ON DISTRIBUTIVITY IN HIGHER ALGEBRA I: THE UNIVERSAL PROPERTY OF BISPANS

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Abstract. Structures where we have both a contravariant (pullback) and a covariant (pushforward) functoriality that satisfy base change can be encoded by functors out of (∞-)categories of spans (or correspondences). In this paper we study the more complicated setup where we have two pushforwards (an "additive" and a "multiplicative" one), satisfying a distributivity relation. Such structures can be described in terms of bispans (or polynomial diagrams). We show that there exist $(\infty, 2)$-categories of bispans, characterized by a universal property: they corepresent functors out of ∞-categories of spans where the pullbacks have left adjoints and certain canonical 2-morphisms (encoding base change and distributivity) are invertible. This gives a universal way to obtain functors from bispans, which amounts to upgrading "monoid-like" structures to "ring-like" ones. For example, symmetric monoidal ∞-categories can be described as product-preserving functors from spans of finite sets, and if the tensor product is compatible with finite coproducts our universal property gives the canonical semiring structure using the coproduct and tensor product. More interestingly, we encode the additive and multiplicative transfers on equivariant spectra as a functor from bispans in finite G-sets, extend the norms for finite étale maps in motivic spectra to a functor from certain bispans in schemes, and make $\mathrm{Perf}(X)$ for X spectral Deligne–Mumford stack a functor of bispans using a multiplicative pushforward for finite étale maps in addition to the usual pullback and pushforward maps.

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1. Introduction

This paper is the first part of a project aimed at better understanding certain sophisticated ring-like structures that occur in "homotopical mathematics". By this we mean not just the theory of E_∞-rings, where additions and multiplications are indexed over finite sets, but also more exotic structures occuring in equivariant and motivic homotopy theory where operations can be indexed over finite G-sets and finite étale morphisms, respectively. Such structures are also relevant to derived algebraic geometry and algebraic K-theory.

In the present paper we construct equivariant and motivic versions of the canonical semiring structure on a symmetric monoidal ∞-category whose tensor product commutes with finite coproducts.

In the G-equivariant case this structure encodes the compatibility of additive and multiplicative transfers (or norms) along maps of finite G-sets. In the case of genuine G-spectra such multiplicative transfers were defined by Hill, Hopkins, and Ravenel [HHR16] (extending a construction on the level of cohomology groups due to Greenlees–May [GM97, Boh14]) and played a key role in their solution of the Kervaire invariant one problem; more recently, they have been considered as the defining structure of an equivariant symmetric monoidal ∞-category in ongoing work of Barwick, Dotto, Glasman, Nardin, and Shah [BDG+16].

In the motivic version, we have multiplicative transfers along finite étale morphisms and additive transfers along smooth morphisms of schemes. Such multiplicative transfers were constructed for motivic spectra (and in a number of related examples) by Bachmann and Hoyois [BaHo18]. These generalize, among other constructions, Fulton–Macpherson's norms on Chow groups [FM87] and Joukhovitski's norms on K_0 [Jou00].

We will also show that the ∞-categories $\mathrm{Perf}(X)$ of perfect quasicoherent sheaves on a spectral Deligne–Mumford stack X have a similar structure given by a multiplicative pushforward for finite étale maps in addition to the usual pushforward and pullback functors. In all these cases we will obtain the canonical "semiring" structures using a universal property of $(\infty, 2)$-categories of *bispans*, which is the main result of this paper.

1.1. Spans and commutative monoids.

Spans and commutative monoids. Before we explain what we mean by bispans, it is helpful to first recall the relation between commutative monoids and spans: if $\mathbb{F}$ denotes the category of finite sets, then we can define a (2,1)-category $\mathrm{Span}(\mathbb{F})$ where

- objects are finite sets,
- morphisms from I to J are *spans* (or correspondences)

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ I & & J, \end{tikzcd}$$

- composition is given by pullback: the composite

$$\left(\begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ J & & K, \end{tikzcd} \right) \circ \left(\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ I & & J, \end{tikzcd} \right)$$

is the outer span in the diagram

$$\begin{tikzcd}
 & & S \times_K T \arrow[dl] \arrow[dr] & & \\
 & S \arrow[dl] \arrow[dr] & & T \arrow[dl] \arrow[dr] & \\
I & & J & & K,
\end{tikzcd}$$

- 2-morphisms are isomorphisms of spans.

If M is a commutative monoid in Set, we can use the monoid structure to define a functor

$$\mathrm{Span}(\mathbb{F}) \to \mathrm{Set}$$

which takes $I \in \mathbb{F}$ to $M^I := \prod_{i\in I} M$ and a span $I \xleftarrow{f} S \xrightarrow{g} J$ to the composite $g_\otimes f^*$ where $f^*\colon M^I \to M^S$ is given by composition with f (so $f^*\phi(s) = \phi(fs)$) and $g_\otimes$ is defined using the product on M by

$$g_\otimes(\phi)(j) = \prod_{s\in g^{-1}(j)} \phi(s).$$

This is compatible with composition of spans, since a pullback square gives a canonical isomorphism of fibres and we have $(gg')_\otimes = g_\otimes g'_\otimes$ as the multiplication is associative.

It can be shown that if $\mathbf{C}$ is any category with finite products, every functor $\Phi\colon \mathrm{Span}(\mathbb{F}) \to \mathbf{C}$ such that $\Phi(I) \cong \Phi(*)^{\times |I|}$ via the canonical maps arises in this way from a commutative monoid in $\mathbf{C}$. More precisely, we can identify commutative monoids in $\mathbf{C}$ with product-preserving functors $\mathrm{Span}(\mathbb{F}) \to \mathbf{C}$. (In other words, (the homotopy category of) $\mathrm{Span}(\mathbb{F})$ is precisely the *Lawvere theory* for commutative monoids.) This is also true homotopically:

Theorem 1.1.1. *Let $\mathcal{C}$ be an ∞-category with finite products. There is a natural equivalence of ∞-categories between commutative monoids in $\mathcal{C}$ and product-preserving functors* $\mathrm{Span}(\mathbb{F}) \to \mathcal{C}$.

The earliest proof of this seems to be the in thesis of Cranch [Cra10, Cra11]; other proofs (as special cases of different generalizations) are due to Bachmann–Hoyois [BaHo18, Appendix C] and Glasman [Gla17, Appendix A]. In addition, it appears in Harpaz [Har20, Section 5.2] as the bottom case of his theory of m-commutative monoids.

1.2. **Bispans and commutative semirings.** We can ask for a similar description for commutative semirings. In this case, we have two operations — addition and multiplication — so we want a (2,1)-category $\mathrm{Bispan}(\mathbb{F})$ whose objects are again finite sets, with a morphism from I to J given by a *bispan* (or polynomial diagram)

$$\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[dl, "p"'] & Y \arrow[dr, "q"] & \\
I & & & J.
\end{tikzcd} \tag{1}$$

If R is a commutative semiring in Set, we want a functor

$$\mathrm{Bispan}(\mathbb{F}) \to \mathrm{Set}$$

that takes a set I to R^I and the bispan (1) to $q_\oplus f_\otimes p^*$ where

- $p^*\colon R^I \to R^X$ is defined by composing with p,

$$p^*(\phi)(x) = \phi(px),$$

- $f_{\otimes} \colon R^X \to R^Y$ is defined by multiplying in R fibrewise,

$$f_{\otimes}(\phi)(y) = \prod_{x \in f^{-1}(y)} \phi(x),$$

- $q_{\oplus}$ is defined by adding in R fibrewise,

$$q_{\oplus}(\phi)(j) = \sum_{y \in q^{-1}(j)} \phi(y).$$

The question is then whether there is a way to define composition of bispans so that this gives a functor. Given a pullback square

$$\begin{tikzcd} I' \arrow[r, "g"] \arrow[d, "i"'] & J' \arrow[d, "j"] \\ I \arrow[r, "f"] & J \end{tikzcd} \tag{2}$$

in $\mathbb{F}$, we have identities $g_{\otimes} i^* = j^* f_{\otimes}$ and $g_{\oplus} i^* = j^* f_{\oplus}$ as before, but now we also need to deal with compositions of the form $v_{\otimes} u_{\oplus}$ for $u \colon I \to J$ and $v \colon J \to K$. Using the distributivity of addition over multiplication, for $\phi \colon I \to R$ and $k \in K$ we can write

$$v_{\otimes} u_{\oplus}(\phi)(k) = \prod_{j \in J_k} \sum_{i \in I_j} \phi(i) = \sum_{(i_j) \in \prod_{j \in J_k} I_j} \prod_{t \in J_k} \phi(i_t). \tag{3}$$

We can interpet this in terms of a *distributivity diagram* in $\mathbb{F}$: if we let $h \colon X \to K$ be the family of sets $X_k = \prod_{j \in J_k} I_j$ (so that $h = v_* u$ where v_* is the right adjoint to pullback along v), then the pullback $v^* X$ has a canonical map to I over J: on the fibre $(v^* X)_j$, which is the product $\prod_{j' \in J_{v(j)}} I_{j'}$, we take the projection to the factor I_j. This gives a commutative diagram

$$\begin{tikzcd} & v^*X \arrow[r, "\tilde{v}"] \arrow[dl, "\epsilon"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "h"] \\ I \arrow[r, "u"'] & J \arrow[r, "v"] & K \end{tikzcd} \tag{4}$$

where the square is cartesian, and we can rewrite the distributivity relation (3) as

$$v_{\otimes} u_{\oplus} = h_{\oplus} \tilde{v}_{\otimes} \epsilon^*.$$

This means we will get a functor $\mathrm{Bispan}(\mathbb{F}) \to \mathrm{Set}$ from the commutative semiring R if we define the composition of two bispans

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J,$$

$$J \xleftarrow{u} F \xrightarrow{q} C \xrightarrow{v} K,$$

as the outer bispan in the following diagram:

$$
\begin{tikzcd}
 & & & G \arrow[rr, "p''"] \arrow[rrr, bend left, "w"] \arrow[dl, "\epsilon'"] \arrow[dddlll, bend right, "r"'] & & X \arrow[r, "\tilde{q}"] \arrow[dl, "\epsilon"] \arrow[dd, "q^*q_*\pi"] & D \arrow[dd, "q_*\pi"] \arrow[dddr, bend left, "x"] & \\
 & & Y \arrow[rr, "p'"] \arrow[dl, "u''"] & & B \times_J F \arrow[dl, "u'"] \arrow[dr, "\pi"] & & & \\
 & E \arrow[rr, "p"] \arrow[dl, "s"] & & B \arrow[dr, "t"] & & F \arrow[dl, "u"] \arrow[r, "q"] & C \arrow[dr, "v"] & \\
I & & & & J & & & K.
\end{tikzcd}
\tag{5}
$$

Here we have used a distributivity diagram for q and the pullback π.

An explicit construction of a (2,1)-category $\mathrm{Bispan}(\mathbb{F})$ with this composition is given in the thesis of Cranch [Cra10], where it is also proved that this has products (given by the disjoint union of sets), so that we can define commutative semirings in $\mathcal{S}$ as functors $\mathrm{Bispan}(\mathbb{F}) \to \mathcal{S}$ that preserve finite products. Alternatively, one can relate bispans of finite sets to *polynomial functors*, which gives an easier definition of $\mathrm{Bispan}(\mathbb{F})$ (as the complicated composition law (5) corresponds to the ordinary composition of such functors); this approach was carried out by Gambino–Kock [GK13], who also show that the homotopy category of $\mathrm{Bispan}(\mathbb{F})$ is the Lawvere theory for commutative semirings, so that commutative semirings in an ordinary category $\mathbf{C}$ with finite products are equivalent to product-preserving functors

$$\mathrm{Bispan}(\mathbb{F}) \to \mathbf{C}.$$

We expect that the homotopical analogue of this statement[1] is also true, but this has not yet been proved.

1.3. **The universal property of spans.** If $\mathcal{C}$ is a symmetric monoidal ∞-category such that $\mathcal{C}$ has finite coproducts and the tensor product preserves these in each variable, then we expect that $\mathcal{C}$ has a canonical semiring structure in Cat_∞ with multiplication and addition given by the tensor product and coproduct, respectively. This follows[2] from the universal property of an $(\infty, 2)$-category of bispans in $\mathbb{F}$, which is a special case of our main result. Before we state this, it is convenient to first recall the simpler universal property of the $(\infty, 2)$-category $\mathrm{SPAN}(\mathbb{F})$ of spans in $\mathbb{F}$, which can be used to prove that an ∞-category with finite products has a canonical symmetric monoidal structure.

Here $\mathrm{SPAN}(\mathbb{F})$ has finite sets as objects, spans as morphisms, and morphisms of spans as 2-morphisms, i.e. 2-morphisms are commutative diagrams

$$
\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] \arrow[dd] & \\
I & & J \\
 & Y. \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Suppose $\mathcal{X}$ is an $(\infty, 2)$-category. A functor $\Phi\colon \mathbb{F}^{\mathrm{op}} \to \mathcal{X}$ is called *right adjointable* if for every morphism $f\colon I \to J$ in $\mathbb{F}$, the morphism $f^* := \Phi(f)\colon \Phi(J) \to \Phi(I)$

[1] Specifically, the definition of commutative semirings in an ∞-category with finite products in terms of $\mathrm{Bispan}(\mathbb{F})$ should be equivalent to that of Gepner–Groth–Nikolaus [GGN15].

[2] This semiring structure is also constructed in [GGN15] by a different method.

in $\mathcal{X}$ has a right adjoint f_*, and for every pullback square (2) in $\mathbb{F}$, the canonical (Beck–Chevalley or mate) transformation

$$j^* f_* \to g_* i^*$$

is an equivalence.

Theorem 1.3.1. *Restricting along the inclusion* $\mathbb{F}^{\mathrm{op}} \to \mathrm{SPAN}(\mathbb{F})$ *(of the subcategory containing only the backwards maps and no non-trivial 2-morphisms) gives a natural equivalence between functors* $\mathrm{SPAN}(\mathbb{F}) \to \mathcal{X}$ *and right adjointable functors* $\mathbb{F}^{\mathrm{op}} \to \mathcal{X}$.

This is a special case of a recent result of Macpherson [Mac20], which we review in more generality below in §2.2. Another proof is sketched in the book of Gaitsgory and Rozenblyum [GR17] where this universal property is used to encode a "six-functor formalism" for various categories of quasicoherent sheaves on derived schemes. Lastly, we note that the analogous result for ordinary 2-categories seems to have been first proved by Hermida [Her00, Theorem A.2].

1.4. **The universal property of bispans.** We now want to consider a 2-category $\mathrm{BISPAN}(\mathbb{F})$ whose objects are finite sets, with morphisms given by bispans and 2-morphisms by commutative diagrams of the form

$$\begin{tikzcd} & E \arrow[r] \arrow[dl] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d] \arrow[dr] & \\ I & & & J \\ & E' \arrow[r] \arrow[ul] & B' \arrow[ur] & \end{tikzcd} \tag{6}$$

where the middle square is cartesian. If we look at the subcategory where the morphisms are bispans whose rightmost leg is invertible and with no non-trivial 2-morphisms, we get an inclusion $\mathrm{Span}(\mathbb{F}) \to \mathrm{BISPAN}(\mathbb{F})$. A special case of our main result gives a universal property of $\mathrm{BISPAN}(\mathbb{F})$ in terms of this subcategory:

Theorem 1.4.1. *Let* $\mathcal{X}$ *be an* $(\infty, 2)$*-category. Restricting along the inclusion* $\mathrm{Span}(\mathbb{F}) \to \mathrm{BISPAN}(\mathbb{F})$ *gives an equivalence between functors* $\mathrm{BISPAN}(\mathbb{F}) \to \mathcal{X}$ *and distributive functors* $\mathrm{Span}(\mathbb{F}) \to \mathcal{X}$.

Here a functor $\Phi\colon \mathrm{Span}(\mathbb{F}) \to \mathcal{X}$ is *distributive* if

- for every morphism $f\colon I \to J$ in $\mathbb{F}$, the morphism $f^{\circledast} := \Phi(J \xleftarrow{f} I = I)$ in $\mathcal{X}$ has a left adjoint $f_{\oplus}$,
- for every pullback square (2) in $\mathbb{F}$, the Beck–Chevalley transformation $g_{\oplus} i^{\circledast} \to j^{\circledast} f_{\oplus}$ is an equivalence,
- for every distributivity diagram (4), the *distributivity transformation*

$$h_{\oplus} \tilde{v}_{\otimes} \epsilon^{\circledast} \to v_{\otimes} u_{\oplus},$$

  which is defined as a certain composite of units and counits, is an equivalence in $\mathcal{X}$.

Note that the only property of $\mathbb{F}$ we have used in the definition of distributive functors is the existence of distributivity diagrams. These exist in any locally cartesian closed ∞-category, and more generally we can consider triples $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ consisting of an ∞-category $\mathcal{C}$ with a pair of subcategories $\mathcal{C}_F$ and $\mathcal{C}_L$ such that

- pullbacks along morphisms in $\mathcal{C}_F$ and $\mathcal{C}_L$ exist in $\mathcal{C}$, and both subcategories are preserved under base change,
- there exist suitable distributivity diagrams in $\mathcal{C}$ for any composable pair of morphisms $l\colon x \to y$ in $\mathcal{C}_L$, $f\colon y \to z$ in $\mathcal{C}_F$.

We can then generalize the notion of distributive functors above to that of *L-distributive functors* $\mathrm{Span}_F(\mathcal{C}) \to \mathcal{X}$, where $\mathrm{Span}_F(\mathcal{C})$ is the ∞-category of spans in $\mathcal{C}$ whose forward legs are required to lie in $\mathcal{C}_F$. Our main result in this paper is then the following generalization of Theorem 1.4.1:

Theorem 1.4.2. *For $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ as above, there exists an $(\infty, 2)$-category*

$$\mathrm{BISPAN}_{F,L}(\mathcal{C})$$

such that

- *objects are objects of $\mathcal{C}$,*
- *morphisms are bispans*
$$x \xleftarrow{p} e \xrightarrow{f} b \xrightarrow{l} y$$
where f is in $\mathcal{C}_F$ and l is in $\mathcal{C}_L$,
- *2-morphisms are diagrams of the form (6),*
- *morphisms compose as in (5).*

The $(\infty, 2)$-category $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ has the universal property that restricting to the subcategory $\mathrm{Span}_F(\mathcal{C})$ gives for any $(\infty, 2)$-category $\mathcal{X}$ an equivalence between functors $\mathrm{BISPAN}_{F,L}(\mathcal{C}) \to \mathcal{X}$ and L-distributive functors $\mathrm{Span}_F(\mathcal{C}) \to \mathcal{X}$.

The analogue of this result for ordinary 2-categories (at least in the case where $\mathcal{C} = \mathcal{C}_F = \mathcal{C}_L$) is due to Walker [Wal19].

1.5. **Equivariant and algebro-geometric bispans.** We will now look briefly at some examples of Theorem 1.4.2 beyond the case of finite sets, coming from equivariant and motivic homotopy theory and derived algebraic geometry. These examples are discussed in more detail in §3.

Let us first consider the equivariant setting, over a finite group G. In all of our discussion above it is straightforward to replace the category $\mathbb{F}$ of finite sets with the category $\mathbb{F}_G$ of *finite G-sets*. The analogue of a commutative monoid is then a functor

$$M \colon \mathrm{Span}(\mathbb{F}_G) \to \mathrm{Set}$$

that preserves products. This is (essentially[3]) the same thing as a *Mackey functor* [Dre71], an algebraic structure where for a subgroup $H \subseteq G$ we have restrictions $M^G \to M^H$ and transfers $M^H \to M^G$ satisfying a base change property that can be interpreted in terms of double cosets. Mackey functors play an important role in group theory, and every genuine G-spectrum E has an underlying Mackey functor $\pi_0 E$.

Similarly, the G-analogue of a commutative semiring is a product-preserving functor $\mathrm{Bispan}(\mathbb{F}_G) \to \mathrm{Set}$, which is essentially[4] a *Tambara functor* [Tam93, Str12, BlHi18]. This is a structure that has both an "additive" and a "multiplicative" transfer, satisfying a distributivity relation. If E is a genuine commutative ring G-spectrum, then $\pi_0 E$ has the structure of a Tambara functor [Bru07].

If we replace the category of sets with the ∞-category of spaces, a theorem of Nardin[5] [Nar16, Corollary A.4.1] shows that connective G-spectra are equivalently

[3]Mackey functors are usually viewed as taking values in Ab rather than Set; since the functor induces commutative monoid structures on its values, this amounts to asking for these monoid structures to be grouplike. The relation between Mackey functors in Ab and Set is thus analogous to that between abelian groups and commutative monoids.

[4]Again, the usual notion of a Tambara functor takes values in Ab, which gives the equivariant version of a commutative ring rather than a semiring.

[5]Building on the description of G-spectra as "spectral Mackey functors", originally due to Guillou and May [GM17].

product-preserving functors $\mathrm{Span}(\mathbb{F}_G) \to \mathcal{S}$ that are grouplike, generalizing the classical description of connective spectra as grouplike commutative monoids in $\mathcal{S}$.[6]

The analogue for ring spectra is also expected to hold: connective genuine commutative G-ring spectra should be equivalent to product-preserving functors $\mathrm{Bispan}(\mathbb{F}_G) \to \mathcal{S}$.

Now we turn to the "categorified" versions of these structures: For H a subgroup of G, the (additive) transfer from H-spectra to G-spectra is classical[7], but the multiplicative transfer or *norm* was only introduced fairly recently by Hill–Hopkins–Ravenel as part of the foundational setup for [HHR16]. This inspired a plethora of work on equivariant symmetric monoidal structures [HH16, GMMO20, Rub17] and its relation to equivariant homotopy-coherent commutativity (in particular [BlHi15] and subsequent work on N_∞-operads), culminating from our point of view in the approach of Barwick, Dotto, Glasman, Nardin, and Shah [BDG+16], where a G-symmetric monoidal ∞-category can be viewed as a product-preserving functor

$$\mathrm{Span}(\mathbb{F}_G) \to \mathrm{Cat}_\infty,$$

i.e. a "categorified Mackey functor".

If the contravariant (restriction) functors have left adjoints that satisfy base change and distributivity, Theorem 1.4.2 allows us to upgrade such G-symmetric monoidal structures to functors from $\mathrm{BISPAN}(\mathbb{F}_G)$, which encodes the distributive compatibility of multiplicative and additive transfers. We will see that this applies in particular to genuine G-spectra, giving a "categorified Tambara functor" structure on G-spectra.

Next, we look at the motivic setting, where it is more instructive to first work in the categorified context. By this we mean Ayoub's construction of a functor from schemes to categories $X \mapsto \mathrm{SH}(X)$ which satisfies a full six functors formalism [Ayo07], vastly expanding Voevodsky's notes in [Voe99]; we also refer the reader to the book of Cisinski and Déglise [CD19] for another exposition, [Hoy17] for an ∞-categorical enhancement of this construction in the more general motivic-equivariant setting, as well as the more recent [DG20] for a universal property of this construction. Here $\mathrm{SH}(X)$ denotes the ∞-category of motivic spectra over a scheme X.

In this context, given a smooth morphism of schemes $f\colon X \to Y$ over a base S, the pullback functor $f^*\colon \mathrm{SH}(Y) \to \mathrm{SH}(X)$ admits a left adjoint, $f_\sharp\colon \mathrm{SH}(X) \to \mathrm{SH}(Y)$. This is a categorified version of the *additive pushforward*: if f is the fold map $\nabla\colon Y^{\amalg I} \to Y$, then $\nabla_\sharp$ computes the I-indexed direct sum. The compatibility of $f_\sharp$ with pullbacks yields a functor

$$\mathrm{SH}\colon \mathrm{Span}_{\mathrm{sm}}(\mathrm{Sch}_S) \to \mathrm{Cat}_\infty. \tag{7}$$

An important additional functoriality of SH was recently discovered by Bachmann and Hoyois in [BaHo18]: given a finite étale morphism $f\colon X \to Y$ we have the *multiplicative pushforward* or *norm* $f_\otimes\colon \mathrm{SH}(X) \to \mathrm{SH}(Y)$ which, in the case when f is the fold map, computes the I-indexed tensor product. This led to the correct notion of a coherent multiplicative structure in motivic homotopy theory — a *normed motivic spectrum* — which is a cartesian section of the unstraightening of (7) and its variants (obtained, say, by restricting to smooth S-schemes).

The motivic bispan category should then encode an additive pushforward along smooth morphisms and a multiplicative pushforward along finite étale morphisms. For technical reasons (due to the non-existence of Weil restriction of schemes in general), for our motivic bispan categories we either have to restrict to morphisms

[6] This can be seen as an ∞-categorical version of more classical descriptions of equivariant infinite loop spaces, cf. [Shi89, Ost16, MMO17, GMMO19].

[7] See e.g. [LMMS86, §II.4]

between schemes that are smooth and quasiprojective or work with algebraic spaces. Thus we consider 2-categories of the form $\mathrm{BISPAN}_{\text{fét,sm}}(\mathrm{AlgSpc}_S)$ where AlgSpc_S means the category of algebraic spaces over S, and we promote SH to a functor

$$\mathrm{SH}\colon \mathrm{BISPAN}_{\text{fét,sm}}(\mathrm{AlgSpc}_S) \to \mathrm{Cat}_\infty; \tag{8}$$

see Theorem 3.5.9.

The decategorification of the above structure has been studied by Bachmann in [Bac18]. Working over a field, and restricting to a category of bispans between smooth schemes, $\mathrm{Bispan}_{\text{fét,sm}}(\mathrm{Sm}_k)$, Bachmann proved that the structure of a normed algebra in the abelian category of homotopy modules (the heart of the so-called homotopy t-structure on motivic spectra) is encoded by certain functors out of this bispan category to abelian groups (appropriately christened *motivic Tambara functors*), at least after inverting the exponential characteristic of k.

We also note that there is a discrepancy with the classical and finite-equivariant story: finite étale transfers are *a priori* not sufficient to encode the structure of a motivic spectrum. Instead, the correct kind of transfers are framed transfers in the sense of [EHK+17]. In particular, the category of framed correspondences, where the backward maps encode framed transfers, is manifestly an ∞-category. In other words, the additive and multiplicative transfers are rather different in the motivic story; for example we do not know if the space of units of a normed motivic spectrum has framed transfers (see [BaHo18, Section 1.5] for a discussion). For us, this means that a more robust theory of bispans in the motivic setting which encodes framed transfers is open for future investigations.

Finally, we consider an example in the context of derived algebraic geometry: If $\mathrm{Perf}(X)$ denotes the ∞-category of perfect quasicoherent sheaves on a spectral Deligne–Mumford stack, Barwick [Bar17] has shown that the pullback and pushforward functors extend to a functor

$$\mathrm{Perf}\colon \mathrm{Span}_{\mathcal{FP}}(\mathrm{SpDM}) \to \mathrm{Cat}_\infty$$

where SpDM is the ∞-category of spectral Deligne–Mumford stacks and $\mathcal{FP}$ is a certain class of morphisms (for which pushforwards preserve perfect objects and base change is satisfied). We promote this to a functor of $(\infty,2)$-categories

$$\mathrm{BISPAN}_{\text{fét},\mathcal{FP}'}(\mathrm{SpDM})^{2\text{-op}} \to \mathrm{CAT}_\infty,$$

using a multiplicative pushforward functor for finite étale maps (which exists by results of Bachmann–Hoyois [BaHo18]), where $\mathcal{FP}'$ is a certain subclass of $\mathcal{FP}$ for which Weil restrictions exist.

1.6. **Notation.** This paper is written in the language of ∞-categories. We use the following reasonably standard notation:

- $\mathcal{S}$ is the ∞-category of spaces, i.e. ∞-groupoids.
- Cat_∞ is the ∞-category of ∞-categories.
- CAT_∞ is the $(\infty,2)$-category of ∞-categories.
- $\mathrm{Cat}_{(\infty,2)}$ is the ∞-category of $(\infty,2)$-categories.
- If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories or $(\infty,2)$-categories, we write $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ for the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$.
- If $\mathcal{C}$ and $\mathcal{D}$ are $(\infty,2)$-categories, we write $\mathrm{FUN}(\mathcal{C},\mathcal{D})$ for the $(\infty,2)$-category of functors from $\mathcal{C}$ to $\mathcal{D}$.
- We write $(-)^{(1)}\colon \mathrm{Cat}_{(\infty,2)} \to \mathrm{Cat}_\infty$ for the functor taking an $(\infty,2)$-category to its underlying ∞-category. (Thus $(-)^{(1)}$ is right adjoint to the inclusion of Cat_∞ into $\mathrm{Cat}_{(\infty,2)}$.)
- We write $(-)^{\simeq}$ for the functors $\mathrm{Cat}_\infty \to \mathcal{S}$ and $\mathrm{Cat}_{(\infty,2)} \to \mathcal{S}$ taking an ∞-category or $(\infty,2)$-category to its underlying ∞-groupoid.

- If $\mathcal{C}$ is an ∞-category and x and y are objects of $\mathcal{C}$, we write $\mathrm{Map}_{\mathcal{C}}(x, y)$ for the space of maps from x to y in $\mathcal{C}$.
- If $\mathcal{C}$ is an $(\infty, 2)$-category and x and y are objects of $\mathcal{C}$, we write $\mathrm{MAP}_{\mathcal{C}}(x, y)$ for the ∞-category of maps from x to y in $\mathcal{C}$.
- If $\mathcal{X}$ is an $(\infty, 2)$-category we write $\mathcal{X}^{1\text{-op}}$ for the $(\infty, 2)$-category obtained by reversing the morphisms in $\mathcal{X}$ and $\mathcal{X}^{2\text{-op}}$ for that obtained by reversing the 2-morphisms.

We also adopt the following standard notation for functors between slices of an ∞-category $\mathcal{C}$:

- If $f\colon x \to y$ is a morphism in $\mathcal{C}$, we have a functor
$$f_!\colon \mathcal{C}_{/x} \to \mathcal{C}_{/y}$$
such that $f_!(t \to x) = t \to x \to y$, i.e. given by composition with f.
- If pullbacks along f exist in $\mathcal{C}$ then $f_!$ has a right adjoint
$$f^*\colon \mathcal{C}_{/y} \to \mathcal{C}_{/x}.$$
- If f^* has a further right adjoint, this will be denoted by
$$f_*\colon \mathcal{C}_{/x} \to \mathcal{C}_{/y}.$$
(This right adjoint exists for all f precisely when $\mathcal{C}$ is locally cartesian closed, for example if $\mathcal{C}$ is an ∞-topos.)

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2. Bispans and distributive functors

2.1. **$(\infty, 2)$-categories and adjunctions.** Throughout this paper we work with $(\infty, 2)$-categories, and in this section we review some basic results we use from the theory of $(\infty, 2)$-categories, particularly regarding adjunctions. There are several equivalent ways to define these objects and their homotopy theory, including Rezk's Θ_n-spaces [Rez10] and Barwick's 2-fold Segal spaces [Bar05]. We can also view $(\infty, 2)$-categories as ∞-categories enriched in ∞-categories, which can be rigidified to categories strictly enriched in quasicategories; the latter is the model used in the papers of Riehl and Verity. We will not review the details of any of these constructions here, as we do not need to refer to any particular model of $(\infty, 2)$-categories in this paper.

We will, however, use the Yoneda lemma for $(\infty, 2)$-categories, which is a special case of Hinich's Yoneda lemma for enriched ∞-categories [Hin20]:

Theorem 2.1.1 (Hinich)**.** *For any $(\infty, 2)$-category $\mathcal{X}$ there exists a fully faithful functor of $(\infty, 2)$-categories*
$$\mathsf{y}_{\mathcal{X}}\colon \mathcal{X} \to \mathrm{FUN}(\mathcal{X}^{\mathrm{op}}, \mathrm{CAT}_\infty)$$
such that for any functor $\Phi\colon \mathcal{X}^{\mathrm{op}} \to \mathrm{CAT}_\infty$ there is a natural equivalence of ∞-categories
$$\Phi(d) \simeq \mathrm{MAP}_{\mathrm{FUN}(\mathcal{X}^{\mathrm{op}}, \mathrm{CAT}_\infty)}(\mathsf{y}_{\mathcal{X}}(d), \Phi).$$

Remark 2.1.2. Hinich's work does use a specific model for $(\infty, 2)$-categories, namely a certain definition of enriched ∞-categories specialized to enrichment in Cat_∞. Hinich's definition has been compared to the original one of Gepner–Haugseng [GH15] by Macpherson [Mac19], and for enrichment in Cat_∞ the latter is equivalent to complete 2-fold Segal spaces [Hau15], which in turn is known by work of Barwick and Schommer-Pries [BSP11] to be equivalent to most other approaches to $(\infty, 2)$-categories (including the complicial sets of Verity by recent work of Gagna–Harpaz–Lanari [GHL20]).

Recall that there is a *universal adjunction*; this is a 2-category ADJ with two objects $-$ and $+$ and generated by 1-morphisms $L\colon \Delta^1 = \{- \to +\} \to \mathrm{ADJ}$ and $R\colon \Delta^1 = \{+ \to -\} \to \mathrm{ADJ}$ such that L is left adjoint to R; see [RV16] for an explicit combinatorial definition of this 2-category. An adjunction in a 2-category can then equivalently be described as a functor from ADJ. This universal property also holds in $(\infty, 2)$-categories, where we can formulate it more precisely as follows:

Theorem 2.1.3 (Riehl–Verity)**.** *Given an $(\infty, 2)$-category $\mathfrak{X}$, the induced maps of spaces*

$$L^*,\, R^* : \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathrm{ADJ}, \mathfrak{X}) \to \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\Delta^1, \mathfrak{X}),$$

are both inclusions of components whose images are the subspaces

$$\mathrm{Map}^L_{\mathrm{Cat}_{(\infty,2)}}(\Delta^1, \mathfrak{X}),\, \mathrm{Map}^R_{\mathrm{Cat}_{(\infty,2)}}(\Delta^1, \mathfrak{X}) \;\subset\; \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\Delta^1, \mathfrak{X})$$

spanned by those functors that are left and right adjoints, respectively.

For details, see [RV16, Theorem 4.3.11 and 4.4.18]. See also [HNP19] for an alternative proof, using the cotangent complex of $(\infty, 2)$-categories.

We need to upgrade this to a statement about $(\infty, 2)$-categories rather than just ∞-groupoids. This follows from the next observation, which identifies the morphisms and 2-morphisms in the $(\infty, 2)$-category of adjunctions using some results from [Hau20]; to state this we need some terminology:

Definition 2.1.4. Let $\mathfrak{X}$ be an $(\infty, 2)$-category and consider a commutative square

$$\begin{tikzcd} x' \arrow[r, "g'"] \arrow[d, "\xi"] & y' \arrow[d, "\eta"] \\ x \arrow[r, "g"] & y \end{tikzcd}$$

in $\mathfrak{X}$. If g and g' are left adjoints, with corresponding right adjoints h and h', then we can use the units and counits of the adjunctions to define a *mate transformation* $\xi h' \to h\eta$ as the composite

$$\xi h' \to hg\xi h' \simeq h\eta g'h' \to h\eta.$$

We say the square is *right adjointable* if this mate transformation is an equivalence. Dually, if g and g' are right adjoints, with left adjoints f and f', we say the square is *left adjointable* if the mate transformation

$$f\eta \to f\eta g'f' \simeq fg\xi f' \to \xi f'$$

is an equivalence.

The next proposition states that for a natural transformation, being an adjoint means being an adjoint pointwise plus an adjointability condition on the naturality squares.

Proposition 2.1.5. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be $(\infty, 2)$-categories. A 1-morphism in the $(\infty, 2)$-category* $\mathrm{FUN}(\mathfrak{X}, \mathfrak{Y})$*, i.e. a natural transformation $\eta\colon F \to G$ of functors $F, G\colon \mathfrak{X} \to \mathfrak{Y}$, is a right (left) adjoint if and only if*

(1) *for every object $x \in \mathcal{X}$, the morphism $\eta_x \colon F(x) \to G(x)$ is a right (left) adjoint in $\mathcal{Y}$,*
(2) *for every morphism $f \colon x \to x'$ in $\mathcal{X}$, the commutative square*

$$\begin{tikzcd} F(x) \arrow[r, "\eta_x"] \arrow[d, "F(f)"] & G(x) \arrow[d, "G(f)"] \\ F(x') \arrow[r, "\eta_{x'}"] & G(x') \end{tikzcd}$$

is left (right) adjointable.

Proof. We consider the case of right adjoints; the left adjoint case can be proved similarly, and also follows by duality. Let $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})_{\mathrm{lax}}$ denote the $(\infty, 2)$-category of functors from $\mathcal{X}$ to $\mathcal{Y}$ with lax natural transformations as morphisms (see [Hau20, §3] for a precise definition). We can view the natural transformation η as a morphism in $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})_{\mathrm{lax}}$. By [Hau20, Theorem 4.6] it has a right adjoint here if and only if η_x has a right adjoint in $\mathcal{Y}$ for every $x \in \mathcal{X}$; this right adjoint is given on morphisms by taking mates. If η has a right adjoint ρ in $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})$ then by uniqueness this is also a right adjoint in $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})_{\mathrm{lax}}$; hence η must be given objectwise by left adjoints and the naturality squares of ρ are the corresponding mate squares — in particular, these mate squares must commute, so conditions (1) and (2) hold. Conversely, if these conditions hold for η then η has a right adjoint ρ in $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})_{\mathrm{lax}}$ and the lax naturality squares of ρ actually commute. By [Hau20, Corollary 3.17] this means that ρ is in the image of the canonical functor $\mathrm{FUN}(\mathcal{X}, \mathcal{Y}) \to \mathrm{FUN}(\mathcal{X}, \mathcal{Y})_{\mathrm{lax}}$; moreover, this functor is locally fully faithful so the unit and counit of the adjunction also lie in $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})$, and so η has a right adjoint in $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})$, as required. □

Notation 2.1.6. Let $\mathcal{X}$ be an $(\infty, 2)$-category. We write $\mathrm{FUN}(\Delta^1, \mathcal{X})^{\mathrm{ladj}}$ for the locally full sub-$(\infty, 2)$-category of $\mathrm{FUN}(\Delta^1, \mathcal{X})$ whose objects are the morphisms that are left adjoints and whose morphisms are the right adjointable squares. Similarly, we write $\mathrm{FUN}(\Delta^1, \mathcal{X})^{\mathrm{radj}}$ for the sub-$(\infty, 2)$-category of right adjoints and left adjointable squares.

Corollary 2.1.7. *The functors $L^*, R^* \colon \mathrm{FUN}(\mathrm{ADJ}, \mathcal{X}) \to \mathrm{FUN}(\Delta^1, \mathcal{X})$ identify the $(\infty, 2)$-category $\mathrm{FUN}(\mathrm{ADJ}, \mathcal{X})$ with the sub-$(\infty, 2)$-categories $\mathrm{FUN}(\Delta^1, \mathcal{X})^{\mathrm{ladj}}$ and $\mathrm{FUN}(\Delta^1, \mathcal{X})^{\mathrm{radj}}$, respectively.*

Proof. We consider the case of left adjoints; the proof for right adjoints is the same. For any $(\infty, 2)$-category $\mathcal{Y}$ we have a natural commutative square

$$\begin{tikzcd} \mathrm{Map}(\mathcal{Y}, \mathrm{FUN}(\mathrm{ADJ}, \mathcal{X})) \arrow[r, "\sim"] \arrow[d, "L^*"] & \mathrm{Map}(\mathrm{ADJ}, \mathrm{FUN}(\mathcal{Y}, \mathcal{X})) \arrow[d, "L^*"] \\ \mathrm{Map}(\mathcal{Y}, \mathrm{FUN}(\Delta^1, \mathcal{X})) \arrow[r, "\sim"] & \mathrm{Map}(\Delta^1, \mathrm{FUN}(\mathcal{Y}, \mathcal{X})). \end{tikzcd}$$

Here Theorem 2.1.3 implies that the right vertical map is a monomorphism of ∞-groupoids with image the components of $\mathrm{Map}(\Delta^1, \mathrm{FUN}(\mathcal{Y}, \mathcal{X}))$ that correspond to left adjoints in $\mathrm{FUN}(\mathcal{Y}, \mathcal{X})$. Using Proposition 2.1.5 we can identify these as precisely those in the image of the subspace $\mathrm{Map}(\mathcal{Y}, \mathrm{FUN}(\Delta^1, \mathcal{X})^{\mathrm{ladj}})$ under the bottom horizontal equivalence. By the Yoneda Lemma it follows that $L^* \colon \mathrm{FUN}(\mathrm{ADJ}, \mathcal{X}) \to \mathrm{FUN}(\Delta^1, \mathcal{X})^{\mathrm{ladj}}$ is an equivalence. □

2.2. **Spans and adjointable functors.** In this section we review the universal property of $(\infty, 2)$-categories of spans in terms of (right) adjointable functors.

Definition 2.2.1. A *span pair* $(\mathcal{C}, \mathcal{C}_F)$ consists of an ∞-category $\mathcal{C}$ together with a subcategory $\mathcal{C}_F$ (containing all objects) such that given morphisms $x \xrightarrow{f} y$ in $\mathcal{C}_F$

and $z \xrightarrow{g} y$ in $\mathcal{C}$, the pullback

$$\begin{tikzcd} x \times_y z \arrow[r, "f'"] \arrow[d, "g'"'] & z \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

exists in $\mathcal{C}$, and moreover f' is also in $\mathcal{C}_F$.

Given a spain pair $(\mathcal{C}, \mathcal{C}_F)$ we can, as in [Bar17, Section 3], define an ∞-category $\mathrm{Span}_F(\mathcal{C})$ whose objects are the objects of $\mathcal{C}$, with morphisms from x to y given by spans

$$\begin{tikzcd} & z \arrow[dl, "g"'] \arrow[dr, "f"] & \\ x & & y, \end{tikzcd}$$

where f is in $\mathcal{C}_F$; we compose spans by taking pullbacks. Following [Hau18, Section 5] we can upgrade this to an $(\infty, 2)$-category $\mathrm{SPAN}_F(\mathcal{C})$ whose 2-morphisms are morphisms of spans, i.e. diagrams

$$\begin{tikzcd} & z \arrow[dl] \arrow[dr] \arrow[dd] & \\ x & & y \\ & z' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

in $\mathcal{C}$, where $z \to z'$ can be any morphism in $\mathcal{C}$.

Warning 2.2.2. In [Hau18], the $(\infty, 2)$-category of spans in $\mathcal{C}$ was denoted $\mathrm{Span}_1^+(\mathcal{C})$, while $\mathrm{SPAN}_n(\mathcal{C})$ was used for an n-uple ∞-category of spans.

The $(\infty, 2)$-category $\mathrm{SPAN}_F(\mathcal{C})$ enjoys a universal property. To state this, we need some definitions:

Definition 2.2.3. Let $(\mathcal{C}, \mathcal{C}_F)$ be a span pair and $\mathcal{X}$ an $(\infty, 2)$-category. A functor $\Phi \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{X}$ is *right F-preadjointable* if for every morphism $f \colon x \to y$ in $\mathcal{C}_F$ the 1-morphism $f^{\circledast} := \Phi(f)$ in $\mathcal{X}$ has a right adjoint $f_{\circledast}$ in $\mathcal{X}$.

Remark 2.2.4. Suppose Φ is right F-preadjointable. Then given any cartesian square

$$\begin{tikzcd} x \times_y z \arrow[r, "f'"] \arrow[d, "g'"] & z \arrow[d, "g"] \\ x \arrow[r, "f"] & y, \end{tikzcd} \tag{9}$$

in $\mathcal{C}$ with f in $\mathcal{C}_F$, we have a 2-morphism

$$g^{\circledast} f_{\circledast} \to f'_{\circledast} g'^{\circledast} \tag{10}$$

(between 1-morphisms $\Phi(x) \to \Phi(z)$), called the *Beck–Chevalley transformation*. This is given by the mate transformation (see also [Lur17, Definition 4.7.4.13])

$$g^{\circledast} f_{\circledast} \to f'_{\circledast} f'^{\circledast} g^{\circledast} f_{\circledast} \simeq f'_{\circledast} (g f')^{\circledast} f_{\circledast} \simeq f'_{\circledast} (f g')^{\circledast} f_{\circledast} \simeq f'_{\circledast} g'^{\circledast} f^{\circledast} f_{\circledast} \to f'_{\circledast} g'^{\circledast},$$

where the first map uses the unit for the adjunction $f'^{\circledast} \dashv f'_{\circledast}$, the equivalences use the functoriality of Ψ and the equivalence $f g' \simeq g f'$ specified by the square, and the last map uses the counit for the adjunction $f^{\circledast} \dashv f_{\circledast}$.

Definition 2.2.5. We say that Φ is *right F-adjointable* if it is right F-preadjointable and the Beck–Chevalley transformation (10) is an equivalence for every pullback square (9) with f in $\mathcal{C}_F$.

Remark 2.2.6. In other words, Φ is right F-adjointable if for every cartesian square (9) with f in $\mathcal{C}_F$, the commutative square

$$\begin{tikzcd}
\Phi(y) \arrow[r, "f^{\circledast}"] \arrow[d, "g^{\circledast}"] & \Phi(x) \arrow[d, "g'^{\circledast}"] \\
\Phi(z) \arrow[r, "f'^{\circledast}"] & \Phi(x \times_y z)
\end{tikzcd}$$

in $\mathcal{X}$ is right adjointable.

Theorem 2.2.7 (Gaitsgory-Rozenblyum [GR17], Macpherson [Mac20])**.** *Let $(\mathcal{C}, \mathcal{C}_F)$ be a span pair, and let $\mathcal{X}$ be an $(\infty, 2)$-category. The inclusion of the backwards maps $\mathcal{C}^{\mathrm{op}} \to \mathrm{SPAN}_F(\mathcal{C})$ gives a monomorphism of ∞-groupoids*

$$\mathrm{Map}(\mathrm{SPAN}_F(\mathcal{C}), \mathcal{X}) \to \mathrm{Map}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$$

with image the components corresponding to the right F-adjointable functors.

Remark 2.2.8. In [GR17], Gaitsgory-Rozenblyum made use of the universal property of spans in order to encode the functoriality of various categories of coherent sheaves on derived schemes. They sketched a proof of Theorem 2.2.7 using a particular construction of $\mathrm{SPAN}_F(\mathcal{C})$. Macpherson has recently given an alternative, complete proof (which is model-independent up to Hinich's Yoneda lemma for ∞-categories [Hin20]), which is the main inspiration behind the techniques of this paper. Roughly speaking, Macpherson's approach is to first show there exists an $(\infty, 2)$-category that represents right adjointable functors and then use the universal property to prove that this representing object has the expected description in terms of spans. One way to show the representing $(\infty, 2)$-category exists is to use the presentability of $\mathrm{Cat}_{(\infty,2)}$; we include here a proof using this approach, as we will extend this later on when we consider bispans.

Proposition 2.2.9. *Let $(\mathcal{C}, \mathcal{C}_F)$ be a span pair, with $\mathcal{C}$ a small ∞-category. Then the functors*

$$\mathrm{Map}_{F\text{-rpreadj}}(\mathcal{C}^{\mathrm{op}}, –), \mathrm{Map}_{F\text{-radj}}(\mathcal{C}^{\mathrm{op}}, –)\colon \mathrm{Cat}_{(\infty,2)} \to \mathcal{S}$$

taking an $(\infty, 2)$-category $\mathcal{X}$ to the spaces of right F-preadjointable and right F-adjointable functors $\mathcal{C} \to \mathcal{X}$, respectively, are accessible and preserve limits.

Proof. By definition, a functor $\mathcal{C} \to \mathcal{X}$ is right F-preadjointable if it takes every morphism in $\mathcal{X}$ to a left adjoint in $\mathcal{X}$. We can therefore write $\mathrm{Map}_{F\text{-rpreadj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ as the pullback

$$\begin{tikzcd}
\mathrm{Map}_{F\text{-rpreadj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \arrow[d, "(f^*)_{f \in S}"] \\
\prod_{f \in S} \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathrm{ADJ}, \mathcal{X}) \arrow[r, "L^*"] & \prod_{f \in S} \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\Delta^1, \mathcal{X})
\end{tikzcd}$$

where the product is over the set S of equivalence classes of morphisms in $\mathcal{C}_F$, ADJ denotes the universal adjunction and $L\colon \Delta^1 \to \mathrm{ADJ}$ the inclusion of its left adjoint. From this description it is immediate that $\mathrm{Map}_{F\text{-rpreadj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ preserves limits in $\mathcal{X}$, since this is clear for the other three corners of the square. The ∞-category $\mathrm{Cat}_{(\infty,2)}$ is presentable, for instance because it can be described as presheaves on Θ_2 satisfying Segal and completeness conditions, which gives an explicit presentation as an accessible localization of an ∞-category of presheaves. We can therefore choose a regular cardinal κ such that S is κ-small and the objects $\mathcal{C}^{\mathrm{op}}$, ADJ, and Δ^1 are all κ-compact in $\mathrm{Cat}_{(\infty,2)}$. Then $\mathrm{Map}_{F\text{-rpreadj}}(\mathcal{C}^{\mathrm{op}}, -)$ preserves κ-filtered colimits, since the other corners of the pullback square do so (as κ-filtered colimits in $\mathcal{S}$ commute with κ-small limits, such as our product over S).

For $\mathrm{Map}_{F\text{-radj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ we impose the additional requirement that every cartesian square (9) in $\mathcal{C}$ where the horizontal maps are in $\mathcal{C}_F$ is taken to a right adjointable square in $\mathcal{X}$. By Proposition 2.1.5 the right adjointable squares are the left adjoints in $\mathcal{X}^{\Delta^1}$, so if S' denotes the set of equivalence classes of relevant cartesian squares in $\mathcal{C}$ we can write $\mathrm{Map}_{F\text{-radj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ as a pullback

$$\begin{tikzcd}
\mathrm{Map}_{F\text{-radj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \arrow[r] \arrow[d] & \mathrm{Map}_{F\text{-rpreadj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \arrow[d] \\
\prod_{S'} \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathrm{ADJ} \times \Delta^1, \mathcal{X}) \arrow[r, "(L\times \mathrm{id})^*"] & \prod_{S'} \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\Delta^1 \times \Delta^1, \mathcal{X}).
\end{tikzcd}$$

The same argument as for right F-preadjointable maps now implies that the presheaf $\mathrm{Map}_{F\text{-radj}}(\mathcal{C}, -)$ is also accessible and preserves limits. $\square$

Since $\mathrm{Cat}_{(\infty,2)}$ is presentable, this immediately implies:

Corollary 2.2.10. *Let $(\mathcal{C}, \mathcal{C}_F)$ be a span pair, with $\mathcal{C}$ a small ∞-category. Then the functor $\mathrm{Map}_{F\text{-radj}}(\mathcal{C}, \mathcal{X})$ is representable by a small $(\infty, 2)$-category $\mathrm{SPAN}_F(\mathcal{C})$, so that there is a natural equivalence*

$$\mathrm{Map}_{F\text{-radj}}(\mathcal{C}, \mathcal{X}) \simeq \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathrm{SPAN}_F(\mathcal{C}), \mathcal{X})$$

for any $\mathcal{X} \in \mathrm{Cat}_{(\infty,2)}$. $\square$

Variant 2.2.11. If $\mathcal{X}$ is an $(\infty, 2)$-category, then we have an equivalence of underlying ∞-categories

$$\mathcal{X}^{(1)} \simeq (\mathcal{X}^{2\text{-op}})^{(1)},$$

while a 1-morphism in $\mathcal{X}$ is a right adjoint if and only if it is a left adjoint in $\mathcal{X}^{2\text{-op}}$. We therefore say that a functor $\Phi \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{X}$ is *left F-adjointable* if the dual functor

$$\mathcal{C}^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})^{2\text{-op}} \xrightarrow{\Phi^{2\text{-op}}} \mathcal{X}^{2\text{-op}}$$

is right F-adjointable. Theorem 2.2.7 then tells us that the left F-adjointable functors correspond to functors $\mathrm{SPAN}_F(\mathcal{C})^{2\text{-op}} \to \mathcal{X}$.

We now upgrade Theorem 2.2.7 to a statement at the level of $(\infty, 2)$-categories, rather than just ∞-groupoids, using Proposition 2.1.5; see Corollary 2.2.14 for a precise formulation.

Definition 2.2.12. Let $(\mathcal{C}, \mathcal{C}_F)$ be a span pair and $\mathcal{X}$ an $(\infty, 2)$-category. We say that a natural transformation $\eta \colon \mathcal{C}^{\mathrm{op}} \times \Delta^1 \to \mathcal{X}$ is *right F-adjointable* if it corresponds to a right F-adjointable functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{FUN}(\Delta^1, \mathcal{X})$. From Proposition 2.1.5 it follows that η is right F-adjointable if and only if the components η_0, η_1 are both right F-adjointable, and for every morphism $f \colon x \to y$ in $\mathcal{C}_F$, the naturality square

$$\begin{tikzcd}
\eta_0(y) \arrow[r, "f^{\circledast}"] \arrow[d, "\eta_y"] & \eta_0(x) \arrow[d, "\eta_x"] \\
\eta_1(y) \arrow[r, "f^{\circledast}"] & \eta_1(x)
\end{tikzcd}$$

is right adjointable. Let $\mathrm{Fun}_{F\text{-radj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ denote the subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ whose objects are the right F-adjointable functors and whose morphisms are the right F-adjointable transformations.

Definition 2.2.13. By Proposition 2.1.5 a morphism $\mathcal{C} \times \mathsf{C}_2 \to \mathcal{X}$ (where C_2 is the 2-cell) corresponds to a right F-adjointable morphism $\mathcal{C} \to \mathrm{FUN}(\mathsf{C}_2, \mathcal{X})$ if and only if the component functors and natural transformations are right F-adjointable. We therefore write $\mathrm{FUN}_{F\text{-radj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ for the locally full sub-$(\infty,2)$-category of $\mathrm{FUN}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ whose underlying ∞-category is $\mathrm{Fun}_{F\text{-radj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$.

Corollary 2.2.14. *Let $(\mathcal{C}, \mathcal{C}_F)$ be a span pair and $\mathcal{X}$ an $(\infty,2)$-category. The inclusion of the backwards maps $\mathcal{C}^{\mathrm{op}} \to \mathrm{SPAN}_F(\mathcal{C})$ gives an equivalence of ∞-categories*

$$\mathrm{FUN}(\mathrm{SPAN}_F(\mathcal{C}), \mathcal{X}) \xrightarrow{\sim} \mathrm{FUN}_{F\text{-radj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}).$$

Proof. For any $(\infty,2)$-category $\mathcal{Y}$ we have a natural equivalence

$$\begin{aligned} \mathrm{Map}(\mathcal{Y}, \mathrm{FUN}(\mathrm{SPAN}_F(\mathcal{C}), \mathcal{X})) &\simeq \mathrm{Map}(\mathrm{SPAN}_F(\mathcal{C}), \mathrm{FUN}(\mathcal{Y}, \mathcal{X})) \\ &\simeq \mathrm{Map}_{F\text{-radj}}(\mathcal{C}^{\mathrm{op}}, \mathrm{FUN}(\mathcal{Y}, \mathcal{X})) \\ &\simeq \mathrm{Map}(\mathcal{Y}, \mathrm{FUN}_{F\text{-radj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})), \end{aligned}$$

where the last equivalence follows from the description of adjoints in functor $(\infty,2)$-categories in Proposition 2.1.5. □

We also briefly comment on the functoriality of $(\infty,2)$-categories of spans:

Definition 2.2.15. If $(\mathcal{C}, \mathcal{C}_F)$ and $(\mathcal{C}', \mathcal{C}'_F)$ are span pairs, then a morphism of spain pairs $(\mathcal{C}, \mathcal{C}_F) \to (\mathcal{C}', \mathcal{C}'_F)$ is a functor $\phi \colon \mathcal{C} \to \mathcal{C}'$ such that $\phi(\mathcal{C}_F) \subseteq \mathcal{C}'_F$ and ϕ preserves pullbacks along morphisms in $\mathcal{C}_F$. We write Pair for the ∞-category of span pairs, which can be defined as a subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$.

Remark 2.2.16. For any $(\infty,2)$-category $\mathcal{X}$, composition with a morphism of span pairs $\phi \colon (\mathcal{C}, \mathcal{C}_F) \to (\mathcal{C}', \mathcal{C}'_{F'})$ restricts to a morphism

$$\mathrm{Map}_{F'\text{-radj}}(\mathcal{C}', \mathcal{X}) \to \mathrm{Map}_{F\text{-radj}}(\mathcal{C}, \mathcal{X}),$$

natural in $\mathcal{X} \in \mathrm{Cat}_{(\infty,2)}$. We obtain a functor

$$\mathrm{Map}_{(-)\text{-radj}}(-,-) \colon \mathrm{Pair}^{\mathrm{op}} \times \mathrm{Cat}_{(\infty,2)} \to \mathcal{S}.$$

Corollary 2.2.10 says that the corresponding functor $\mathrm{Pair}^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{Cat}_{(\infty,2)}, \mathcal{S})$ takes values in corepresentable copresheaves, and so by the Yoneda lemma factors through a canonical functor $\mathrm{SPAN} \colon \mathrm{Pair} \to \mathrm{Cat}_{(\infty,2)}$.

2.3. **Distributive functors.**

We now start our discussion of bispans. In this section we define the required input data for our construction of bispans, and prove that this gives a representable functor. For this we need the notion of a distributivity diagram, which dictates how the multiplicative and additive pushforwards should interact:

Definition 2.3.1. Let $x \xrightarrow{l} y \xrightarrow{f} z$ be morphisms in an ∞-category $\mathcal{C}$. A *distributivity diagram* for l and f is a commutative diagram

(11)
$$\begin{tikzcd} & w \times_z y \arrow[r, "\tilde{f}"] \arrow[dl, "\epsilon"'] \arrow[d, "\tilde{g}"] & w \arrow[d, "g"] \\ x \arrow[r, "l"'] & y \arrow[r, "f"'] & z, \end{tikzcd}$$

where the square is cartesian, with the property that for any morphism $\phi \colon u \to z$, the composite map

$$\mathrm{Map}_{/z}(\phi, g) \to \mathrm{Map}_{/y}(f^*\phi, \tilde{g}) \xrightarrow{\epsilon_*} \mathrm{Map}_{/y}(f^*\phi, l) \tag{12}$$

is an equivalence. The distributivity diagram is necessarily unique if it exists.

Remark 2.3.2. Consider the ∞-category of diagrams of shape (11) (with the square cartesian). If all pullbacks along f exist in $\mathcal{C}$, then this is equivalent an object of the fibre product of ∞-categories

$$\mathcal{C}_{/z} \times_{\mathcal{C}_{/y}} \mathcal{C}_{/x},$$

with the functors in the pullback being $f^*\colon \mathcal{C}_{/z} \to \mathcal{C}_{/y}$ and $l_!\colon \mathcal{C}_{/x} \to \mathcal{C}_{/y}$. The universal property of the distributivity diagram can then be reformulated as that of being a terminal object in this ∞-category.

Definition 2.3.3. A *bispan triple* $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ consists of an ∞-category $\mathcal{C}$ together with two subcategories $\mathcal{C}_F, \mathcal{C}_L$ such that the following assumptions hold:

(a) $(\mathcal{C}, \mathcal{C}_F)$ is a span pair.
(b) $(\mathcal{C}, \mathcal{C}_L)$ is a span pair.
(c) For $l\colon x \to y$ in $\mathcal{C}_L$ and $f\colon y \to z$ in $\mathcal{C}_F$ there exists a distributivity diagram (11) where g is in $\mathcal{C}_L$ (and hence $\tilde{f}$ is in $\mathcal{C}_F$ by (a) and $\tilde{g}$ is in $\mathcal{C}_L$ by (b)).

Notation 2.3.4. For $x \in \mathcal{C}$ we write $\mathcal{C}^L_{/x}$ for the full subcategory of $\mathcal{C}_{/x}$ spanned by morphisms $y \to x$ in $\mathcal{C}_L$.

Remark 2.3.5. For a fixed $f\colon y \to z$ in $\mathcal{C}_F$, if the distributivity diagram (11) exists for all l in $\mathcal{C}_L$ then the functor $f^*\colon \mathcal{C}^L_{/z} \to \mathcal{C}^L_{/y}$ given by pullback along f has a right adjoint f_*. Indeed, from (12) we see that we have $(w \xrightarrow{g} z) \simeq f_*(x \xrightarrow{l} y)$ which determines the rest of the diagram. Note, however, that if the category $\mathcal{C}_L$ is not all of $\mathcal{C}$ then the property of (12) is slightly stronger than the existence of the right adjoint: for this to exist it suffices to consider maps from ϕ in $\mathcal{C}^L_{/z}$, while (12) asks for an equivalence on maps from any ϕ in $\mathcal{C}_{/z}$. We can characterize the additional assumption on these right adjoints needed to have a bispan triple in terms of a base change property described in the next lemma; we will make use of this criterion to construct examples of bispans later in the paper.

Lemma 2.3.6. *Suppose we have a triple $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ consisting of an ∞-category $\mathcal{C}$ together with two subcategories $\mathcal{C}_F, \mathcal{C}_L$ such that*

(a) $(\mathcal{C}, \mathcal{C}_F)$ is a span pair,
(b) $(\mathcal{C}, \mathcal{C}_L)$ is a span pair,
(c) for any $f\colon x \to y$ in $\mathcal{C}_F$ the functor $f^*\colon \mathcal{C}^L_{/y} \to \mathcal{C}^L_{/x}$ given by pullback along f has a right adjoint f_*.

Then $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ is a bispan triple if and only if for every cartesian square

$$\begin{tikzcd} x' \arrow[r, "f'"] \arrow[d, "\xi"] & y' \arrow[d, "\eta"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

with f in $\mathcal{C}_F$ the commutative square

$$\begin{tikzcd} \mathcal{C}^L_{/y} \arrow[r, "f^*"] \arrow[d, "\eta^*"] & \mathcal{C}^L_{/x} \arrow[d, "\xi^*"] \\ \mathcal{C}^L_{/y'} \arrow[r, "f'^*"] & \mathcal{C}^L_{/x'} \end{tikzcd}$$

is right adjointable, i.e. the mate transformation

$$\eta^* f_* \to f'_* \xi^*$$

is invertible.

Proof. First suppose we have a bispan triple. Then for $l \in \mathcal{C}^L_{/x}$ and $l' \in \mathcal{C}^L_{/y'}$ we have natural equivalences

$$\begin{aligned}\mathrm{Map}_{/y'}(l', f'_*\xi^* l) &\simeq \mathrm{Map}_{/x'}(f'^* l', \xi^* l)\\ &\simeq \mathrm{Map}_{/x}(\xi_! f'^* l', l)\\ &\simeq \mathrm{Map}_{/x}(f^*\eta_! l', l)\\ &\simeq \mathrm{Map}_{/y}(\eta_! l', f_* l)\\ &\simeq \mathrm{Map}_{/y}(l', \eta^* f_* l),\end{aligned}$$

using the functors $\eta_!$ and $\xi_!$ given by composition with η and ξ, respectively, which act as left adjoints to η^* and ξ^* when pullbacks along η and ξ exist, and the full strength of condition (12) for distributivity diagrams, which implies that we have the second-to-last equivalence even though $\eta_! l'$ is not necessarily in $\mathcal{C}_L$.

Now suppose our triple satisfies the assumption on Beck–Chevalley transformations. To check that it is a bispan triple we must show that for $l: c \to x$ in $\mathcal{C}_L$ and $f: x \to y$ in $\mathcal{C}_F$ the pushforward $f_* l$ has the universal property (12), i.e. that for every morphism $\eta: y' \to y$ we have a natural equivalence

$$\mathrm{Map}_{/y}(\eta, f_* l) \simeq \mathrm{Map}_{/x}(f^*\eta, l).$$

Denoting the pullback square containing η and f as above, we have

$$\begin{aligned}\mathrm{Map}_{/y}(\eta, f_* l) &\simeq \mathrm{Map}_{/y}(\eta_!\mathrm{id}_{y'}, f_* l)\\ &\simeq \mathrm{Map}_{/y'}(\mathrm{id}_{y'}, \eta^* f_* l)\\ &\simeq \mathrm{Map}_{/y'}(\mathrm{id}_{y'}, f'_*\xi^* l)\\ &\simeq \mathrm{Map}_{/x'}(f'^*\mathrm{id}_{y'}, \xi^* l)\\ &\simeq \mathrm{Map}_{/x'}(\mathrm{id}_{x'}, \xi^* l)\\ &\simeq \mathrm{Map}_{/x}(\xi_!\mathrm{id}_{x'}, l)\\ &\simeq \mathrm{Map}_{/x}(f^*\eta, l),\end{aligned}$$

where the fourth equivalence holds because $\mathrm{id}_{y'}$ is in $\mathcal{C}_L$. □

Remark 2.3.7. In particular, if $\mathcal{C}$ is locally cartesian closed, then all distributivity diagrams exist in $\mathcal{C}$ for any choice of $\mathcal{C}_L$; this is the case of $\mathcal{C}$ is an ∞-topos, for example.

Notation 2.3.8. For any functor $\Phi\colon \mathrm{Span}_F(\mathcal{C}) \to \mathcal{X}$ we use the following notation:

- For $f: x \to y$ a morphism in $\mathcal{C}$, we write $f^{\circledast}: \Phi(y) \to \Phi(x)$ for the value of Φ at $y \xleftarrow{f} x = x$.
- For $f: x \to y$ a morphism in $\mathcal{C}_F$, we write $f_{\otimes}: \Phi(x) \to \Phi(y)$ for the value of Φ at $x = x \xrightarrow{f} y$.

For the next definitions we fix a bispan triple $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$, an $(\infty, 2)$-category $\mathcal{X}$, and a functor

$$\Phi\colon \mathrm{Span}_F(\mathcal{C}) \to \mathcal{X}.$$

Definition 2.3.9. We say that Φ is *L-adjointable* if the restriction of Φ to a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{X}$ is left L-adjointable in the sense of Variant 2.2.11. In other words, Φ is L-adjointable if for every morphism $l: x \to y$ in $\mathcal{C}_L$, the 1-morphism $l^{\circledast}$ in $\mathcal{X}$ has a left adjoint, which we denote $l_{\oplus}$, and for every pullback square

$$\begin{tikzcd} x \times_y z \arrow[r, "l'"] \arrow[d, "f'"] & z \arrow[d, "f"] \\ x \arrow[r, "l"] & y, \end{tikzcd}$$

in $\mathcal{C}$ with l in $\mathcal{C}_L$, the induced Beck–Chevalley transformation

$$l'_{\oplus} f'^{\circledast} \to f^{\circledast} l_{\oplus}$$

is an equivalence. If the context is clear, we simply say that Φ is *adjointable.*

Definition 2.3.10. Given $l\colon x \to y$ in $\mathcal{C}_L$ and $f\colon y \to z$ in $\mathcal{C}_F$, we have by assumption a distributivity diagram as in (11) in $\mathcal{C}$. If Φ is L-adjointable we can then define the *distributivity transformation* for l and f as the composite

$$g_{\oplus}\tilde{f}_{\otimes}\epsilon^{\circledast} \to g_{\oplus}\tilde{f}_{\otimes}\epsilon^{\circledast}l^{\circledast}l_{\oplus} \simeq g_{\oplus}\tilde{f}_{\otimes}\tilde{g}^{\circledast}l_{\oplus} \simeq g_{\oplus}g^{\circledast}f_{\otimes}l_{\oplus} \to f_{\otimes}l_{\oplus}, \tag{13}$$

where the first map uses the unit for the adjunction $l_{\oplus} \dashv l^{\circledast}$, the second equivalence uses the functoriality of Φ for compositions of spans, and the last map uses the counit of the adjunction $g_{\oplus} \dashv g^{\circledast}$.

Definition 2.3.11. We say the functor Φ is *L-distributive* if it is L-adjointable and the distributivity transformation (13) is an equivalence for all l in $\mathcal{C}_L$ and f in $\mathcal{C}_F$. If the context is clear, we simply call Φ *distributive.* We write

$$\mathrm{Map}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X}) \subset \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X}),$$

for the subspace spanned by the L-distributive functors.

Variant 2.3.12. Let $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ be a bispan triple and $\mathcal{X}$ an $(\infty,2)$-category. We say that a functor $\Phi\colon \mathrm{Span}_F(\mathcal{C})^{\mathrm{op}} \to \mathcal{X}$ is *L-codistributive* if the opposite functor

$$\Phi^{\mathrm{op}}\colon\ \mathrm{Span}_F(\mathcal{C}) \to \mathcal{X}^{\mathrm{op}}$$

is L-distributive. Unpacking this definition, it says: for every morphism $f \in \mathcal{C}$ we have the pushforward $f_{\circledast}$, and if l is in $\mathcal{C}_L$ the functor $l_{\circledast}$ has a right adjoint, $l^{\oplus}$, satisfying the Beck-Chevalley condition. For f in $\mathcal{C}_F$, we have a pullback $f^{\otimes}$. Given a distributivity diagram as in (11) with l in $\mathcal{C}_L$ and f in $\mathcal{C}_F$, we then have the codistributivity transformation

$$\epsilon_{\circledast}\tilde{f}^{\otimes}g^{\oplus} \to l^{\oplus}f^{\otimes},$$

which we demand to be an equivalence.

Variant 2.3.13. Let $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ be a bispan triple and $\mathcal{X}$ an $(\infty,2)$-category. We say that a functor $\Phi\colon \mathrm{Span}_F(\mathcal{C})^{\mathrm{op}} \to \mathcal{X}$ is *right L-distributive* if the functor

$$\mathrm{Span}_F(\mathcal{C}) \simeq \mathrm{Span}_F(\mathcal{C})^{2\text{-op}} \xrightarrow{\Phi^{2\text{-op}}} \mathcal{X}^{2\text{-op}}$$

is L-distributive. Since left adjoints in $\mathcal{X}^{2\text{-op}}$ correspond to right adjoints in $\mathcal{X}$, this amounts to: for every morphism $l \in \mathcal{C}_L$ the morphism $l^{\circledast}$ has a *right* adjoint $l_{\oplus}$, and the restriction of Φ to $\mathcal{C}^{\mathrm{op}} \to \mathcal{X}$ is right L-adjointable. Moreover, given a distributivity diagram as in (11) with l in $\mathcal{C}_L$ and f in $\mathcal{C}_F$, we have a right distributivity transformation

$$f_{\otimes}l_{\oplus} \to g_{\oplus}g^{\circledast}f_{\otimes}l_{\oplus} \simeq g_{\oplus}\tilde{f}_{\otimes}\epsilon^{\circledast}l^{\circledast}l_{\oplus} \to g_{\oplus}\tilde{f}_{\otimes}\epsilon^{\circledast},$$

which is required to be an equivalence.

Proposition 2.3.14. *The functor*

$$\mathrm{Map}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), -)\colon \mathrm{Cat}_{(\infty,2)} \to \mathcal{S}$$

is accessible and preserves limits.

Proof. Let $\mathrm{Map}_{L\text{-adj}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})$ denote the space of functors $\mathrm{Span}_F(\mathcal{C}) \to \mathcal{X}$ that are L-adjointable in the sense of Definition 2.3.11. Then we have a pullback square

$$\begin{tikzcd}
\mathrm{Map}_{L\text{-adj}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X}) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X}) \arrow[d] \\
\mathrm{Map}_{L\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \arrow[r] & \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}),
\end{tikzcd}$$

where $\mathrm{Map}_{L\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ denotes the space of left L-adjointable functors (which is equivalent to $\mathrm{Map}_{L\text{-radj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}^{2\text{-op}})$ by Variant 2.2.11). It now follows that the presheaf $\mathrm{Map}_{L\text{-adj}}(\mathrm{Span}_F(\mathcal{C}), -)$ is accessible and preserves limits, since this holds for the other three corners of the square by Proposition 2.2.9.

Given an appropriate distributivity diagram in $\mathcal{C}$, there is a natural morphism

$$\mathrm{Map}_{L\text{-adj}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X}) \to \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathsf{C}_2, \mathcal{X}),$$

where C_2 denotes the 2-cell, taking an L-adjointable functor to the distributivity 2-morphism for this distributivity diagram, since this is built by composing unit and counit morphisms that are themselves natural. If S denotes the set of equivalence classes of relevant distributivity diagrams, we can therefore write $\mathrm{Map}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})$ as a pullback

$$\begin{tikzcd}
\mathrm{Map}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X}) \arrow[r] \arrow[d] & \mathrm{Map}_{L\text{-adj}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X}) \arrow[d] \\
\prod_S \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathsf{C}_2^{\mathrm{eq}}, \mathcal{X}) \arrow[r] & \prod_S \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathsf{C}_2, \mathcal{X}),
\end{tikzcd}$$

where $\mathsf{C}_2^{\mathrm{eq}}$ denotes the universal invertible 2-morphism. Proceeding as in the proof of Proposition 2.2.9, it follows that $\mathrm{Map}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), -)$ is accessible and preserves limits. $\square$

Since $\mathrm{Cat}_{(\infty,2)}$ is a presentable ∞-category, we immediately get:

Corollary 2.3.15. *Let $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ be a bispan triple. There exists an $(\infty, 2)$-category $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ and an L-distributive functor*

$$i\colon \mathrm{Span}_F(\mathcal{C}) \to \mathrm{BISPAN}_{F,L}(\mathcal{C})$$

such that the restriction morphism

$$i^*\colon \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathrm{BISPAN}_{F,L}(\mathcal{C}), \mathcal{X}) \xrightarrow{\sim} \mathrm{Map}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X}) \tag{14}$$

is an equivalence for all $\mathcal{X} \in \mathrm{Cat}_{(\infty,2)}$. $\square$

We call $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ the $(\infty, 2)$-category of *bispans* with additive forward maps in $\mathcal{C}_L$ and multiplicative forward maps in $\mathcal{C}_F$. We will justify this terminology in §2.5.

Notation 2.3.16. In keeping with our conventions so far, we will denote the underlying ∞-category of $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ by

$$\mathrm{Bispan}_{F,L}(\mathcal{C}) := \mathrm{BISPAN}_{F,L}(\mathcal{C})^{(1)}.$$

In examples we will often have $\mathcal{C}_L \simeq \mathcal{C}$, in which case we abbreviate $\mathrm{BISPAN}_F(\mathcal{C}) := \mathrm{BISPAN}_{F,L}(\mathcal{C})$. If we also have $\mathcal{C}_F \simeq \mathcal{C}$, we write $\mathrm{BISPAN}(\mathcal{C})$ for $\mathrm{BISPAN}_F(\mathcal{C})$. We also adopt the same conventions for the ∞-category of bispans $\mathrm{Bispan}_{F,L}(\mathcal{C})$.

We now want to upgrade the equivalence (14) from an equivalence of ∞-groupoids to one of $(\infty, 2)$-categories, which requires the following definition of morphisms and 2-morphisms between distributive functors:

Definition 2.3.17. By Proposition 2.1.5, an L-distributive functor $\mathrm{Span}_F(\mathcal{C}) \to \mathcal{X}^{\Delta^1}$ corresponds to a natural transformation

$$\eta\colon \mathrm{Span}_F(\mathcal{C}) \times \Delta^1 \to \mathcal{X}$$

such that both η_0 and η_1 are L-distributive functors, and the mate square for the required left adjoints commutes. Such a natural transformation will be called an *L-distributive transformation.* We let $\mathrm{Fun}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})$ denote the subcategory of $\mathrm{Fun}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})$ consisting of L-distributive functors and L-distributive transformations.

The ∞-category $\mathrm{Fun}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})$ admits a natural $(\infty, 2)$-categorical enhancement described as follows:

Definition 2.3.18. From Proposition 2.1.5 we also know that a functor $\mathrm{Span}_F(\mathcal{C}) \to \mathcal{X}^{\mathsf{C}_2}$ is L-distributive if and only if its underlying functors and natural transformations to $\mathcal{X}$ are L-distributive, i.e. if and only if the adjoint morphism

$$\mathsf{C}_2 \to \mathrm{FUN}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})$$

factors through $\mathrm{Fun}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})$ on underlying ∞-categories. We therefore write $\mathrm{FUN}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})$ for the locally full subcategory of $\mathrm{FUN}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})$ whose underlying ∞-category is $\mathrm{Fun}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})$.

Proposition 2.3.19. *For any $(\infty, 2)$-category $\mathcal{Y}$ there is a natural equivalence*

$$\mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{Y}, \mathrm{FUN}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})) \simeq \mathrm{Map}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathrm{FUN}(\mathcal{Y}, \mathcal{X})).$$

Proof. We claim that the two sides are identified under the natural equivalence

$$\mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{Y}, \mathrm{FUN}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})) \simeq \mathrm{Map}(\mathrm{Span}_F(\mathcal{C}), \mathrm{FUN}(\mathcal{Y}, \mathcal{X})).$$

Indeed, the subspace $\mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{Y}, \mathrm{FUN}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X}))$ of the left-hand side consists of those functors $\Phi\colon \mathcal{Y} \to \mathrm{FUN}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})$ such that for every object $y \in \mathcal{Y}$ the image $\Phi(y)$ is an L-distributive functor and for every morphism $f\colon y \to y'$ the image $\Phi(f)$ is an L-distributive natural transformation. Since the distributivity transformation is given pointwise by distributivity transformations in $\mathcal{X}$, and equivalences in $\mathrm{FUN}(\mathcal{Y}, \mathcal{X})$ are detected by evaluation at all objects of $\mathcal{Y}$, these conditions precisely correspond to L-distributivity for the adjoint functor $\mathrm{Span}_F(\mathcal{C}) \to \mathrm{FUN}(\mathcal{Y}, \mathcal{X})$ by Proposition 2.1.5. □

We then obtain the following $(\infty, 2)$-categorical upgrade of Corollary 2.3.15:

Corollary 2.3.20. *Composition with $i\colon \mathrm{Span}_F(\mathcal{C}) \to \mathrm{BISPAN}_{F,L}(\mathcal{C})$ induces an equivalence*

$$\mathrm{FUN}(\mathrm{BISPAN}_{F,L}(\mathcal{C}), \mathcal{X}) \xrightarrow{\sim} \mathrm{FUN}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X}),$$

natural in $\mathcal{X} \in \mathrm{Cat}_{(\infty,2)}$.

Proof. For any $(\infty, 2)$-category $\mathcal{Y}$ we have natural equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{Y}, \mathrm{FUN}(\mathrm{BISPAN}_{F,L}(\mathcal{C}), \mathcal{X}) &\simeq \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathrm{BISPAN}_{F,L}(\mathcal{C}), \mathrm{FUN}(\mathcal{Y}, \mathcal{X}))\\
&\simeq \mathrm{Map}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathrm{FUN}(\mathcal{Y}, \mathcal{X}))\\
&\simeq \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{Y}, \mathrm{FUN}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})),
\end{aligned}$$

where the second equivalence follows from Corollary 2.3.15 and the third from Proposition 2.3.19. □

Variant 2.3.21. Using analogous notation for codistributive functors, we have equivalences

$$\begin{aligned}\mathrm{Map}_{L\text{-codist}}(\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}},\mathcal{X}) &\simeq \mathrm{Map}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}),\mathcal{X}^{\mathrm{op}})\\ &\simeq \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathrm{BISPAN}_{F,L}(\mathcal{C}),\mathcal{X}^{\mathrm{op}})\\ &\simeq \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathrm{BISPAN}_{F,L}(\mathcal{C})^{\mathrm{op}},\mathcal{X}),\end{aligned}$$

$$\begin{aligned}\mathrm{Fun}_{L\text{-codist}}(\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}},\mathcal{X}) &\simeq \mathrm{Fun}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}),\mathcal{X}^{\mathrm{op}})^{\mathrm{op}}\\ &\simeq \mathrm{Fun}(\mathrm{BISPAN}_{F,L}(\mathcal{C}),\mathcal{X}^{\mathrm{op}})^{\mathrm{op}}\\ &\simeq \mathrm{Fun}(\mathrm{BISPAN}_{F,L}(\mathcal{C})^{\mathrm{op}},\mathcal{X}),\end{aligned}$$

and similarly we have

$$\mathrm{FUN}_{L\text{-codist}}(\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}},\mathcal{X}) \simeq \mathrm{FUN}(\mathrm{BISPAN}_{F,L}(\mathcal{C})^{\mathrm{op}},\mathcal{X}).$$

The universal property also implies that our $(\infty,2)$-categories of bispans are functorial for morphisms of bispan triples, in the following sense:

Definition 2.3.22. A morphism of bispan triples $(\mathcal{C},\mathcal{C}_F,\mathcal{C}_L)\to(\mathcal{C}',\mathcal{C}'_{F'},\mathcal{C}'_{L'})$ is a functor $\phi\colon \mathcal{C}\to\mathcal{C}'$ such that $\phi(\mathcal{C}_F)\subseteq\mathcal{C}'_{F'}$, $\phi(\mathcal{C}_L)\subseteq\mathcal{C}'_{L'}$, and ϕ preserves pullbacks along $\mathcal{C}_F$ and $\mathcal{C}_L$ as well as distributivity diagrams. We define Trip to be the subcategory of $\mathrm{Fun}(\Lambda^2_2,\mathrm{Cat}_\infty)$ containing the bispan triples and the morphisms thereof.

Proposition 2.3.23. *There is a functor* $\mathrm{BISPAN}\colon \mathrm{Trip}\to\mathrm{Cat}_{(\infty,2)}$ *that takes* $(\mathcal{C},\mathcal{C}_F,\mathcal{C}_L)$ *to* $\mathrm{BISPAN}_{F,L}(\mathcal{C})$.

Proof. Composition with a morphism of bispan triples $\phi\colon(\mathcal{C},\mathcal{C}_F,\mathcal{C}_L)\to(\mathcal{C}',\mathcal{C}'_{F'},\mathcal{C}'_{L'})$ restricts for any $(\infty,2)$-category $\mathcal{X}$ to a morphism

$$\mathrm{Map}_{L'\text{-dist}}(\mathrm{Span}_{F'}(\mathcal{C}'),\mathcal{X})\to\mathrm{Map}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}),\mathcal{X}),$$

so we get a functor

$$\mathrm{Map}_{(-)\text{-dist}}(\mathrm{Span}_{(-)}(-),-)\colon \mathrm{Trip}^{\mathrm{op}}\times\mathrm{Cat}_{(\infty,2)}\to\mathcal{S}.$$

By Corollary 2.3.15 the associated functor $\mathrm{Trip}^{\mathrm{op}}\to\mathrm{Fun}(\mathrm{Cat}_{(\infty,2)},\mathcal{S})$ factors through the full subcategory of corepresentable copresheaves, and so by the Yoneda lemma arises from a functor $\mathrm{BISPAN}\colon\mathrm{Trip}\to\mathrm{Cat}_{(\infty,2)}$, as required. □

2.4. **More on distributivity transformations.** In this section we prove some formal properties of distributivity transformations that will be useful later on, especially for Proposition 2.5.11. Specifically, we want to know that distributivity transformations are compatible with composition and pullbacks. First we need to decompose the corresponding distributivity diagrams:

Lemma 2.4.1. *Let* $(\mathcal{C},\mathcal{C}_F,\mathcal{C}_L)$ *be a bispan triple, and suppose given morphisms* $l_1\colon x\to y$ *and* $l_2\colon y\to z$ *in* L *and* $f\colon z\to w$ *in* F*. Then we can make the following* *diagram:*

$$
\begin{tikzcd}
 & & \bullet \arrow[dl, "\epsilon_1"] \arrow[dd, "g_1'"] \arrow[r, "f''"] & \bullet \arrow[dd, "g_1"] \\
 & \bullet \arrow[dl, "\epsilon_2'"] \arrow[dr, "l_1'"] & & \\
x \arrow[dr, "l_1"] & & \bullet \arrow[dl, "\epsilon_2"] \arrow[dd, "g_2'"] \arrow[r, "f'"] & \bullet \arrow[dd, "g_2"] \\
 & y \arrow[dr, "l_2"] & & \\
 & & z \arrow[r, "f"] & w.
\end{tikzcd}
\tag{15}
$$

Here all three squares are cartesian and the two rightmost give distributivity diagrams (so $g_2 = f_*l_2$, $g_1 = f'_*l'_1$). Then the outer diagram is a distributivity diagram for l_2l_1 and f.

Proof. Set $l := l_2l_1$, $g := g_2g_1$ and $\epsilon := \epsilon'_2\epsilon_1$. Then we must show that ϵ, regarded as a morphism $f^*g \to l_2l_1$ over z, has the required universal property. Given a map $\phi\colon \bullet \to w$, we have a commutative diagram

$$
\begin{tikzcd}
\mathrm{Map}_{/w}(\phi, g) \arrow[r] \arrow[d] & \mathrm{Map}_{/y}(f^*\phi, f^*g) \arrow[r, "\epsilon_*"] \arrow[d] & \mathrm{Map}_{/y}(f^*\phi, l) \arrow[d] \\
\mathrm{Map}_{/w}(\phi, g_2) \arrow[r] & \mathrm{Map}_{/y}(f^*\phi, f^*g_2) \arrow[r, "\epsilon_{2,*}"] & \mathrm{Map}_{/y}(f^*\phi, l_2),
\end{tikzcd}
$$

where the left vertical map can be depicted as

$$
\left(
\begin{tikzcd}
\bullet \arrow[r, "\alpha"] \arrow[ddr, "\phi"'] & \bullet \arrow[d, "g_1"] \\
 & \bullet \arrow[d, "g_2"] \\
 & w
\end{tikzcd}
\right)
\mapsto
\left(
\begin{tikzcd}
\bullet \arrow[r, "g_1\alpha"] \arrow[dr, "\phi"'] & \bullet \arrow[d, "g_2"] \\
 & w
\end{tikzcd}
\right)
$$

and the others do the same thing using g'_1 and g'_2 and l_1 and l_2, respectively. Here the bottom horizontal composite is an equivalence, so to prove the top horizontal composite is an equivalence it suffices to show that it gives an equivalence on fibres over each point of $\mathrm{Map}_{/w}(\phi, g_2)$. We can thus fix a factorization of ϕ as

$$\bullet \xrightarrow{\psi} \bullet \xrightarrow{g_2} w.$$

The map on fibres over ψ can be identified as

$$\mathrm{Map}_{/\bullet}(\psi, g_1) \to \mathrm{Map}_{/\bullet}(f'^*\psi, f'^*g_1) \xrightarrow{\epsilon_{1,*}} \mathrm{Map}_{/\bullet}(f'^*\psi, l'_1)$$

using the pullback square defining l'_1, so this is indeed also an equivalence. $\square$

Lemma 2.4.2. *Let $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ be a bispan triple, and suppose given morphisms $l\colon x \to y$ in L and $f_1\colon y \to z, f_2\colon z \to w$ in F. Then we can make the following* *diagram:*

(16)
$$\begin{tikzcd}
 & & \bullet \arrow[rr, "f_1''"] \arrow[dl, "\epsilon_2'"'] \arrow[dddl, "g_2''"] & & \bullet \arrow[r, "f_2'"] \arrow[dl, "\epsilon_2"'] \arrow[dddl, "g_2'"] & \bullet \arrow[ddd, "g_2"] \\
 & \bullet \arrow[rr, "f_1'"] \arrow[dl, "\epsilon_1"'] \arrow[dd, "g_1'"] & & \bullet \arrow[dd, "g_1"] & & \\
x \arrow[dr, "l"] & & & & & \\
 & y \arrow[rr, "f_1"] & & z \arrow[rr, "f_2"] & & w.
\end{tikzcd}$$

Here all three squares are cartesian and two give distributivity diagrams (exhibiting $g_1 = f_{1,*}l$ *and* $g_2 = f_{2,*}g_1$*. Then the outer diagram is a distributivity diagram for* l *and* f_2f_1*. (Note that the composite square is indeed cartesian since the two left squares are cartesian and the middle triangle just exhibits* g_2' *as a composite.)*

Proof. Set $f := f_2f_1$ and $\epsilon := \epsilon_1\epsilon_2'$ Then we must show that ϵ, regarded as a morphism $f^*g_2 \to l$ over y, has the required universal property, i.e. for a map $\phi\colon \bullet \to w$ we must show that the composite map

$$\mathrm{Map}_{/w}(\phi, g_2) \to \mathrm{Map}_{/y}(f^*\phi, f^*g_2) \xrightarrow{\epsilon_*} \mathrm{Map}_{/y}(f^*\phi, l)$$

is an equivalence. This is implied by the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Map}_{/w}(\phi, g_2) \arrow[r] \arrow[dr, "\sim"] & \mathrm{Map}_{/z}(f_2^*\phi, f_2^*g_2) \arrow[r] \arrow[d, "\epsilon_{2,*}"] & \mathrm{Map}_{/y}(f^*\phi, f^*g_2) \arrow[d, "\epsilon'_{2,*}"] \\
 & \mathrm{Map}_{/z}(f_2^*\phi, g_1) \arrow[r] \arrow[dr, "\sim"] & \mathrm{Map}_{/y}(f^*\phi, f_1^*g_1) \arrow[d, "\epsilon_{1,*}"] \\
 & & \mathrm{Map}_{/y}(f^*\phi, l),
\end{tikzcd}$$

where the two diagonal morphisms are equivalences by the universal properties of the distributivity diagrams for the pairs (g_1, f_2) and (l, f_1). □

Lemma 2.4.3. *Let* $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ *be a bispan triple, and suppose we have morphisms* $l\colon x \to y$ *in* $\mathcal{C}_L$*,* $f\colon y \to z$ *in* $\mathcal{C}_F$ *and an arbitary morphism* $\zeta\colon z' \to z$*. Then we can form the following diagram*

(17)
$$\begin{tikzcd}
 & w'\times_z y \arrow[rrr, "h"] \arrow[drr, "\omega\times_\zeta\eta"] \arrow[ddl, "\epsilon'"'] \arrow[dddd, "f'^*g'"] & & & w' \arrow[dr, "\omega"] \arrow[dddd, "g'"] & \\
 & & & w\times_z y \arrow[rr, "h"] \arrow[ddl, "\epsilon"] \arrow[dddd, "f^*g"] & & w \arrow[dddd, "g"] \\
x' \arrow[drr, "\xi"] \arrow[ddr, "l'"'] & & & & & \\
 & & x \arrow[ddr, "l"] & & & \\
 & y' \arrow[rrr, "f'"] \arrow[drr, "\eta"'] & & & z' \arrow[dr, "\zeta"] & \\
 & & & y \arrow[rr, "f"] & & z
\end{tikzcd}$$

where the front face is a distributivity diagram for l and f, and the rest of the diagram is obtained by pulling this back along ζ. Then the back face in (17) is a distributivity diagram for l' and f'.

Proof. Given a map $\phi\colon \bullet \to z'$ we must show that the composite map

$$\mathrm{Map}_{/z'}(\phi, g') \to \mathrm{Map}_{/y'}(f'^*\phi, f'^*g') \xrightarrow{\epsilon'_*} \mathrm{Map}_{/y'}(f'^*\phi, l')$$

is an equivalence. But we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{/z'}(\phi, g') \arrow[r] \arrow[d, "\sim"] & \mathrm{Map}_{/y'}(f'^*\phi, f'^*g') \arrow[r, "\epsilon'_*"] \arrow[d, "\sim"] & \mathrm{Map}_{/y'}(f'^*\phi, l') \arrow[d, "\sim"] \\
\mathrm{Map}_{/z}(\zeta_!\phi, g) \arrow[r] & \mathrm{Map}_{/y}(f^*\zeta_!\phi, f^*g) \arrow[r, "\epsilon_*"] & \mathrm{Map}_{/y}(f^*\zeta_!\phi, l),
\end{tikzcd}$$

where the first vertical map is the adjunction equivalence

$$\mathrm{Map}_{/z'}(\phi, \zeta^*g) \xrightarrow{\sim} \mathrm{Map}_{/z}(\zeta_!\phi, g),$$

the second is the composite

$$\mathrm{Map}_{/y'}(f'^*\phi, f'^*\zeta^*g) \simeq \mathrm{Map}_{/y'}(f'^*\phi, \eta^*f^*g) \xrightarrow{\sim} \mathrm{Map}_{/y}(\eta_!f'^*\phi, f^*g) \simeq \mathrm{Map}_{/y}(f^*\zeta_!\phi, f^*g)$$

(with the equivalence $\eta_!f'^*\phi \simeq f^*\zeta_!\phi$ implied by η being the pullback $f^*\zeta$), and the third is

$$\mathrm{Map}_{/y'}(f'^*\phi, \eta^*l) \xrightarrow{\sim} \mathrm{Map}_{/y}(\eta_!\phi, l) \simeq \mathrm{Map}_{/y}(f^*\zeta_!\phi, l).$$

The composite in the bottom row in this diagram is an equivalence since the front face of (17) was by assumption a distributivity diagram, hence so is the composite in the top row. □

Now we show that these three constructions of distributivity diagrams give corresponding descriptions of distributivity transformations:

Proposition 2.4.4. *Let $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ be a bispan triple, and suppose given morphisms $l_1\colon x \to y$ and $l_2\colon y \to z$ in L and $f\colon z \to w$ in F. If $\Phi\colon \mathrm{Span}_F(\mathcal{C}) \to \mathcal{X}$ is an adjointable functor, then in terms of the diagram (15) the distributivity transformation for l_2l_1 and f is the composite*

$$g_{2,\oplus}g_{1,\oplus}f''_{\otimes}\epsilon_1^{\circledast}\epsilon_2'^{\circledast} \to g_{2,\oplus}f'_{\otimes}l'_{1,\oplus}\epsilon_2'^{\circledast} \xrightarrow{\sim} g_{2,\oplus}f'_{\otimes}\epsilon_2^{\circledast}l_{1,\oplus} \to f_{\otimes}l_{2,\oplus}l_{1,\oplus},$$

using the distributivity transformations for the pairs (l'_1, f') and (l_2, f) and the Beck–Chevalley transformation $l'_{1,\oplus}\epsilon_2'^{\circledast} \to \epsilon_2^{\circledast}l_{1,\oplus}$.

Proof. This is implied by the following commutative diagram:

$$\begin{tikzcd}
g_{2,\oplus}g_{1,\oplus}f''_{\otimes}\epsilon_1^{\circledast}\epsilon_2'^{\circledast} \arrow[r] \arrow[d] & g_{2,\oplus}g_{1,\oplus}f''_{\otimes}\epsilon_1^{\circledast}l_1'^{\circledast}l'_{1,\oplus}\epsilon_2'^{\circledast} \arrow[r, "\sim"] \arrow[d] & g_{2,\oplus}g_{1,\oplus}g_1^{\circledast}f'_{\otimes}l'_{1,\oplus}\epsilon_2'^{\circledast} \arrow[r] \arrow[d] & g_{2,\oplus}f'_{\otimes}l'_{1,\oplus}\epsilon_2'^{\circledast} \arrow[d] & \\
g_{2,\oplus}g_{1,\oplus}f''_{\otimes}\epsilon_1^{\circledast}\epsilon_2'^{\circledast}l_1^{\circledast}l_{1,\oplus} \arrow[r] \arrow[d] \arrow[dr, "\sim"] & g_{2,\oplus}g_{1,\oplus}f''_{\otimes}\epsilon_1^{\circledast}l_1'^{\circledast}l'_{1,\oplus}\epsilon_2'^{\circledast}l_1^{\circledast}l_{1,\oplus} \arrow[r, "\sim"] \arrow[dr, "\sim"] & g_{2,\oplus}g_{1,\oplus}g_1^{\circledast}f'_{\otimes}l'_{1,\oplus}\epsilon_2'^{\circledast}l_1^{\circledast}l_{1,\oplus} \arrow[r] \arrow[dr, "\sim"] & g_{2,\oplus}f'_{\otimes}l'_{1,\oplus}\epsilon_2'^{\circledast}l_1^{\circledast}l_{1,\oplus} \arrow[dr, "\sim"] & \\
g_{2,\oplus}g_{1,\oplus}f''_{\otimes}\epsilon_1^{\circledast}\epsilon_2'^{\circledast}l_1^{\circledast}l_2^{\circledast}l_{2,\oplus}l_{1,\oplus} \arrow[dr, "\sim"] & g_{2,\oplus}g_{1,\oplus}f''_{\otimes}\epsilon_1^{\circledast}l_1'^{\circledast}\epsilon_2^{\circledast}l_{1,\oplus} \arrow[r] \arrow[d] \arrow[dr, equal] & g_{2,\oplus}g_{1,\oplus}f''_{\otimes}\epsilon_1^{\circledast}l_1'^{\circledast}l'_{1,\oplus}l_1'^{\circledast}\epsilon_2^{\circledast}l_{1,\oplus} \arrow[r, "\sim"] \arrow[d] & g_{2,\oplus}g_{1,\oplus}g_1^{\circledast}f'_{\otimes}l'_{1,\oplus}l_1'^{\circledast}\epsilon_2^{\circledast}l_{1,\oplus} \arrow[r] \arrow[d] & g_{2,\oplus}f'_{\otimes}l'_{1,\oplus}l_1'^{\circledast}\epsilon_2^{\circledast}l_{1,\oplus} \arrow[d] \\
& g_{2,\oplus}g_{1,\oplus}f''_{\otimes}\epsilon_1^{\circledast}l_1'^{\circledast}\epsilon_2^{\circledast}l_2^{\circledast}l_{2,\oplus}l_{1,\oplus} \arrow[dr, equal] & g_{2,\oplus}g_{1,\oplus}f''_{\otimes}\epsilon_1^{\circledast}l_1'^{\circledast}\epsilon_2^{\circledast}l_{1,\oplus} \arrow[r, "\sim"] \arrow[d] & g_{2,\oplus}g_{1,\oplus}g_1^{\circledast}f'_{\otimes}\epsilon_2^{\circledast}l_{1,\oplus} \arrow[r] \arrow[d] & g_{2,\oplus}f'_{\otimes}\epsilon_2^{\circledast}l_{1,\oplus} \arrow[d] \\
& & g_{2,\oplus}g_{1,\oplus}f''_{\otimes}\epsilon_1^{\circledast}l_1'^{\circledast}\epsilon_2^{\circledast}l_2^{\circledast}l_{2,\oplus}l_{1,\oplus} \arrow[r, "\sim"] & g_{2,\oplus}g_{1,\oplus}g_1^{\circledast}f'_{\otimes}\epsilon_2^{\circledast}l_2^{\circledast}l_{2,\oplus}l_{1,\oplus} \arrow[r] \arrow[d, "\sim"] & g_{2,\oplus}f'_{\otimes}\epsilon_2^{\circledast}l_2^{\circledast}l_{2,\oplus}l_{1,\oplus} \arrow[d, "\sim"] \\
& & & g_{2,\oplus}g_{1,\oplus}g_1^{\circledast}g_2^{\circledast}f_{\otimes}l_{2,\oplus}l_{1,\oplus} \arrow[r] & g_{2,\oplus}g_2^{\circledast}f_{\otimes}l_{2,\oplus}l_{1,\oplus} \arrow[d] \\
& & & & f_{\otimes}l_{2,\oplus}l_{1,\oplus}.
\end{tikzcd}$$

Here we have used one of the adjunction identities for $l'_{1,\oplus} \dashv l_1'^{\circledast}$ and naturality. □

Proposition 2.4.5. *Let $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ be a bispan triple, and suppose given morphisms $l \colon x \to y$ in L and $f_1 \colon y \to z$ and $f_2 \colon z \to w$ in F. If $\Phi \colon \mathrm{Span}_F(\mathcal{C}) \to \mathcal{X}$ is an adjointable functor, then in terms of the diagram (16) the distributivity transformation for l and $f_2 f_1$ is the composite*

$$g_{2,\oplus} f'_{2,\otimes} f''_{1,\otimes} \epsilon_2'^{\circledast} \epsilon_1^{\circledast} \simeq g_{2,\oplus} f'_{2,\otimes} \epsilon_2^{\circledast} f'_{1,\otimes} \epsilon_1^{\circledast} \to f_{2,\otimes} g_{1,\oplus} f'_{1,\otimes} \epsilon_1^{\circledast} \to f_{2,\otimes} f_{1,\otimes} l_{\oplus}$$

using the distributivity transformations for the pairs (l, f_1) and (g_1, f_2).

Proof. This is implied by the following commutative diagram:

$$\begin{tikzcd}
g_{2,\oplus} f'_{2,\otimes} f''_{1,\otimes} \epsilon_2'^{\circledast} \epsilon_1^{\circledast} \arrow[r, "\sim"] \arrow[d] & g_{2,\oplus} f'_{2,\otimes} \epsilon_2^{\circledast} f'_{1,\otimes} \epsilon_1^{\circledast} \arrow[r] \arrow[d] & g_{2,\oplus} f'_{2,\otimes} \epsilon_2^{\circledast} g_1^{\circledast} g_{1,\oplus} f'_{1,\otimes} \epsilon_1^{\circledast} \arrow[r, "\sim"] \arrow[d] & g_{2,\oplus} g_2^{\circledast} f_{2,\otimes} g_{1,\oplus} f'_{1,\otimes} \epsilon_1^{\circledast} \arrow[r] \arrow[d] & f_{2,\otimes} g_{1,\oplus} f'_{1,\otimes} \epsilon_1^{\circledast} \arrow[d] \\
g_{2,\oplus} f'_{2,\otimes} f''_{1,\otimes} \epsilon_2'^{\circledast} \epsilon_1^{\circledast} l^{\circledast} l_{\oplus} \arrow[r, "\sim"] & g_{2,\oplus} f'_{2,\otimes} \epsilon_2^{\circledast} f'_{1,\otimes} \epsilon_1^{\circledast} l^{\circledast} l_{\oplus} \arrow[r] \arrow[d, "\sim"] & g_{2,\oplus} f'_{2,\otimes} \epsilon_2^{\circledast} g_1^{\circledast} g_{1,\oplus} f'_{1,\otimes} \epsilon_1^{\circledast} l^{\circledast} l_{\oplus} \arrow[r, "\sim"] \arrow[d, "\sim"] & g_{2,\oplus} g_2^{\circledast} f_{2,\otimes} g_{1,\oplus} f'_{1,\otimes} \epsilon_1^{\circledast} l^{\circledast} l_{\oplus} \arrow[r] \arrow[d, "\sim"] & f_{2,\otimes} g_{1,\oplus} f'_{1,\otimes} \epsilon_1^{\circledast} l^{\circledast} l_{\oplus} \arrow[d, "\sim"] \\
 & g_{2,\oplus} f'_{2,\otimes} \epsilon_2^{\circledast} g_1^{\circledast} f_{1,\otimes} l_{\oplus} \arrow[r] \arrow[dr, equal] & g_{2,\oplus} f'_{2,\otimes} \epsilon_2^{\circledast} g_1^{\circledast} g_{1,\oplus} g_1^{\circledast} f_{1,\otimes} l_{\oplus} \arrow[r, "\sim"] \arrow[d] & g_{2,\oplus} g_2^{\circledast} f_{2,\otimes} g_{1,\oplus} g_1^{\circledast} f_{1,\otimes} l_{\oplus} \arrow[r] \arrow[d] & f_{2,\otimes} g_{1,\oplus} g_1^{\circledast} f_{1,\otimes} l_{\oplus} \arrow[d] \\
 & & g_{2,\oplus} f'_{2,\otimes} \epsilon_2^{\circledast} g_1^{\circledast} f_{1,\otimes} l_{\oplus} \arrow[r, "\sim"] & g_{2,\oplus} g_2^{\circledast} f_{2,\otimes} f_{1,\otimes} l_{\oplus} \arrow[r] & f_{2,\otimes} f_{1,\otimes} l_{\oplus}
\end{tikzcd}$$

Here we have used one of the adjunction identities for $g_{1,\oplus} \dashv g_1^{\circledast}$ and naturality. $\square$

Proposition 2.4.6. *Let $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ be a bispan triple, and suppose we have the diagram (17). Then the distributivity transformations for (l, f) and (l', f') are related by a commutative square*

$$\begin{tikzcd}
g'_{\oplus} h'_{\otimes} \epsilon'^{\circledast} \xi^{\circledast} \arrow[r] \arrow[d, "\sim"] & f'_{\otimes} l'_{\oplus} \xi^{\circledast} \arrow[d, "\sim"] \\
g'_{\oplus} \omega^{\circledast} h_{\otimes} \epsilon^{\circledast} \arrow[d, "\sim"] & f'_{\otimes} \eta^{\circledast} l_{\oplus} \arrow[d, "\sim"] \\
\zeta^{\circledast} g_{\oplus} h_{\otimes} \epsilon^{\circledast} \arrow[r] & \zeta^{\circledast} f_{\otimes} l_{\oplus}
\end{tikzcd} \tag{18}$$

Proof. The diagram we want can be extracted from the following larger diagram:

$$\begin{tikzcd}
 & g'_{\oplus} h'_{\otimes} \epsilon'^{\circledast} l'^{*} l'_{\oplus} \xi^{\circledast} \arrow[r, "\sim"] \arrow[d, "\sim"] & g'_{\oplus} g'^{\circledast} f'_{\otimes} l'_{\oplus} \xi^{\circledast} \arrow[r] \arrow[d, "\sim"] & f'_{\otimes} l'_{\oplus} \xi^{\circledast} \arrow[d, "\sim"] \\
g'_{\oplus} h'_{\otimes} \epsilon'^{\circledast} \xi^{\circledast} \arrow[ur] \arrow[r] \arrow[d, "\sim"] & g'_{\oplus} h'_{\otimes} \epsilon'^{\circledast} \xi^{\circledast} l^{\circledast} l_{\oplus} \arrow[r, "\sim"] \arrow[d, "\sim"] & g'_{\oplus} g'^{\circledast} f'_{\otimes} \eta^{\circledast} l_{\oplus} \arrow[r] \arrow[d, "\sim"] & f'_{\otimes} \eta^{\circledast} l_{\oplus} \arrow[dd, "\sim"] \\
g'_{\oplus} \omega^{\circledast} h_{\otimes} \epsilon^{\circledast} \arrow[r] \arrow[d, "\sim"] & g'_{\oplus} \omega^{\circledast} h_{\otimes} \epsilon^{\circledast} l^{\circledast} l_{\oplus} \arrow[r, "\sim"] \arrow[d, "\sim"] & g'_{\oplus} \omega^{\circledast} g^{\circledast} f_{\otimes} l_{\oplus} \arrow[d, "\sim"] & \\
\zeta^{\circledast} g_{\oplus} h_{\otimes} \epsilon^{\circledast} \arrow[r] & \zeta^{\circledast} g_{\oplus} h_{\otimes} \epsilon^{\circledast} l^{\circledast} l_{\oplus} \arrow[r, "\sim"] & \zeta^{\circledast} g_{\oplus} g^{\circledast} f_{\otimes} l_{\oplus} \arrow[r] & \zeta^{\circledast} f_{\otimes} l_{\oplus}.
\end{tikzcd}$$

Here all the squares clearly commute except for the bottom right one, so it remains to check that this and the top left triangle do in fact commute.

For the triangle, this is implied by the following commutative diagram:

$$\begin{tikzcd}
\xi^{\circledast} \arrow[r] \arrow[d] & \xi^{\circledast} l^{\circledast} l_{\oplus} \arrow[r, "\sim"] \arrow[d] & l'^{\circledast} \eta^{\circledast} l_{\oplus} \arrow[d] \arrow[dr, equal] & \\
l'^{\circledast} l'_{\oplus} \xi^{\circledast} \arrow[r] \arrow[rrr, bend right, "\sim"'] & l'^{\circledast} l'_{\oplus} \xi^{\circledast} l^{\circledast} l_{\oplus} \arrow[r, "\sim"] & l'^{\circledast} l'_{\oplus} l'^{\circledast} \eta^{\circledast} l_{\oplus} \arrow[r] & l'^{\circledast} \eta^{\circledast} l_{\oplus}.
\end{tikzcd}$$

Here we have used one of the adjunction identities for $l'_{\oplus} \dashv l'^{\circledast}$.

We can divide the bottom right square into the following diagram:

$$
\begin{tikzcd}
g'_{\oplus} g'^{\circledast} f'_{\otimes} \eta^{\circledast} \arrow[r] \arrow[d, "\sim"] & f'_{\otimes} \eta^{\circledast} \arrow[d, "\sim"] \\
g'_{\oplus} g'^{\circledast} \zeta^{\circledast} f_{\otimes} \arrow[r] \arrow[d, "\sim"] & \zeta^{\circledast} f_{\otimes} \arrow[dd, equal] \\
g'_{\oplus} \omega^{\circledast} g^{\circledast} f_{\otimes} \arrow[d] & \\
\zeta^{\circledast} g_{\oplus} g^{\circledast} f_{\otimes} \arrow[r] & \zeta^{\circledast} f_{\otimes}.
\end{tikzcd}
$$

Here the top square commutes by naturality and the bottom part commutes because of the commutative diagram

$$
\begin{tikzcd}
g'_{\oplus} \omega^{\circledast} g^{\circledast} \arrow[d] \arrow[dr, equal] & \\
g'_{\oplus} \omega^{\circledast} g^{\circledast} g_{\oplus} g^{\circledast} \arrow[r] \arrow[d, "\sim"] & g'_{\oplus} \omega^{\circledast} g^{\circledast} \arrow[d, "\sim"] \\
g'_{\oplus} g'^{\circledast} \zeta^{\circledast} g_{\oplus} g^{\circledast} \arrow[r] \arrow[d] & g'_{\oplus} g'^{\circledast} \zeta^{\circledast} \arrow[d] \\
\zeta^{\circledast} g_{\oplus} g^{\circledast} \arrow[r] & \zeta^{\circledast},
\end{tikzcd}
$$

where we have used one of the adjunction identities for $g_{\oplus} \dashv g^{\circledast}$. □

2.5. **The $(\infty, 2)$-category of bispans.** We now come to the technical heart of the paper: In this section we will explicitly describe the $(\infty, 2)$-category $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ using Hinich's Yoneda lemma for $(\infty, 2)$-categories. In other words, we will:

(1) identify the objects of $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ with the objects of $\mathcal{C}$;
(2) given $c, d \in \mathcal{C}$, compute the ∞-category of maps between c and d explicitly as bispans between c and d;
(3) describe the composition law in this $(\infty, 2)$-category.

As a first step toward getting a handle on the $(\infty, 2)$-category $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ we have the following observation:

Lemma 2.5.1. *The functor $i\colon \mathrm{Span}_F(\mathcal{C}) \to \mathrm{BISPAN}_{F,L}(\mathcal{C})$ is essentially surjective.*

Proof. Let $\mathcal{I}$ denote the full sub-$(\infty, 2)$-category of $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ spanned by the objects in the image of i. Then i factors through $i'\colon \mathrm{Span}_F(\mathcal{C}) \to \mathcal{I}$, and i' is again L-distributive (since the relevant adjoints and 2-morphisms all live in $\mathcal{I}$). Hence i' corresponds to a functor $\mathrm{BISPAN}_{F,L}(\mathcal{C}) \to \mathcal{I}$ such that the composite

$$\mathrm{BISPAN}_{F,L}(\mathcal{C}) \to \mathcal{I} \to \mathrm{BISPAN}_{F,L}(\mathcal{C})$$

is the identity. It follows that the inclusion of $\mathcal{I}$ must be essentially surjective, which means that i is also essentially surjective. □

We will see later that we can in fact identify objects of $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ with objects of $\mathcal{C}$ (more precisely, i gives an equivalence between their underlying ∞-groupoids).

To get a handle on the ∞-categories of morphisms in $\mathrm{BISPAN}_{F,L}(\mathcal{C})$, we will use two ingredients: (1) the Yoneda Lemma for $(\infty,2)$-categories (Theorem 2.1.1) due to Hinich, and (2) the construction of the free (co)cartesian fibrations due to Gepner, Nikolaus, and the second author [GHN17]. Applying the Yoneda lemma to $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ we get a canonical family of L-codistributive functors to CAT_∞:

Proposition 2.5.2. *There is an L-distributive functor*

$$\mathsf{Y}\colon \mathrm{Span}_F(\mathcal{C}) \to \mathrm{FUN}_{L\text{-codist}}(\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}, \mathrm{CAT}_\infty)$$

such that for any L-codistributive functor $\Phi\colon \mathrm{Span}_F(\mathcal{C})^{\mathrm{op}} \to \mathrm{CAT}_\infty$ there is a natural equivalence

$$\Phi(c) \simeq \mathrm{MAP}_{L\text{-codist}}(\mathsf{Y}(c), \Phi),$$

where the latter denotes the ∞-category of L-codistributive natural transformations.

Proof. Applying Hinich's Yoneda embedding (Theorem 2.1.1) to $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ we get a functor

$$\mathsf{y}\colon \mathrm{BISPAN}_{F,L}(\mathcal{C}) \to \mathrm{FUN}(\mathrm{BISPAN}_{F,L}(\mathcal{C})^{\mathrm{op}}, \mathrm{CAT}_\infty).$$

By Corollary 2.3.15 this corresponds to an L-distributive functor

$$\mathsf{Y}\colon \mathrm{Span}_{F,L} \to \mathrm{FUN}(\mathrm{BISPAN}_{F,L}(\mathcal{C})^{\mathrm{op}}, \mathrm{CAT}_\infty) \simeq \mathrm{FUN}_{L\text{-codist}}(\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}, \mathrm{CAT}_\infty)$$

via the equivalence of Variant 2.3.21. Translating the universal property of representable presheaves through the latter equivalence now gives the result. $\square$

To proceed further, we will work unstraightened. To motivate this maneuver recall that if $\mathcal{C}$ is an ∞-category and $c \in \mathcal{C}$, then the functor $\mathrm{Map}(-,c) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is classified by a cocarteseian fibration in spaces $\mathcal{C}_{/c} \to \mathcal{C}$ and accessing the ∞-category $\mathcal{C}_{/c}$ is as good as understanding all maps in $\mathcal{C}$ with target c. Similarly, we access the ∞-category of maps in bispans by displaying it as a certain cocartesian fibration.

Notation 2.5.3. We let

$$\mathrm{Cat}^{\mathrm{cocart}+L\text{-cart}}_{\infty/\,\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}} := \mathrm{Cat}^{\mathrm{cocart}}_{\infty/\,\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}} \times_{\mathrm{Cat}_{\infty/\mathcal{C}_L}} \mathrm{Cat}^{\mathrm{cart}}_{\infty/\mathcal{C}_L}$$

denote the ∞-category of cocartesian fibrations $\mathcal{E} \to \mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}$ that have cartesian morphisms over $\mathcal{C}_L$, and whose morphisms are functors over $\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}$ that preserve cocartesian morphisms as well as cartesian morphisms over $\mathcal{C}_L$.

Remark 2.5.4. Here we have used that for a cocartesian fibration to $\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}$, the condition of having cartesian morphisms over $\mathcal{C}_L$ is equivalent to the pullback to $\mathcal{C}_L$ being a cartesian fibration, which follows from [Lur09, Corollary 5.2.2.4].

Lemma 2.5.5. *The straightening equivalence*

$$\mathrm{Fun}(\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}, \mathrm{Cat}_\infty) \simeq \mathrm{Cat}^{\mathrm{cocart}}_{\infty/\,\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}}$$

identifies the subcategory $\mathrm{Fun}_{L\text{-codist}}(\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}, \mathrm{Cat}_\infty)$ *with a full subcategory*

$$\mathrm{Cat}^{L\text{-codist}}_{\infty/\,\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}} \subseteq \mathrm{Cat}^{\mathrm{cocart}+L\text{-cart}}_{\infty/\,\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}}.$$

Proof. By [Lur09, Corollary 5.2.2.4] a cocartesian fibration to $\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}$ has cartesian morphisms over $\mathcal{C}_L$ if and only if it has locally cartesian morphisms over $\mathcal{C}_L$, which is equivalent to the corresponding morphisms in Cat_∞ having right adjoints (cf. [Lur09, Definition 5.2.2.1]). Thus the unstraightening of a codistributive functor gives an object of $\mathrm{Cat}^{\mathrm{cocart}+L\text{-cart}}_{\infty/\,\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}}$. Moreover, by [Lur17, Proposition 4.7.4.17] a morphism over $\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}$ that preserves cocartesian morphisms and cartesian morphisms over $\mathcal{C}_L$ corresponds to a natural transformation whose naturality squares over $\mathcal{C}_L$ are right adjointable, which is precisely the requirement for codistributive natural transformations. $\square$

Definition 2.5.6. For $c \in \mathcal{C}$, let $\mathcal{Y}_c \to \mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}$ denote the cocartesian fibration classified by the codistributive functor $\mathsf{Y}(c) : \mathrm{Span}_F(\mathcal{C})^{\mathrm{op}} \to \mathrm{Cat}_\infty$ that corresponds via Proposition 2.5.2 to the functor $\mathrm{BISPAN}_{F,L}(\mathcal{C})^{\mathrm{op}} \to \mathrm{CAT}_\infty$ represented by $i(c)$.

The idea is now to construct a "candidate" for $\mathcal{Y}_c$ using a result from [GHN17] together with the following observation:

Lemma 2.5.7. *If $p\colon \mathcal{E} \to \mathcal{C}$ is a functor such that $\mathcal{E}$ has p-cartesian morphisms over morphisms in $\mathcal{C}_L$, then the functor*

$$\mathrm{Fun}^{L\text{-cart}}_{/\mathcal{C}}(\mathcal{C}^L_{/x}, \mathcal{E}) \to \mathrm{Fun}_{/\mathcal{C}}(\{x\}, \mathcal{E}) \simeq \mathcal{E}_x$$

given by restriction along the inclusion

$$\{x\} \simeq \{\mathrm{id}_x\} \hookrightarrow \mathcal{C}^L_{/x},$$

is an equivalence, where $\mathrm{Fun}^{L\text{-cart}}_{/\mathcal{C}}(\mathcal{C}^L_{/x}, \mathcal{E})$ is the full subcategory of $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}^L_{/x}, \mathcal{E})$ spanned by functors that preserve cartesian morphisms over $\mathcal{C}_L$.

Proof. We use the results on relative Kan extensions from [Lur09, §4.3.2]. For every object $l\colon y \to x$ of $\mathcal{C}^L_{/x}$ the ∞-category

$$\{x\}_{l/} := \{x\} \times_{\mathcal{C}^L_{/x}} (\mathcal{C}^L_{/x})_{l/} \simeq \mathrm{Map}_{\mathcal{C}^L_{/x}}(l, \mathrm{id}_x)$$

is contractible, since id_x is a terminal object in $\mathcal{C}^L_{/x}$. Hence a morphism

$$\{x\}^{\triangleleft}_{l/} \simeq \Delta^1 \to \mathcal{E}$$

is a p-limit if and only if it's a cartesian morphism by [Lur09, Example 4.3.1.4]. It follows that a functor $\Phi\colon \mathcal{C}^L_{/x} \to \mathcal{E}$ over $\mathcal{C}$ is a p-right Kan extension from $\{x\}$ if and only if for every $l\colon y \to x$ in $\mathcal{C}_L$ it takes the unique morphism $l \to \mathrm{id}_x$ (which is cartesian over f) to a cartesian morphism in $\mathcal{E}$. By the 3-for-2 property of cartesian morphisms this is equivalent to Φ preserving cartesian morphisms over $\mathcal{C}_L$. Hence [Lur09, Proposition 4.3.2.15] implies that, since $\mathcal{E}$ has p-cartesian morphisms, the functor $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}^L_{/x}, \mathcal{E}) \to \mathrm{Fun}_{/\mathcal{C}}(\{x\}, \mathcal{E})$ restricts to an equivalence from the full subcategory $\mathrm{Fun}^{L\text{-cart}}_{/\mathcal{C}}(\mathcal{C}^L_{/x}, \mathcal{E})$. □

Construction 2.5.8. Since $\mathcal{Y}_c$ corresponds to a codistributive functor, its restriction to $\mathcal{C}$ has cartesian morphisms over $\mathcal{C}_L$. Applying Lemma 2.5.7 to $\mathcal{E} = \mathcal{Y}_c \times_{\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}} \mathcal{C} \to \mathcal{C}$ and $x = c$, we see that there is a unique commutative square

$$\begin{tikzcd}
\mathcal{C}^L_{/c} \arrow[r, "\alpha_c^-"] \arrow[d] & \mathcal{Y}_c \arrow[d] \\
\mathcal{C} \arrow[r] & \mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}
\end{tikzcd}$$

such that the top horizontal functor preserves cartesian morphisms over $\mathcal{C}_L$ and takes id_c in $\mathcal{C}^L_{/c}$ to the identity morphism $\mathrm{id}_{i(c)}$ in $\mathrm{BISPAN}_{F,L}(\mathcal{C})(i(c), i(c)) \simeq \mathcal{Y}_{c,c}$.

Now since $\mathcal{Y}_c \to \mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}$ is also a cocartesian fibration, we can extend this to a unique functor from the free cocartesian fibration [GHN17, Theorem 4.5]

$$\mathcal{B}_c := \mathcal{C}_{L/c} \times_{\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}} (\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}})^{\Delta^1} \to \mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}, \tag{19}$$

giving a unique commutative triangle

$$\begin{tikzcd}
\mathcal{B}_c \arrow[rr, "\alpha_c"] \arrow[dr] & & \mathcal{Y}_c \arrow[dl] \\
& \mathrm{Span}_F(\mathcal{C})^{\mathrm{op}} &
\end{tikzcd}$$

where the horizontal functor preserves cocartesian morphisms and restricts to α_c^- on $\mathcal{C}^L_{/c}$.

Remark 2.5.9. To understand the $(\infty, 2)$-category $\mathrm{BISPAN}_{F,L}(\mathcal{C})$, we are going to show that the functor α_c is an equivalence. The explicit construction of the ∞-category $\mathcal{B}_c$ via the domain (19) allows us to unpack it easily, revealing the expected definition of bispans. We carry this out:

- An object of $\mathcal{B}_c$ consists of an object $x \xrightarrow{h} c$ in $\mathcal{C}^L_{/c}$, i.e. a morphism h to c in $\mathcal{C}_L$, together with a morphism to x in $\mathrm{Span}_F(\mathcal{C})$, i.e. a span $y \xleftarrow{f} z \xrightarrow{g} x$ with g in $\mathcal{C}_F$; in other words, it is precisely a *bispan*
$$y \xleftarrow{f} z \xrightarrow{g} x \xrightarrow{h} c,$$
where $g \in \mathcal{C}_F, h \in \mathcal{C}_L$. The functor (19) to $\mathrm{Span}_F(\mathcal{C})$ takes this to the object y.
- A morphism from this object to another object
$$y \leftarrow z' \to x' \to c;$$
in the fibre $\mathcal{B}_{y,c}$ consists of a morphism
$$\begin{tikzcd} x \arrow[dd, "\xi"] \arrow[dr, "h"] & \\ & c \\ x' \arrow[ur, "h'"'] & \end{tikzcd}$$
in $\mathcal{C}_{/c}$ and a commutative triangle in $\mathrm{Span}_F(\mathcal{C})$ which we can depict as
$$\begin{tikzcd} & & z \arrow[dl] \arrow[dr] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\ & z' \arrow[dl] \arrow[dr] & & x \arrow[dl, "\xi"] \arrow[dr, equal] & \\ y & & x' & & x \end{tikzcd}$$
where the square is cartesian. In other words, we have precisely a morphism of bispans in the form of a diagram
$$\begin{tikzcd} & z \arrow[r] \arrow[dl] \arrow[dd] & x \arrow[dr] \arrow[dd] & \\ y & & & c \\ & z' \arrow[r] \arrow[ul] & x', \arrow[ur] & \end{tikzcd}$$
where the square is cartesian.
- Moreover, the cocartesian morphism over the span $y' \leftarrow w \to y$ is given by composition in $(\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}})^{\Delta^1}$, and so takes the bispan $y \xleftarrow{f} z \to x \to c$ to

the outer bispan in

$$\begin{tikzcd}
 & & w \times_y z \arrow[dl] \arrow[dr] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & & \\
 & w \arrow[dl] \arrow[dr] & & z \arrow[dl] \arrow[dr] & & \\
y' & & y & & x \arrow[r] & c.
\end{tikzcd}$$

In particular, for the span $y' \xleftarrow{\eta} y = y$, we simply get the bispan

$$y' \xleftarrow{\eta f} z \to x \to c.$$

Notation 2.5.10. We introduce the following abbreviated notation for bispans in $\mathcal{C}$ with only one non-trivial leg:

- For any morphism $f\colon x \to y$, we write $[f]_B$ for the bispan
$$y \xleftarrow{f} x = x = x.$$
- For any morphism $f\colon x \to y$ in $\mathcal{C}_F$, we write $[f]_F$ for the bispan
$$x = x \xrightarrow{f} y = y.$$
- For any morphism $f\colon x \to y$ in $\mathcal{C}_L$, we write $[f]_L$ for the bispan
$$x = x = x \xrightarrow{f} y.$$

Our goal is now to prove that our functor $\mathcal{B}_c \to \mathcal{Y}_c$ is an equivalence. To do this we want to use the universal property of $\mathcal{Y}_c$ (i.e. the Yoneda embedding) to produce a functor $\beta_c\colon \mathcal{Y}_c \to \mathcal{B}_c$, which will be the inverse of α_c. This requires knowing the following:

Proposition 2.5.11. *The functor $B_c\colon \mathrm{Span}_F(\mathcal{C})^{\mathrm{op}} \to \mathrm{Cat}_\infty$ classifying the cocartesian fibration $\mathcal{B}_c \to \mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}$ is L-codistributive.*

Notation 2.5.12. Before we embark on this proof, we will adopt the following notation for the various immediate functorialities of B_c:

(1) For $f\colon x \to y$ in $\mathcal{C}$, we denote the value of B_c at $y \xleftarrow{f} x = x$ by
$$f_{\circledast}\colon \mathcal{B}_{x,c} \to \mathcal{B}_{y,c};$$
this is given by
$$x \xleftarrow{p} z \to w \to c \quad \mapsto \quad y \xleftarrow{f \circ p} z \to w \to c.$$

(2) For $g\colon y \to z$ in $\mathcal{C}_F$ we denote the value at $y = y \xrightarrow{g} z$ by
$$g^{\otimes}\colon \mathcal{B}_{z,c} \to \mathcal{B}_{y,c};$$
this is given by
$$z \leftarrow x \to w \to c \quad \mapsto \quad y \leftarrow y \times_z x \to w \to c.$$

Proof of Proposition 2.5.11. Unwinding the definition of L-codistributive functors we see that we need to establish the following three claims:

(A) The functor $\mathcal{B}_c \to \mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}$ is cartesian over $\mathcal{C}_L$. In other words, for every morphism $\eta\colon y' \to y$ in $\mathcal{C}_L$, the functor
$$\eta_{\circledast}\colon \mathcal{B}_{y',c} \to \mathcal{B}_{y,c},$$
has a right adjoint, which we denote by $\eta^{\oplus}$.

(B) Given a pullback square

$$
\begin{tikzcd}
\tilde{y}' \arrow[r, "\tilde{\eta}"] \arrow[d, "\tilde{\phi}"] & \tilde{y} \arrow[d, "\phi"] \\
y' \arrow[r, "\eta"] & y
\end{tikzcd}
\tag{20}
$$

in $\mathcal{C}$ with η in $\mathcal{C}_L$, the Beck-Chevalley transformation

$$\phi_{\circledast}\eta^{\oplus} \to \tilde{\eta}^{\oplus}\tilde{\phi}_{\circledast}$$

is an equivalence.

(C) Given a distributivity diagram

$$
\begin{tikzcd}
 & u \arrow[r, "\tilde{\gamma}"] \arrow[dl, "\epsilon"'] \arrow[dd, "\tilde{\psi}"] \arrow[ddr, "k"] & w \arrow[dd, "\psi\simeq\gamma_*\phi"] \\
y \arrow[dr, "\phi"'] & & \\
 & y' \arrow[r, "\gamma"] & y''
\end{tikzcd}
\tag{21}
$$

in $\mathcal{C}$ with ϕ in $\mathcal{C}_L$ and γ in $\mathcal{C}_F$, the codistributivity transformation (prescribed by the dual of (13))

$$\epsilon_{\circledast}\tilde{\gamma}^{\otimes}\psi^{\oplus}B \to \phi^{\oplus}\gamma^{\otimes}B \tag{22}$$

is an equivalence for any $B \in \mathcal{B}_{c,y''}$.

We first prove (A). Proving that the functor $\eta_{\circledast}$ has a right adjoint for $\eta\colon y' \to y$ in $\mathcal{C}_L$ amounts to showing that for any bispan $B = (y \xleftarrow{f} z \xrightarrow{g} x \xrightarrow{h} c)$, the ∞-category $\mathcal{B}_{y',c/B} := \mathcal{B}_{y',c} \times_{\mathcal{B}_{y,c}} \mathcal{B}_{y,c/B}$ has a terminal object (a reformulation of [Lur09, Lemma 5.2.4.1]). An object of $\mathcal{B}_{y',c/B}$ is a commutative diagram

$$
\begin{tikzcd}
y' \arrow[d, "\eta"] & z' \arrow[l, "f'"'] \arrow[r, "g'"] \arrow[d, "\zeta"] & x' \arrow[d, "\xi"] \arrow[dr, "h'"] & \\
 & & & c \\
y & z \arrow[l, "f"] \arrow[r, "g"'] & x, \arrow[ur, "h"'] &
\end{tikzcd}
\tag{23}
$$

where the middle square is cartesian.

Let $\tilde{z} := y' \times_y z$ and write u for the projection $\tilde{z} \to z$; this is a pullback of η and so lies in $\mathcal{C}_L$. We then have the distributivity diagram

$$
\begin{tikzcd}
 & w \times_x z \arrow[r, "g'"] \arrow[dl, "\epsilon"'] \arrow[dd, "v'"] & w \arrow[dd, "v"] \\
\tilde{z} \arrow[dr, "u"'] & & \\
 & z \arrow[r, "g"] & x
\end{tikzcd}
\tag{24}
$$

for the pair (u, g), where the morphism v is $g_*u \simeq g_*f^*\eta$. The universal property of (24) says that

$$\mathrm{Map}_{/x}(x', w) \to \mathrm{Map}_{/z}(z', w \times_x z) \to \mathrm{Map}_{/z}(z', \tilde{z}) \simeq \mathrm{Map}_{/y}(z', y')$$

is an equivalence. We can then insert the diagram (24) into the diagram (23) in the sense that there exist fillers to make the next diagram commute:

$$\begin{tikzcd}
 & & z' \arrow[r] \arrow[dll] \arrow[d, dashed] & x' \arrow[d, dashed] \arrow[ddr] & \\
y' \arrow[d] & \tilde{z} \arrow[l] \arrow[dr] & z \times_x w \arrow[l] \arrow[r] \arrow[d] & w \arrow[d] & \\
y & & z \arrow[ll] \arrow[r] & x \arrow[r] & c.
\end{tikzcd}$$

In other words, there is a unique morphism in $\mathcal{B}_{y',c/B}$ to the bispan

$$y' \leftarrow z \times_x w \to w \to c$$

over B. Hence this is a terminal object as required. We denote the right adjoint to $\eta_\circledast$ by

$$\eta^\oplus \colon \mathcal{B}_{y,c} \to \mathcal{B}_{y',c}.$$

We now prove (B). Given a pullback square (20) we must check that the Beck-Chevalley transformation

$$\phi_\circledast \eta^\oplus \to \tilde{\eta}^\oplus \tilde{\phi}_\circledast$$

is an equivalence. For a bispan $\tilde{y} \leftarrow z \to x \to c$ this is immediate from the commutative diagram

$$\begin{tikzcd}
y' \arrow[d] & \tilde{y}' \arrow[l] \arrow[d] & \tilde{z} \arrow[l] \arrow[d] & w \times_x z \arrow[l] \arrow[r] \arrow[dl] & w \arrow[d] \arrow[dr] & \\
y & \tilde{y} \arrow[l] & z \arrow[l] \arrow[rr] & & x \arrow[r] & c,
\end{tikzcd}$$

where the two left-hand squares are cartesian and the middle is a distributivity diagram, since this shows the same distributivity occurs for $\eta^\oplus$ and $\tilde{\eta}^\oplus \tilde{\phi}_\circledast$. Thus B_c is adjointable.

We will prove (C) in three steps:

(C.1) We prove that all codistributivity transformations (22) give equivalences when evaluated at the bispan

$$[l]_L := (x = x = x \xrightarrow{l} y)$$

for $l \colon x \to y$ in $\mathcal{C}_L$.

(C.2) We prove that if all codistributivity transformations give equivalences when evaluated at a certain object X, then they also give equivalences when evaluated at $g^\otimes X$.

(C.3) We prove that if all codistributivity transformations give equivalences when evaluated at a certain object X, then they also give equivalences when evaluated at $f_\circledast X$.

Since we know an arbitrary bispan $B = (y \xleftarrow{p} z \xrightarrow{f} x \xrightarrow{l} c)$ can be written as as

$$B = p_\circledast f^\otimes [l],$$

this will prove (C).

For (C.1), the codistributivity transformation $\epsilon_\circledast \tilde{\gamma}^\otimes \psi^\oplus [l] \to \phi^\oplus \gamma^\otimes [l]$ can be viewed as a morphism $\epsilon_\circledast \tilde{\gamma}^\otimes \psi^\oplus [l] \to \gamma^\otimes [l]$ over ϕ, and it suffices to show that this is a cartesian morphism. Unpacking the definition of codistributivity transformations, we see that this morphism decomposes in two steps as

$$\begin{tikzcd}
\bullet \arrow[d, "\phi"] & \bullet \arrow[l, "\epsilon"'] \arrow[r, "\tilde{\gamma}"] \arrow[d, equal] & \bullet \arrow[r, "\psi l"] \arrow[d, equal] & \bullet \arrow[d, equal] \\
\bullet \arrow[d, equal] & \bullet \arrow[l, "\tilde{\psi}"'] \arrow[r, "\tilde{\gamma}"] \arrow[d, "\tilde{\psi}"] & \bullet \arrow[r, "\psi l"] \arrow[d, "\psi"] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, equal] & \bullet \arrow[r, "\gamma"] & \bullet \arrow[r, "l"] & \bullet,
\end{tikzcd}$$

where the top row of morphisms is the cocartesian arrow

$$\epsilon_{\circledast}\tilde{\gamma}^{\otimes}\psi^{\oplus}[l] \to \phi_{\circledast}\epsilon_{\circledast}\tilde{\gamma}^{\otimes}\psi^{\oplus}[l] \simeq \tilde{\psi}_{\circledast}\tilde{\gamma}^{\otimes}\psi^{\oplus}[l]$$

over ϕ and the bottom row is the composite

$$\tilde{\psi}_{\circledast}\tilde{\gamma}^{\otimes}\psi^{\oplus}[l] \simeq \gamma^{\otimes}\psi_{\circledast}\psi^{\oplus}[l] \to \gamma^{\otimes}[l].$$

It then remains to see that the composite

$$\begin{tikzcd}
\bullet \arrow[d, "\phi"] & \bullet \arrow[l, "\epsilon"'] \arrow[r, "\tilde{\gamma}"] \arrow[d, "\tilde{\psi}"] & \bullet \arrow[r, "\psi l"] \arrow[d, "\psi"] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, equal] & \bullet \arrow[r, "\gamma"] & \bullet \arrow[r, "l"] & \bullet
\end{tikzcd}$$

is a cartesian morphism over ϕ, which is clear from our description of cartesian morphisms in the proof of step (A) and the definition of the top bispan in terms of the distributivity diagram for ϕ and γ.

For (C.2), Proposition 2.4.5 implies that the codistributivity transformation

$$\epsilon_{\circledast}\tilde{\gamma}^{\otimes}\psi^{\oplus}g^{\otimes}X \to \phi^{\oplus}\gamma^{\otimes}g^{\otimes}X$$

fits in a commutative triangle with the codistributivity transformations for $\psi^{\oplus}g^{\otimes}X$ and for $\phi^{\oplus}(g\gamma)^{\otimes}X$, both of which are equivalences by assumption.

For (C.3), Proposition 2.4.6 implies that the codistributivity transformation

$$\epsilon_{\circledast}\tilde{\gamma}^{\otimes}\psi^{\oplus}f_{\circledast}X \to \phi^{\oplus}\gamma^{\otimes}f_{\circledast}X$$

fits in a commutative square where the other sides are two equivalences and the codistributivity transformation for the pullback of (21) along f evaluated at X and composed with $f'_{\circledast}$, where f' is the pullback of f along $\gamma\phi$. The latter is an equivalence by assumption, hence so is the transformation for $f_{\circledast}X$. □

Translating Proposition 2.5.2 through the equivalence of Lemma 2.5.5, we see that Proposition 2.5.11 implies that any object $X \in \mathcal{B}_c$ over $c' \in \mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}$ corresponds to a morphism $\mathcal{Y}_{c'} \to \mathcal{B}_c$. In particular, we have:

Corollary 2.5.13. *There is a canonical functor $\beta_c \colon \mathcal{Y}_c \to \mathcal{B}_c$ over $\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}$ corresponding to the identity bispan of c; this preserves cocartesian morphisms and cartesian morphisms over $\mathcal{C}_L^{\mathrm{op}}$.* □

We now need to prove that the functor α_c has the same property:

Proposition 2.5.14. *The functor $\alpha_c : \mathcal{B}_c \to \mathcal{Y}_c$ over $\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}$ preserves cocartesian morphisms and cartesian morphisms over $\mathcal{C}_L^{\mathrm{op}}$.*

Remark 2.5.15. For the proof we first need to discuss the naturality of Bech–Chevalley transformations in the following situation: Suppose we have a commutative triangle of ∞-categories

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "F"] \arrow[dr, "p"'] & & \mathcal{F} \arrow[dl, "q"] \\
& \mathcal{B}, &
\end{tikzcd}$$

where p and q have both cartesian and cocartesian morphisms over $f\colon a \to b$ in $\mathcal{B}$, but F does not necessarily preserve these. We write $f_!$ for the cocartesian pushforward and f^* for the cartesian pullback along f for both p and q (so the functor $f_!$ is left adjoint to f^*). Then we can make the following commutative diagrams for $x \in \mathcal{E}_a$, $y \in \mathcal{E}_b$:

$$\begin{tikzcd}
Fx \arrow[r] \arrow[d] & Ff^*f_!x \arrow[d] \\
f^*f_!Fx \arrow[r] \arrow[d] & f^*Ff_!x \arrow[d] \\
f_!Fx \arrow[r] & Ff_!x,
\end{tikzcd}
\qquad
\begin{tikzcd}
Ff^*y \arrow[r] \arrow[d] & f^*Fy \arrow[d] \\
f_!Ff^*y \arrow[r] \arrow[d] & f_!f^*Fy \arrow[d] \\
Ff_!f^*y \arrow[r] & Fy.
\end{tikzcd} \tag{25}$$

In particular, the top left and bottom right squares here encode the compatibility of F with the units and counits of the two adjunctions $f_! \dashv f^*$. Now suppose we have a commutative square

$$\begin{tikzcd}
a' \arrow[r, "f'"] \arrow[d, "g'"] & b' \arrow[d, "g"] \\
a \arrow[r, "f"] & b,
\end{tikzcd}$$

where p and q have cocartesian morphisms over f, f' and both cartesian and cocartesian morphisms over g, g'. Then we claim that the two Beck–Chevalley transformations $f'_!g'^* \to g^*f_!$, intertwined by F, are related by a commutative diagram

$$\begin{tikzcd}
 & f'_!F(g'^*X) \arrow[dl] \arrow[dr] & \\
f'_!g'^*F(X) \arrow[d] & & F(f'_!g'^*X) \arrow[d] \\
g^*f_!FX \arrow[dr] & & F(g^*f_!X) \arrow[dl] \\
 & g^*F(f_!X). &
\end{tikzcd} \tag{26}$$

This can be extracted from the following diagram, where we have used (25) together with naturality:

$$
\begin{tikzcd}
 & & & f'_!Fg'^*X \arrow[dll] \arrow[dd] \arrow[dr] & & \\
 & f'_!g'^*FX \arrow[d] & & & Ff'_!g'^*X \arrow[dd] \arrow[ddr] & \\
 & g^*g_!f'_!g'^*FX \arrow[dl, "\sim"] & & g^*g_!f'_!Fg'^*X \arrow[ll] \arrow[dl, "\sim"'] \arrow[dr] & & \\
g^*f_!g'_!g'^*FX \arrow[d] & & g^*f_!g'_!Fg'^*X \arrow[ll] \arrow[d] & & g^*g_!Ff'_!g'^*X \arrow[d] & Fg^*g_!f'_!g'^*X \arrow[dl] \arrow[d, "\sim"] \\
g^*f_!FX \arrow[ddrrr, bend right] & & g^*f_!Fg'_!g'^*X \arrow[ll] \arrow[dr] & & g^*Fg_!f'_!g'^*X \arrow[dl, "\sim"'] & Fg^*f_!g'_!g'^*X \arrow[dll] \arrow[d] \\
 & & & g^*Ff_!g'_!g'^*X \arrow[d] & & Fg^*f_!X \arrow[dll] \\
 & & & g^*Ff_!X. & &
\end{tikzcd}
$$

Proof of Proposition 2.5.14. The universal property we used to define α_c implies that it preserves cocartesian morphisms. Moreover, since α_c was extended from a functor $\mathcal{C}^L_{/c} \to \mathcal{Y}_c$ that preserved cartesian morphisms over $\mathcal{C}_L$, we know α_c preserves cartesian morphisms in the image of $\mathcal{C}^L_{/c}$. In other words, for $h\colon x \to c$ in $\mathcal{C}_L$ the map $\alpha_c(\phi^\oplus[h]_L) \to \phi^\oplus\alpha_c([h]_L)$ is an equivalence for all $\phi\colon x' \to x$ in $\mathcal{C}_L$.

More generally, for a bispan $B = (y \xleftarrow{f} z \xrightarrow{g} x \xrightarrow{h} c)$, we need to show that $\alpha_c(\eta^\oplus B) \simeq \eta^\oplus\alpha_c(B)$ for any morphism $\eta\colon y' \to y$. To proceed let us view B as $f_\circledast g^\otimes[h]$. Forming the pullback square,

$$
\begin{tikzcd}
z' \arrow[r, "f'"] \arrow[d, "\zeta"] & y' \arrow[d, "\eta"] \\
z \arrow[r, "f"] & y,
\end{tikzcd}
$$

the Beck-Chevalley transformation yields an equivalence:

$$f'_\circledast\zeta^\oplus g^\otimes[h] \xrightarrow{\sim} \eta^\oplus f_\circledast g^\otimes[h].$$

Moreover, from (26) we get a natural commutative diagram

$$
\begin{tikzcd}
 & f'_\circledast\alpha_c(\zeta^\oplus X) \arrow[dl] \arrow[dr] & \\
f'_\circledast\zeta^\oplus\alpha_c(X) \arrow[d, "\sim"] & & \alpha_c(f'_\circledast\zeta^\oplus X) \arrow[d, "\sim"] \\
\eta^\oplus f_\circledast\alpha_c(X) \arrow[dr] & & \alpha_c(\eta^\oplus f_\circledast X) \arrow[dl] \\
 & \eta^\oplus\alpha_c(f_\circledast X), &
\end{tikzcd}
$$

where the labelled equivalences come from the Beck-Chevalley transformations.

Next form the distributivity diagram

$$\begin{tikzcd}
& \tilde{z} \arrow[dl, "\epsilon"] \arrow[dd, "\zeta'"] \arrow[r, "\tilde{g}"] & x' \arrow[dd, "\xi = g_*\zeta"] \\
z' \arrow[dr, "\zeta"] & & \\
& z \arrow[r, "g"] & x.
\end{tikzcd}$$

Then using the diagrams (25) as in the construction of (26) we can show that the two distributivity transformations are also related by a natural commutative diagram

$$\begin{tikzcd}
& \epsilon_{\circledast}\tilde{g}^{\otimes}\alpha_c(\xi^{\oplus}X) \arrow[dl] \arrow[dr] & \\
\epsilon_{\circledast}\tilde{g}^{\otimes}\xi^{\oplus}\alpha_c(X) \arrow[d, "\sim"] & & \alpha_c(\epsilon_{\circledast}\tilde{g}^{\otimes}\xi^{\oplus}X) \arrow[d, "\sim"] \\
\zeta^{\oplus}g^{\otimes}\alpha_c(X) \arrow[dr] & & \alpha_c(\zeta^{\oplus}g^{\otimes}X) \arrow[dl] \\
& \zeta^{\oplus}\alpha_c(g^{\otimes}X), &
\end{tikzcd}$$

where the labelled equivalences come from the codistributivity transformations.

Combining these two diagrams we get the following one:

$$\begin{tikzcd}
& f'_{\circledast}\epsilon_{\circledast}\tilde{g}^{\otimes}\alpha_c(\xi^{\oplus}[h]) \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & & \\
f'_{\circledast}\epsilon_{\circledast}\tilde{g}^{\otimes}\xi^{\oplus}\alpha_c([h]) \arrow[d, "\sim"] & & f'_{\circledast}\alpha_c(\epsilon_{\circledast}\tilde{g}^{\otimes}\xi^{\oplus}[h]) \arrow[d, "\sim"] & \\
f'_{\circledast}\zeta^{\oplus}g^{\otimes}\alpha_c([h]) \arrow[dr, "\sim"] & & f'_{\circledast}\alpha_c(\zeta^{\oplus}g^{\otimes}[h]) \arrow[dl] \arrow[dr, "\sim"] & \\
& f'_{\circledast}\zeta^{\oplus}\alpha_c(g^{\otimes}[h]) \arrow[d, "\sim"] & & \alpha_c(f'_{\circledast}\zeta^{\oplus}g^{\otimes}[h]) \arrow[d, "\sim"] \\
& \eta^{\oplus}f_{\circledast}g^{\otimes}\alpha_c([h]) \arrow[dr, "\sim"] & & \alpha_c(\eta^{\oplus}f_{\circledast}g^{\otimes}[h]) \arrow[dl] \\
& & \eta^{\oplus}\alpha_c(f_{\circledast}g^{\otimes}[h]), &
\end{tikzcd}$$

Here we know all but two arrows are equivalences, as indicated, by the explanations above and the fact that we already know α_c preserves cocartesian morphisms as well as the cartesian morphism $\xi^{\oplus}[h] \to [h]$. It follows that the last two arrows are also equivalences, and so the natural map

$$\alpha_c(\eta^{\oplus}B) \to \eta^{\oplus}\alpha_c(B)$$

is an equivalence, as required. $\square$

Corollary 2.5.16. *The functors $\beta_c \colon \mathcal{Y}_c \to \mathcal{B}_c$ and $\alpha_c \colon \mathcal{B}_c \to \mathcal{Y}_c$ satisfy*

$$\beta_c\alpha_c \simeq \mathrm{id}_{\mathcal{B}_c}, \qquad \alpha_c\beta_c \simeq \mathrm{id}_{\mathcal{Y}_c}.$$

Thus α_c is an equivalence with inverse β_c.

Proof. By construction α_c takes the identity bispan of c to

$$\mathrm{id}_c \in \mathcal{Y}_{c,c} \simeq \mathrm{MAP}_{\mathrm{BISPAN}_{F,L}(\mathcal{C})}(i(c), i(c)).$$

The composite $\beta_c\alpha_c$ is a functor $\mathcal{B}_c \to \mathcal{B}_c$ that preserves cocartesian morphisms, hence it's determined by its restriction to $\mathcal{C}^L_{/c} \to \mathcal{B}_c$. This restriction preserves cartesian morphisms over $\mathcal{C}_L$ and so by Lemma 2.5.7 it is determined by its value at id_c, which is the identity bispan in $\mathcal{B}_c$. The same holds for the identity of $\mathcal{B}_c$ and so $\mathrm{id}_{\mathcal{B}_c} \simeq \beta_c\alpha_c$. Conversely, $\alpha_c\beta_c$ is a functor $\mathcal{Y}_c \to \mathcal{Y}_c$ that preserves cocartesian morphisms and cartesian morphisms over $\mathcal{C}_L$ by Proposition 2.5.14. By Proposition 2.5.2, interpreted in terms of fibrations, this functor is determined by where it sends the identity of c; since we know this is taken to itself, this functor must be the identity $\mathrm{id}_{\mathcal{Y}_c}$. □

The equivalence $\mathcal{Y}_c \simeq \mathcal{B}_c$ allows us to identify morphisms $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ with bispans, and 2-morphisms with morphisms of bispans. We now check that composition of bispans works as expected:

Proposition 2.5.17. *Composition of bispans in* $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ *is given by the composition law (5).*

Proof. By construction, the cocartesian morphisms in $\mathcal{Y}_c$ encode composition with the images of spans in $\mathrm{Span}_F(\mathcal{C})$ under the functor i: Given such a span $x \xleftarrow{f} y \xrightarrow{g} z$ with g in $\mathcal{C}_F$, we have

$$i(x \xleftarrow{f} y \xrightarrow{g} z) \circ B \simeq f_{\circledast}g^{\otimes}B$$

for any bispan B. In particular, from our description of the right-hand side we have

$$i(x \xleftarrow{f} y \xrightarrow{g} z) \simeq i(x \xleftarrow{f} y \xrightarrow{g} z) \circ \mathrm{id}_z \simeq x \xleftarrow{f} y \xrightarrow{g} z = z,$$

and more generally

$$x \xleftarrow{f} y \xrightarrow{g} z \xrightarrow{l} w \simeq f_{\circledast}g^{\otimes}[l]_L \simeq (x \xleftarrow{f} y \xrightarrow{g} z = z) \circ (z = z = z \xrightarrow{l} w),$$

which agrees with the composition of these bispans according to (5).

This means that to describe an arbitrary composition in $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ it suffices (by associativity of composition) to understand compositions of the form

$$(\bullet \leftarrow \bullet \rightarrow \bullet = \bullet) \circ B$$

and

$$(\bullet = \bullet = \bullet \rightarrow \bullet) \circ B.$$

The first case is, as already mentioned, given by the cocartesian morphisms in $\mathcal{Y}_c$ if B is a bispan with target c, which we know is obtained from the composition law in spans, i.e. the composite

$$(\bullet \leftarrow a \rightarrow y = y) \circ (y \leftarrow z \rightarrow x \rightarrow c)\,,$$

is given by

$$\bullet \leftarrow a \times_y z \rightarrow x \rightarrow c.$$

By inspection, this agrees with the composition law in (5).

It remains to describe composition with bispans of the form $[\eta]_L$ with $\eta \in \mathcal{C}_L$. First consider the bispan

$$[\eta]_B := y \xleftarrow{\eta} y' = y' = y'.$$

Now the bispan $[\eta]_B$ is in the image of $\mathcal{C}_L^{\mathrm{op}} \to \mathrm{Span}_F(\mathcal{C}) \to \mathrm{BISPAN}_{F,L}(\mathcal{C})$ and therefore admits a left adjoint in $\mathrm{BISPAN}_{F,L}(\mathcal{C})$, which we denote by $[\eta]'_L$. We claim that

$$[\eta]'_L \simeq [\eta]_L.$$

To see this, we note that composition on the left with $[\eta]_B$ is exactly the functor $\eta_{\circledast}$ which admits a right adjoint $\eta^{\oplus}$ (see the beginning of the proof of Proposition 2.5.11). Therefore, by uniqueness of adjoints we conclude that composition with $[\eta]'_L$ coincides with the right adjoint of $\eta_{\circledast}$, i.e., with $\eta^{\oplus}$. Therefore the bispan $[\eta]'_L$ is computed as

$$[\eta]'_L \simeq \eta^{\oplus}(\mathrm{id}) \simeq \eta^{\oplus}(y = y = y = y) \simeq [\eta]_L.$$

Furthermore, this also shows that we can compute composition on the left with $[\eta]_L$ as applying $\eta^{\oplus}$. Examining the formula for $\eta^{\oplus}$ described in the proof of Proposition Proposition 2.5.11 now shows agreement with the composition law described in (5). $\square$

Remark 2.5.18. This proof also shows that the bispan $[\eta]_L$ for $\eta \in \mathcal{C}_L$ is left adjoint to $[\eta]_B$ in $\mathrm{BISPAN}_{F,L}(\mathcal{C})$. Unpacking the unit and counit for this adjunction, we see that they correspond to the diagrams

$$\begin{tikzcd}
 & y' \arrow[r, equal] \arrow[dl, equal] \arrow[dd, "\Delta"] \arrow[dr, phantom, "\lrcorner", very near start] & y' \arrow[dr, equal] \arrow[dd, "\Delta"] & \\
y' & & & y' \\
 & y' \times_y y' \arrow[r, equal] \arrow[ul, "\pi_1"] & y', \times_y y', \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & y' \arrow[r, equal] \arrow[dl, "\eta"'] \arrow[dd, "\eta"] \arrow[dr, phantom, "\lrcorner", very near start] & y' \arrow[dr, "\eta"] \arrow[dd, "\eta"] & \\
y \arrow[dr, equal] & & & y \arrow[dl, equal] \\
 & y \arrow[r, equal] & y, &
\end{tikzcd}$$

respectively, where $\Delta \colon y' \to y' \times_y y'$ is the diagonal and $\pi_1, \pi_2 \colon y' \times_y y' \to y'$ are the two projections.

Proposition 2.5.19. *A bispan $B = (x \xleftarrow{f} y \xrightarrow{g} z \xrightarrow{h} w)$ is invertible as a morphism in $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ if and only if the components f, g, h are all invertible in $\mathcal{C}$.*

Proof. For this proof it is convenient to use the abbreviated notation $B = (f, g, h)$ for a bispan, with the names of the objects involved suppressed. It is clear that a bispan is invertible if all its components are equivalences in $\mathcal{C}$, so suppose that B is an invertible bispan with inverse $A = (a, b, c)$, a bispan $w \to x$. As a special case of (5) we have that the bispan $B \colon x \to w$ decomposes as $(\mathrm{id}, \mathrm{id}, h) \circ (f, g, \mathrm{id})$.

Let $A' = A \circ (\mathrm{id}, \mathrm{id}, h)$ with $A' = (a', b', c')$. Then on the one hand

$$A' \circ (f, g, \mathrm{id}) \simeq A \circ B \simeq (\mathrm{id}, \mathrm{id}, \mathrm{id}),$$

but on the other hand (5) implies that this is (α, β, c') where (α, β) is the composite of the spans (a', b') and (f, g). Thus $c' \simeq \mathrm{id}$.

Now we consider the composite $B \circ A'$. On the one hand we have

$$B \circ A' \simeq B \circ A \circ (\mathrm{id}, \mathrm{id}, h) \simeq (\mathrm{id}, \mathrm{id}, h).$$

On the other hand, (5) implies that

$$B \circ A' \simeq (f, g, h) \circ (a', b', \mathrm{id}) \simeq (\phi, \gamma, h)$$

where (ϕ, γ) is the composite of the spans (f, g) and (a', b'). Thus we have shown that the span (f, g) is invertible in $\mathrm{Span}_F(\mathcal{C})$; by [Hau18, Lemma 8.2] this implies that f and g is invertible.

Similarly, if we set $B' = B \circ (\mathrm{id}, \mathrm{id}, c)$ with $B' = (f', g', h')$ then $h' \simeq \mathrm{id}$, and analyzing the composite $A \circ B'$ we see that (a, b) is an invertible span, and hence a and b are equivalences too.

The compositions $A \circ B$ and $B \circ A$ now degenerate to ordinary compositions in $\mathcal{C}$, and so h and c must also be equivalences. $\square$

Corollary 2.5.20. *The functor $i\colon \mathrm{Span}_F(\mathcal{C}) \to \mathrm{BISPAN}_{F,L}(\mathcal{C})$ gives an equivalence on underlying ∞-groupoids*

$$\mathcal{C}^{\simeq} \simeq \mathrm{Span}_F(\mathcal{C})^{\simeq} \xrightarrow{\sim} \mathrm{BISPAN}_{F,L}(\mathcal{C})^{\simeq}.$$

Proof. It follows from [Hau18, Proposition 8.1] that $\mathcal{C}^{\simeq} \xrightarrow{\sim} \mathrm{Span}_F(\mathcal{C})^{\simeq}$. We know the functor i is essentially surjective on objects, so it is enough to show that for any objects x, y the map

$$\mathrm{Map}_{\mathcal{C}}(x, y)^{\mathrm{eq}} \to \mathrm{Map}_{\mathrm{Bispan}_{F,L}(\mathcal{C})}(ix, iy)^{\mathrm{eq}}$$

is an equivalence, where we are taking the components of the mapping spaces that correspond to equivalences. This is immediate from Proposition 2.5.19 and our description of the mapping spaces in $\mathrm{Bispan}_{F,L}(\mathcal{C})$. $\square$

Combining the results of this section, we get the following precise version of Theorem 1.4.2:

Theorem 2.5.21. *Let $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ be a bispan triple. Then L-distributive functors out of $\mathrm{Span}_F(\mathcal{C})$ are corepresented by an $(\infty, 2)$-category $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ via an L-distributive functor $i\colon \mathrm{Span}_F(\mathcal{C}) \to \mathrm{BISPAN}_{F,L}(\mathcal{C})$ with the following properties:*

- *i gives an equivalence*
$$\mathrm{BISPAN}_{F,L}(\mathcal{C})^{\simeq} \simeq \mathcal{C}^{\simeq}$$
on underlying ∞-groupoids,
- *morphisms from $i(x)$ to $i(y)$ can be identified with bispans*
$$x \xleftarrow{p} e \xrightarrow{f} b \xrightarrow{l} y$$
where f is in $\mathcal{C}_F$ and l is in $\mathcal{C}_L$,
- *2-morphisms correspond to diagrams of the form (6),*
- *and composition of morphisms is as in (5).* $\square$

We end this section by deducing a description of the functor of $(\infty, 2)$-categories corresponding to a distributive functor:

Proposition 2.5.22. *For an L-distributive functor $\phi\colon \mathrm{Span}_F(\mathcal{C}) \to \mathcal{X}$, the corresponding functor $\Phi\colon \mathrm{BISPAN}_{F,L}(\mathcal{C}) \to \mathcal{X}$ can be described as follows:*

(1) On objects, $\Phi(c) \simeq \phi(c)$ for $c \in \mathcal{C}$.

(2) On morphisms, Φ takes a bispan
$$B = (x \xleftarrow{s} e \xrightarrow{f} b \xrightarrow{l} y)$$
to the composite $l_{\oplus} f_{\otimes} s^{\circledast}\colon \phi(x) \to \phi(y)$, where $l_{\oplus}$ is the left adjoint to $l^{\circledast}$.

(3) On 2-morphisms, Φ takes the 2-morphism $\beta\colon B \to B'$ given by the commutative diagram

$$\begin{tikzcd}
& e \arrow[r, "f"] \arrow[dl, "s"'] \arrow[dd, "g"] \arrow[ddr, phantom, "\lrcorner", very near start] & b \arrow[dr, "l"] \arrow[dd, "h"] & \\
x & & & y \\
& e' \arrow[ul, "s'"] \arrow[r, "f'"'] & b' \arrow[ur, "l'"'] &
\end{tikzcd}$$

to the composite

$$l_\oplus f_\otimes s^\circledast \simeq l_\oplus f_\otimes g^\circledast s'^\circledast \simeq l_\oplus h^\circledast f'_\otimes s'^\circledast \to l_\oplus h^\circledast l'^\circledast l'_\oplus f'_\otimes s'^\circledast \simeq l_\oplus l^\circledast l'_\oplus f'_\otimes s'^\circledast \to l'_\oplus f'_\otimes s'^\circledast,$$

where the first noninvertible arrow is a unit and the second noninvertible arrow is a counit.

Proof. We know that $\Phi \circ i \simeq \phi$ and that i is an equivalence on underlying ∞-groupoids by Corollary 2.5.20, which gives (1). To prove (2), observe that the bispan B is the composite in $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ of $B' = (x \xleftarrow{s} e \xrightarrow{f} b = b)$ and $[l] = (b = b = b \xrightarrow{l} y)$. Here B' is the image of the span $S = (x \xleftarrow{s} e \xrightarrow{f} b)$ in $\mathrm{Span}_F(\mathcal{C})$, and so we have

$$\Phi(B') \simeq \Phi(i(S)) \simeq \phi(S) \simeq f_\otimes s^\circledast.$$

By Remark 2.5.18 the bispan $[l]_L$ is left adjoint to $[l]_B := (y \xleftarrow{l} b = b = b)$, hence its image $\Phi([l]_L)$ is the left adjoint to $l^\circledast$. Thus we have $\Phi(B) \simeq \Phi([l]_L) \circ \Phi(B') \simeq l_\oplus f_\otimes s^\circledast$.

To prove (3), first observe that the 2-morphism β is the composite ("whiskering") of the morphism $x \xleftarrow{s'} e' \xrightarrow{f'} b' = b'$ with the 2-morphism λ given by

$$\begin{tikzcd}
 & b \arrow[r, equal] \arrow[dl, "h"'] \arrow[dd, "h"] \arrow[dr, phantom, "\lrcorner", very near start] & b \arrow[dd, "h"] \arrow[dr, "l"] & \\
b' \arrow[dr, equal] & & & y \\
 & b' \arrow[r, equal] & b', \arrow[ur, "l'"'] &
\end{tikzcd}$$

and this whiskering corresponds to the first two equivalences in (3).

It thus suffices to show that Φ takes λ to the composite

$$l_\oplus h^\circledast \to l_\oplus h^\circledast l'^\circledast l'_\oplus \simeq l_\oplus l^\circledast l'_\oplus \to l'_\oplus$$

using the unit for $l'_\oplus \dashv l'^\circledast$ and the counit for $l_\oplus \dashv l^\circledast$. To show this we will check that the morphism λ has the corresponding decomposition in $\mathrm{BISPAN}_{F,L}(\mathcal{C})$. Indeed, we can decompose λ as the composite

$$\begin{tikzcd}
 & b \arrow[r, equal] \arrow[dl, "h"'] \arrow[d, "\phi"] & b \arrow[d, "\phi"] \arrow[dr, "l"] & \\
b' & b' \times_y b \arrow[l, "\pi'"'] \arrow[r, equal] \arrow[d, "\pi'"] & b' \times_y b \arrow[r, "l\pi"] \arrow[d, "\pi'"] & y \\
 & b' \arrow[ul, equal] \arrow[r, equal] & b', \arrow[ur, "l'"'] &
\end{tikzcd}$$

where π, π' are the projections from $b' \times_y b$ to b and b', respectively, and ϕ is the unique morphism such that $\pi\phi \simeq \mathrm{id}$, $\pi'\phi \simeq h$. Now the description of units and counits in Remark 2.5.18 implies that the top morphism in this decomposition is the composite of the unit for $[l'] \dashv [l']'$ with the morphism $b' \xleftarrow{h} b = b \xrightarrow{l} y$ and the bottom is the composite of $[l']$ with the counit for $[l]_L \dashv [l]_B$. $\square$

2.6. **Symmetric monoidal structures.** In this section we will prove that the functors $\mathrm{SPAN}\colon \mathrm{Pair} \to \mathrm{Cat}_{(\infty,2)}$ and $\mathrm{BISPAN}\colon \mathrm{Trip} \to \mathrm{Cat}_{(\infty,2)}$ both preserve products. As a consequence, in some cases a symmetric monoidal structure on $\mathcal{C}$ will induce a symmetric monoidal structure on $\mathrm{BISPAN}_{F,L}(\mathcal{C})$. In the case of spans, this is discussed in [GR17, Part III, Chapter 9] (where it is used to encode Serre duality for Ind-coherent sheaves) and [Mac20, §3.2]. f An explicit construction (not relying on the universal property) of symmetric monoidal structures on ∞-categories of spans is also given in [BGS20].

The technical input to this discussion is Corollary 2.6.12 which proves that bispans preserves products. We will work towards proving this by first establishing the same result for spans.

Proposition 2.6.1. *The ∞-categories* Pair *and* Trip *have finite products, given by*

$$(\mathcal{C}, \mathcal{C}_F) \times (\mathcal{C}', \mathcal{C}'_{F'}) \simeq (\mathcal{C} \times \mathcal{C}', \mathcal{C}_F \times \mathcal{C}'_{F'}),$$

$$(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L) \times (\mathcal{C}', \mathcal{C}'_{F'}, \mathcal{C}'_{L'}) \simeq (\mathcal{C} \times \mathcal{C}', \mathcal{C}_F \times \mathcal{C}'_{F'}, \mathcal{C}_L \times \mathcal{C}'_{L'}).$$

Proof. It follows from the definition of Pair that the morphism

$$\mathrm{Map}_{\mathrm{Pair}}((\mathcal{C}, \mathcal{C}_F), (\mathcal{C}', \mathcal{C}'_{F'})) \to \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathcal{C}')$$

is a monomorphism (and similarly for Trip), so it suffices to show that for any span pair $(\mathcal{D}, \mathcal{D}_F)$ (or bispan triple $(\mathcal{D}, \mathcal{D}_F, \mathcal{D}_L)$) a functor $\mathcal{D} \to \mathcal{C} \times \mathcal{C}'$ is a morphism of span pairs (or bispan triples) if and only if the functors $\mathcal{D} \to \mathcal{C}$ and $\mathcal{D} \to \mathcal{C}'$ are both morphisms of span pairs (or bispan triples). This is clear, since a pair of cartesian squares in $\mathcal{C}$ and $\mathcal{C}'$ gives a cartesian square in $\mathcal{C} \times \mathcal{C}'$, and similarly for distributivity diagrams. □

Proposition 2.6.2. *Suppose $(\mathcal{C}, \mathcal{C}_F)$ and $(\mathcal{C}', \mathcal{C}'_{F'})$ are span pairs. Then a functor $\Phi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C}'^{\mathrm{op}} \to \mathcal{X}$ is right (F, F')-adjointable if and only if it is right adjointable in each variable, i.e.*

- *for every $c \in \mathcal{C}$, the functor $\Phi(c, -)$ is right F'-adjointable,*
- *for every $c' \in \mathcal{C}'$, the functor $\Phi(-, c')$ is right F-adjointable,*
- *for every morphism $c_1 \to c_2$ in $\mathcal{C}$, the transformation $\Phi(c_2, -) \to \Phi(c_1, -)$ is right F'-adjointable,*
- *for every morphism $c'_1 \to c'_2$ in $\mathcal{C}'$, the transformation $\Phi(c'_2, -) \to \Phi(c'_1, -)$ is right F-adjointable.*

Proof. We first observe that Φ is right (F, F')-*preadjointable* if and only if $(f, \mathrm{id}_{c'})^{\circledast}$ has a right adjoint for all f in F and c' in $\mathcal{C}'$, and $(\mathrm{id}_c, f')^{\circledast}$ has a right adjoint for all c in $\mathcal{C}$ and f' in F', since $(f, f')^{\circledast} \simeq (f, \mathrm{id})^{\circledast}(\mathrm{id}, f')^{\circledast}$ and right adjoints compose. Thus Φ is right (F, F')-preadjointable if and only if $\Phi(c, -)$ is right F'-preadjointable for all $c \in \mathcal{C}$ and $\Phi(-, c')$ is right F-preadjointable for all c' in $\mathcal{C}'$.

A pair of cartesian squares gives a cartesian square in $\mathcal{C} \times \mathcal{C}'$, so if Φ is right (F, F')-preadjointable then it is right (F, F')-adjointable if and only if for cartesian squares

$$\begin{tikzcd} w \arrow[r, "v"] \arrow[d, "u"] & z \arrow[d, "g"] \\ x \arrow[r, "f"] & y, \end{tikzcd} \qquad \begin{tikzcd} w' \arrow[r, "v'"] \arrow[d, "u'"] & z' \arrow[d, "g'"] \\ x' \arrow[r, "f'"] & y', \end{tikzcd}$$

with f in F and f' in F', the square

$$\begin{tikzcd} \Phi(y, y') \arrow[r, "(f,f')^{\circledast}"] \arrow[d, "(g,g')^{\circledast}"] & \Phi(x, x') \arrow[d, "(u,u')^{\circledast}"] \\ \Phi(z, z') \arrow[r, "(v,v')^{\circledast}"] & \Phi(w, w') \end{tikzcd} \tag{27}$$

is right adjointable. Taking $f = g = \mathrm{id}_c$ this implies that $\Phi(c, -)$ is right F'-adjointable, while taking $f = \mathrm{id}_y$ we see that the transformation $\Phi(g, -)\colon \Phi(y, -) \to \Phi(z, -)$ is right F'-adjointable. The same goes in the other variable, so if Φ is right (F, F')-adjointable then the four given conditions hold.

Conversely, if these conditions hold then we want to show that the square (27) is right adjointable. We can decompose this square into the commutative diagram

$$\begin{tikzcd} \Phi(y,y') \arrow[r, "{(f,\mathrm{id})^{\circledast}}"] \arrow[d, "{(g,\mathrm{id})^{\circledast}}"] & \Phi(x,y') \arrow[r, "{(\mathrm{id},f')^{\circledast}}"] \arrow[d, "{(u,\mathrm{id})^{\circledast}}"] & \Phi(x,x') \arrow[d, "{(u,\mathrm{id})^{\circledast}}"] \\ \Phi(z,y') \arrow[r, "{(v,\mathrm{id})^{\circledast}}"] \arrow[d, "{(\mathrm{id},g')^{\circledast}}"] & \Phi(w,y') \arrow[r, "{(\mathrm{id},f')^{\circledast}}"] \arrow[d, "{(\mathrm{id},g')^{\circledast}}"] & \Phi(w,x') \arrow[d, "{(\mathrm{id},u')^{\circledast}}"] \\ \Phi(z,z') \arrow[r, "{(v,\mathrm{id})^{\circledast}}"] & \Phi(w,z') \arrow[r, "{(\mathrm{id},v')^{\circledast}}"] & \Phi(w,w'). \end{tikzcd}$$

Here all four squares are right adjointable:

- the top left square since $\Phi(-,y')$ is right F-adjointable,
- the top right square since $\Phi(u,-)$ is a right F'-adjointable transformation,
- the bottom left square since $\Phi(-,g')$ is a right F-adjointable transformation,
- the bottom right square since $\Phi(w,-)$ is right F'-adjointable.

Since mate transformations are compatible with horizontal and vertical compositions of squares, right adjointable squares are closed under both horizontal and vertical compositions. Thus the outer square (27) is right adjointable, which completes the proof. □

Corollary 2.6.3. *Suppose $(\mathcal{C},\mathcal{C}_F)$ and $(\mathcal{C}',\mathcal{C}'_{F'})$ are span pairs. Then there is a natural equivalence*

$$\mathrm{Map}_{(F,F')\text{-radj}}(\mathcal{C}^{\mathrm{op}}\times\mathcal{C}'^{\mathrm{op}},\mathcal{X})\simeq\mathrm{Map}_{F\text{-radj}}(\mathcal{C}^{\mathrm{op}},\mathrm{FUN}_{F'\text{-radj}}(\mathcal{C}'^{\mathrm{op}},\mathcal{X}))$$

for $\mathcal{X}\in\mathrm{Cat}_{(\infty,2)}$.

To prove this we also need the following lemma:

Lemma 2.6.4. *Suppose $(\mathcal{C},\mathcal{C}_F)$ is a span pair. A morphism $\eta\colon\phi\to\psi$ in the $(\infty,2)$-category $\mathrm{FUN}_{F\text{-radj}}(\mathcal{C},\mathcal{X})$ has a right adjoint if and only if*

- *the morphism $\eta_c\colon\phi(c)\to\psi(c)$ has a right adjoint in $\mathcal{X}$ for every $c\in\mathcal{C}$,*
- *the commutative square*

$$\begin{tikzcd} \phi(c) \arrow[r, "\eta_c"] \arrow[d] & \psi(c) \arrow[d] \\ \phi(c') \arrow[r, "\eta_{c'}"] & \psi(c') \end{tikzcd}$$

  is right adjointable for every morphism $c\to c'$ in $\mathcal{C}$.

Moreover, a commutative square

$$\begin{tikzcd} \phi \arrow[r, "\eta"] \arrow[d, "\lambda"] & \psi \arrow[d, "\mu"] \\ \phi' \arrow[r, "\eta'"] & \psi' \end{tikzcd}$$

in $\mathrm{FUN}_{F\text{-radj}}(\mathcal{C},\mathcal{X})$ is right adjointable if and only if the commutative square in $\mathcal{X}$ obtained by evaluation at c is right adjointable in $\mathcal{X}$ for every $c\in\mathcal{C}$.

Proof. By the universal property of spans, we have an equivalence

$$\mathrm{FUN}_{F\text{-radj}}(\mathcal{C},\mathcal{X})\simeq\mathrm{FUN}(\mathrm{SPAN}_F(\mathcal{C}),\mathcal{X});$$

suppose $H\colon\Phi\to\Psi$ is the morphism corresponding to $\eta\colon\phi\to\psi$ under this equivalence. Then we know that H has a right adjoint if and only if

- $H_c\colon\Phi(c)\to\Psi(c)$ has a right adjoint for every $c\in\mathcal{C}$,

- the square

$$\begin{tikzcd} \Phi(c_1) \arrow[r, "H_{c_1}"] \arrow[d] & \Psi(c_1) \arrow[d] \\ \Phi(c_2) \arrow[r, "H_{c_2}"] & \Psi(c_2) \end{tikzcd}$$

 is right adjointable for every morphism $c_1 \to c_2$ in $\mathrm{SPAN}_F(\mathcal{C})$.

In terms of η, these conditions say that η_c has a right adjoint for every $c \in \mathcal{C}$, and for every span $c_1 \xleftarrow{g} x \xrightarrow{f} c_2$ with f in F, the outer square in the diagram

$$\begin{tikzcd} \phi(c_1) \arrow[r, "\eta_{c_1}"] \arrow[d, "g^{\circledast}"] & \psi(c_1) \arrow[d, "g^{\circledast}"] \\ \phi(x) \arrow[r, "\eta_x"] \arrow[d, "f_{\circledast}"] & \psi(x) \arrow[d, "f_{\circledast}"] \\ \phi(c_2) \arrow[r, "\eta_{c_2}"] & \psi(c_2) \end{tikzcd}$$

is right adjointable. Since right adjointable squares compose, and the two squares here are those associated to spans where one leg is the identity, it is equivalent to require these two squares to be right adjointable. For the top square this is the condition we want, while the bottom square is automatically right adjointable since its mate is obtained by passing to right adjoints everywhere in the commutative square

$$\begin{tikzcd} \phi(c_2) \arrow[r, "\eta_{c_2}"] \arrow[d, "f^{\circledast}"] & \psi(c_2) \arrow[d, "f^{\circledast}"] \\ \phi(x) \arrow[r, "\eta_x"] & \psi(x). \end{tikzcd}$$

Since the mate of a square of natural transformations is given by taking mates objectwise, the characterization of right adjointable squares is immediate. □

Proof of Corollary 2.6.3. Unpacking definitions, a functor $\Phi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C}'^{\mathrm{op}} \to \mathcal{X}$ corresponds to a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}_{F'\text{-radj}}(\mathcal{C}'^{\mathrm{op}}, \mathcal{X})$ if and only if $\Phi(c, -)$ is a right F'-adjointable functor for every $c \in \mathcal{C}$, and $\Phi(c_2, -) \to \Phi(c_1, -)$ is a right F'-adjointable transformation for every morphism $c_1 \to c_2$ in $\mathcal{C}$.

Moreover, it follows from Lemma 2.6.4 that such a functor

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}_{F'\text{-radj}}(\mathcal{C}', \mathcal{X}) \simeq \mathrm{Fun}(\mathrm{SPAN}_{F'}(\mathcal{C}'), \mathcal{X})$$

is right F-adjointable precisely when the following conditions hold:

- for every morphism $f\colon c_1 \to c_2$ in F and every object $c' \in \mathcal{C}'$, the morphism $(f, \mathrm{id})^{\circledast}\colon \Phi(c_2, c') \to \Phi(c_1, c')$ has a right adjoint,
- for every morphism $f\colon c_1 \to c_2$ in F and every morphism $c'_1 \to c'_2$ in $\mathcal{C}'$, the commutative square

$$\begin{tikzcd} \Phi(c_2, c'_2) \arrow[r, "(f,\mathrm{id})^{\circledast}"] \arrow[d] & \Phi(c_1, c'_2) \arrow[d] \\ \Phi(c_2, c'_1) \arrow[r, "(f,\mathrm{id})^{\circledast}"] & \Phi(c_1, c'_1) \end{tikzcd}$$

 is right adjointable,

- for every pullback square
$$\begin{tikzcd} w \arrow[r, "v"] \arrow[d, "u"] & z \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$
in $\mathcal{C}$ with f in F, the commutative square
$$\begin{tikzcd} \Phi(y, c') \arrow[r, "(f,\mathrm{id})^{\circledast}"] \arrow[d, "(g,\mathrm{id})^{\circledast}"] & \Phi(x, c') \arrow[d, "(u,\mathrm{id})^{\circledast}"] \\ \Phi(z, c') \arrow[r, "(v,\mathrm{id})^{\circledast}"] & \Phi(w, c') \end{tikzcd}$$
is right adjointable.

These conditions say precisely that $\Phi(-, c')$ is a right F-adjointable functor for every $c' \in \mathcal{C}'$ and $\Phi(-, c'_2) \to \Phi(-, c'_1)$ is a right F-adjointable transformation for every morphism $c'_1 \to c'_2$ in $\mathcal{C}'$. We have thus shown that a functor $\Phi\colon \mathcal{C} \times \mathcal{C}' \to \mathcal{X}$ corresponds to a right F-adjointable functor $\mathcal{C} \to \mathrm{Fun}_{F'\text{-radj}}(\mathcal{C}', \mathcal{X})$ if and only if it satisfies the four conditions that we saw characterized right (F, F')-adjointable functors in Proposition 2.6.2. □

From Corollary 2.6.3 we deduce that spans preserves products, which is enough to deduce a sufficient condition for a symmetric monoidal structure on SPAN.

Corollary 2.6.5. *Suppose $(\mathcal{C}, \mathcal{C}_F)$ and $(\mathcal{C}', \mathcal{C}'_{F'})$ are span pairs. Then the natural morphism*
$$\mathrm{SPAN}_{(F,F')}(\mathcal{C} \times \mathcal{C}') \to \mathrm{SPAN}_F(\mathcal{C}) \times \mathrm{SPAN}_{F'}(\mathcal{C}')$$
is an equivalence.

Proof. For $\mathcal{X}$ an $(\infty, 2)$-category we have natural equivalences
$$\begin{aligned} \mathrm{Map}(\mathrm{SPAN}_{(F,F')}(\mathcal{C} \times \mathcal{C}'), \mathcal{X}) &\simeq \mathrm{Map}_{(F,F')\text{-radj}}(\mathcal{C} \times \mathcal{C}', \mathcal{X}) \\ &\simeq \mathrm{Map}_{F\text{-radj}}(\mathcal{C}, \mathrm{FUN}_{F'\text{-radj}}(\mathcal{C}', \mathcal{X})) \\ &\simeq \mathrm{Map}_{F\text{-radj}}(\mathcal{C}, \mathrm{FUN}(\mathrm{SPAN}_{F'}(\mathcal{C}'), \mathcal{X})) \\ &\simeq \mathrm{Map}(\mathrm{SPAN}_F(\mathcal{C}), \mathrm{FUN}(\mathrm{SPAN}_{F'}(\mathcal{C}'), \mathcal{X})) \\ &\simeq \mathrm{Map}(\mathrm{SPAN}_F(\mathcal{C}) \times \mathrm{SPAN}_{F'}(\mathcal{C}'), \mathcal{X}). \end{aligned}$$
□

Corollary 2.6.6. *Suppose $(\mathcal{C}, \mathcal{C}_F)$ is a span pair and $\mathcal{C}$ has a (symmetric) monoidal structure such that the tensor product functor is a morphism of span pairs*
$$\otimes\colon (\mathcal{C} \times \mathcal{C}, \mathcal{C}_F \times \mathcal{C}_F) \to (\mathcal{C}, \mathcal{C}_F),$$
i.e. given morphisms $f\colon x \to y$ and $f'\colon x' \to y'$ in F, the morphism $f \otimes f'\colon x \otimes x' \to y \otimes y'$ is also in F, and given a pair of pullback squares
$$\begin{tikzcd} w \arrow[r, "v"] \arrow[d, "u"] & z \arrow[d, "g"] \\ x \arrow[r, "f"] & y, \end{tikzcd} \qquad \begin{tikzcd} w' \arrow[r, "v'"] \arrow[d, "u'"] & z' \arrow[d, "g'"] \\ x' \arrow[r, "f'"] & y', \end{tikzcd}$$
with f and f' in $\mathcal{C}_F$, the commutative square
$$\begin{tikzcd} w \otimes w' \arrow[r] \arrow[d] & z \otimes z' \arrow[d] \\ x \otimes x' \arrow[r] & y \otimes y' \end{tikzcd}$$
is cartesian. Then $\mathrm{SPAN}_F(\mathcal{C})$ inherits a (symmetric) monoidal structure from that on $\mathcal{C}$.

Proof. Since the functor SPAN preserves products, it takes (commutative) algebras in span pairs to (commutative) algebras in $(\infty, 2)$-categories. □

Examples 2.6.7.

(i) Suppose $(\mathcal{C}, \mathcal{C}_F)$ is a span pair where $\mathcal{C}$ has finite products and morphisms in $\mathcal{C}_F$ are closed under products. Products of cartesian squares are always again cartesian, so in this case Corollary 2.6.6 implies that the cartesian product induces a symmetric monoidal structure on $\mathrm{SPAN}_F(\mathcal{C})$. This recovers the discussion in [GR17, Chapter 9; 2.1] and some cases of [BGS20, Theorem 2.15].

(ii) Suppose $(\mathcal{C}, \mathcal{C}_F)$ is a span pair where $\mathcal{C}$ has finite coproducts and morphisms in $\mathcal{C}_F$ are closed under coproducts. If finite coproducts in $\mathcal{C}$ satisfy *descent* in the sense that the coproduct functor

$$\amalg\colon \prod_{i=1}^{n} \mathcal{C}_{/x_i} \to \mathcal{C}_{/\coprod_{i=1}^{n} x_i}$$

is an equivalence, then coproducts of cartesian squares are again cartesian. Hence in this case the coproduct induces a symmetric monoidal structure on $\mathrm{SPAN}_F(\mathcal{C})$ by Corollary 2.6.6. The descent condition is satisfied, for instance, if $\mathcal{C}$ is an ∞-topos, or in the category of sets. See also [Bar17, §4], where $\mathcal{C}$ is called "disjunctive" if $\mathcal{C}$ has pullbacks and the descent condition for coproducts holds; in this case the coproduct in $\mathcal{C}$ gives both the product and coproduct in $\mathrm{Span}(\mathcal{C})$ by [Bar17, Proposition 4.3].

Now we turn to the analogous results for bispans:

Proposition 2.6.8. *Suppose $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ and $(\mathcal{C}', \mathcal{C}'_F, \mathcal{C}'_L)$ are bispan triples. Then a functor $\Phi\colon \mathrm{Span}_F(\mathcal{C}) \times \mathrm{Span}_{F'}(\mathcal{C}') \to \mathcal{X}$ is (L, L')-distributive if and only if it is distributive in each variable, i.e.*

- *for each $c \in \mathcal{C}$, the functor $\Phi(c, \text{–})\colon \mathrm{Span}_{F'}(\mathcal{C}') \to \mathcal{X}$ is L'-distributive,*
- *for each $c' \in \mathcal{C}'$, the functor $\Phi(\text{–}, c')\colon \mathrm{Span}_F(\mathcal{C}) \to \mathcal{X}$ is L-distributive,*
- *for each morphism $c_1 \leftarrow x \to c_2$ in $\mathrm{Span}_F(\mathcal{C})$, the transformation $\Phi(c_1, \text{–}) \to \Phi(c_2, \text{–})$ is L'-distributive,*
- *for each morphism $c'_1 \leftarrow x' \to c'_2$ in $\mathrm{Span}_{F'}(\mathcal{C}')$, the transformation $\Phi(\text{–}, c'_1) \to \Phi(\text{–}, c'_2)$ is L-distributive.*

Remark 2.6.9. We can reformulate these conditions as follows:

(1) For each c in $\mathcal{C}$, the functor $\Phi(c, -)\colon \mathcal{C}'^{\mathrm{op}} \to \mathcal{X}$ is left L'-adjointable.
(2) For each c' in $\mathcal{C}'$, the functor $\Phi(-, c')\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{X}$ is left L-adjointable.
(3) For each morphism $\gamma\colon c_1 \to c_2$ in $\mathcal{C}$ and $l'\colon x' \to y'$ in L', the commutative square

$$\begin{tikzcd}
\Phi(c_2, y') \arrow[r, "(\mathrm{id}_{c_2}, l')^{\circledast}"] \arrow[d, "(\gamma, \mathrm{id}_{y'})^{\circledast}"] & \Phi(c_2, x') \arrow[d, "(\gamma, \mathrm{id}_{x'})^{\circledast}"] \\
\Phi(c_1, y') \arrow[r, "(\mathrm{id}_{c_2}, l')^{\circledast}"] & \Phi(c_1, x')
\end{tikzcd}$$

is left adjointable.
(4) For each morphism $\gamma'\colon c'_1 \to c'_2$ in $\mathcal{C}'$ and $l\colon x \to y$ in L, the commutative square

$$\begin{tikzcd}
\Phi(y, c'_2) \arrow[r, "(l, \mathrm{id}_{c'_2})^{\circledast}"] \arrow[d, "(\mathrm{id}_y, \gamma')^{\circledast}"] & \Phi(x, c'_2) \arrow[d, "(\mathrm{id}_x, \gamma')^{\circledast}"] \\
\Phi(y, c'_1) \arrow[r, "(l, \mathrm{id}_{c_2})^{\circledast}"] & \Phi(x, c'_1)
\end{tikzcd}$$

is left adjointable.

(5) For each morphism $f\colon c_2 \to c_1$ in $\mathcal{C}_F$ and $l'\colon x' \to y'$ in L', the commutative square

$$\begin{tikzcd} \Phi(c_2, y') \arrow[r, "(\mathrm{id}_{c_2}, l')^{\circledast}"] \arrow[d, "(f, \mathrm{id}_{y'})_{\otimes}"] & \Phi(c_2, x') \arrow[d, "(f, \mathrm{id}_{x'})_{\otimes}"] \\ \Phi(c_1, y') \arrow[r, "(\mathrm{id}_{c_2}, l')^{\circledast}"] & \Phi(c_1, x') \end{tikzcd}$$

is left adjointable.

(6) For each morphism $f'\colon c_2' \to c_1'$ in F' and $l\colon x \to y$ in L, the commutative square

$$\begin{tikzcd} \Phi(y, c_2') \arrow[r, "(l, \mathrm{id}_{c_2'})^{\circledast}"] \arrow[d, "(\mathrm{id}_y, f')_{\otimes}"] & \Phi(x, c_2') \arrow[d, "(\mathrm{id}_x, f')_{\otimes}"] \\ \Phi(y, c_1') \arrow[r, "(l, \mathrm{id}_{c_2})^{\circledast}"] & \Phi(x, c_1') \end{tikzcd}$$

is left adjointable.

(7) For c in $\mathcal{C}$, $f\colon x' \to y'$ in L', and $g\colon y' \to z'$ in F', the distributivity transformation

$$(\mathrm{id}_c, h)_{\oplus}(\mathrm{id}_c, \tilde{g})_{\otimes}(\mathrm{id}_c, \epsilon)^{\circledast} \to (\mathrm{id}_c, g)_{\otimes}, (\mathrm{id}_c, f)_{\oplus}$$

is an equivalence.

(8) For c' in $\mathcal{C}'$, $f\colon x \to y$ in L, and $g\colon y \to z$ in F, the distributivity transformation

$$(h, \mathrm{id}_{c'})_{\oplus}(\tilde{g}, \mathrm{id}_{c'})_{\otimes}(\epsilon, \mathrm{id}_{c'})^{\circledast} \to (g, \mathrm{id}_{c'})_{\otimes}, (f, \mathrm{id}_{c'})_{\oplus}$$

is an equivalence.

Proof of Proposition 2.6.8. By Proposition 2.6.2 we know that the first four conditions in Remark 2.6.9 are equivalent to the restriction of Φ to $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}'^{\mathrm{op}} \to \mathcal{X}$ being left (L, L')-adjointable. It remains to show that the last four conditions are equivalent to the distributivity transformation

$$(h, h')_{\oplus}(\tilde{g}, \tilde{g}')_{\otimes}(\epsilon, \epsilon')^{\circledast} \to (g, g')_{\otimes}(f, f')_{\oplus} \tag{28}$$

being an equivalence for any $(f\colon x \to y, f'\colon x' \to y')$ in (L, L') and $(g\colon y \to z, g'\colon y' \to z')$ in (F, F').

First suppose (28) is always an equivalence. Conditions (7) and (8) are special cases of this. Moreover, if we specialize to $f' = \mathrm{id}_{y'}$, $g = \mathrm{id}_y$ then the distributivity transformation reduces to the mate transformation

$$(f, \mathrm{id}_{z'})_{\oplus}(\mathrm{id}_x, g')_{\otimes} \to (\mathrm{id}_y, g')_{\otimes}(f, \mathrm{id}_{y'})_{\oplus},$$

which gives condition (6), while putting f and g' to be identities gives condition (5).

For the other direction we want to decompose the distributivity transformation for (f, f') and (g, g'). We first consider the case where $f = \mathrm{id}$; decomposing $(g, g')_{\otimes}$ as $(g, \mathrm{id})_{\otimes}(\mathrm{id}, g')_{\otimes}$, we can apply Proposition 2.4.5 to factor the distributivity transformation as

$$(\mathrm{id}, h')_{\oplus}(g, \mathrm{id})_{\otimes}(\mathrm{id}, \tilde{g}')_{\otimes}(\mathrm{id}, \epsilon')^{\circledast} \to (g, \mathrm{id})_{\otimes}(\mathrm{id}, h')_{\oplus}(\mathrm{id}, \tilde{g}')_{\otimes}(\mathrm{id}, \epsilon')^{\circledast} \to (g, \mathrm{id})_{\otimes}(\mathrm{id}, g')_{\otimes}(\mathrm{id}, f')_{\otimes},$$

where the first map uses the distributivity transformation for (g, id) and (id, h') (which is a mate transformation) and the second uses the distributivity transformation for (id, g') and (id, f'). These are both equivalences by assumption, hence this composite is also an equivalence. Similarly, the distributivity transformation

$$(h, \mathrm{id})_{\oplus}(\tilde{g}, g')_{\otimes}(\epsilon, \mathrm{id})^{\circledast} \to (g, g')_{\otimes}(f, \mathrm{id})_{\otimes}$$

is also an equivalence.

Now we decompose $(g,g')_{\otimes}(f,f')_{\oplus}$ as $(g,g')_{\otimes}(\mathrm{id},f')_{\oplus}(f,\mathrm{id})_{\oplus}$; applying Proposition 2.4.4 we can decompose (28) as

$$(\mathrm{id},h')_{\oplus}(h,\mathrm{id})_{\oplus}(\tilde{g},\tilde{g}')_{\otimes}(\epsilon,\mathrm{id})^{\circledast}(\mathrm{id},\epsilon')^{\circledast} \to (\mathrm{id},h')_{\oplus}(g,\tilde{g}')_{\otimes}(f,\mathrm{id})_{\oplus}(\mathrm{id},\epsilon')^{\circledast} \to (\mathrm{id},h')_{\oplus}(g,\tilde{g}')_{\otimes}(\mathrm{id},\epsilon')^{\circledast}(f,\mathrm{id})_{\oplus} \to (g,g')_{\otimes}(\mathrm{id},f')_{\oplus}(f,\mathrm{id})_{\oplus},$$

where the first map uses the distributivity transformation for $(g,\tilde{g}')$ and (f,id), the second uses a mate transformation, and the third uses the distributivity transformation for (g,g') and (id,f'). All three of these are equivalences, hence so is the composite (28), as required. □

Corollary 2.6.10. *Suppose $(\mathcal{C},\mathcal{C}_F,\mathcal{C}_L)$ and $(\mathcal{C}',\mathcal{C}'_F,\mathcal{C}'_L)$ are bispan triples. Then there is a natural equivalence*

$$\mathrm{Map}_{(L,L')\text{-dist}}(\mathrm{Span}_F(\mathcal{C})\times\mathrm{Span}_{F'}(\mathcal{C}'),\mathcal{X}) \simeq \mathrm{Map}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}),\mathrm{FUN}_{L'\text{-dist}}(\mathrm{Span}_{F'}(\mathcal{C}'),\mathcal{X})).$$

For the proof we need the following observation:

Lemma 2.6.11. *Suppose $(\mathcal{C},\mathcal{C}_F,\mathcal{C}_L)$ is a bispan triple. A morphism $\eta\colon \phi\to\psi$ in $\mathrm{FUN}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}),\mathcal{X})$ has a left adjoint if and only if*

- *the morphism $\eta_c\colon \phi(c)\to\psi(c)$ has a left adjoint for every $c\in\mathcal{C}$,*
- *the commutative square*

$$\begin{tikzcd} \phi(c_2) \arrow[r, "\eta_{c_2}"] \arrow[d, "f^{\circledast}"] & \psi(c_2) \arrow[d, "f^{\circledast}"] \\ \phi(c_1) \arrow[r, "\eta_{c_1}"] & \psi(c_1) \end{tikzcd}$$

  is left adjointable for every morphism $f\colon c_1\to c_2$ in $\mathcal{C}$,
- *the commutative square*

$$\begin{tikzcd} \phi(c_2) \arrow[r, "\eta_{c_2}"] \arrow[d, "f_{\otimes}"] & \psi(c_2) \arrow[d, "f_{\otimes}"] \\ \phi(c_1) \arrow[r, "\eta_{c_1}"] & \psi(c_1) \end{tikzcd}$$

  is left adjointable for every morphism $f\colon c_2\to c_1$ in $\mathcal{C}_F$.

Proof. We have an equivalence

$$\mathrm{FUN}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}),\mathcal{X}) \simeq \mathrm{FUN}(\mathrm{BISPAN}_{F,L}(\mathcal{C}),\mathcal{X});$$

suppose $H\colon \Phi\to\Psi$ is the morphism corresponding to $\eta\colon\phi\to\psi$ under this equivalence. Then we know that H has a left adjoint if and only if

- $H_c\colon \Phi(c)\to\Psi(c)$ has a left adjoint for every $c\in\mathcal{C}$,
- the square

$$\begin{tikzcd} \Phi(c_1) \arrow[r, "H_{c_1}"] \arrow[d] & \Psi(c_1) \arrow[d] \\ \Phi(c_2) \arrow[r, "H_{c_2}"] & \Psi(c_2) \end{tikzcd}$$

  is left adjointable for every morphism $c_1\to c_2$ in $\mathrm{BISPAN}_{F,L}(\mathcal{C})$.

As in the proof of Lemma 2.6.4 the second condition reduces to the case where the vertical morphisms are of the form $f^{\circledast}$ or $f_{\otimes}$ with f in F. □

Proof of Corollary 2.6.10. According to Proposition 2.6.8 an (L,L')-distributive functor $\Phi\colon \mathrm{Span}_F(\mathcal{C})\times\mathrm{Span}_{F'}(\mathcal{C}')\to\mathcal{X}$ corresponds to a functor $\Phi'\colon \mathrm{Span}_F(\mathcal{C})\to\mathrm{FUN}_{L'\text{-dist}}(\mathrm{Span}_{F'}(\mathcal{C}'),\mathcal{X})$ such that the functor $\Phi(-,c')$ is L-distributive for every $c'\in\mathcal{C}'$ and the transformation $\Phi(-,c'_1)\to\Phi(-,c'_2)$ is L-distributive for every morphism $c'_1\to c'_2$ in $\mathrm{Span}_{F'}(\mathcal{C}')$. By Lemma 2.6.11 these are precisely the conditions for Φ' to be L-distributive. □

Corollary 2.6.12. *Suppose $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ and $(\mathcal{C}', \mathcal{C}'_F, \mathcal{C}'_L)$ are bispan triples. Then the natural morphism*

$$\mathrm{BISPAN}_{(F,F'),(L,L')}(\mathcal{C} \times \mathcal{C}') \to \mathrm{BISPAN}_{F,L}(\mathcal{C}) \times \mathrm{BISPAN}_{F',L'}(\mathcal{C}')$$

is an equivalence.

Proof. We have natural equivalences

$$\begin{aligned}
\mathrm{Map}(\mathrm{BISPAN}_{(F,F'),(L,L')}(\mathcal{C} \times \mathcal{C}'), \mathcal{X}) &\simeq \mathrm{Map}_{(L,L')\text{-dist}}(\mathrm{Span}_{(F,F')}(\mathcal{C} \times \mathcal{C}'), \mathcal{X}) \\
&\simeq \mathrm{Map}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathrm{FUN}_{L'\text{-dist}}(\mathrm{Span}_{F'}(\mathcal{C}'), \mathcal{X})) \\
&\simeq \mathrm{Map}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathrm{FUN}(\mathrm{BISPAN}_{F',L'}(\mathcal{C}'), \mathcal{X})) \\
&\simeq \mathrm{Map}(\mathrm{BISPAN}_{F,L}(\mathcal{C}), \mathrm{FUN}(\mathrm{BISPAN}_{F',L'}(\mathcal{C}'), \mathcal{X})) \\
&\simeq \mathrm{Map}(\mathrm{BISPAN}_{F,L}(\mathcal{C}) \times \mathrm{BISPAN}_{F',L'}(\mathcal{C}'), \mathcal{X}),
\end{aligned}$$

as required. □

From this, the main result on symmetric monoidal structures on bispans follows.

Theorem 2.6.13. *Suppose $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ is a bispan triple and $\mathcal{C}$ has a (symmetric) monoidal structure such that the tensor product is a morphism of bispan triples*

$$\otimes \colon (\mathcal{C} \times \mathcal{C}, \mathcal{C}_F \times \mathcal{C}_F, \mathcal{C}_L \times \mathcal{C}_L) \to (\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L),$$

i.e.

- *both $\mathcal{C}_F$ and $\mathcal{C}_L$ are closed under $\otimes$,*
- *given a pair of pullback squares*

$$\begin{tikzcd} w \arrow[r, "v"] \arrow[d, "u"] & z \arrow[d, "g"] \\ x \arrow[r, "f"] & y, \end{tikzcd} \qquad \begin{tikzcd} w' \arrow[r, "v'"] \arrow[d, "u'"] & z' \arrow[d, "g'"] \\ x' \arrow[r, "f'"] & y', \end{tikzcd}$$

  with f and f' either both in F or both in L, the commutative square

$$\begin{tikzcd} w \otimes w' \arrow[r] \arrow[d] & z \otimes z' \arrow[d] \\ x \otimes x' \arrow[r] & y \otimes y' \end{tikzcd}$$

  is cartesian,
- *given morphisms $f \colon x \to y$, $f' \colon x' \to y'$ in $\mathcal{C}_L$ and $g \colon y \to z$, $g' \colon y' \to z'$, the diagram*

$$\begin{tikzcd} & \bullet \otimes \bullet \arrow[r] \arrow[dl] \arrow[dd] & \bullet \otimes \bullet \arrow[dd, "g_* f \otimes g'_* f'"] \\ x \otimes x' \arrow[dr, "f \otimes f'"'] & & \\ & y \otimes y' \arrow[r, "g \otimes g'"] & z \otimes z' \end{tikzcd}$$

  obtained as the tensor product of the distributivity diagrams for (f, g) and (f', g'), is a distributivity diagram for $(f \otimes f', g \otimes g')$.[8]

Then $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ inherits a (symmetric) monoidal structure from that on $\mathcal{C}$. □

[8]Note that the previous condition implies that the square in this diagram is cartesian; the condition can therefore be interpreted as asking for the natural map

$$g_* f \otimes g'_* f' \to (g \otimes g')_*(f \otimes f')$$

arising from this cartesian square to be an equivalence.

Example 2.6.14. Suppose $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ is a bispan triple such that $\mathcal{C}$ has finite coproducts, morphisms in $\mathcal{C}_F$ and $\mathcal{C}_L$ are closed under coproducts, and finite coproducts satisfy descent as in Examples 2.6.7(ii). Then the coproduct satisfies the conditions of Theorem 2.6.13: The descent condition implies that coproducts of cartesian squares are cartesian, and the condition on distributivity diagrams amounts to asking for the natural map

$$g_*f \amalg g'_*f' \to (g \amalg g')_*(f \amalg f')$$

to be an equivalence; this is true because by descent we have

$$\begin{aligned}
\mathrm{Map}_{/z\amalg z'}(u \amalg u', g_*f \amalg g'_*f') &\simeq \mathrm{Map}_{/z}(u, g_*f) \times \mathrm{Map}_{/z'}(u', g'_*f') \\
&\simeq \mathrm{Map}_{/y}(g^*u, f) \times \mathrm{Map}_{/y'}(g'^*u', f') \\
&\simeq \mathrm{Map}_{/y\amalg y'}(g^*u \amalg g'^*u', f \amalg f') \\
&\simeq \mathrm{Map}_{/y\amalg y'}((g \amalg g')^*(u \amalg u'), f \amalg f') \\
&\simeq \mathrm{Map}_{/y\amalg y'}(u \amalg u', (g \amalg g')_*(f \amalg f')).
\end{aligned}$$

for an object $u \amalg u'$ over $z \amalg z'$. The coproduct therefore induces a symmetric monoidal structure on $\mathrm{BISPAN}_{F,L}(\mathcal{C})$.[9]

Remark 2.6.15. Suppose $\mathcal{C}$ is a locally cartesian closed ∞-category with descent for finite coproducts. Then the symmetric monoidal structure on $\mathrm{Bispan}(\mathcal{C})$ induced by the coproduct in $\mathcal{C}$ is a cartesian product: we have

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Bispan}(\mathcal{C})}(c, x \amalg y) &\simeq \{c \leftarrow a \to b \to x \amalg y\} \\
&\simeq \{c \leftarrow a_x \amalg a_y \to b_x \amalg b_y \to x \amalg y\} \\
&\simeq \{c \leftarrow a_x \to b_x \to x\} \times \{c \leftarrow a_y \to b_y \to y\} \\
&\simeq \mathrm{Map}_{\mathrm{Bispan}(\mathcal{C})}(c, x) \times \mathrm{Map}_{\mathrm{Bispan}(\mathcal{C})}(c, y).
\end{aligned}$$

Moreover, we have the same identification for ∞-categories of morphisms, so this is actually an $(\infty, 2)$-categorical product. However, this is *not* a coproduct in bispans: in particular, $\emptyset$ is not an initial object, since we have

$$\mathrm{Map}_{\mathrm{Bispan}(\mathcal{C})}(\emptyset, x) \simeq \{\emptyset \leftarrow \emptyset \to b \to x\} \simeq \mathcal{C}^{\simeq}_{/x}$$

which is not in general contractible.

Remark 2.6.16. For any ∞-category $\mathcal{C}$ we can consider the minimal bispan triple $\mathcal{C}^\flat := (\mathcal{C}, \mathcal{C}^{\simeq}, \mathcal{C}^{\simeq})$ where the morphisms in $\mathcal{C}_F$ and $\mathcal{C}_L$ are just the equivalences. Any functor gives a morphism of minimial bispan triples, so we have a functor

$$(-)^\flat \colon \mathrm{Cat}_\infty \to \mathrm{Trip}$$

that is moreover fully faithful. The ∞-category Trip is then a Cat_∞-module via cartesian products with $(-)^\flat$. We also have a natural equivalence $\mathrm{BISPAN}(\mathcal{C}^\flat) \simeq \mathcal{C}$ as all functors are distributive. This means the functor $\mathrm{BISPAN} \colon \mathrm{Trip} \to \mathrm{Cat}_{(\infty,2)}$ is a morphism of Cat_∞-modules, which we can view as a functor of $(\infty, 2)$-categories using the recent results of Heine [Hei20]. Moreover, the natural transformation $\mathrm{Span}_F(\mathcal{C}) \to \mathrm{BISPAN}_{F,L}(\mathcal{C})$ is a transformation of Cat_∞-modules, which means the universal property of $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ is actually Cat_∞-natural.

[9]However, the cartesian product in $\mathcal{C}$ does not typically give a symmetric monoidal structure on bispans — we do *not* in general have an equivalence between $g_*f \times g'_*f'$ and $(g \times g')_*(f \times f')$.

3. Examples of distributivity

3.1. **Bispans in finite sets and symmetric monoidal ∞-categories.** In this section we consider the relationship between symmetric monoidal ∞-categories and bispans in finite sets. We first recall that symmetric monoidal ∞-categories can be described in terms of functors from spans of finite sets, and then show that the resulting functor is distributive if and only if the tensor product commutes with finite coproducts in each variable. Our universal property then gives a (product-preserving) functor from bispans in finite sets, which we can interpret as a semiring structure with the coproduct as addition and the tensor product as multiplication.

Notation 3.1.1. We write $\mathbb{F}$ for the category of finite sets and $\mathbb{F}_*$ for the category of finite pointed sets and base-point preserving maps; every object of $\mathbb{F}_*$ is isomorphic to one of the form $\langle n\rangle := (\{0,\ldots,n\},0)$. For $I \in \mathbb{F}$ we write I_+ for the pointed set $(I \amalg \{*\}, *)$ obtained by adding a disjoint base point to I.

Definition 3.1.2. If $\mathcal{C}$ is an ∞-category with finite products, a *commutative monoid* in $\mathcal{C}$ is a functor $\Phi\colon \mathbb{F}_* \to \mathcal{C}$ such that for every $n = 0, 1, \ldots$ the map

$$\Phi(\langle n\rangle) \xrightarrow{(\Phi(\rho_i))_{i=1,\ldots,n}} \prod_{i=1}^{n} \Phi(\langle 1\rangle)$$

is an equivalence, where $\rho_i\colon \langle n\rangle \to \langle 1\rangle$ is defined by

$$\rho_i(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

We write $\mathrm{CMon}(\mathcal{C})$ for the full subcategory of $\mathrm{Fun}(\mathbb{F}_*, \mathcal{C})$ spanned by the commutative monoids.

Notation 3.1.3. If $\mathcal{C}, \mathcal{D}$ are ∞-categories with finite products, we write $\mathrm{Fun}^\times(\mathcal{C}, \mathcal{D})$ for the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the functors that preserve finite products.

Remark 3.1.4. The category $\mathbb{F}_*$ can be identified with the subcategory of $\mathrm{Span}(\mathbb{F})$ whose morphisms are the spans $I \xleftarrow{f} J \xrightarrow{g} K$ where f is injective, with this corresponding to the morphism $I_+ \to K_+$ given by

$$i \mapsto \begin{cases} g(j), & i = f(j), \\ *, & \text{otherwise;} \end{cases}$$

we write $j\colon \mathbb{F}_* \to \mathrm{Span}(\mathbb{F})$ for this subcategory inclusion.

The following description of commutative monoids in terms of spans seems to have been first proved by Cranch [Cra10, Cra11]; other proofs, as special cases of various generalizations, can be found in [Gla17, Har20, BaHo18].

Proposition 3.1.5. *Let $\mathcal{C}$ be an ∞-category with finite products. Restriction along the inclusion $j\colon \mathbb{F}_* \to \mathrm{Span}(\mathbb{F})$ gives an equivalence*

$$\mathrm{Fun}^\times(\mathrm{Span}(\mathbb{F}), \mathcal{C}) \xrightarrow{\sim} \mathrm{CMon}(\mathcal{C}),$$

with the inverse given by right Kan extension along the functor j.

Proof. In the stated form, this is [BaHo18, Proposition C.1]. □

Remark 3.1.6. The functor $\Phi\colon \mathrm{Span}(\mathbb{F}) \to \mathrm{Cat}_\infty$ corresponding to a symmetric monoidal ∞-category $\mathcal{C}$ admits the following description:

- $\Phi(I) \simeq \prod_{i \in I} \mathcal{C} \simeq \mathrm{Fun}(I, \mathcal{C})$,
- for $f\colon I \to J$, the functor $f^{\circledast}\colon \mathrm{Fun}(J, \mathcal{C}) \to \mathrm{Fun}(I, \mathcal{C})$ is that given by composition with f,

- for $f\colon I \to J$, the functor

$$f_{\otimes}\colon \mathrm{Fun}(I, \mathcal{C}) \simeq \prod_{j\in J}\prod_{i\in I_j} \mathcal{C} \to \prod_{j \in J} \mathcal{C}$$

  is given by tensoring the components corresponding to the preimages of each $j \in J$.

In particular, for $q\colon I \to *$ the functor $q_{\otimes}\colon \mathrm{Fun}(I, \mathcal{C}) \to \mathcal{C}$ takes $\phi\colon I \to \mathcal{C}$ to $\bigotimes_{i\in I} \phi(i)$, while $q^{\circledast}\colon \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$ is the diagonal functor.

Proposition 3.1.7. *Suppose $\Phi\colon \mathrm{Span}(\mathbb{F}) \to \mathrm{Cat}_\infty$ is a product-preserving functor, corresponding to a symmetric monoidal structure on $\mathcal{C} = \Phi(*)$. Then Φ is distributive if and only if $\mathcal{C}$ has finite coproducts and the symmetric monoidal structure is compatible with these, i.e. the tensor product preserves finite coproducts in each variable.*

Proof. For $I \in \mathbb{F}$, the functor $q^{\circledast}\colon \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$ corresponding to the morphism $q\colon I \to *$ is the diagonal. This has a left adjoint if and only if $\mathcal{C}$ admits all I-indexed colimits, i.e. $\mathcal{C}$ has I-indexed coproducts. Moreover, if $\mathcal{C}$ has finite coproducts, then the functor $f^{\circledast}\colon \mathrm{Fun}(J, \mathcal{C}) \to \mathrm{Fun}(I, \mathcal{C})$ given by composition with $f\colon I \to J$ has a left adjoint for any f, since all pointwise left Kan extensions along f exist in $\mathcal{C}$. Given a cartesian square

$$\begin{tikzcd} I' \arrow[r, "f'"] \arrow[d, "g'"] & J' \arrow[d, "g"] \\ I \arrow[r, "f"] & J \end{tikzcd}$$

in $\mathbb{F}$, the mate transformation

$$g'_{\oplus} f'^{\circledast} \to f^{\circledast} g_{\oplus}$$

is then automatically an equivalence, since for $\phi\colon J' \to \mathcal{C}$ this is given at $i \in I$ by the natural map

$$\coprod_{x \in I'_i} \phi(f'x) \to \coprod_{y \in J'_{f(i)}} \phi(y),$$

which is an equivalence since these fibres are canonically isomorphic. This proves that Φ is left adjointable if and only if $\mathcal{C}$ admits finite coproducts.

Given morphisms $l\colon I \to J$ and $f\colon J \to K$ in $\mathbb{F}$, we have the distributivity square

$$\begin{tikzcd} & J' \arrow[r, "f'"] \arrow[dl, "\epsilon"'] \arrow[d] & K' \arrow[d, "h=f_*l"] \\ I \arrow[r, "l"'] & J \arrow[r, "f"] & K, \end{tikzcd}$$

where $K'_k \cong \prod_{j\in J_k} I_j$. The distributivity transformation $h_{\oplus} f'_{\otimes} \epsilon^{\circledast} \to f_{\otimes} l_{\oplus}$ is given for $\phi\colon I \to \mathcal{C}$ at $k \in K$ by the canonical map

$$\coprod_{(i_j)_j \in \prod_{j\in J_k} I_j} \bigotimes_{j \in J_k} \phi(i_j) \to \bigotimes_{j\in J_k} \left(\coprod_{i \in I_j} \phi(i) \right).$$

This is an equivalence if $\otimes$ preserves finite coproducts in each variable. Conversely, for $K \in \mathbb{F}$ we have in particular the distributivity diagram

$$\begin{tikzcd} & K \amalg K \arrow[r, "\nabla"] \arrow[dl, "\epsilon"'] \arrow[d, "q\amalg q"] & K \arrow[d, "q"] \\ K \amalg * \arrow[dr, "q\amalg \mathrm{id}"'] & & \\ & * \amalg * \arrow[r, "\nabla"] & *, \end{tikzcd} \tag{29}$$

where q is the unique map $K \to *$ and ∇ are fold maps. The corresponding distributivity transformation is given for $\phi\colon K \amalg * \to \mathcal{C}$ by

$$\coprod_{k\in K} (\phi(k) \otimes \phi(*)) \to \left(\coprod_{k\in K} \phi(k) \right) \otimes \phi(*).$$

If Φ is distributive then this is an equivalence for all K and ϕ, which is precisely the condition that $\otimes$ preserves finite coproducts in each variable. □

Corollary 3.1.8. *Product-preserving functors* $\mathrm{BISPAN}(\mathbb{F}) \to \mathrm{CAT}_\infty$ *correspond to symmetric monoidal ∞-categories that are compatible with finite coproducts.*

Proof. By Corollary 2.3.15, functors $\mathrm{BISPAN}(\mathbb{F}) \to \mathrm{CAT}_\infty$ correspond to distributive functors $\mathrm{Span}(\mathbb{F}) \to \mathrm{CAT}_\infty$. Moreover, from Remark 2.6.15 we know that the product in $\mathrm{BISPAN}(\mathbb{F})$ is given by the coproduct in $\mathbb{F}$, just as in $\mathrm{Span}(\mathbb{F})$, so product-preserving functors from $\mathrm{BISPAN}(\mathbb{F})$ correspond to product-preserving distributive functors under this equivalence. By Proposition 3.1.7 and Proposition 3.1.5, the latter are equivalent to symmetric monoidal ∞-categories that are compatible with finite coproducts. □

3.2. **Bispans in spaces and symmetric monoidal ∞-categories.** In this section we consider a variant of the results of the preceding one: symmetric monoidal ∞-categories can also be described in terms of spans of spaces, and we will prove that the resulting functor is distributive (with respect to all morphisms of spaces) if and only if the tensor product commutes with colimits indexed by ∞-groupoids. This applies in many examples, since most naturally occurring tensor products are compatible with all colimits.

Notation 3.2.1. We write $\mathcal{S}_{\mathrm{fin}}$ for the subcategory of $\mathcal{S}$ containing only the morphisms whose fibres are equivalent to finite sets. Then $(\mathcal{S}, \mathcal{S}_{\mathrm{fin}})$ is a span pair.

Remark 3.2.2. If $f\colon X \to I$ is a morphism in $\mathcal{S}_{\mathrm{fin}}$ and I is a finite set, then the straightening equivalence

$$\mathrm{colim}_I \colon \mathrm{Fun}(I, \mathcal{S}) \xrightarrow{\sim} \mathcal{S}_{/I}$$

implies that X is an I-indexed coproduct of finite sets, and so is itself a finite set. It follows that the functor $\mathrm{Span}(\mathbb{F}) \to \mathrm{Span}_{\mathrm{fin}}(\mathcal{S})$ induced by the morphism of span pairs $(\mathbb{F}, \mathbb{F}) \to (\mathcal{S}, \mathcal{S}_{\mathrm{fin}})$ is fully faithful.

Proposition 3.2.3. *Let $\mathcal{C}$ be a complete ∞-category. Right Kan extension along the fully faithful functor* $\mathrm{Span}(\mathbb{F}) \to \mathrm{Span}_{\mathrm{fin}}(\mathcal{S})$ *identifies* $\mathrm{Fun}^\times(\mathrm{Span}(\mathbb{F}), \mathcal{C})$ *with the full subcategory* $\mathrm{Fun}^{\mathrm{RKE}}(\mathrm{Span}_{\mathrm{fin}}(\mathcal{S}), \mathcal{C})$ *of* $\mathrm{Fun}(\mathrm{Span}_{\mathrm{fin}}(\mathcal{S}), \mathcal{C})$ *spanned by functors Φ such that $\Phi|_{\mathcal{S}^{\mathrm{op}}}$ is right Kan extended from $\{*\}$.*

Proof. This is a special case of [BaHo18, Proposition C.18]. □

Combining this with Proposition 3.1.5, we have:

Corollary 3.2.4. *Let $\mathcal{C}$ be a complete ∞-category. There is an equivalence*
$$\mathrm{CMon}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{RKE}}(\mathrm{Span}_{\mathrm{fin}}(\mathcal{S}), \mathcal{C}),$$
given by right Kan extension along $\mathbb{F}_* \to \mathrm{Span}(\mathbb{F}) \hookrightarrow \mathrm{Span}_{\mathrm{fin}}(\mathcal{S})$. □

Remark 3.2.5. In particular, symmetric monoidal ∞-categories can be identified with functors $\mathrm{Span}_{\mathrm{fin}}(\mathcal{S}) \to \mathrm{Cat}_\infty$ whose restrictions to $\mathcal{S}^{\mathrm{op}}$ are right Kan extended from $\{*\}$. If $\mathcal{C}$ is a symmetric monoidal ∞-category, the corresponding functor $\Phi\colon \mathrm{Span}_{\mathrm{fin}}(\mathcal{S}) \to \mathrm{Cat}_\infty$ admits the following description (analogous to Remark 3.1.6):

- for $X \in \mathcal{S}$, $\Phi(X) \simeq \lim_{x\in X} \Phi(\{x\}) \simeq \mathrm{Fun}(X, \mathcal{C})$,
- for $f\colon X \to Y$ in $\mathcal{S}$, the functor $f^\circledast\colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ is given by composition with f,
- for $g\colon E \to B$ in $\mathcal{S}_{\mathrm{fin}}$, the functor $g_\otimes\colon \mathrm{Fun}(E, \mathcal{C}) \to \mathrm{Fun}(B, \mathcal{C})$ is given by tensoring fibrewise along g, i.e. for $\phi\colon E \to B$ we have
$$(g_\otimes\phi)(b) \simeq \bigotimes_{e\in E_b} \phi(e).$$

In particular, for $q\colon X \to *$ the functor $q^\circledast$ is the diagonal functor, while if X is a finite set the functor $q_\otimes$ is the X-indexed tensor product.

We will identify when such functors from $\mathrm{Span}_{\mathrm{fin}}(\mathcal{S})$ are distributive with respect to bispan triples of the following form:

Lemma 3.2.6. *Let $\mathcal{K}$ be a full subcategory of $\mathcal{S}$ with the following properties:*

- *if $p\colon E \to B$ is a morphism in $\mathcal{S}$ such that $B \in \mathcal{K}$ and $E_b \in \mathcal{K}$ for all $b \in B$, then $E \in \mathcal{K}$,*
- *$\mathcal{K}$ is closed under finite products.*

This implies that morphisms in $\mathcal{S}$ whose fibres lie in $\mathcal{K}$ are closed under composition, giving a subcategory $\mathcal{S}_{\mathcal{K}}$ of $\mathcal{S}$. Then $(\mathcal{S}, \mathcal{S}_{\mathrm{fin}}, \mathcal{S}_{\mathcal{K}})$ is a bispan triple.

Proof. Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms whose fibres lie in $\mathcal{K}$. We have a morphism $X_z \to Y_z$ between the fibres of gf and g at $z \in Z$, whose fibre at $y \in Y_z$ is equivalent to X_y. Since Y_z and X_y lie in $\mathcal{K}$ for all $z \in Z, y \in Y$, it follows that X_z also lies in $\mathcal{K}$. Thus we do indeed have a subcategory $\mathcal{S}_{\mathcal{K}}$ of morphisms whose fibres lie in $\mathcal{K}$. Such morphisms are obviously preserved under base change, and so $(\mathcal{S}, \mathcal{S}_{\mathcal{K}})$ is a span pair.

All distributivity diagrams exist in $\mathcal{S}$ since this is a locally cartesian closed ∞-category; see Remark 2.3.7. To show that $(\mathcal{S}, \mathcal{S}_{\mathrm{fin}}, \mathcal{S}_{\mathcal{K}})$ is a bispan triple it therefore only remains to check that if $l\colon X \to Y$ is a morphism in $\mathcal{S}_{\mathcal{K}}$ and $f\colon Y \to Z$ is a morphism in $\mathcal{S}_{\mathrm{fin}}$, then f_*l is also a morphism in $\mathcal{S}_{\mathcal{K}}$. But we have
$$(f_*l)_z \simeq \prod_{y\in Y_z} X_y,$$
which is a finite product of objects of $\mathcal{K}$, and so again lies in $\mathcal{K}$ by assumption. □

Examples 3.2.7. We can take $\mathcal{K}$ in Lemma 3.2.6 to consist of

- finite sets,
- all spaces,
- π-finite spaces[10], as follows by examining the long exact sequence in homotopy groups associated to a fiber sequence,

[10] These are the spaces X such that (1) X is n-truncated for some n, (2) $\pi_0(X)$ is finite, and (3) for each $x \in X$, the homotopy group $\pi_k(X, x)$ is finite for each $k \geq 1$.

- κ-compact spaces[11] for any regular cardinal κ, since κ-filtered colimits in $\mathcal{S}$ commute with κ-small limits, and the κ-compact spaces are precisely the (retracts of) κ-small ∞-groupoids.

Proposition 3.2.8. *Suppose $\Phi\colon \mathrm{Span}_{\mathrm{fin}}(\mathcal{S}) \to \mathrm{Cat}_\infty$ corresponds to a symmetric monoidal structure on $\mathcal{C} = \Phi(*)$, and let $\mathcal{K}$ be as in Lemma 3.2.6. Then Φ is $\mathcal{K}$-distributive if and only if $\mathcal{C}$ has $\mathcal{K}$-indexed colimits (meaning K-indexed colimits for all $K \in \mathcal{K}$), and the tensor product on $\mathcal{C}$ is compatible with such colimits (i.e. preserves them in each variable).*

Proof. For $K \in \mathcal{K}$, the functor $q^{\circledast}\colon \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$ corresponding to the morphism $q\colon K \to *$ is the diagonal. This has a left adjoint if and only if $\mathcal{C}$ admits all K-indexed colimits. Moreover, if $\mathcal{C}$ has $\mathcal{K}$-indexed colimits then the functor $f^{\circledast}\colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ given by composition with $f\colon X \to Y$ has a left adjoint for any f in $\mathcal{S}_{\mathcal{K}}$, since all pointwise left Kan extensions along f then exist in $\mathcal{C}$. Given a cartesian square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{S}$ with g in $\mathcal{S}_{\mathcal{K}}$, the mate transformation

$$g'_{\oplus} f'^{\circledast} \to f^{\circledast} g_{\oplus}$$

is then automatically an equivalence, since for $\phi\colon Y' \to \mathcal{C}$ this is given at $x \in X$ by the natural map

$$\mathrm{colim}_{p \in X'_x} \phi(f'p) \to \mathrm{colim}_{y \in Y'_{f(y)}} \phi(y),$$

which is an equivalence since these fibres are canonically equivalent. This proves that Φ is left adjointable if and only if $\mathcal{C}$ admits $\mathcal{K}$-indexed colimits.

Given morphisms $l\colon X \to Y$ in $\mathcal{S}_{\mathcal{K}}$ and $f\colon Y \to Z$ in $\mathcal{S}_{\mathrm{fin}}$, we have the distributivity square

$$\begin{tikzcd} & Y' \arrow[dl, "\epsilon"'] \arrow[r, "f'"] \arrow[d] & Z' \arrow[d, "h=f_*l"] \\ X \arrow[r, "l"'] & Y \arrow[r, "f"] & Z, \end{tikzcd}$$

where $Z'_z \simeq \prod_{y \in Y_z} X_y$. The distributivity transformation $h_{\oplus} f'_{\otimes} \epsilon^{\circledast} \to f_{\otimes} l_{\oplus}$ is given for $\phi\colon X \to \mathcal{C}$ at $z \in Z$ by the canonical map

$$\mathrm{colim}_{(x_y)_y \in \prod_{y \in Y_z} X_y} \bigotimes_{y \in Y_z} \phi(x_y) \to \bigotimes_{y \in Y_z} \left(\mathrm{colim}_{x \in X_y} \phi(x)\right).$$

This is an equivalence if $\otimes$ preserves $\mathcal{K}$-indexed colimits in each variable. Conversely, for $K \in \mathcal{K}$ we have in particular the distributivity diagram

$$\begin{tikzcd} & K \amalg K \arrow[dl, "\epsilon"'] \arrow[r, "\nabla"] \arrow[d, "q \amalg q"] & K \arrow[d, "q"] \\ K \amalg * \arrow[r, "q \amalg \mathrm{id}"'] & * \amalg * \arrow[r, "\nabla"] & *, \end{tikzcd}$$

[11] Meaning κ-compact objects of the ∞-category $\mathcal{S}$ of spaces.

where q is the unique map $K \to *$ and ∇ are fold maps. The corresponding distributivity transformation is given for $\phi\colon K \amalg * \to \mathcal{C}$ by

$$\operatorname{colim}_{k \in K}(\phi(k) \otimes \phi(*)) \to (\operatorname{colim}_{k \in K} \phi(k)) \otimes \phi(*).$$

If Φ is distributive then this is an equivalence for all $K \in \mathcal{K}$ and ϕ, which is precisely the condition that $\otimes$ preserves $\mathcal{K}$-indexed colimits in each variable. □

Corollary 3.2.9. *Let $\mathcal{K}$ be as in Lemma 3.2.6. Then functors $\Phi\colon \mathrm{BISPAN}_{\mathrm{fin},\mathcal{K}}(\mathcal{S}) \to \mathrm{CAT}_\infty$ such that the restriction to $\mathcal{S}^{\mathrm{op}}$ is right Kan extended from $\{*\}$ correspond to symmetric monoidal ∞-categories that are compatible with $\mathcal{K}$-indexed colimits.*

Proof. By Corollary 2.3.15, functors $\mathrm{BISPAN}_{\mathrm{fin},\mathcal{K}} \to \mathrm{CAT}_\infty$ correspond to $\mathcal{K}$-distributive functors $\mathrm{Span}_{\mathrm{fin}}(\mathcal{S}) \to \mathrm{CAT}_\infty$. On the other hand, we know from Corollary 3.2.4 that such functors whose restriction to $\mathcal{S}^{\mathrm{op}}$ is right Kan extended from $\{*\}$ correspond to symmetric monoidal ∞-categories, and in this case the functor is $\mathcal{K}$-distributive if and only if the tensor product is compatible with $\mathcal{K}$-indexed colimits by Proposition 3.2.8. □

3.3. **Bispans in spaces and analytic monads.** Our goal in this section is to relate bispans in the ∞-category of spaces to the polynomial and analytic functors studied in [GHK17], where it is shown that analytic monads are equivalent to ∞-operads. Using the results of the previous section we will see that there is a canonical action of every analytic monad on any symmetric monoidal ∞-category compatible with ∞-groupoid-indexed colimits.

We first give a general construction of a functor from bispans to ∞-categories using slice ∞-categories:

Proposition 3.3.1. *Suppose $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ is a bispan triple. Then there is a functor of $(\infty,2)$-categories $\mathrm{Sl}\colon \mathrm{BISPAN}_{F,L}(\mathcal{C}) \to \mathrm{CAT}_\infty$ such that*

- *$\mathrm{Sl}(c) \simeq \mathcal{C}^L_{/c}$, the full subcategory of $\mathcal{C}_{/c}$ spanned by the morphisms to c that lie in $\mathcal{C}_L$.*
- *for $f\colon c \to c'$ in $\mathcal{C}$, the functor $f^{\circledast}$ is the functor $f^*\colon \mathcal{C}^L_{/c'} \to \mathcal{C}^L_{/c}$ given by pullback along f,*
- *for $f\colon c \to c'$ in $\mathcal{C}_L$, the functor $f_\oplus$ is the functor $f_!\colon \mathcal{C}^L_{/c} \to \mathcal{C}^L_{/c'}$ given by composition with f,*
- *for $f\colon c \to c'$ in $\mathcal{C}_F$, the functor $f_\otimes$ is the functor $f_*\colon \mathcal{C}^L_{/c} \to \mathcal{C}^L_{/c'}$ given by the partial right adjoint to $f^*\colon \mathcal{C}_{/c'} \to \mathcal{C}_{/c}$.*

Proof. Let $\mathcal{X}$ be the full subcategory of $\mathcal{C}^{\Delta^1}$ spanned by the morphisms in $\mathcal{C}_L$. Then the restriction of $\mathrm{ev}_1\colon \mathcal{C}^{\Delta^1} \to \mathcal{C}$ to a functor $\mathcal{X} \to \mathcal{C}$ is a cartesian fibration, with cocartesian morphisms over morphisms in $\mathcal{C}_L \subseteq \mathcal{C}$. This corresponds to a functor $\lambda\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$, which takes $c \in \mathcal{C}$ to the ∞-category $\mathcal{C}^L_{/c}$ and $f\colon x \to y$ to the functor $f^*\colon \mathcal{C}^L_{/y} \to \mathcal{C}^L_{/x}$ given by pullback along f. The functor λ is right F-adjointable: since $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ is a bispan triple, the functor f^* for $f\colon x \to y$ in $\mathcal{C}_F$ has a right adjoint $f_*\colon \mathcal{C}^L_{/x} \to \mathcal{C}^L_{/y}$ by Remark 2.3.5, and given a cartesian square

$$\begin{tikzcd} x' \arrow[r, "f'"] \arrow[d, "\xi"] & y' \arrow[d, "\eta"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

with f and f' in $\mathcal{C}_F$, the mate transformation

$$\eta^* f_* \to f'_* \xi^*$$

is an equivalence by Lemma 2.3.6. The functor λ therefore extends canonically to a functor

$$\Lambda\colon \mathrm{SPAN}_F(\mathcal{C}) \to \mathrm{CAT}_\infty$$

by Theorem 2.2.7. We claim the underlying functor of ∞-categories $\lambda'\colon \mathrm{Span}_F(\mathcal{C}) \to \mathrm{Cat}_\infty$ is L-distributive. Certainly if $l\colon x \to y$ is a morphism in $\mathcal{C}_L$, then the pullback functor $l^*\colon \mathcal{C}^L_{/y} \to \mathcal{C}^L_{/x}$ has a left adjoint $l_!$ given by composition with l; to see that the restriction of λ' to $\mathcal{C}^{\mathrm{op}}$ is left L-adjointable it remains to observe that for any cartesian square

$$\begin{tikzcd}
x' \arrow[r, "l'"] \arrow[d, "\xi"] & y' \arrow[d, "\eta"] \\
x \arrow[r, "l"] & y
\end{tikzcd}$$

with l and l' in $\mathcal{C}_L$, the natural transformation $l'_!\xi^* \to \eta^* l_!$ is an equivalence, since for $g\colon z \to x$ in $\mathcal{C}^L_{/x}$ in the diagram

$$\begin{tikzcd}
z' \arrow[r, "g'"] \arrow[d, "\zeta"] & x' \arrow[r, "l'"] \arrow[d, "\xi"] & y' \arrow[d, "\eta"] \\
z \arrow[r, "g"] & y \arrow[r, "l"] & x,
\end{tikzcd}$$

the left square is cartesian if and only if the composite square is cartesian. To see that λ' is also L-distributive, consider $l\colon x \to y$ in $\mathcal{C}_L$ and $f\colon y \to z$ in $\mathcal{C}_F$ and form a distributivity diagram (11). The distributivity transformation

$$g_!\tilde{f}_*\epsilon^* \to f_* l_!$$

evaluated at $l'\colon c \to x$ is a canonical map

$$f_* l \circ \tilde{f}_*\epsilon^* l \to f_*(l \circ l');$$

this is an equivalence by Lemma 2.4.1. It follows by Corollary 2.3.15 that λ' extends uniquely to a functor $\mathrm{Sl}\colon \mathrm{BISPAN}_{F,L}(\mathcal{C}) \to \mathrm{CAT}_\infty$, which by construction has the required properties. □

Applying this to the bispan triple $(\mathcal{S}, \mathcal{S}, \mathcal{S})$ we get in particular:

Corollary 3.3.2. *There is a functor* $\mathrm{Sl}\colon \mathrm{BISPAN}(\mathcal{S}) \to \mathrm{CAT}_\infty$ *taking* $X \in \mathcal{S}$ *to* $\mathcal{S}_{/X}$ *and a bispan*

$$X \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} Y$$

to the functor $t_!p_*s^*\colon \mathcal{S}_{/X} \to \mathcal{S}_{/Y}$*, where* s^* *is given by pullback along* s*,* p_* *is the right adjoint to* p^**, and* $t_!$ *is given by composition with* t*.* □

Definition 3.3.3. A *polynomial functor* $F\colon \mathcal{S}_{/X} \to \mathcal{S}_{/Y}$ is an accessible functor that preserves weakly contractible limits. By [GHK17, Theorem 2.2.3] the polynomial functors are equivalently those functors obtained as composites $t_!p_*s^*$ for some bispan of spaces

$$X \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} Y.$$

Let $\mathrm{PolyFun}(\mathcal{S})$ be the sub-$(\infty,2)$-category of CAT_∞ whose objects are the slices $\mathcal{S}_{/X}$ for $X \in \mathcal{S}$, whose 1-morphisms are the polynomial functors, and whose 2-morphisms are the cartesian natural transformations.

Remark 3.3.4. The $(\infty,2)$-category $\mathrm{PolyFun}(\mathcal{S})$ is the underlying $(\infty,2)$-category of the double ∞-category of polynomial functors considered in [GHK17].

Corollary 3.3.5. *The functor* $\mathrm{Sl}\colon \mathrm{BISPAN}(\mathcal{S}) \to \mathrm{CAT}_\infty$ *restricts to an equivalence*

$$\mathrm{BISPAN}(\mathcal{S}) \xrightarrow{\sim} \mathrm{PolyFun}(\mathcal{S}).$$

Proof. We apply the description from Proposition 2.5.22 to understand Sl: On objects, Sl takes $X \in \mathcal{S}$ to the ∞-category $\mathcal{S}_{/X}$, and on morphisms it takes the bispan

$$X \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} Y$$

to the functor $t_!p_*s^*$. The functors of this form are precisely the polynomial functors, by [GHK17, Theorem 2.2.3]. The description of Sl on 2-morphisms from Proposition 2.5.22(3) implies that they are sent to composites of equivalences and (co)unit transformations that are cartesian by [GHK17, Lemma 2.1.5]. Hence Sl factors through $\mathrm{PolyFun}(\mathcal{S})$, and is essentially surjective on objects and morphisms. To show that Sl factors through an equivalence it then suffices to show it gives an equivalence

$$\mathrm{MAP}_{\mathrm{BISPAN}(\mathcal{S})}(X,Y) \to \mathrm{MAP}_{\mathrm{PolyFun}(\mathcal{S})}(X,Y)$$

on mapping ∞-categories for all X, Y in $\mathcal{S}$. Using Corollary 2.5.16 we can identify this with the functor shown to be an equivalence in [GHK17, Proposition 2.4.13]. □

Remark 3.3.6. We expect that there is an analogue of Corollary 3.3.5 for bispans in any ∞-topos $\mathcal{X}$, but this requires working with *internal* ∞-categories in $\mathcal{X}$ (or equivalently sheaves of ∞-categories on $\mathcal{X}$): A key step in the identification of polynomial functors with bispans is the description of colimit-preserving functors between slices of $\mathcal{S}$ as spans, i.e. the equivalence

$$\mathrm{Fun}^L(\mathcal{S}_{/X}, \mathcal{S}_{/Y}) \simeq \mathrm{Fun}(X, \mathcal{S}_{/Y}) \simeq \mathrm{Fun}(X \times Y, \mathcal{S}) \simeq \mathcal{S}_{/X\times Y}.$$

This certainly fails for any other ∞-topos $\mathcal{X}$, but an analogous statement should hold if we view $\mathcal{X}$ instead as an ∞-category internal to itself.

Definition 3.3.7. An *analytic functor* $F\colon \mathcal{S}_{/X} \to \mathcal{S}_{/Y}$ is a functor that preserves sifted colimits and weakly contractible limits. By [GHK17, Proposition 3.1.9] the analytic functors are equivalently those functors obtained as composites $t_!p_*s^*$ for some bispan of spaces

$$X \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} Y,$$

where p has finite discrete fibres.

As a consequence, we get:

Corollary 3.3.8. *The functor* $\mathrm{Sl}\colon \mathrm{BISPAN}(\mathcal{S}) \to \mathrm{CAT}_\infty$ *factors through an equivalence*

$$\mathrm{BISPAN}_{\mathrm{fin}}(\mathcal{S}) \xrightarrow{\sim} \mathrm{AnFun}(\mathcal{S}), \tag{30}$$

where $\mathrm{AnFun}(\mathcal{S})$ *is the locally full sub-*$(\infty,2)$*-category of* $\mathrm{PolyFun}(\mathcal{S})$ *containing all objects, with the analytic functors as morphisms, as well as all 2-morphisms between these.* □

Definition 3.3.9. An *analytic monad* is a monad in the $(\infty,2)$-category $\mathrm{AnFun}(\mathcal{S})$, i.e. an associative algebra in the monoidal ∞-category $\mathrm{MAP}_{\mathrm{AnFun}(\mathcal{S})}(X,X)$ of endomorphisms of some object X, or a functor $\mathfrak{mnd} \to \mathrm{AnFun}(\mathcal{S})$, where $\mathfrak{mnd}$ is the universal 2-category containing a monad. In other words, it is a monad on the ∞-category $\mathcal{S}_{/X}$ whose underlying endofunctor is analytic and whose unit and multiplication transformations are cartesian. From the equivalence (30) we see that analytic monads are equivalently monads in the $(\infty,2)$-category $\mathrm{BISPAN}_{\mathrm{fin}}(\mathcal{S})$.

Corollary 3.3.10. *Suppose T is an analytic monad on $\mathcal{S}_{/X}$ and $\mathcal{V}$ is a symmetric monoidal ∞-category compatible with ∞-groupoid-indexed colimits. Then T induces a canonical monad $T_{\mathcal{V}}$ on* $\mathrm{Fun}(X,\mathcal{V})$.

Proof. We can identify T with a monad on $X \in \mathrm{BISPAN}_{\mathrm{fin}}(\mathcal{S})$. By Corollary 3.2.9 $\mathcal{V}$ induces a functor $\mathrm{BISPAN}_{\mathrm{fin}}(\mathcal{S}) \to \mathrm{CAT}_\infty$ that takes $Y \in \mathcal{S}$ to $\mathrm{Fun}(Y, \mathcal{V})$. Any functor of $(\infty, 2)$-categories preserves monads, since they can be described as simply functors of $(\infty, 2)$-categories from $\mathfrak{mnd}$. Hence under the functor induced by $\mathcal{V}$ the monad T maps to a monad in CAT_∞ which indeed acts on $\mathrm{Fun}(X, \mathcal{V})$. □

Remark 3.3.11. Suppose the underlying bispan of the monad T is

$$X \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} X.$$

Then the underlying endofunctor of the monad $T_{\mathcal{V}}$ is given by

$$(T_{\mathcal{V}}\phi)(x) \simeq \mathrm{colim}_{b \in B_x} \bigotimes_{e \in E_b} \phi(s(e)).$$

This has the same form as the formula for the free algebra monad of an ∞-operad, and the main result of [GHK17] is that analytic monads are equivalent to ∞-operads in the form of dendroidal Segal spaces. We therefore expect that if $\mathcal{O}$ is the ∞-operad corresponding to T, then the monad $T_{\mathcal{V}}$ is the free $\mathcal{O}$-algebra monad for $\mathcal{O}$-algebras in $\mathcal{V}$.

3.4. **Equivariant bispans and G-symmetric monoidal ∞-categories.** In this section we look at the G-equivariant version of our results from §3.1 on symmetric monoidal ∞-categories compatible with finite coproducts, where G is a finite group: we replace the category of finite sets by the category $\mathbb{F}_G$ of finite G-sets, and consider when a G-symmetric monoidal ∞-category, defined as a product-preserving functor $\mathrm{Span}(\mathbb{F}_G) \to \mathrm{Cat}_\infty$, is distributive, and so extends to a functor $\mathrm{BISPAN}(\mathbb{F}_G) \to \mathrm{CAT}_\infty$.

Definition 3.4.1. Let G be a finite group, and BG the corresponding 1-object groupoid. We write $\mathbb{F}_G$ for the category $\mathrm{Fun}(BG, \mathbb{F})$ of finite G-sets, and $\mathcal{O}_G$ for the full subcategory of *orbits*, i.e. finite G-sets of the form G/H where H is a subgroup of G. Then $\mathbb{F}_G$ is obtained from $\mathcal{O}_G$ by freely adding finite coproducts, so that for any ∞-category $\mathcal{C}$ with finite products, restriction along the inclusion $\mathcal{O}_G \hookrightarrow \mathbb{F}_G$ gives an equivalence

$$\mathrm{Fun}^\times(\mathbb{F}_G^{\mathrm{op}}, \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{C}).$$

If $\mathcal{C}$ is the ∞-category of spaces, this says that the ∞-category $\mathcal{S}_G := \mathcal{P}(\mathcal{O}_G)$ of *G-spaces*[12] is equivalent to $\mathrm{Fun}^\times(\mathbb{F}_G^{\mathrm{op}}, \mathcal{S})$. By analogy with the case of G-spaces, we can think of a functor $\mathcal{F}\colon \mathcal{O}_G^{\mathrm{op}} \to \mathcal{C}$ as an "object of $\mathcal{C}$ with G-action", with $\mathcal{F}^H := \mathcal{F}(G/H)$ the object of "H-fixed points" of $\mathcal{F}$. We will in particular apply this notation for functors $\mathcal{O}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$, which we will call *$G$-$\infty$-categories*.

Remark 3.4.2. The category $\mathbb{F}_G$ satisfies descent for finite coproducts, and so by [Bar17, Proposition 4.3] the coproduct in $\mathbb{F}_G$ gives both the product and coproduct in $\mathrm{Span}(\mathbb{F}_G)$.

Definition 3.4.3. Let $\mathcal{C}$ be an ∞-category with finite products. A *G-commutative monoid* in $\mathcal{C}$ is a product-preserving functor $\mathrm{Span}(\mathbb{F}_G) \to \mathcal{C}$. We write

$$\mathrm{CMon}_G(\mathcal{C}) := \mathrm{Fun}^\times(\mathrm{Span}(\mathbb{F}_G), \mathcal{C})$$

for the ∞-category of G-commutative monoids in $\mathcal{C}$. A *G-symmetric monoidal ∞-category* is a G-commutative monoid in Cat_∞.

[12]By Elmendorf's theorem [Elm83] the ∞-category $\mathcal{S}_G$ is equivalent to that obtained from the category of topological spaces with G-action by inverting the maps that give weak homotopy equivalences on all spaces of fixed points.

Remark 3.4.4. When G is the trivial group this is equivalent to the usual definition of commutative monoids (in terms of functors from $\mathbb{F}_*$ satisfying a Segal condition) by Proposition 3.1.5. More generally, see [Nar16, Theorem 6.5] for an alternative description of G-commutative monoids in terms of "finite pointed G-sets" (where this must be read in a non-trivial parametrized sense).

Remark 3.4.5. A functor $\mathcal{F}\colon \mathrm{Span}(\mathbb{F}_G) \to \mathcal{C}$ preserves products if and only if the restriction to $\mathbb{F}_G^{\mathrm{op}} \to \mathcal{C}$ preserves products, and so is determined by its restriction to $\mathcal{O}_G^{\mathrm{op}} \to \mathcal{C}$. The additional structure given by the forwards maps in $\mathrm{Span}(\mathbb{F}_G)$ can be decomposed into

- multiplication maps $\mathcal{F}^H \times \mathcal{F}^H \to \mathcal{F}^H$ for each subgroup H of G, coming from the fold map $G/H \amalg G/H \to G/H$,
- multiplicative *transfer* maps $\mathcal{F}^H \to \mathcal{F}^K$ for each inclusion $H \subseteq K$ of subgroups, coming from the quotient map $G/H \to G/K$,

together with various homotopy-coherent compatibilities that in particular make each $\mathcal{F}^H$ a commutative monoid.

Remark 3.4.6. Grouplike G-commutative monoids in $\mathcal{S}$ can be identified with connective genuine G-spectra, by [Nar16, Corollary A.4.1]. Applying π_0, such a grouplike G-commutative monoid induces a G-commutative monoid in Set, which factors through a product-preserving functor

$$\mathrm{Span}(\mathbb{F}_G) \to \mathrm{Ab}$$

because it is grouplike — this is precisely a *Mackey functor*, which is well-known as the structure appearing as π_0 of a genuine G-spectrum.

For a functor $\mathrm{Span}(\mathbb{F}_G) \to \mathrm{Cat}_\infty$ we can simplify the condition that it is left adjointable as follows:

Proposition 3.4.7. *Suppose $\mathcal{F}\colon \mathbb{F}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is a product-preserving functor. Then $\mathcal{F}$ is left adjointable if and only if*

(1) for every subgroup $H \subseteq G$, the ∞-category $\mathcal{F}^H$ has finite coproducts,
(2) for every inclusion of subgroups $H \subseteq K$ the functor $(q_H^K)^\circledast \colon \mathcal{F}^K \to \mathcal{F}^H$, corresponding to the quotient map $q_H^K\colon G/H \to G/K$, has a left adjoint $(q_H^K)_\oplus$,
(3) for every inclusion of subgroups $H \subseteq K$ the functor $(q_H^K)^\circledast$ preserves finite coproducts,
(4) for subgroups $H, K \subseteq L$, let X be defined by the pullback

$$\begin{tikzcd} X \arrow[r, "f_K"] \arrow[d, "f_H"] & G/K \arrow[d, "q_K^L"] \\ G/H \arrow[r, "q_H^L"] & G/L; \end{tikzcd}$$

then the square

$$\begin{tikzcd} \mathcal{F}^L \arrow[r, "(q_H^L)^\circledast"] \arrow[d, "(q_K^L)^\circledast"] & \mathcal{F}^H \arrow[d, "f_H^\circledast"] \\ \mathcal{F}^K \arrow[r, "f_K^\circledast"] & \mathcal{F}(X) \end{tikzcd}$$

is left adjointable, i.e. the mate transformation

$$f_{K,\oplus} f_H^\circledast \to (q_K^L)^\circledast (q_H^L)_\oplus$$

is an equivalence.

Remark 3.4.8. The pullback X in condition (4) can be decomposed into a sum of orbits indexed by double cosets:

$$X \cong \coprod_{[g]\in H\backslash L/K} H \cap K_g,$$

where K_g denotes the conjugate gKg^{-1}. The left adjointability in (4) then amounts to the following *double coset formula*:

$$(q_K^L)^{\circledast}(q_H^L)_{\oplus} \simeq \coprod_{[g]\in H\backslash L/K} c_{g,\oplus}(q_{H\cap K_g}^{K_g})_{\oplus}(q_{H\cap K_g}^{H})^{\circledast},$$

where c_g is the isomorphism $G/K \cong G/K_g$.

Proof of Proposition 3.4.7. Since $\mathbb{F}_G$ satisfies descent for finite coproducts, a morphism $\phi\colon X \to Y$ in $\mathbb{F}_G$ where $Y \cong \coprod_i G/H_i$ decomposes as a coproduct $\coprod_i \phi_i$ for $\phi_i\colon X_i \to G/H_i$. Since $\mathcal{F}$ is product-preserving, to show that $\phi^{\circledast}$ has a left adjoint it suffices to consider the case where Y is an orbit G/H. Moreover, for $\phi\colon X \to G/H$ where $X \cong \coprod_j G/K_j$ we can decompose ϕ as

$$\coprod_j G/K_j \xrightarrow{\coprod_j \phi|_{G/K_j}} \coprod_j G/H \xrightarrow{\nabla} G/H$$

where ∇ denotes the fold map. Since adjunctions compose, to prove that left adjoints exist it is enough to consider fold maps and morphisms between orbits. In the first case, the functor $\nabla^{\circledast}$ induced by the fold map $\nabla\colon \coprod_{j\in J} G/H \to G/H$ can be identified with the diagonal functor $\mathcal{F}^H \to \mathrm{Fun}(J, \mathcal{F}^H)$ and so has a left adjoint for all finite sets J if and only if $\mathcal{F}^H$ admits finite coproducts, i.e. if and only if assumption (1) holds. In the second case, a morphism $\phi\colon G/K \to G/H$ can be decomposed as $G/K \xrightarrow{\sim} G/gKg^{-1} \xrightarrow{q^H_{gKg^{-1}}} G/H$ and so $\phi^{\circledast}$ has a left adjoint for all such maps ϕ if and only if assumption (2) holds.

Now we consider the adjointability condition. Using descent again, a pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

where $W \cong \coprod_i G/H_i$ decomposes as a coproduct of pullback squares

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] & Y_i \arrow[d] \\ Z_i \arrow[r] & G/H_i. \end{tikzcd}$$

Since $\mathcal{F}$ preserves products and taking mate squares commutes with products, we see that $\mathcal{F}$ is left adjointable if and only if it is left adjointable for pullback squares

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & G/H \end{tikzcd}$$

over an orbit. If $Y \cong \coprod_i G/K_i$ and $Z \cong \coprod_j G/L_j$ then we can decompose our pullback square into the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \coprod_{i,j} G/K_i \arrow[r] \arrow[d] & \coprod_i G/K_i \arrow[d] \\ \coprod_{i,j} G/L_j \arrow[r] \arrow[d] & \coprod_{i,j} G/H \arrow[r] \arrow[d] & \coprod_i G/H \arrow[d] \\ \coprod_j G/L_j \arrow[r] & \coprod_j G/H \arrow[r] & G/H. \end{tikzcd}$$

where the top left square decomposes as a coproduct of pullback squares

$$\begin{tikzcd} X_{i,j} \arrow[r] \arrow[d] & G/K_i \arrow[d] \\ G/L_j \arrow[r] & G/H \end{tikzcd}$$

of the form considered in (4) and the other squares are defined using fold maps. Since mate squares are compatible with both vertical and horizontal composition of squares, the functor $\mathcal{F}$ will be left adjointable if the images of the four squares in such decompositions are left adjointable. Using again the assumption that $\mathcal{F}$ preserves finite products, we see that this holds if and only if (4) holds and we have left adjointability for squares of the form

$$\begin{tikzcd} \coprod_{i\in I} G/K \arrow[r, "\nabla_K"] \arrow[d, "\coprod_{i\in I}\phi"] & G/K \arrow[d, "\phi"] \\ \coprod_{i\in I} G/H \arrow[r, "\nabla_H"] & G/H. \end{tikzcd}$$

The latter means the canonical map $\nabla_{K,\oplus}(\prod_i \phi^{\circledast}) \to \phi^{\circledast}\nabla_{H,\oplus}$ is an equivalence, i.e. the functor $\phi^{\circledast}$ preserves I-indexed coproducts where ϕ is a map between orbits in $\mathbb{F}_G$. Since such maps are composites of isomorphisms and maps coming from subgroup inclusions, this is equivalent to condition (3). □

Definition 3.4.9. We say a G-∞-category $\mathcal{F}$ has *additive transfers* if it satisfies the conditions of Proposition 3.4.7 when viewed as a product-preserving functor $\mathbb{F}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$.

Remark 3.4.10. In the terminology of [Sha18, Nar16] a G-∞-category has *finite G-coproducts* if and only if it has additive transfers in our sense, cf. [Nar16, Proposition 2.11].

Proposition 3.4.11. *Suppose $\mathcal{F}\colon \mathrm{Span}(\mathbb{F}_G) \to \mathrm{Cat}_\infty$ is a G-symmetric monoidal ∞-category whose underlying G-∞-category has additive transfers. Then $\mathcal{F}$ is distributive if and only if for all morphisms $\phi\colon X \to Y, \psi\colon Y \to G/H$ in $\mathbb{F}_G$, the distributivity transformation*

$$g_\oplus \tilde{\psi}_\otimes \epsilon^{\circledast} \to \psi_\otimes \phi_\oplus$$

from the distributivity square

$$
\begin{tikzcd}
 & W \arrow[r, "\tilde{\psi}"] \arrow[dl, "\epsilon"'] \arrow[dd, "\tilde{g}"] & Z \arrow[dd, "g"] \\
X \arrow[dr, "\phi"'] & & \\
 & Y \arrow[r, "\psi"] & G/H,
\end{tikzcd}
$$

is an equivalence.

Remark 3.4.12. Decomposing $Z = \psi_* X$ in such a distributivity diagram as a coproduct of orbits is in general a non-trivial problem in finite group theory, so we do not expect that this distributivity condition can be simplified further.

Proof of Proposition 3.4.11. Since $\mathbb{F}_G$ satisfies descent for finite coproducts, we know from Example 2.6.14 that finite coproducts of distributivity diagrams are again distributivity diagrams. Since $\mathcal{F}$ preserves products, distributivity transformations associated to such coproducts decompose as products, hence the distributivity condition reduces to the case where the target of the second map is an orbit. □

Definition 3.4.13. We say a G-symmetric monoidal ∞-category $\mathcal{F}\colon \mathrm{Span}(\mathbb{F}_G) \to \mathrm{Cat}_\infty$ is *compatible with additive transfers* if it satisfies the condition of Proposition 3.4.11.

Corollary 3.4.14. *Product-preserving functors* $\mathrm{BISPAN}(\mathbb{F}_G) \to \mathrm{Cat}_\infty$ *correspond to G-symmetric monoidal ∞-categories that are compatible with additive transfers.*

Proof. By Corollary 2.3.15, functors $\mathrm{BISPAN}(\mathbb{F}_G) \to \mathrm{CAT}_\infty$ correspond to distributive functors $\mathrm{Span}(\mathbb{F}_G) \to \mathrm{CAT}_\infty$. Moreover, from Remark 2.6.15 we know that the product in $\mathrm{BISPAN}(\mathbb{F}_G)$ is given by the coproduct in $\mathbb{F}_G$, just as in $\mathrm{Span}(\mathbb{F}_G)$, so product-preserving functors from $\mathrm{BISPAN}(\mathbb{F}_G)$ correspond to product-preserving distributive functors under this equivalence. By Proposition 3.4.7 and Proposition 3.4.11, the latter are equivalent to G-symmetric monoidal ∞-categories that are compatible with additive transfers. □

We now consider some examples of G-symmetric monoidal ∞-categories compatible with additive transfers:

Example 3.4.15 (Finite G-sets)**.** As a special case of Proposition 3.3.1 we get a functor

$$\mathrm{BISPAN}(\mathbb{F}_G) \to \mathrm{CAT}$$

taking $X \in \mathbb{F}_G$ to the slice $(\mathbb{F}_G)_{/X}$. Here we can identify $(\mathbb{F}_G)_{/(G/H)}$ with $\mathbb{F}_H$, so the underlying G-category is given by $(G/H) \mapsto \mathbb{F}_H$. Since $\mathbb{F}_G$ has descent for finite coproducts, this is a product-preserving functor; the underlying G-symmetric monoidal category encodes the cartesian products of finite G-sets and their compatibility with the left and right adjoints to the restriction functor $\mathbb{F}_G \to \mathbb{F}_H$ for H a subgroup of G.

Example 3.4.16 (G-spaces)**.** As a variant of the previous example, we can consider the G-∞-category of G-spaces. Since $\mathcal{S}_G$ is locally cartesian closed (being a (presheaf) ∞-topos), we can apply Proposition 3.3.1 to it and then restrict to bispans in $\mathbb{F}_G$ to get a functor

$$\mathrm{BISPAN}(\mathbb{F}_G) \to \mathrm{CAT}_\infty$$

that takes $X \in \mathbb{F}_G$ to $(\mathcal{S}_G)_{/X}$; here we can identify $(\mathcal{S}_G)_{/(G/H)}$ with $\mathcal{S}_H$. Since $\mathcal{S}_G$ satisfies descent, this is a product-preserving functor. The underlying G-symmetric monoidal category (compatible with additive transfers) encodes the cartesian products of H-spaces for all subgroups H of G and their compatibility with the left and right adjoints to the restriction functor $\mathcal{S}_G \to \mathcal{S}_H$.

Example 3.4.17 (G-actions in a symmetric monoidal ∞-category)**.** Let $\mathcal{C}$ be a symmetric monoidal ∞-category. By Proposition 3.2.3 this determines a functor $\mathrm{Span}_{\mathrm{fin}}(\mathcal{S}) \to \mathrm{Cat}_\infty$ taking $X \in \mathcal{S}$ to $\mathrm{Fun}(X, \mathcal{C})$. For a finite group G we have a functor $\mathcal{G}\colon \mathbb{F}_G \to \mathcal{S}$ by restricting the colimit functor $\mathrm{Fun}(BG, \mathcal{S}) \to \mathcal{S}$; this takes a finite G-set X to the groupoid $X/\!/G$. The functor $\mathcal{G}$ preserves pullbacks since the colimit functor factors as the straightening equivalence $\mathrm{Fun}(BG, \mathcal{S}) \xrightarrow{\sim} \mathcal{S}_{/BG}$ followed by the forgetful functor to $\mathcal{S}$, which preserves all weakly contractible limits. Moreover, $\mathcal{G}$ takes values in $\mathcal{S}_{\mathrm{fin}}$, and so yields a functor $\mathrm{Span}(\mathcal{G})\colon \mathrm{Span}(\mathbb{F}_G) \to \mathrm{Span}_{\mathrm{fin}}(\mathcal{S})$. This functor preserves products, since $\mathcal{G}$ preserves coproducts. It follows that we can restrict along $\mathrm{Span}(\mathcal{G})$ and obtain for any symmetric monoidal ∞-category $\mathcal{C}$ a G-symmetric monoidal structure on $\mathrm{Fun}(BG, \mathcal{C})$. Moreover, if the tensor product in $\mathcal{C}$ is compatible with finite coproducts, this G-symmetric monoidal structure will be compatible with additive transfers.

Example 3.4.18 (G-representations)**.** As a special case of the previous example, we can take $\mathcal{C}$ to be the category Vect_k of k-vector spaces with the tensor product as symmetric monoidal structure. We then obtain a functor

$$\rho_G\colon \mathrm{BISPAN}(\mathbb{F}_G) \to \mathrm{CAT}$$

such that $\rho_G(G/H)$ is the category $\mathrm{Rep}_H(k) := \mathrm{Fun}(BH, \mathrm{Vect}_k)$ of H-representations and for for subgroups $H \subseteq K \subseteq G$,

- $(q_H^K)^{\circledast}$ is the restriction functor $\mathrm{Res}_H^K\colon \mathrm{Rep}_K(k) \to \mathrm{Rep}_H(k)$,
- $(q_H^K)_\oplus$ is the induced representation functor $\mathrm{Ind}_H^K\colon \mathrm{Rep}_H(k) \to \mathrm{Rep}_H(k)$, left adjoint to Res_H^K, and given on objects by taking an H-representation V to $\bigoplus_{K/H} V$ with induced action of K,
- $(q_H^K)_\otimes$ is the *tensor-induction functor* $\mathrm{Rep}_H(k) \to \mathrm{Rep}_H(k)$, given on objects by taking an H-representation V to $\bigotimes_{K/H} V$ with induced action of K.

Our final example of a G-symmetric monoidal ∞-category compatible with additive transfers is the ∞-category of genuine G-spectra. This is less formal than our previous examples, but the input we need is already in the literature:

Proposition 3.4.19. *The ∞-category Sp_G of genuine G-spectra is a G-symmetric monoidal ∞-category compatible with additive transfers.*

Proof. Taking fixed point spectra for subgroups of G gives a functor $\mathcal{O}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$ that takes G/H to the ∞-category Sp_H of genuine H-spectra; this is the G-∞-category of G-spectra. The corresponding product-preserving functor $\mathbb{F}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$ extends to a functor $\sigma_G\colon \mathrm{Span}(\mathbb{F}_G) \to \mathrm{Cat}_\infty$ such that for $H \subseteq K \subseteq G$ the functor $(q_H^K)_\otimes\colon \mathrm{Sp}_H \to \mathrm{Sp}_K$ is the multiplicative norm of [HHR16]; this follows from the results of [BaHo18, §9] by restricting the functor from spans of profinite groupoids defined there. For $\phi\colon X \to Y$ in $\mathbb{F}_G$, the functor $\phi^{\circledast}\colon \sigma_G(Y) \to \sigma_G(X)$ has a left adjoint by [BaHo18, Lemma 9.7(2)] — for $H \subseteq K \subseteq G$ the left adjoint $(q_H^K)_\oplus$ is the classical (additive) *transfer* or induction functor. To see that the functor σ_G is left adjointable we check the 3 remaining conditions in Proposition 3.4.7: conditions (1) and (3) hold since the ∞-categories of G-spectra are stable (hence any right adjoint functor between them automatically preserves finite colimits). To check condition (4) we use that any H-spectrum is a sifted colimit of desuspensions of suspension spectra $\Sigma^\infty_+ X$ with X a finite H-set, and the functors involve preserve (sifted) colimits and desuspensions. Hence it suffices to check that the canonical map

$$f_{K,\oplus}f_H^{\circledast}\Sigma^\infty_+ X \to (q_K^L)^{\circledast}(q_H^L)_{\oplus}\Sigma^\infty_+ X$$

is an equivalence for $X \in \mathbb{F}_H$. But here all the functors are given on suspension spectra of finite H-sets by the suspension spectra on the corresponding functors for finite G-sets, so this follows from Example 3.4.15. The same argument works for distributivity, since we also have $f_{\otimes}\Sigma^\infty_+ X \simeq \Sigma^\infty_+ f_* X$. □

Remark 3.4.20. The adjointability condition here reduces by Remark 3.4.8 to a double coset formula for additive transfers. This is a basic fact in equivariant stable homotopy theory that has surely long been well-known to the experts, but the only explicit references we could find are [HHR16, Proposition A.30] (applied to the direct sum in orthogonal spectra) and [Pat16, Corollary 5.2]. The distributivity condition also appears (in terms of orthogonal spectra) as [HHR16, Proposition A.37].

Variant 3.4.21. Following Blumberg and Hill [BlHi18] we can consider subcategories $\mathbb{F}_{G,\mathcal{I}}$ where $\mathcal{I}$ is an *indexing system* as in [BlHi18, Definition 1.2]; by [BlHi18, Theorem 1.4] these are precisely the subcategories of $\mathbb{F}_G$ such that $(\mathbb{F}_G, \mathbb{F}_{G,\mathcal{I}})$ is a span pair. Since $\mathbb{F}_G$ is locally cartesian closed, we then have a bispan triple $(\mathbb{F}_G, \mathbb{F}_{G,\mathcal{I}}, \mathbb{F}_G)$. Product-preserving functors out of $\mathrm{Span}_{\mathcal{I}}(\mathbb{F}_G)$ are G-symmetric monoidal ∞-categories where only some subclass of multiplicative norms exist, and we can characterize distributivity for such functors by the analogue of Proposition 3.4.11 with the map ψ restricted to lie in $\mathbb{F}_{G,\mathcal{I}}$.

Variant 3.4.22. We can also consider a G-equivariant analogue of §3.2: Using [BaHo18, Proposition C.18] a G-symmetric monoidal ∞-category determines by right Kan extension a functor $\mathrm{Span}_{\mathrm{fin}}(\mathcal{S}_G) \to \mathrm{Cat}_\infty$, where $\mathcal{S}_{G,\mathrm{fin}}$ denotes the subcategory of $\mathcal{S}_G$ containing the maps $\phi\colon X \to Y$ such that for every map $G/H \to Y$, the pullback $X \times_Y G/H$ is a finite G-set. Here $(\mathcal{S}_G, \mathcal{S}_{G,\mathrm{fin}}, \mathcal{S}_G)$ is a bispan triple, and we might say that the G-symmetric monoidal ∞-category is "compatible with G-space-indexed G-colimits" if this is distributive. We expect that this should hold for the G-symmetric monoidal ∞-category of genuine G-spectra and, by analogy with [GHK17], that monads in the $(\infty,2)$-category $\mathrm{BISPAN}_{\mathrm{fin}}(\mathcal{S}_G)$ should be related to a notion of *G-∞-operads*.

Variant 3.4.23. In [BaHo18, Chapter 9], Bachmann and Hoyois define ∞-categories of equivariant spectra for *profinite groupoids*, and we can also consider distributivity in this setting. We can take the $(2,1)$-category of finite groupoids $\mathrm{FinGpd} \subset \mathcal{S}$ to be the full subcategory of spaces spanned by 1-truncated spaces with finite π_0, π_1. We then form the $(2,1)$-category of profinite groupoids by taking pro-objects: $\mathrm{ProfGpd} := \mathrm{Pro}(\mathrm{FinGpd})$. Let $\mathrm{ProfGpd}_{\mathrm{fp}}$ be the subcategory containing only the *finitely presented* maps as in [BaHo18, §9.1]. It then follows from [BaHo18, Lemmas 9.3 and 9.5] and Lemma 2.3.6 that we have a bispan triple

$$(\mathrm{ProfGpd}, \mathrm{ProfGpd}_{\mathrm{fp}}, \mathrm{ProfGpd}).$$

Here $\mathrm{ProfGpd}_{\mathrm{fp}/BG} \simeq \mathbb{F}_G$. In [BaHo18, Chapter 9] equivariant spectra are defined as a functor

$$\mathrm{Span}_{\mathrm{fp}}(\mathrm{ProfGpd}) \to \mathrm{Cat}_\infty;$$

we expect that this is distributive, giving a functor of $(\infty,2)$-categories

$$\mathrm{BISPAN}_{\mathrm{fp}}(\mathrm{ProfGpd}) \to \mathrm{CAT}_\infty.$$

3.5. **Motivic bispans and normed ∞-categories.** In this section we will relate the *normed ∞-categories* of Bachmann–Hoyois to functors from certain bispans in schemes (and more generally algebraic spaces) to CAT_∞. As an example of this, we will see that the results of [BaHo18] imply that ∞-categories of motivic spectra give such a functor. We begin by describing some bispan triples on schemes.

Warning 3.5.1. Throughout this section, schemes and algebraic spaces are always assumed to be quasi-compact and quasi-separated (qcqs).[13]

Notation 3.5.2. If S is a (qcqs) scheme, we write Sch_S for the category of (qcqs) schemes over S. (This has pullbacks since qcqs morphisms are closed under base change, see [Stacks, Tags 01KU and 01K5].)

Proposition 3.5.3. *The following are bispan triples:*

(i) $(\mathrm{Sch}_S, \mathrm{Sch}_S^{\mathrm{flf}}, \mathrm{Sch}_S^{\mathrm{qp}})$ *for any scheme S, where $\mathrm{Sch}_S^{\mathrm{flf}}$ consists of finite locally free (meaning finite, flat, and of finite presentation) morphisms of S-schemes and $\mathrm{Sch}_S^{\mathrm{qp}}$ of quasiprojective morphisms of S-schemes,*

(ii) $(\mathrm{Sch}_S, \mathrm{Sch}_S^{\mathrm{fét}}, \mathrm{Sch}_S^{\mathrm{qp}})$ *for any scheme S, where $\mathrm{Sch}_S^{\mathrm{fét}}$ consists of finite étale morphisms of S-schemes,*

(iii) $(\mathrm{Sch}_S, \mathrm{Sch}_S^{\mathrm{flf}}, \mathrm{Sch}_S^{\mathrm{smqp}})$ *for any scheme S, where $\mathrm{Sch}_S^{\mathrm{smqp}}$ consists of smooth and quasiprojective morphisms of S-schemes,*

(iv) $(\mathrm{Sch}_S, \mathrm{Sch}_S^{\mathrm{fét}}, \mathrm{Sch}_S^{\mathrm{smqp}})$ *for any scheme S.*

(v) $(\mathrm{Sch}_S, \mathrm{Sch}_S^{\mathrm{fét}}, \mathrm{Sch}_S^{\mathrm{proj}})$ *for any scheme S, where $\mathrm{Sch}_S^{\mathrm{proj}}$ consists of projective morphisms of S-schemes.*

Proof. The classes of morphisms of schemes that are finite locally free, quasiprojective, smooth, finite, and étale are all closed under base change by [Stacks, Tags 02KD,0B3G,01VB,01WL,02GO], respectively. Hence the subcategories $\mathrm{Sch}_S^{\mathrm{flf}}$, $\mathrm{Sch}_S^{\mathrm{qp}}$, $\mathrm{Sch}_S^{\mathrm{smqp}}$, and $\mathrm{Sch}_S^{\mathrm{fét}}$ of Sch_S all give span pairs.

Now the main point is the existence of *Weil restrictions* for schemes: if $f\colon S' \to S$ is a morphism of schemes and X is an S'-scheme, the Weil restriction $R_f X$, if it exists, is an S-scheme that represents the functor

$$\mathrm{Hom}_{/S'}((-) \times_{S'} S, X)\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Set};$$

note that this is exactly the requirement (12) for a distributivity diagram for $X \to S'$ and f.

By [BLR90, Theorem 7.6.4], the Weil restriction $R_f X$ exists if f is a finite locally free morphism and X is quasiprojective. Moreover, $R_f X$ is quasiprojective over S by [BaHo18, Lemma 2.13]. This gives the bispan triple (i), from which (ii) is trivially obtained by restricting from finite locally free morphisms to the subclass of finite étale ones. For (iii) and (iv) the only additional input needed is that R_f takes smooth morphisms to smooth morphisms, which holds by [BLR90, Proposition 7.6.5(h)], while for (v) we use that R_f preserves proper morphisms if f is finite étale by [BLR90, Proposition 7.6.5(f)] and that a morphism is projective if and only if it is proper and quasiprojective by [Stacks, Tag 0BCL]. □

We now review the construction of a distributive functor for the bispan triple (v) from motivic spectra, due to Bachmann and Hoyois.

Notation 3.5.4. We write $\mathrm{SH}(S)$ for the ∞-category of motivic spectra over a base scheme S and $\mathrm{H}(S)$ for that of motivic spaces over S. For any morphism of

[13] Note that every morphism between qcqs schemes is automatically a qcqs morphism (see [Stacks, Tags 01KV and 03GI]); this means we do not need to distinguish between morphisms of finite presentation and locally of finite presentation, since the additional qcqs assumption is automatic.

schemes $f\colon S \to S'$ we have a pullback functor $f^*\colon \mathrm{SH}(S') \to \mathrm{SH}(S)$, and similarly in the unstable case. This gives functors $\mathrm{SH}, \mathrm{H}\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}_\infty$.

In [BaHo18], Bachmann and Hoyois promoted the contravariant functor $X \mapsto \mathrm{SH}(X)$ to include a *multiplicative pushforward* for finite étale morphisms, encoded as a functor out of a span category

$$\mathrm{SH}\colon \mathrm{Span}_{\text{fét}}(\mathrm{Sch}) \to \mathrm{Cat}_\infty, \qquad X \xleftarrow{f} Z \xrightarrow{g} Y \mapsto g_\otimes f^*. \tag{31}$$

Given a finite étale morphism $g\colon X \to Y$, the functor

$$g_\otimes\colon \mathrm{SH}(X) \to \mathrm{SH}(Y)$$

is first constructed unstably as the functor on the level of the pointed unstable motivic homotopy ∞-category

$$g_\otimes\colon \mathrm{H}(X)_* \to \mathrm{H}(Y)_*.$$

This functor is in turn induced by the functor of Weil restriction [BLR90, §7.6], $R_g\colon \mathrm{SmQP}_X \to \mathrm{SmQP}_Y$, where SmQP_X denotes the full subcategory of Sch_X spanned by smooth and quasiprojective X-schemes, using the fact that the inclusion $\mathrm{SmQP}_X \subset \mathrm{Sm}_X$ into the full subcategory of smooth X-schemes induces equivalent motivic unstable categories (since every smooth X-scheme is Zariski-locally also quasiprojective). We refer to [BaHo18, §1.6] for a summary of the construction and [BaHo18, §6.1] for a detailed construction of (31).

Given a smooth morphism $f\colon X \to Y$ of schemes, the pullback functor f^* admits a left adjoint

$$f_\sharp\colon \mathrm{SH}(X) \to \mathrm{SH}(Y).$$

This left adjoint should be thought of as an *additive pushforward* along f; indeed, if I is a finite set and $\nabla_I\colon \coprod_I X \to X$ is the fold map then, under the identification $\mathrm{SH}(\coprod_I X) \simeq \mathrm{SH}(X)^{\times I}$, the functor $(\nabla_I)_\sharp$ is given by

$$(X_i)_{i\in I} \mapsto \bigoplus_{i\in I} X_i.$$

The functor $f_\sharp$ is first constructed unstably as a functor

$$f_\sharp\colon \mathrm{H}(X) \to \mathrm{H}(Y),$$

which in turn is induced by the functor $\mathrm{Sm}_X \to \mathrm{Sm}_Y$ given by composition with f (i.e. the functor that sends a smooth X-scheme T to itself regarded as a Y-scheme); see [Hoy17, Section 4.1, Lemma 6.2] for a construction in the language of this paper in the more general context of equivariant motivic homotopy theory.

The importance of this additional left adjoint functoriality in formulating smooth base change was first pointed out by Voevodsky [Voe99] and worked out by Ayoub in [Ayo07]; see [Hoy17, Section 6.1, Proposition 4.2] for an ∞-categorical formulation (in the more general equivariant context). In our language, this says that the functor $\mathrm{SH}\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is left adjointable with respect to smooth maps. Moreover, combining this with [BaHo18, Proposition 5.10(1)] we get that the functor (31) is smqp-distributive. Applying Corollary 2.3.15 these results imply:

Theorem 3.5.5. *Motivic spectra give a* smqp*-distributive functor*

$$\mathrm{SH}\colon \mathrm{Span}_{\text{fét}}(\mathrm{Sch}) \to \mathrm{Cat}_\infty,$$

and so a functor of $(\infty, 2)$-categories

$$\mathrm{BISPAN}_{\text{fét},\text{smqp}}(\mathrm{Sch}) \to \mathrm{CAT}_\infty.$$

As indicated already in [BaHo18, Remark 2.14], the restriction to those smooth morphisms that are quasiprojective here is an artifact of the restriction of SH to schemes instead of algebraic spaces. We will therefore extend this result by working with algebraic spaces.

Notation 3.5.6. For a (qcqs) scheme S, we write AlgSpc_S for the category of (qcqs) algebraic spaces over S.[14] This category has pullbacks since qcqs morphisms of algebraic spaces are closed under base change by [Stacks, Tags 03KL,03HF]; note also that for an S-scheme S' we have an equivalence

$$\mathrm{AlgSpc}_{S'} \simeq (\mathrm{AlgSpc}_S)_{/S'}$$

by [Stacks, Tag 04SG].

Here we again have several bispan triples:

Proposition 3.5.7. *The following are bispan triples:*

(i) $(\mathrm{AlgSpc}_S, \mathrm{AlgSpc}_S^{\mathrm{flf}}, \mathrm{AlgSpc}_S)$ *for any scheme S, where $\mathrm{AlgSpc}_S^{\mathrm{flf}}$ consists of finite locally free morphisms of algebraic spaces over S,*

(ii) $(\mathrm{AlgSpc}_S, \mathrm{AlgSpc}_S^{\text{fét}}, \mathrm{AlgSpc}_S)$ *for any scheme S, where $\mathrm{AlgSpc}_S^{\text{fét}}$ consists of finite étale morphisms of algebraic spaces over S,*

(iii) $(\mathrm{AlgSpc}_S, \mathrm{AlgSpc}_S^{\mathrm{flf}}, \mathrm{AlgSpc}_S^{\mathrm{sm}})$ *for any scheme S, where $\mathrm{AlgSpc}_S^{\mathrm{sm}}$ consists of smooth morphisms of algebraic spaces over S,*

(iv) $(\mathrm{AlgSpc}_S, \mathrm{AlgSpc}_S^{\text{fét}}, \mathrm{AlgSpc}_S^{\mathrm{sm}})$ *for any scheme S.*

(v) $(\mathrm{AlgSpc}_S, \mathrm{AlgSpc}_S^{\text{fét}}, \mathrm{AlgSpc}_S^{\mathrm{prop}})$ *for any scheme S, where $\mathrm{AlgSpc}_S^{\mathrm{prop}}$ consists of proper morphisms of algebraic spaces over S.*

Proof. Morphisms of algebraic spaces that are finite locally free, finite, étale, and smooth are closed under base change by [Stacks, Tags 03ZY, 03ZS, 0466, 03ZE], respectively. Thus the subcategories $\mathrm{AlgSpc}_S^{\mathrm{flf}}$, $\mathrm{AlgSpc}_S^{\text{fét}}$, $\mathrm{AlgSpc}_S^{\mathrm{sm}}$ of AlgSpc_S all give span pairs.

Suppose $f\colon Y \to Z$ is a finite locally free morphism of algebraic spaces. Then the functor

$$f^*\colon \mathrm{AlgSpc}_Z \to \mathrm{AlgSpc}_Y$$

admits a right adjoint R_f, given by Weil restriction of algebraic spaces, by a result of Rydh [Ryd11, Theorem 3.7]; note that R_f preserves the qcqs property we require by [Ryd11, Proposition 3.8(xiii,xix)]. This gives the bispan triples (i) and (ii) (since finite étale morphisms are in particular finite locally free).

To obtain (iii) and (iv) it suffices to note that R_f converts smooth morphisms to smooth morphisms. If $f\colon X \to Y$ is a finite locally free morphism of schemes this follows from [Ryd11, Proposition 3.5(i,iv)] and [Stacks, Tag 0DP0]. The extension to the general case is easy: First note that for W over X we have that $R_f W \to Y$ is smooth if and only if its pullback along any morphism $g\colon T \to Y$ with T a scheme is smooth, by [Stacks, Tag 03ZF]. In the pullback square

$$\begin{tikzcd} U \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ T \arrow[r, "g"] & Y \end{tikzcd}$$

the algebraic space U is a scheme since f is finite and so by definition representable. We also have the base change equivalence $g^* R_f W \cong R_{f'} g'^* W$, and if W is smooth over X then $R_{f'} g'^* W$ is smooth over T since the base change $g'^* W$ is smooth over

[14] Every morphism between qcqs algebraic spaces is a qcqs morphism by [Stacks, Tag 03KR,03KS]; thus we can still ignore the distinction between morphisms of finite presentation and locally of finite presentation.

U and f' is a finite locally free morphism of schemes. Finally, (v) holds since by the same argument starting with [Ryd11, Remark 3.9] R_f preserves proper morphisms if f is finite étale. □

In order to work with motivic spectra over algebraic spaces effectively we record the following lemma which amounts to saying that any Nisnevich sheaf on algebraic spaces is right Kan extended from schemes.

Lemma 3.5.8. *Let $\mathcal{C}$ be a complete ∞-category, S be a scheme and let*

$$\iota\colon \mathrm{Sch}_S \hookrightarrow \mathrm{AlgSpc}_S$$

be the inclusion. Then the restriction functor $\iota^*\colon \mathrm{PShv}(\mathrm{AlgSpc}_S, \mathcal{C}) \to \mathrm{PShv}(\mathrm{Sch}_S, \mathcal{C})$ induces an equivalence of ∞-categories:

$$\iota^*\colon \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{AlgSpc}_S, \mathcal{C}) \xrightarrow{\sim} \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sch}_S, \mathcal{C}),$$

with the inverse given by right Kan extension.

Proof. Let

$$\iota_*\colon \mathrm{PShv}(\mathrm{Sch}_S, \mathcal{C}) \to \mathrm{PShv}(\mathrm{AlgSpc}_S, \mathcal{C})$$

denote the right adjoint to ι^* which is computed by right Kan extension. We first claim that ι_* preserves Nisnevich sheaves, i.e., there exists a filler in the following diagram:

$$\begin{tikzcd} \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sch}_S, \mathcal{C}) \arrow[r, dashed] \arrow[d, hook] & \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{AlgSpc}_S, \mathcal{C}) \arrow[d, hook] \\ \mathrm{PShv}(\mathrm{Sch}_S, \mathcal{C}) \arrow[r, "\iota_*"] & \mathrm{PShv}(\mathrm{AlgSpc}_S, \mathcal{C}) \end{tikzcd}$$

It suffices to verify that the functor ι is topologically cocontinuous[15], for the Nisnevich topology on Sch_S and on AlgSpc_S.

Unwinding definitions, this is the following claim:

- for any $X \in \mathrm{Sch}_S$ and any Nisnevich sieve $R' \hookrightarrow \iota(X)$ of algebraic spaces, the sieve on Sch_S generated by morphisms of schemes $X' \to X$ such that $\iota(X') \to \iota(X)$ factors through R' is a Nisnevich sieve of X.

This condition is verified by [Knu71, Chapter II, Theorem 6.4]; indeed if $x \in X$ and $f\colon Y \to X$ an étale morphism such that we have a lift

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ \operatorname{Spec} \kappa(x) \arrow[ur] \arrow[r, "x"] & X. \end{tikzcd}$$

then this result tells us that we can find a completely decomposed étale morphism $U \to Y$ with U an affine scheme such that $\operatorname{Spec} \kappa(x) \to Y$ factors through U. Since the composite $U \to X$ is an étale morphism, the desired claim is verified.

Therefore we have an adjunction on the level of Nisnevich sheaves

$$\iota^*\colon \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{AlgSpc}_S, \mathcal{C}) \rightleftarrows \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sch}_S, \mathcal{C})\colon \iota_*,$$

where ι_* is fully faithful since it is given by right Kan extension along the fully faithful functor ι. Equivalently, the counit transformation $\iota^*\iota_* \to \mathrm{id}$ is an equivalence. Moreover, the functor ι^* is conservative, since any algebraic space has a cover by schemes. To see that ι^* is an equivalence it now suffices to show that

[15]In the sense of [SGA4, Éxpose III, Définition 2.1] where this is called "continuous"; to avoid confusion with the notion of a functor that preserves limits, we borrow this terminology from [Kha19].

the unit transformation $\mathrm{id} \to \iota_*\iota^*$ is also an equivalence, which we can check after applying ι^*. But the composite

$$\iota^* \to \iota^*\iota_*\iota^* \xrightarrow{\sim} \iota^*$$

is an equivalence by one of the adjunction identities, and we already know the second map is an equivalence, so this holds by the 2-of-3 property. □

We can now prove:

Theorem 3.5.9. *The functor* (31) *extends canonically to a* sm-*distributive functor*

$$\mathrm{SH}\colon \mathrm{Span}_{\text{fét}}(\mathrm{AlgSpc}_S) \to \mathrm{Cat}_\infty$$

and hence to a functor of $(\infty, 2)$*-categories*

$$\mathrm{SH}\colon \mathrm{BISPAN}_{\text{fét},\mathrm{sm}}(\mathrm{AlgSpc}_S) \to \mathrm{CAT}_\infty.$$

Proof. First, by Lemma 3.5.8, the right Kan extension of the Nisnevich sheaf

$$\mathrm{SH}\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAT}_\infty$$

to algebraic spaces defines a Nisnevich sheaf

$$\mathrm{SH}\colon \mathrm{AlgSpc}^{\mathrm{op}} \to \mathrm{CAT}_\infty.$$

Therefore we can apply [BaHo18, Proposition C.18] to obtain an extension

$$\mathrm{SH}\colon \mathrm{Span}_{\text{fét}}(\mathrm{AlgSpc}) \to \mathrm{Cat}_\infty$$

of (31).

Now in order to use Corollary 2.3.15, we need to verify that this extension satisfies the distributivity property with respect to smooth morphisms. Since the functors involved in the adjointability and distributivity transformations are stable under base change, we are reduced to the case of schemes we already discussed above. □

Remark 3.5.10. In the more general setting of spectral algebraic spaces, Khan has constructed the unstable motivic homotopy ∞-category (defined in [Kha19, Definition 2.4.1]). By Nisnevich descent, this agrees with the right Kan extended version appearing in the proof of Theorem 3.5.9, using the uniqueness part of Lemma 3.5.8.

3.6. **Bispans in spectral Deligne-Mumford stacks and** Perf**.** In this final section, we promote the functor $\mathrm{Perf}\colon \mathrm{SpDM} \to \mathrm{Cat}_\infty$ to a functor out of a category of bispans. This extends a result of Barwick [Bar17, Example D], which gives a functor

$$\mathrm{Perf}\colon \mathrm{Span}_{\mathcal{FP}}(\mathrm{SpDM}) \to \mathrm{Cat}_\infty$$

encoding the usual pullback f^* and pushforward f_* for a morphism in SpDM, with $\mathcal{FP}$ a class of morphisms for which f_* restricts to perfect objects and satisfies base change. Our version adds a multiplicative pushforward $f_\otimes$ where f is finite étale (at the cost of restricting the class $\mathcal{FP}$ in order to guarantee the existence of Weil restrictions). We note that the multiplicative pushforward in this situation is right Kan extended from the symmetric monoidal structure in Perf and is thus not as complicated as in the motivic and equivariant cases we considered above. However, we include this section with a view towards applications in algebraic K-theory which we hope to discuss elsewhere.

In this section, we will freely use the language of spectral Deligne-Mumford (DM) stacks introduced in [Lur18]. We denote by SpDM_S the ∞-category of spectral Deligne-Mumford stacks over a base S. We also adopt the following terminology from [Bar17, Example D]:

Definition 3.6.1. Recall that for $X \in \mathrm{SpDM}$ an object $\mathcal{E} \in \mathrm{QCoh}(X)$ is called *perfect* if for every map $x\colon \operatorname{Spec} A \to X$, where A is an E_∞-ring spectrum, the A-module $x^*\mathcal{E}$ is perfect (i.e. dualizable or equivalently compact in the symmetric monoidal ∞-category Mod_A). Now suppose that $f\colon X \to Y$ is a morphism in SpDM. We say that f is *perfect* if the pushforward functor

$$f_*\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y),$$

takes perfect objects to perfect objects.

The following theorem of Lurie furnishes a large class of perfect morphisms:

Theorem 3.6.2 ([Lur18, Theorem 6.1.3.2])**.** *Let $f\colon X \to Y$ be a morphism in* SpDM*. If f is proper, locally almost of finite presentation, and of finite Tor-amplitude, then f is perfect.*

Notation 3.6.3. Following [Bar17, Notation D.17], we label the class of morphisms in Theorem 3.6.8 by $\mathcal{FP}$.

We will also make use the existence of the spectral version of Weil restriction; see the discussion of [Lur18, Section 19.1] and note that the definitions are completely analogous with the classical situation. There Lurie proves the following existence theorem:

Theorem 3.6.4 ([Lur18, Theorem 19.1.0.1])**.** *Suppose that $f\colon X \to Y$ is a morphism in* SpDM *that is proper, flat, and locally almost of finite presentation. Let $p\colon Z \to X$ be a relative spectral algebraic space that is quasi-separated and locally almost of finite presentation. Then the Weil restriction $R_f(p) \in \mathrm{SpDM}_Y$ exists.*

Notation 3.6.5. In light of this, let us write

- $\mathcal{W}$ for the class of morphisms in SpDM that are proper, flat, and locally almost of finite presentation,
- $\mathcal{Q}$ for the class of morphisms in SpDM that are relative spectral algebraic spaces, quasi-separated, and locally almost of finite presentation,
- $\mathcal{FP}' \subset \mathcal{FP}$ for the class of morphisms in $\mathcal{FP}$ which are furthermore relative spectral algebraic spaces,
- fét for the class of finite étale morphisms in SpDM.

Then Weil restrictions of morphisms in $\mathcal{Q}$ along ones in $\mathcal{W}$ exist in SpDM. Here $\mathcal{FP}' \subseteq \mathcal{Q}$ since proper morphisms are always quasi-separated, and fét $\subseteq \mathcal{W}$.

Lemma 3.6.6. *Suppose that $f\colon X \to Y$ is a morphism in* SpDM *of class $\mathcal{W}$ and $p\colon Z \to X$ is one of class $\mathcal{Q}$. Then $R_f(p)$ is again of class $\mathcal{Q}$. Assuming f is moreover finite étale, we also have:*

(a) if p is quasi-compact, then $R_f(p) \to Y$ is quasi-compact;
(b) if p is proper, then $R_f(p) \to Y$ is proper;
(c) if p is of finite Tor-amplitude, then $R_f(p) \to Y$ is of finite Tor-amplitude.

Proof. The statement that $R_f(p)$ is of class $\mathcal{Q}$ is part of [Lur18, Theorem 19.1.0.1].

Let us now verify properties (a)–(c). To verify (a) and (c), note that quasi-compactness and having finite Tor-amplitude can be detected étale locally on the target (for the former, this is [Lur18, Remark 2.3.2.5] and the equivalences of [Lur18, Proposition 2.3.2.1] and for the latter this is [Lur18, Proposition 6.1.2.2]). Therefore, we may work étale locally on Y. Since f was assumed to be finite étale, it is étale locally a fold map $f\colon X \simeq \coprod_{i=1}^n Y \to Y$. In this case, we can write $p\colon Z \to X$ as a coproduct $\coprod_i f_i\colon \coprod_i Z_i \to \coprod_i Y$. Therefore the Weil restriction takes the form $R_f(p) \simeq Z_1 \times_Y Z_2 \times_Y \cdots \times_Y Z_n \to Y$. To conclude (a), we note that quasi-compactness is stable under base change [Lur18, Proposition 2.3.3.1], while for (c), we note that Tor-amplitudes add up under base change [Lur18, Lemma 6.1.1.6].

To prove (b), we use the valuative criterion for properness [Lur18, Corollary 5.3.1.2], which applies since we have already verified (a) and (1)–(3), together with the functor-of-points description of the Weil restriction. □

Proposition 3.6.7. *Let S be a spectral Deligne-Mumford stack. Then*

$$(\mathrm{SpDM}_S, \mathrm{SpDM}_S^{\text{fét}}, \mathrm{SpDM}_S^{\mathcal{FP}'})$$

is a bispan triple.

Proof. After Lemma 3.6.6 it suffices to note that morphisms in fét and $\mathcal{FP}'$ are stable under base change. This follows from [Lur18, Proposition 5.1.3.1, Proposition 4.2.1.6, Proposition 6.1.2.2, Proposition 1.4.1.11(2), Propisition 3.3.1.8]. □

Theorem 3.6.8. *Let S be a spectral Deligne-Mumford stack. The functor*

$$\mathrm{Perf}\colon \mathrm{SpDM}_S \to \mathrm{Cat}_\infty,$$

canonically extends to a functor

$$\mathrm{Perf}\colon\ \mathrm{Span}_{\text{fét}}(\mathrm{SpDM}_S) \to \mathrm{Cat}_\infty.$$

Moreover, this is right $\mathcal{FP}'$-distributive (in the sense of Variant 2.3.13), and so canonically extends further to a functor of $(\infty, 2)$-categories

$$\mathrm{Perf}\colon \mathrm{BISPAN}_{\text{fét},\mathcal{FP}'}(\mathrm{SpDM}_S)^{\text{2-op}} \to \mathrm{CAT}_\infty.$$

Proof. We first apply [BaHo18, Proposition C.9] to extend Perf to a functor

$$\mathrm{Span}_{\text{fold}}(\mathrm{SpDM}_S) \to \mathrm{Cat}_\infty,$$

where $\mathrm{SpDM}_S^{\text{fold}}$ consists of the finite fold maps, i.e. the maps $\coprod_I X \to \coprod_J X$ with $I \to J$ a map of finite sets. Here the pushforward

$$\nabla_\otimes\colon \mathrm{Perf}(\coprod_I X) \cong \mathrm{Perf}(X)^{\times I} \to \mathrm{Perf}(X)$$

is just the tensor product, and the base change simply encodes the fact that the pullback functors are symmetric monoidal.

Next, we use [BaHo18, Corollary C.13] for $\mathcal{C} = \mathrm{SpDM}$, t the étale topology and m the class of finite étale map to obtain a functor

$$\mathrm{Perf}\colon\ \mathrm{Span}_{\text{fét}}(\mathrm{SpDM}_S) \to \mathrm{Cat}_\infty.$$

The content of this result is that since Perf is an étale sheaf and finite étale morphisms are étale-locally contained in the class of fold maps, we can extend the symmetric monoidal structure to norms along finite étale morphisms.

In order to show that this functor is right $\mathcal{FP}'$-distributive, we first check that its restriction $\mathrm{Perf}\colon \mathrm{SpDM}_S \to \mathrm{Cat}_\infty$ is right $\mathcal{FP}'$-adjointable. For any morphism $f\colon X \to Y$ in SpDM_S the functor $f^*\colon \mathrm{QCoh}(Y) \to \mathrm{QCoh}(Y)$ has a right adjoint f_*. Given a pullback square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

the commutative square

$$\begin{tikzcd} \mathrm{QCoh}(Y) \arrow[r, "f^*"] \arrow[d, "g^*"] & \mathrm{QCoh}(X) \arrow[d, "g'^*"] \\ \mathrm{QCoh}(Y') \arrow[r, "f'^*"] & \mathrm{QCoh}(X') \end{tikzcd}$$

is right adjointable by [Lur18, Corollary 3.4.2.2] provided f is quasi-compact and quasi-separated. This is true by definition [Lur18, Definition 5.1.2.1] for any proper morphism and so for any morphism in $\mathcal{FP}'$. Moreover, if f is of finite Tor amplitude then f_* preserves perfect complexes by Theorem 3.6.8, so in this case the adjunction restricts to an adjunction

$$f^* : \mathrm{Perf}(Y) \rightleftarrows \mathrm{Perf}(X) : f_*$$

on the full subcategories of perfect objects, which still satisfies the right adjointability condition if f is also quasi-compact and quasi-separated. This holds in particular if f is in $\mathcal{FP}'$, so that Perf is indeed right $\mathcal{FP}'$-adjointable.

It remains to check the (right) distributivity condition for $p\colon X \to Y$ in $\mathcal{FP}'$ and $f\colon Y \to Z$ finite étale: given a distributivity diagram

(32)
$$\begin{tikzcd} & f^*R_f(p) \arrow[r, "\tilde{f}"] \arrow[dl, "\epsilon"'] \arrow[d, "\tilde{g}"] & R_f(p) \arrow[d, "g"] \\ X \arrow[r, "p"'] & Y \arrow[r, "f"] & Z, \end{tikzcd}$$

the (right) distributivity transformation

$$f_\otimes p_* \to g_* \tilde{f}_\otimes \epsilon^*$$

must be invertible. Since Perf is an étale sheaf and distributivity transformations satisfy base change by Proposition 2.4.6, we may check this étale-locally on Z. Since finite étale morphisms are étale-locally given by finite fold maps, this means we may assume that f is a fold map

$$\nabla\colon Y \simeq \coprod_{i=1}^{n} Z \to Z.$$

Since SpDM has descent for finite coproducts, we get a decomposition of p as

$$\coprod_{i=1}^{n} p_i\colon \coprod_{i=1}^{n} X_i \to \coprod_{i=1}^{n} Z,$$

and an equivalence

$$R_\nabla(p) \simeq X_1 \times_Z X_2 \times_Z \cdots \times_Z X_n,$$

since the universal property of $R_\nabla(p)$ is equivalent to that of this iterated fibre product:

$$\begin{aligned} \mathrm{Map}_{\mathrm{SpDM}_{S/\coprod_i Z}}(W, R_\nabla(p)) &\simeq \mathrm{Map}_{\mathrm{SpDM}_{S/\coprod_i Z}}(\nabla^* W, X) \\ &\simeq \mathrm{Map}_{\mathrm{SpDM}_{S/\coprod_i Z}}(\coprod_i W, \coprod_i X_i) \\ &\simeq \prod_i \mathrm{Map}_{\mathrm{SpDM}_{S/Z}}(W, X_i) \\ &\simeq \mathrm{Map}_{\mathrm{SpDM}_{S/Z}}(W, X_1 \times_Z \cdots \times_Z X_n). \end{aligned}$$

If π_i denotes the projection $X_1 \times_Z \cdots \times_Z X_n \to X_i$, then $\epsilon \simeq \coprod_i \pi_i$. Now given $\mathcal{F} \in \mathrm{Perf}(X)$ corresponding to $\mathcal{F}_i \in \mathrm{Perf}(X_i)$ under the equivalence $\mathrm{Perf}(X) \simeq \prod_i \mathrm{Perf}(X_i)$, we can write

$$\nabla_\otimes p_* \mathcal{F} \simeq p_{1,*}\mathcal{F}_1 \otimes \cdots \otimes p_{n,*}\mathcal{F}_n,$$

$$g_* \tilde{\nabla}_\otimes \epsilon^* \mathcal{F} \simeq g_*(\pi_1^*\mathcal{F}_1 \otimes \cdots \otimes \pi_n^*\mathcal{F}_n),$$

with the distributivity map $\nabla_{\otimes} p_* \mathcal{F} \to g_* \tilde{\nabla}_{\otimes} \epsilon^* \mathcal{F}$ given by the composite

$$\begin{aligned} p_{1,*}\mathcal{F}_1 \otimes \cdots \otimes p_{n,*}\mathcal{F}_n &\to g_* g^*(p_{1,*}\mathcal{F}_1 \otimes \cdots \otimes p_{n,*}\mathcal{F}_n) \\ &\simeq g_*(g^* p_{1,*}\mathcal{F}_1 \otimes \cdots \otimes g^* p_{n,*}\mathcal{F}_n) \\ &\simeq g_*(\pi_1^* p_1^* p_{1,*}\mathcal{F}_1 \otimes \cdots \otimes \pi_n^* p_n^* p_{n,*}\mathcal{F}_n) \\ &\to g_*(\pi_1^*\mathcal{F}_1 \otimes \cdots \otimes \pi_n^*\mathcal{F}_n). \end{aligned}$$

That this is an equivalence now follows from base change and the projection formula (which applies for maps in $\mathcal{FP}'$ by [Lur18, Remark 3.4.2.6]). To keep the notation bearable we spell this out only in the case $n = 2$, where we have the following commutative diagram (here all the squares are naturality squares and the triangle in the right-hand column is an adjunction identity):

$$\begin{tikzcd}
p_{1,*}\mathcal{F}_1 \otimes p_{2,*}\mathcal{F}_2 \arrow[r] \arrow[ddr, bend right, "\mathrm{PF}"'] & p_{1,*}p_1^*(p_{1,*}\mathcal{F}_1 \otimes p_{2,*}\mathcal{F}_2) \arrow[r] \arrow[d, "\sim"] & p_{1,*}\pi_{1,*}\pi_1^*p_1^*(p_{1,*}\mathcal{F}_1 \otimes p_{2,*}\mathcal{F}_2) \arrow[d, "\sim"] & & \\
& p_{1,*}(p_1^*p_{1,*}\mathcal{F}_1 \otimes p_1^*p_{2,*}\mathcal{F}_2) \arrow[r] \arrow[d] & p_{1,*}\pi_{1,*}\pi_1^*(p_1^*p_{1,*}\mathcal{F}_1 \otimes p_1^*p_{2,*}\mathcal{F}_2) \arrow[r, "\sim"] \arrow[d] & p_{1,*}\pi_{1,*}(\pi_1^*p_1^*p_{1,*}\mathcal{F}_1 \otimes \pi_1^*p_1^*p_{2,*}\mathcal{F}_2) \arrow[d] & \\
& p_{1,*}(\mathcal{F}_1 \otimes p_1^*p_{2,*}\mathcal{F}_2) \arrow[r] \arrow[d] \arrow[ddd, bend right=60, "\mathrm{BC}"'] & p_{1,*}\pi_{1,*}\pi_1^*(\mathcal{F}_1 \otimes p_1^*p_{2,*}\mathcal{F}_2) \arrow[r, "\sim"] \arrow[d] & p_{1,*}\pi_{1,*}(\pi_1^*\mathcal{F}_1 \otimes \pi_1^*p_1^*p_{2,*}\mathcal{F}_2) \arrow[d] \arrow[dr, equal] & \\
& p_{1,*}(\mathcal{F}_1 \otimes \pi_{1,*}\pi_1^*p_1^*p_{2,*}\mathcal{F}_2) \arrow[r] \arrow[d, "\sim"] & p_{1,*}\pi_{1,*}\pi_1^*(\mathcal{F}_1 \otimes \pi_{1,*}\pi_1^*p_1^*p_{2,*}\mathcal{F}_2) \arrow[r, "\sim"] \arrow[d, "\sim"] & p_{1,*}\pi_{1,*}(\pi_1^*\mathcal{F}_1 \otimes \pi_1^*\pi_{1,*}\pi_1^*p_1^*p_{2,*}\mathcal{F}_2) \arrow[r] \arrow[d, "\sim"] & p_{1,*}\pi_{1,*}(\pi_1^*\mathcal{F}_1 \otimes \pi_1^*p_1^*p_{2,*}\mathcal{F}_2) \arrow[d, "\sim"] \\
& p_{1,*}(\mathcal{F}_1 \otimes \pi_{1,*}\pi_2^*p_2^*p_{2,*}\mathcal{F}_2) \arrow[r] \arrow[d] & p_{1,*}\pi_{1,*}\pi_1^*(\mathcal{F}_1 \otimes \pi_{1,*}\pi_2^*p_2^*p_{2,*}\mathcal{F}_2) \arrow[r, "\sim"] \arrow[d] & p_{1,*}\pi_{1,*}(\pi_1^*\mathcal{F}_1 \otimes \pi_1^*\pi_{1,*}\pi_2^*p_2^*p_{2,*}\mathcal{F}_2) \arrow[r] \arrow[d] & p_{1,*}\pi_{1,*}(\pi_1^*\mathcal{F}_1 \otimes \pi_2^*p_2^*p_{2,*}\mathcal{F}_2) \arrow[d] \\
& p_{1,*}(\mathcal{F}_1 \otimes \pi_{1,*}\pi_2^*\mathcal{F}_2) \arrow[r] \arrow[rrr, bend right, "\mathrm{PF}"'] & p_{1,*}\pi_{1,*}\pi_1^*(\mathcal{F}_1 \otimes \pi_{1,*}\pi_2^*\mathcal{F}_2) \arrow[r, "\sim"] & p_{1,*}\pi_{1,*}(\pi_1^*\mathcal{F}_1 \otimes \pi_1^*\pi_{1,*}\pi_2^*\mathcal{F}_2) \arrow[r] & p_{1,*}\pi_{1,*}(\pi_1^*\mathcal{F}_1 \otimes \pi_2^*\mathcal{F}_2).
\end{tikzcd}$$

Here the maps labelled PF are equivalences by the projection formula for p_1 and π_1, the map labelled BC is an equivalence by base change for the pullback square

$$\begin{tikzcd}
X_1 \times_Z X_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X_1 \arrow[d, "p_1"] \\
X_2 \arrow[r, "p_2"] & Z,
\end{tikzcd}$$

and the composite along the top and right edges is the distributivity map (since $g \simeq p_1\pi_1 \simeq p_2\pi_2$).

We have shown that the functor

$$\mathrm{Perf}\colon \mathrm{Span}_{\text{fét}}(\mathrm{SpDM}_S) \to \mathrm{Cat}_\infty$$

is right $\mathcal{FP}'$-adjointable, and it therefore extends canonically to a functor of $(\infty, 2)$-categories

$$\mathrm{BISPAN}_{\text{fét},\mathcal{FP}'}(\mathrm{SpDM}_S) \to \mathrm{CAT}_\infty$$

by Corollary 2.3.15. □

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