A report on the framed motive program.

(University of Michigan, Oct 12)

- Bachmann
- Hoyois
- Khan
- Sosnilo
- Yakerson.

Motivic S^1-spectrum over S

$$X : Sm_S^{op} \longrightarrow Spectra$$

- [Homotopy inv.]

- [Nis descent]

Contraction

- $X_{-1}(T) = \mathrm{fib}\left(X(T \times \mathbb{G}_m) \xrightarrow{t=1} X(T)\right)$

$\left(= \qquad\qquad - \text{loops} \right)$

- $X(T \times \mathbb{G}_m) \simeq X_{-1}(T) \oplus X(T)$

- also write: $\Omega_{\mathbb{G}_m} X$

A motivic spectrum over S

is :

- $(X_n)_{n \in \mathbb{Z}}$

- $(X_n)_{-1} \simeq X_{n-1}$

$\rightsquigarrow SH(S)$

Examples:

I. Weibel's KH

$$(KH)_{-1}(T) \simeq KH(T)$$

II motivic cohomology (Voevodsky, Spitzweck)

$$\mathbb{Z}(q)_{-1} \simeq \mathbb{Z}(q-1)[-1]$$

III $X \in Sm_S$.

$$\Sigma^\infty_+ X \in SH(S).$$

$$\Omega^\infty \Sigma^\infty_+ X$$

$$\simeq \operatorname*{colim}_{n \to \infty} \Omega^{\wedge n}_{\mathbb{G}_m}\left((\mathbb{G}_m, 1)^{\wedge n} \wedge X_+\right).$$

Q_n: if X is an

S^1-spectrum, when is

$$X \simeq \Omega^\infty\left(\left(X_n\right)_{n\in\mathbb{Z}}\right).$$

$$\simeq \mathrm{Ev}_0\left(\left(X_n\right)_{n\in\mathbb{Z}}\right)$$

Voevodsky's lemma:

(unpublished circa 2001)

$$\mathrm{Hom}_{\mathrm{Shv}_{Nis}(\mathrm{Sm}_S)_\bullet}\left((\mathbb{P}^1,\infty)\wedge T_+,\ \frac{\mathbb{A}^1}{\mathbb{A}^1\setminus 0}\wedge X_+\right)$$

"

" "

(T)

$\underline{\qquad\qquad}$

$$\frac{\mathbb{P}^1 \times T}{\infty \times T} \simeq (\mathbb{P}^1, \infty) \wedge T_+ .$$

$$\begin{tikzcd}
U \setminus Z \arrow[r] \arrow[d] & U \arrow[d, "\text{étale neighborhood}"] & \\
\mathbb{A}^1_T \setminus Z \arrow[r, hook] & \mathbb{A}^1_T \left(= \mathbb{P}^1_T - \infty_T \right) \arrow[d] & Z \arrow[l, hook'] \arrow[dl, "\text{finite}"] \\
 & T &
\end{tikzcd}$$

$$\begin{tikzcd}
Z \setminus U \arrow[r, hook] \arrow[d] & U \arrow[d] \\
Z \arrow[r, hook] & \mathbb{A}^1_T
\end{tikzcd}$$

$$\frac{\mathbb{A}^1 \times X}{(\mathbb{A}^1 \setminus 0) \times X}$$

Def:

$T, Y \in Schs$ on $(\mathbb{A}^n_T \dashrightarrow \mathbb{A}^n_Y)$.

equationally framed correspondence

of level n is :

- $Z \hookrightarrow \mathbb{A}^n_T$ finite $/T$
- $U \to \mathbb{A}^n$ étale neighborhood of Z
- $U \xrightarrow{\varphi} \mathbb{A}^n$, $\varphi^{-1}(0) \cong Z$
- $U \to Y$.

$$\begin{tikzcd}
& & Z \arrow[dl, hook'] \arrow[dr] & \\
& U \arrow[dl] \arrow[dr, "{(\varphi,f)}"] & & 0\times Y \arrow[dl, hook'] \\
\mathbb{A}^n_T & & \mathbb{A}^n_Y. &
\end{tikzcd}$$

T, $Y_\cdot$, $n \geq 0$

Def (normally framed correspondence)

$$\begin{tikzcd}
\mathbb{A}^n_X \arrow[dr] & & Z \arrow[ll, hook', "i"'] \arrow[dl, "f"'] \arrow[dr, "h_\cdot"] & \\
& X & & Y_\cdot
\end{tikzcd}$$

- f is finite, lci and flat.
- $\tau : \mathcal{O}_Z^n \simeq N_i$

Def (tangentially framed correspondence.) $(X \dashrightarrow Y)$

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

- f is fsyn.
- $0 \overset{\tau}{\simeq} \mathbb{L}_f \in K(Z)$

$\mathrm{Corr}^{fr}(S)$

- $X \in Sm_S$

- $$\begin{tikzcd}
 & & W \arrow[dl, "f"'] \arrow[dr] \arrow[dd, phantom, "\vee"] & & \\
 & Y_0 \arrow[dl, "g"'] \arrow[dr] & & Y_1 \arrow[dl] \arrow[dr] & \\
X_0 & & X_1 & & X_2
\end{tikzcd}$$

$$f^* \mathbb{L}_g \to \mathbb{L}_{g \circ f} \to \mathbb{L}_f$$

Thm (EHKSY '17)

k = perfect field.

$$\mathrm{Fun}_{\mathrm{Nis}, \mathbb{A}^1}\left(\mathrm{Corr}^{\mathrm{fr}}(k), \mathrm{Spc} \right)$$

$$\simeq \mathrm{SH}^{\mathrm{eff}}(k).$$

- $\Omega^\infty_{\mathbb{G}_m}(X_n)$ has unique str of framed trfs.

Thm (H.18)

$$SH^{fr}(S) \simeq SH(S).$$

$\forall$ qcqs S.

Ex. $L_{Nis}\mathbb{Z}$ is motivic cohomology (Spitzweck, Voevodsky.)

finite locally free

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

$$\mathrm{Corr}^{fr}(S) \longrightarrow \mathrm{Corr}^{flf}(S).$$

Thm (EHKSY 18)

- $Z \to X$ f syn. $/S$

$$f_! : E(Z, \mathbb{L}_f) \longrightarrow E(X)$$

(Déglise - Jin - Khan)

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[dr, "\tau : 0 \simeq \mathbb{L}_f"] & \\ Y & & X \end{tikzcd}$$

$$E(Y) \xrightarrow{g^*} E(Z) \simeq E(Z, \mathbb{L}_f) \xrightarrow{f_!} E(X)$$

- $\mathrm{Corr}^{fr}_S(Y, X) \longrightarrow \mathrm{Maps}(\Sigma^\infty_+ Y, \Sigma^\infty_+ X)$

Thm: • Both agree.

• Corr^{fr} useful for computations.

Ex:

1) $\mathcal{F}$ is a PST (Voevodsky)

- $\mathrm{Corr}^{fr} \longrightarrow \mathrm{SmCor}$

2) $\mathcal{F}$ has Milnor-Witt transfers (Calmès-Fasel)

$$X \xrightarrow{f} Y$$

$$\widetilde{CH}^{\bullet}(X, \omega_f) \xrightarrow{f_*} \widetilde{CH}^{\bullet}(Y).$$

3) Flat trsfs in KH

4) MGL.

App I

- $$\begin{tikzcd} T \arrow[d] \arrow[r, phantom, "\subseteq"] & \mathbb{A}^1_S \arrow[dl] & V(f) \arrow[ll, bend right, dash] \\ S & & \end{tikzcd}$$

f = monic degree d.

$$\begin{tikzcd} E(S,0) \arrow[r, "f^*"] \arrow[drr, bend right=60, "d_\ell"'] \arrow[dr, phantom, "\Downarrow"] & E(T,0) \arrow[d, phantom, "\wr"] & \\ & E(T, \mathbb{L}_f) \arrow[r, "f_!"] & E(S,0) \end{tikzcd}$$

Thm (E-Khan.)

- $X \longrightarrow Y$ universal homeomorphism

  $\mathcal{P}$ = primes such that if $q \notin \mathcal{P}$

  then q is invertible in res. fields of Y.

- Then

  $$SH(Y)[\mathcal{P}^{-1}] \xrightarrow{\sim} SH(X)[\mathcal{P}^{-1}]$$

- $R \longrightarrow R^{perf}$ char $p>0$.

  $$SH(R)[\tfrac{1}{p}] \xrightarrow{\sim} SH(R^{perf})[\tfrac{1}{p}]$$

App II. k = perfect field.

$E \in SH(k)$.

$$\cdots \to f_1 E \to f_0 E \to f_{-1} E \to \cdots \to E$$

<u>Example</u> $E = KGL$

then get AHSS.

<u>Thm</u> (Levine, Voevodsky's conjectures)

- $s_0 \mathbb{1}_k \simeq H\mathbb{Z}$
- $\Omega^\infty_{\mathbb{G}_m} : SH(k) \longrightarrow SH^{S^1}(k)$

respects slice.

- $\mathrm{Hilb}_d^{lci}(\mathbb{A}^n) \subseteq \mathrm{Hilb}_d(\mathbb{A}^n)$

- $\operatorname{colim}_{n \to \infty} \mathrm{Hilb}_d^{lci}(\mathbb{A}^\infty) = \mathrm{Hilb}_d(\mathbb{A}^\infty)$

- $\left(\coprod_{d \in \mathbb{N}} \mathrm{Hilb}_d^{lci}(\mathbb{A}^\infty)\right)^{gp} = \mathrm{Hilb}^{lci}$

<u>Thm</u> (EHKSY 19, BEHKSY 19.)

$$\Omega^\infty_{\mathbb{P}^1} MGL \simeq \left(\mathrm{Hilb}^{lci}\right)^{gp}$$

$$\simeq \mathbb{Z} \times (\mathrm{Hilb}^{lci})^+$$

$$\begin{tikzcd}
\Omega^\infty_{\mathbb{P}^1} MGL \arrow[r] \arrow[d, dash, "2."] & \Omega^\infty_{\mathbb{P}^1} H\mathbb{Z} \arrow[d] \\
\mathbb{Z} \times (\mathrm{Hilb}^{lci})^+ \arrow[r] & \mathbb{Z}.
\end{tikzcd}$$

- suffices to prove : rational variety (Kahn-Sujatha, Pelaez)

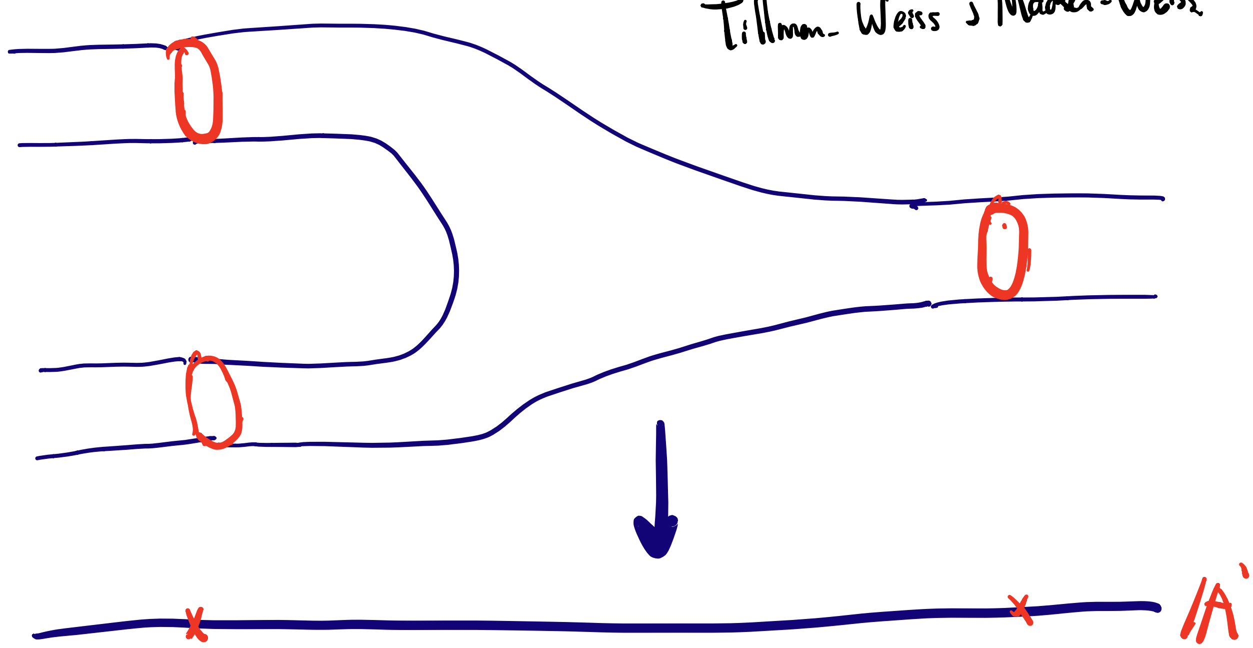
<u>Thm</u> (B-E) This is the case, reproves Levine.

What next?

- $\mathrm{FSyn} \xrightarrow[\mathbb{A}^1]{\sim} \Omega^\infty_{\mathbb{P}^1} \mathrm{MGL}$

- $\mathcal{PQS}m^{d.} \xrightarrow{?} \Omega^\infty \Sigma^{-n}_{\mathbb{P}^1} \mathrm{MGL}$

Galatius-Madsen-Tillman-Weiss ; Madsen-Weiss



- $S \longmapsto \mathrm{Pic}(SH(S))$

$J : K \longrightarrow \mathrm{Pic}(SH)$

? deloop $\mathrm{Pic}(SH)$

- Study

$\mathrm{id} \longrightarrow \Omega_G$