{id}_{\mathrm{A}}.$$

where the right hand side is given the structure of a B*-module by forgetting strucutre.*

(2) *Show by example that such a morphism is not, in general, an isomorphism.*

(3) *Prove that if* $\mathcal{F}$ *is a vector bundle, then the map is always an isomorphism.*

Exercise 11.0.7. *Prove all equivalences of (1)-(4) of Lemma 10.1.1*

12. Lecture 10: Nakayama's lemma, leftover on vector bundles

12.1. **Nakayama's lemma revisited.** We all have seen many versions of Nakayama's lemma. Here's another one:

Lemma 12.1.1. *Let* $\mathcal{F}$ *be a finite type quasicoherent sheaf on* X*. Let* k *be a field and* $x : \operatorname{Spec} k \to \mathrm{X}$ *be a* k*-point of* X *such that* $x^*\mathcal{F} = 0$*, then there exists an open immersion* $\mathrm{U} \hookrightarrow \mathrm{X}$ *containing* x *such that* $j^*\mathcal{F} = 0$*.*

This is a geometric statement: if $\mathcal{F}$ vanishes on a point, it vanishes in a neighborhood.

Remark 12.1.2. Consider $\mathbf{Q}$ as a $\mathbf{Z}$-module. Then for any prime p consider the $\mathbf{F}_p$-point of $\mathbf{Z}$:

$$i : \operatorname{Spec} \mathbf{F}_p \to \operatorname{Spec} \mathbf{Z}.$$

Then $i^*\mathbf{Q} = \mathbf{F}_p \otimes_{\mathbf{Z}} \mathbf{Q} = 0$. However we note that $\mathbf{Z}_{(p)} \otimes_{\mathbf{Z}} \mathbf{Q} \neq 0$ and $\operatorname{Spec} \mathbf{Z}_{(p)}$ is the "smallest open" containing $\operatorname{Spec} \mathbf{F}_p$ in $\operatorname{Spec} \mathbf{Z}$. In particular $\mathbf{Q}$ does not vanish on any open of $\operatorname{Spec} \mathbf{Z}$. This shows that we really do need the finite generation of $\mathbf{Z}$.

Remark 12.1.3. In contrast, the easiest finitely generated $\mathbf{Z}$-module imaginable is of the form $\mathbf{Z}/q$. Say q is a prime which is not p. Then we note that

$$\mathbf{F}_p \otimes_{\mathbf{Z}} \mathbf{Z}/q = 0,$$

and furthermore

$$\mathbf{Z}_{(p)} \otimes_{\mathbf{Z}} \mathbf{Z}/q = 0,$$

exactly since q is invertible in $\mathbf{Z}_{(p)}$ so that

$$1 \otimes 1 = q/q \otimes 1 = 1/q \otimes 0 = 0.$$

There isn't really anything special about this argument, $\mathbf{Z}$ or $\mathbf{Z}/q$ and other friends. Say A is a local ring with maximal ideal $\mathfrak{m}$. Suppose that M is an A-module which is generated by a single generator; this means that we have a surjection of A-modules

$$\mathrm{A} \to \mathrm{M} = \mathrm{A}/\mathrm{I} \to 0$$

If we do know that $A/\mathfrak{m} \otimes_A M = 0$ then for any $\overline{x}$

$$\overline{x} = \tfrac{y}{y}\overline{x} = \tfrac{\overline{x}}{y}y = 0 \cdot y = 0.$$

This simple algebraic observation is the basis of Nakayama's lemma.

Proof. We again break the proof down into several steps:

(Affine case) Let $X = \operatorname{Spec} A$ so that $\mathcal{F}$ corresponds to a finitely generated A-module M. In this case a k-point of X is the same thing as a map $A \to k$. By hypothesis, $M \otimes_A k = 0$. We induct on the number of generators of M. If M is generated by no elements (so the zero module), then we are done. Now, say M is generated by n-elements. Therefore we can write M as an extension

$$0 \to N \to M \to A/I \to 0,$$

where N is generated by $n-1$-elements. We do know that $M \otimes_A k = 0$. The tensor product is right exact so that we have a surjection

$$0 \to A/I \otimes_A k \to 0,$$

hence $A/I \otimes_A k = 0$ and thus $k/Ik = 0$ and thus

$$k = Ik.$$

Since k is not the zero ring, this means that we can find an $f \in A$ such that its image in k is nonzero and therefore the map $A \to k$ factors as

$$A[f^{-1}] \to k.$$

Localizing is exact and thus we have an exact sequence

$$0 \to N[f^{-1}] \to M[f^{-1}] \to A/I[f^{-1}] \to 0,$$

But now $A/I[f^{-1}] = 0$ and thus $N[f^{-1}] \to M[f^{-1}]$ is an isomorphism. As A_f-modules, these are generated by $n-1$-elements and so the inductive hypothesis applies.

(Assembling) We have the open immersion $j : \operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$. We note that $j^*N \cong j^*M$ and are modules over A_f generated by $n-1$-elements. We further note that the map $x : \operatorname{Spec} k \to \operatorname{Spec} A$ factors through $\operatorname{Spec} A_f$ as noted above so that $x^*j^*N \cong x^*j^*M = 0$. Therefore we can find an open U

$$x \in U \subset \operatorname{Spec} A_f \subset \operatorname{Spec} A$$

such that $M|_U = 0$. This is the desired open.

(Globalizing) By the definition of a scheme, there exists an affine scheme $\operatorname{Spec} A$ such that $\operatorname{Spec} A \times_X \operatorname{Spec} k \neq \emptyset$. We can then replace X by $\operatorname{Spec} A$.

□

We now prove

Proof of (some parts of) Lemma 10.1.1. Everything in sight is local so we assume that $X = \operatorname{Spec} A$.

First, let us assume that $\mathcal{E}$ is finitely generated and projective. Our goal is to produce a cover of X on which $\mathcal{E}$ is finitely generated and free. Let $\mathfrak{m}$ be a maximal ideal of A so that we obtain a field point

$$i_\kappa : \operatorname{Spec} A/\mathfrak{m} = \kappa \to \operatorname{Spec} A$$

of the scheme $\operatorname{Spec} A$. Then we note that $i_\kappa^*\mathcal{E} = \mathcal{E}/\mathfrak{m}$ is a finitely generated $A/\mathfrak{m}$-module, i.e., a vector space over κ of finite dimension. Let's say the dimension is n we can choose a basis

$$\overline{v}_1, \cdots \overline{v}_n \in \mathcal{E}/\mathfrak{m},$$

which we can then lift to elements of the A-module

$$v_1, \cdots, v_n \in \mathcal{E}.$$

In particular we obtain a morphism

$$v : \mathrm{A}^{\oplus} \to \mathcal{E}.$$

I claim:

- there exists an open affine $\mathrm{U} \hookrightarrow \operatorname{Spec} \mathrm{A}$ such that i_κ factors through it:

$$\operatorname{Spec} \kappa \xrightarrow{i_\kappa} \mathrm{U} \hookrightarrow \operatorname{Spec} \mathrm{A}$$

  such that the map $v|_{\mathrm{U}} : \widetilde{\mathrm{A}^{\oplus n}}|_{\mathrm{U}} \to \mathcal{E}|_{\mathrm{U}}$ is an isomorphism.

From this we are done: extract the open cover of affines

$$\{\mathrm{U}_\kappa \to \operatorname{Spec} \mathrm{A}\},$$

and furthermore can be refined to a finite collection.

The claim breaks down into injectivity and surjectivity (of course we need to find such a U):

(Surj.) We have the exact sequence of A-modules (equivalently an exact sequence in **QCoh**(Spec A)):

$$\mathrm{A}^{\oplus n} \to \mathcal{E} \to \operatorname{coker}(v) \to 0.$$

Observe that $\operatorname{coker}(v)$ is finitely generated as well since it is the quotient of a map between finitely generated modules. Furthermore we know that $\operatorname{coker}(v)|_{\operatorname{Spec} \kappa} = 0$ and therefore Nakayama's lemma furnishes an open $\mathrm{U} \hookrightarrow \operatorname{Spec} \mathrm{A}$ such that $\operatorname{coker}(v)|_{\mathrm{U}} = 0$. By shrinking U further we may assume that U is actually affine.

(Inj.) Here we need to use exactness: the map v splits since $\mathcal{E}$ is projective and therefore $\ker(v)$ is finitely generated. Furthermore $\ker(v)|_{\operatorname{Spec} \kappa} = 0$ and therefore, by Nakayama again, we can find an affine open V of Spec A through which i_κ factors and v is injective. Taking $\mathrm{U} \times_{\operatorname{Spec} \mathrm{A}} \mathrm{V}$ we are done.

The converse trickier. We first leave an exercise the following assertion:

- for a ring A and $\{\mathrm{A} \to \mathrm{A}_{f_i}\}$ a Zariski open cover. Let M be an A-module and assume that each M_{f_i} is finitely generated (resp. finitely presented) as an A_{f_i}-module, then M is a finitely generated (finitely presented) A-module.

Assuming this exercise, let's prove the result. To prove projectivity take

$$0 \to \mathrm{N}' \to \mathrm{N} \to \mathrm{N}'' \to 0$$

a short exact sequence of A-modules. We want to prove that

$$0 \to \operatorname{Hom}_{\mathrm{A}}(\mathrm{M}, \mathrm{N}') \to \operatorname{Hom}_{\mathrm{A}}(\mathrm{M}, \mathrm{N}) \to \operatorname{Hom}_{\mathrm{A}}(\mathrm{M}, \mathrm{N}'') \to 0$$

is a short exact sequence of R-modules. It suffices, from a previous lemma, to prove that

$$0 \to \operatorname{Hom}_{\mathrm{A}}(\mathrm{M}, \mathrm{N}')[f_i^{-1}] \to \operatorname{Hom}_{\mathrm{A}}(\mathrm{M}, \mathrm{N})[f_i^{-1}] \to \operatorname{Hom}_{\mathrm{A}}(\mathrm{M}, \mathrm{N}'')[f_i^{-1}] \to 0$$

is exact for all f_i's such that $\{\mathrm{A} \to \mathrm{A}_{f_i}\}$ an Zariski open cover. The next lemma finishes the proof.

□

The next lemma is kind of underrated.

Lemma 12.1.4. *Let* A *be a ring,* M *finitely presented* A*-module and* N *an* R*-module. Then for any* $f \in \mathrm{A}$

$$\operatorname{Hom}_{\mathrm{A}}(\mathrm{M}, \mathrm{N})[f^{-1}] \cong \operatorname{Hom}_{\mathrm{A}}(\mathrm{M}[f^{-1}], \mathrm{N}[f^{-1}]).$$

Proof. If M is finitely presented, then we can present M as

$$\bigoplus_{i=1}^{n} \mathrm{A} \to \bigoplus_{j=1}^{m} \mathrm{A} \to \mathrm{M} \to 0.$$

In particular we must prove this in the case that M is just finitely generated and free. For this, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{A}}(\mathrm{A}^{\oplus n}, \mathrm{N})[f^{-1}] &\cong \mathrm{N}^{\oplus n}[f^{-1}] \\ &\cong \mathrm{N}[f^{-1}]^{\oplus n} \\ &\cong \mathrm{Hom}_{\mathrm{A}_f}(\mathrm{A}[f^{-1}]^{\oplus n}, \mathrm{N}[f^{-1}]).\end{aligned}$$

The proof then finishes off by examining the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathrm{A}}(\mathrm{M},\mathrm{N})[f^{-1}] \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{A}}(\bigoplus_{j=1}^m \mathrm{A},\mathrm{N})[f^{-1}] \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathrm{A}}(\bigoplus_{j=1}^n \mathrm{A},\mathrm{N})[f^{-1}] \arrow[d, "\cong"] \\ 0 \arrow[r] & \mathrm{Hom}_{\mathrm{A}}(\mathrm{M}[f^{-1}],\mathrm{N}[f^{-1}]) \arrow[r] & \mathrm{Hom}_{\mathrm{A}}(\bigoplus_{j=1}^m \mathrm{A}[f^{-1}],\mathrm{N}[f^{-1}]) \arrow[r] & \mathrm{Hom}_{\mathrm{A}}(\bigoplus_{j=1}^n \mathrm{A}[f^{-1}],\mathrm{N}[f^{-1}]) \end{tikzcd}$$

□

12.2. **Line bundles and examples.** One of the things algebraic geometry is obsessed about (for good reasons) is the idea of defining *numerical invariants*: we associate to an algebro-geometric object a number which varies well (to varying degrees) in families. This last requirement is more difficult to satisfy than you think in the sense that invariants are usually not easy to define globally but easy to define locally. The task is then to give a local definition then prove that it globalizes well.

Definition 12.2.1. Let X be a scheme and suppose that $\mathcal{E}$ is a vector bundle on X. Let $x : \operatorname{Spec} k \to \mathrm{X}$ be a field point, then the **rank of $\mathcal{E}$ at x** is defined as follows: by Lemma **??** there exists an affine open containing x,

$$\mathrm{U} \hookrightarrow \operatorname{Spec} \mathrm{X}$$

such that $\mathcal{E}|_{\mathrm{U}} \simeq \mathcal{O}_{\mathrm{U}}^{\oplus n}$. We define the the rank to be n:

$$\mathrm{rank}_x(\mathcal{E}) := n.$$

We say that a vector bundle $\mathcal{E}$ on X is **of constant rank** n or, if the context is clear, is **rank** n if for all $x : \operatorname{Spec} k \to \mathrm{X}$, it is indeed of rank n.

It is left an exercises to check that the notion of rank at a point is well-defined.

Remark 12.2.2. Suppose that $\mathrm{X} = \operatorname{Spec} \mathrm{A}$ and we have a decomposition

$$\operatorname{Spec} \mathrm{A} \cong \operatorname{Spec} \mathrm{B} \sqcup \operatorname{Spec} \mathrm{C}.$$

Then we see that a vector bundle on X is the same thing as a vector bundle on Spec B and another on Spec C. They do not have to have same rank. For now, rank is an intrinsically local notion.

Definition 12.2.3. A **line bundle** on a scheme X is a vector bundle of rank 1.

Example 12.2.4. We would like to define a vector bundle on $\mathbf{A}^2 \setminus 0$. Here is one way to do it.

- set M on $\mathbf{Z}[x, x^{-1}, y]$ to be $\mathcal{O}$ and N to be $\mathcal{O}$ on $\mathbf{Z}[x, y, y^{-1}]$.
- Now on $\mathbf{Z}[x, x^{-1}, y, y^{-1}]$ we need to specify an isomorphism; we can take any automorphism

$$\mathcal{O} \to \mathcal{O};$$

we note that this automorphism must be $\mathbf{Z}[x, x^{-1}, y, y^{-1}]$-linear and hence is determined by an element of

$$(\mathbf{Z}[x, x^{-1}, y, y^{-1}])^{\times} = \{\pm x^{\pm k}, \pm y^{\pm j}, \}$$

and this will specify the desired line bundle. Weirdly, any such line bundle is trivial (exercise).

13. Lecture 11: the projective line

We continue trying to give examples of line bundles.

Example 13.0.1. Let k be a field of characteristic zero. We will consider the following affine scheme: let R be

$$\mathrm{R} = k[x,y]/(y^2 - x^3 - 1).$$

In other words, it is the closed subscheme of $\mathbf{A}^2$ cut out by the equation $y^2 = x^3 + 1$. This is an example of a **punctured elliptic curve** but more on that later. Consider

$$\mathrm{X} = \operatorname{Spec} \mathrm{R}.$$

Our goal is to produce a nontrivial vector bundle on R. In order to do so, let us pick a Zariski open cover of $\operatorname{Spec}\mathrm{R}$; so we give two elements of R, g_1, g_2 such that the ideal

$$(g_1, g_2) = \mathrm{R}.$$

Consider $g_1 = y - 1, g_2 = y + 1$. In this case, we have that

$$g_1 - g_2 = y - 1 - (y + 1) = -2$$

so that indeed $(g_1, g_2) = \mathrm{R}$.

Consider the ideal

$$\mathcal{L} = (x, y + 1).$$

Geometrically, this coincides with the point $(0, -1) \in \mathrm{X}(k)$. Now, I claim that $\mathcal{L}$ is free on the charts $\operatorname{Spec}\mathrm{R}_{g_1}, \operatorname{Spec}\mathrm{R}_{g_2}$. Indeed

$$\mathcal{L}|_{\mathrm{R}_{g_2}} = (x, y+1)[\tfrac{1}{y+1}] = (y + 1),$$

while

$$\mathcal{L}|_{\mathrm{R}_{g_1}} = (x, y+1)[\tfrac{1}{y-1}] = (x)$$

since

$$x^3 = 1 - y^2 = (1 - y)(1 + y) \Rightarrow \tfrac{x^3}{1-y} = 1 + y.$$

Therefore we see that $\mathcal{L}$ is a locally free sheaf.

13.1. **An attempted definition.** For this discussion, we suppose that we are working in Sch_k where k is a field. Our goal is to define $\mathbf{P}^n_k$, the projective space over k. We have an idea of what this is: its k-points should be should be the set of lines in the $n+1$-dimensional vector space k^{n+1} or, equivalently,

$$\mathbf{P}^n_k(k) = (k^{n+1} \smallsetminus 0)/k^\times$$

where $k^\times$ act on the vector space k^{n+1} by scaling. In topology, we can make this set and declare a topology on it by giving it the quotient topology. This misses so much of the point of algebraic geometry of course.

Let us attempt to define the $\operatorname{Spec}\mathrm{A}$-points of $\mathbf{P}^n$ in a functorial way: given a k-morphism $\mathrm{A} \to \mathrm{B}$, we need a map

$$\mathbf{P}^n_k(\mathrm{A}) \to \mathbf{P}^n_k(\mathrm{B});$$

in particular given a map $\mathrm{A} \to k$ (thought of as $\operatorname{Spec} k \to \operatorname{Spec}\mathrm{A}$) we need to get

$$\mathbf{P}^n_k(\mathrm{A}) \to \mathbf{P}^n_k(k).$$

If we had rolled with something like $\mathbf{A} \smallsetminus 0/\mathrm{A}^\times$ then note that we might accidentally "pull points" back to zero: indeed say $\mathrm{A} = k[\epsilon]$ and we consider the map $k[\epsilon] \to k, \epsilon \mapsto 0$, then the class of ϵ will be sent to zero.

This says that the naive formulation of projective space is not quite right. Here's another thing we can do:

Definition 13.1.1. Define the prestack

$$(\mathbf{P}^n_k)^{\mathrm{naive}} : \mathrm{CAlg}_k \to \mathrm{Set}$$
$$\mathrm{A} \mapsto \{v \in \mathrm{A}^{n+1} : \forall x : \operatorname{Spec}\kappa \to \mathrm{A}, x^*(v) \in k^{n+1} \neq 0\}/\{v = aw : a \in \mathrm{A}^\times\}.$$

This is not such a bad definition but this does not satisfy Zariski descent.

13.2. **Line bundles as a solution.** Here is the problem: nontrivial line bundles exist on affine schemes! Indeed the equivalence class of v above is basically given by a map

$$\mathcal{O}_{\mathrm{Spec\,A}} \to \mathrm{A}^{\oplus n+1}.$$

Now, a line bundle is, in particular, a quasicoherent sheaf which is locally of the form $\mathcal{O}$ but *not* globally so. In other words, as A varies, there is no assurance that we can glue together the maps $\mathcal{O}_{\mathrm{A}} \to \mathrm{A}^{n+1}$ as the $\mathcal{O}$'s need not glue. To make this precise let us go right into the solution. First we need to define the above "nondegeneracy condition" in a more elegant manner.

13.3. **Nondegeneracy conditions and the definition of projective space.**

Lemma 13.3.1. *Let* A *be a ring and* M, N *be* A*-modules which are locally free and* $\varphi : \mathrm{M} \to \mathrm{N}$ *be a morphism. The following are equivalent:*

(1) *for each maximal ideal* $\mathfrak{m} \subset \mathrm{A}$*, the induced map*

$$\mathrm{M}/\mathfrak{m}\mathrm{M} \to \mathrm{N}/\mathfrak{m}\mathrm{N}$$

is injective,

(2) *the dual map*

$$\varphi^{\vee} : \mathrm{M}^{\vee} \to \mathrm{N}^{\vee}$$

surjects,

(3) *the image of* M *in* N *can be split off* M *as a summand,*

(4) *the map* φ *is injective with locally free cokernel.*

This lemma is an exercise.

Lemma 13.3.2. *Let* $\varphi : \mathcal{L} \to \mathcal{E}$ *be a morphism of vector bundles over* X *where* $\mathcal{L}$ *is a line bundle and* $\mathcal{E}$ *is of constant rank* n*. The following are equivalent:*

(1) *for each point* $\mathrm{Spec}\, k \to \mathrm{X}$*, the map*

$$x^*\varphi : x^*\mathcal{L} \to k^{\oplus n}$$

is injective,

(2) *the dual map*

$$\varphi^{\vee} : \mathcal{E}^{\vee} \to \mathcal{L}^{\vee}$$

is surjective,

(3) *the image of* $\mathcal{L}$ *under* φ *can be split off* $\mathcal{E}$ *as a summand,*

(4) *the map* φ *is injective and cokernel is locally free,*

(5) *there exists a Zariski open cover* $\{\mathrm{U} \hookrightarrow \mathrm{X}\}$ *such that on each* $j : \mathrm{U} \hookrightarrow \mathrm{X}$*,* $j^*\mathcal{E}$ *is trivial and*

$$\mathcal{L}|_{\mathrm{U}} \to \mathcal{E}|_{\mathrm{U}} \cong \mathcal{O}^{\oplus n} \to \mathcal{O}_{\mathrm{U}}$$

is an isomorphism.

Proof. We assume (1). Let us prove that for each $\mathrm{Spec}\, k \to \mathrm{X}$, there exists an affine open $\mathrm{U} \hookrightarrow \mathrm{X}$ over which $\mathcal{E}$ is trivial and such such that

$$\mathcal{L}|_{\mathrm{U}} \to \mathcal{E}|_{\mathrm{U}} \to \mathcal{O}_{\mathrm{U}}$$

is an isomorphism. Indeed, since the map $x^*\mathcal{L} \to x^*\mathcal{E} \cong k^{\oplus n}$ is an isomorphism, Nakayama's lemma furnishes an open affine containing x such that

$$\mathcal{L}|_{\mathrm{U}} \cong \mathcal{O}_{\mathrm{U}}$$

is an isomorphism; we can of course arrange this map compatibly with the trivialization of $\mathcal{E}$ by letting U be small enough. □

Any morphism $\mathcal{L} \to \mathcal{E}$ of the above form will be called a **bundle injection**. Finally:

Construction 13.3.3. Let $n \geqslant 0$. We define

$$\mathbf{P}^n : \mathrm{CAlg} \to \mathrm{Set}$$

as sending A to

$$\{(\mathcal{L}, \varphi) : \varphi : \mathcal{L} \to \mathrm{A}^{\oplus n} \text{ a bundle injection}\}$$

Theorem 13.3.4. *The prestack* $\mathbf{P}^n$ *is a scheme.*

14. Lecture 12: vector bundles, affine morphisms and projective bundles

15. Lecture 13: quasicompact and quasiseparated schemes

16. Lecture 14: more geometric properties

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