ON NILPOTENT EXTENSIONS OF ∞-CATEGORIES AND THE CYCLOTOMIC TRACE

ELDEN ELMANTO AND VLADIMIR SOSNILO

ABSTRACT. We do three things in the paper: (1) study the analog of localization sequences (in the sense of algebraic K-theory of stable ∞-categories) for additive ∞-categories, (2) define the notion of square zero extensions for suitable ∞-categories, and (3) use (1) and (2) to extend the Dundas-Goodwillie-McCarthy theorem for stable ∞-categories which are not monogenically generated (such as the stable ∞-category of Voevodsky's motives or the stable ∞-category of perfect complexes on some algebraic stacks). The key input in our paper is Bondarko's notion of weight structures which provides a "ring-with-many-objects" analog of a connnective $\mathbb{E}_1$-ring spectrum.

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1. INTRODUCTION

Every application of trace methods in algebraic K-theory goes through the celebrated Dundas-Goodwillie-McCarthy (DGM) theorem [Goo86, Dun97, McC97, DGM13]:

Theorem 1.0.1 (Dundas-Goodwillie-McCarthy). *Suppose that* $\mathrm{A} \to \mathrm{B}$ *be a morphism of connective* $\mathbb{E}_1$*-ring spectra such that the induced map* $\pi_0(\mathrm{A}) \to \pi_0(\mathrm{B})$ *has a nilpotent kernel. Then the trace map*

$$\mathrm{tr} : \mathrm{K} \to \mathrm{TC}$$

induces a cartesian square

$$\begin{array}{ccc} \mathrm{K(A)} & \longrightarrow & \mathrm{TC(A)} \\ \downarrow & & \downarrow \\ \mathrm{K(B)} & \longrightarrow & \mathrm{TC(B)}. \end{array}$$

The goal of this paper is to extend Theorem 1.0.1 to a more general setting. Our point of departure is the viewpoint that a dg/spectral category should be viewed as a dg/$\mathbb{E}_1$-algebra with many objects. Often, (small) categories of interest are generated by a single compact generator (for example, **Perf** over a quasicompact, quasiseparated spectral/derived scheme). Therefore Morita invariants such as K-theory and TC view these categories and End of the generators as one and the same thing. However, there are many examples for when we cannot use Morita invariance to access categories of interest. Here are two classes of examples.

(1) Let $\mathcal{X}$ be a quotient stack $[\mathrm{Spec\,B/G}]$. Then the category of perfect complexes is compactly generated by G-equivariant bundles over Spec B. It's more generally compactly generated for good enough stacks; see Section 4.2 for a review.
(2) Let R be a ring k be a field whose exponential characteristic is invertible in k. Then $\mathbf{DM}(\mathrm{Spec}\,k; \mathrm{R})$ is compactly generated by Tate twists of M(X) where X is a smooth projective k-scheme; see Section 4.1 for a review. This is again the case for other "motivic categories" such as the relative version of **DM**, modules over various motivic ring spectra or constructible ℓ-adic sheaves.

In these cases, if we want to deploy trace methods to study the algebraic K-theory of these categories, then Theorem 1.0.1 does not immediately apply. Our primary goal is to rectify this situation.

Theorem 1.0.2 (Theorems 4.3.4 and 4.3.5)**.** *Let $f : (\mathcal{A}, w) \to (\mathcal{B}, w)$ be a nilpotent extension of boundedly weighted ∞-categories. Then the diagram*

$$\begin{array}{ccc} \mathrm{K}(\mathcal{A}) & \longrightarrow & \mathrm{TC}(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathrm{K}(\mathcal{B}) & \longrightarrow & \mathrm{TC}(\mathcal{B}) \end{array}$$

is a cartesian square. More generally, if E *is a truncating invariant in the sense of* [LT19]*, then* E *converts a nilpotent extension of boundedly weighted ∞-categories to an equivalence.*

Weight structures, in the sense of Bondarko [Bon10], are needed here in order to define a nilpotent extension in a reasonable way. Thinking along the philosophy that spectral categories are many objects-rings, the word "weighted" corresponds to the notion of a "many objects-*connective* ring." The examples listed above are weighted, and so we obtain the following applications of our results:

Corollary 1.0.3 (See Example 4.3.8)**.** *Let k be a commutative ring and let* R $\to$ S *be a nilpotent extension of* G*-equivariant $\mathbb{E}_1$-k-algebras, where* G *is a linearly reductive embeddable group scheme over k. Then the square*

$$\begin{array}{ccc} \mathrm{K^G(R)} & \longrightarrow & \mathrm{TC^G(R)} \\ \downarrow & & \downarrow \\ \mathrm{K^G(S)} & \longrightarrow & \mathrm{TC^G(S)} \end{array}$$

is cartesian.

Corollary 1.0.4 (See Example 4.3.7)**.** *Let k be a field of exponential characteristic c. Then the square*

$$\begin{array}{ccc} \mathrm{K}(\mathbf{DM}_{\mathrm{gm}}(k)[1/c]) & \longrightarrow & \mathrm{TC}(\mathbf{DM}_{\mathrm{gm}}(k)[1/c]) \\ \downarrow & & \downarrow \\ \mathrm{K}(\mathbf{Chow}(k)[1/c]) & \longrightarrow & \mathrm{TC}(\mathbf{Chow}(k)[1/c]) \end{array}$$

is cartesian.

1.1. **Remarks on the proof.** The proof of our result uses the DGM theorem. Instead of passing through a many objects version of the original argument, we deduce that any localizing invariant that induces equivalences on nilpotent extensions of $\mathbb{E}_1$-rings, also induces equivalences on nilpotent extensions of boundedly weighted ∞-categories.

The idea of the proof can be expressed in these steps:

- Any boundedly weighted category $\mathcal{A}$ can be embedded into the category of perfect complexes over a connective $\mathbb{E}_1$-ring spectrum $\mathrm{R}(\mathcal{A})$.
- The idempotent completion (also often called Karoubi completion) of the Verdier quotient of such embedding is also the category of modules over a connective $\mathbb{E}_1$-ring spectrum $\mathrm{S}(\mathcal{A})$.
- The value of any localizing invariant E on $\mathcal{A}$ can be expressed as the fiber of a map $\mathrm{E}(\mathrm{R}(\mathcal{A})) \to \mathrm{E}(\mathrm{S}(\mathcal{A}))$.
- Using $\mathrm{E} = \mathrm{K}^{\mathrm{inv}}$ we reduce Theorem 1.0.2 to showing that the corresponding maps

  $$\mathrm{R}(\mathcal{A}) \to \mathrm{R}(\mathcal{B}) \text{ and } \mathrm{S}(\mathcal{A}) \to \mathrm{S}(\mathcal{B})$$

  are nilpotent extensions of rings and applying the classical version of the Dundas-Goodwillie-McCarthy theorem.

The main technical problem behind proving a result like Theorem 1.0.2 for general categories is the fact that TC *does not* commute with filtered colimits of categories even after modding out by p; we note that [CMM, Theorem G] proves the remarkable result TC/p does commute with filtered colimits of connective $\mathbb{E}_1$-rings. If TC did, then we could express a stable ∞-category as a colimit of stable ∞-categories which are modules over the endomorphism algebra of some generators and invoke the DGM theorem without recourse to additive ∞-categories and boundedly weighted ∞-categories. We refer the reader to Remark 3.0.11 for a more extensive discussion of this phenomena and reference to an explicit counterexample. In principle, this paper *isolates* a large class functors of stable ∞-categories $f : \mathcal{C} \to \mathcal{D}$ for which we can define to be nilpotent in a sensible manner (see the discussion at the beginning of Section 4), and for which we can prove the DGM theorem. It is conceivable that the DGM theorem could hold for more general functors than the ones considered here.

1.2. **Application to truncating invariants.** We can apply this idea to other localizing invariants and not just $\mathrm{K}^{\mathrm{inv}}$. In particular, we can show the following result. Recall that a truncating invariant, in the sense of Land-Tamme [LT19], is any localizing invariant E for which $\mathrm{E}(\mathrm{R}) \to \mathrm{E}(\pi_0(\mathrm{R}))$ for all connective $\mathbb{E}_1$-ring spectra R.

Corollary 1.2.1. *Let* E *be a truncating invariant. Let k be a commutative ring and let* $\mathrm{R} \to \mathrm{S}$ *be a nilpotent extension of* G*-equivariant* $\mathbb{E}_1$*-k-algebras, where* G *is a linearly reductive embeddable group scheme over k. Then* $\mathrm{E}^{\mathrm{G}}(\mathrm{R}) \to \mathrm{E}^{\mathrm{G}}(\mathrm{S})$ *is an equivalence.*

The last result together with the pro-excision result of [BKRS] implies cdh-excision for all truncating invariants on a quite general class of stacks. This recovers, as a special instance, the case of KH which was proved by Hoyois and Krishna in [HK19]) using equivariant motivic homotopy theory; see Corollary 4.4.5.

1.3. **Summary.** In Section 2, we clarify certain aspects of additive ∞-catgeories which we believe are of independent interest. In particular, we discuss localizations and Verdier quotients of additive ∞-categories in Section 2.2 in the style of Blumberg-Gepner-Tabuada [BGT13, Section 5] or Nikolaus-Scholze [NS17, Chapter I]; we note that Bondarko and the second author had studied the "triangulated" analogs in [BS18b]. Another aspect of additive ∞-categories we study is the notion of nilpotent extensions which we introduce in Section 2.3. Using lax pullbacks we define the ∞-categorical versions of constructions of Tabuada [Tab09] and Dotto [Dot18], defining the square zero extensions of an additive ∞-category by a bimodule in Section 2.4.

In Section 3, we prove the DGM theorem in the context of additive ∞-categories, while in Section 4 we prove our main results on the DGM theorem for boundedly weighted ∞-categories. This last section also contains a review of weight structures on motivic categories and on stacks, and applications to truncating invariants are given in Section 4.4. In Appendix A we discuss the possible meanings of K-theory of additive ∞-categories.

1.4. **Conventions.** We use relatively standard terminology on ∞-categories following [Lur17b, Lur17a, Lur18]. Additionally, we use the following notations:

(1) if $\mathcal{C}$ is an ∞-category we write $\mathcal{C}^{\simeq} \subset \mathcal{C}$ for its core aka maximal subgroupid;
(2) if $\mathcal{C}$ is an stable (resp. additive) ∞-category, then for each $x, y \in \mathcal{C}$ we denote its mapping (resp. connective) spectrum by $\mathrm{maps}(x, y)$, while $\mathrm{Maps}(x, y)$ denote the underlying mapping spaces so that $\Omega^{\infty}\mathrm{maps}(x, y) \simeq \mathrm{Maps}(x, y)$; the relationship between the notation $\mathrm{end}(x)$ and $\mathrm{End}(x)$ is analogous.
(3) If $\mathcal{C}$ is an ∞-category with finite coproducts, then we write
$$\mathrm{PSh}_{\Sigma}(\mathcal{C}) := \mathrm{Fun}^{\times}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spc}),$$
the ∞-category of functors which converts finite coproducts to products.

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2. On additive ∞-categories

2.1. **Localization sequences of additive ∞-categories.** Recall that an additive category is, in particular, enriched in the category of abelian groups. Hence, $x \in \mathrm{Obj}(\mathcal{A})$ then $\mathrm{Hom}(x, x)$ is naturally an associative ring with $\circ$ acting as the multiplication. From this point of view, it is natural to view an additive category as the generalization of an associative ring. In higher algebra, the analog of an additive category is an additive ∞-category [GGN15a]:

Definition 2.1.1. A **semi-additive ∞-category** $\mathcal{A}$ is an ∞-category with finite products and coproducts such that for any pair of objects x, y, the canonical map
$$x \sqcup y \to x \times y$$
is an equivalence. We write this object as the **biproduct** $x \oplus y$. The biproduct admits a **shear map** $s = (\pi_1, \nabla) : x \oplus x \to x \oplus x$ where π_1 is the first projection and ∇ is the fold map. If this map is an equivalence for all $x \in \mathrm{Obj}(\mathcal{A})$, then we say that $\mathcal{A}$ is an **additive ∞-category**. An **additive functor** of additive ∞-categories is a functor $\mathrm{F} : \mathcal{A} \to \mathcal{B}$ that preserves all biproducts.

We refer to [GGN15a, Section 2] for a more extensive discussion and the following result which is [GGN15a, Proposition 2.8]:

Proposition 2.1.2. *Let $\mathcal{C}$ be an ∞-category with finite coproducts and products. Then the following are equivalent:*

(1) the ∞-category $\mathcal{A}$ is additive,
(2) the homotopy category $h\mathcal{A}$ is additive,

(3) the forgetful functor

$$\mathrm{Gp}_{\mathbb{E}_\infty}(\mathcal{C}) \to \mathcal{C}$$

is an equivalence.

2.2. **Verdier quotients of additive ∞-categories.** Our first goal is to develop the theory of exact sequences of additive ∞-categories following the case of stable ∞-categories following [BGT13, Section 5] and [NS17, Chapter I]. In the 1-categorical context, this was studied by the the second author and Bondarko in [BS18b].

Before we proceed, let us gather the necessary ingredients. First, for an ∞-category $\mathcal{C}$ and $\mathrm{W} \subset \mathcal{C}$ a collection of arrows we can form the ∞-category $\mathcal{C}[\mathrm{W}^{-1}]$ equipped with a functor $\gamma : \mathcal{C} \to \mathcal{C}[\mathrm{W}^{-1}]$ which is the universal ∞-category under $\mathcal{C}$ where all elements in W are invertible (more precisely, see the universal property [Cis19, Definition 7.1.2]). This localization exists and is (essentially) unique [Cis19, Proposition 7.1.3] up to equivalence of categories under $\mathcal{C}$. Moreover, the functor can be chosen to be identical on objects [Cis19, Remark 7.1.4]. Furthermore, the functor $h(\mathrm{C}) \to h(\mathrm{C}[\mathrm{W}^{-1}])$ witnesses $h(\mathrm{C}[\mathrm{W}^{-1}])$ as a localization of the 1-category $h(\mathrm{C})$ in the sense of ordinary category theory (as reviewed, for example, in [Cis19, Definition 2.2.8]).

Second, let us recall the passage from "small categories" to "big categories." In the context of stable ∞-categories what we mean by thus is the passage to ind-completion: suppose that $\mathcal{C}$ is a small stable ∞-category, then the Yoneda functor

$$よ : \mathcal{C} \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spt}),$$

identifies the codomain with $\mathrm{Ind}(\mathcal{C})$; see, for example, [BGT13, Proposition 3.2]. The upshot of going to $\mathrm{Ind}(\mathcal{C})$ is that the resulting ∞-category is a stable presentable ∞-category. Therefore localizations of this ∞-category can often be constructed via adjoint functor theorems [Lur17b, Corollary 5.5.2.9]. In fact, the theory of localization sequences in algebraic K-theory as pioneered by Thomason [TT90], Neeman [Nee92], and later in [BGT13] is to pass to large ∞-categories first before going back to small categories by taking compact objects — which requires one to consider categories up to idempotent completion. In the world of additive ∞-categories we should consider the passage to Quillen's "nonabelian derived categories."

$$よ : \mathcal{C} \to \mathrm{PSh}_\Sigma(\mathcal{C}).$$

Lemma 2.2.1. *Let $\mathcal{C}$ be an ∞-category with finite coproducts. Consider the Yoneda functor*

$$よ : \mathcal{C} \to \mathrm{PSh}_\Sigma(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spc}).$$

Then:

(1) the ∞-category $\mathcal{C}$ is additive if and only if $\mathrm{PSh}_\Sigma(\mathcal{C})$ is prestable.
(2) in this case, we have an equivalence:

$$\mathrm{PSh}_\Sigma(\mathcal{C}) \simeq \mathrm{Fun}^\times(\mathcal{C}^{\mathrm{op}}, \mathrm{Spt}_{\geqslant 0}), \textit{and}$$

(3) the ∞-category $\mathrm{PSh}_\Sigma(\mathcal{C})$ is a presentable, prestable ∞-category.

Proof. See [Lur18, Proposition C.1.5.7 and Remark C.1.5.9]. □

In the next theorem, we address the theory of Verdier quotients in the context of additive ∞-categories.

Theorem 2.2.2. *Let $\mathcal{A}$ be a small additive ∞-category and $\mathcal{B} \subset \mathcal{A}$ be an additive full subcategory. Then*

(1) Define

$$\mathrm{W} := \{f \oplus g \colon b \oplus a' \to b' \oplus a' \textit{ where } f \colon b \to b' \in \mathcal{B} \textit{ and } g \textit{ is invertible}\}.$$

$\mathcal{A}/\mathcal{B} := \mathcal{A}[\mathrm{W}^{-1}]$ is an additive ∞-category equipped with an additive functor

$$\overline{(-)} : \mathcal{A} \to \mathcal{A}/\mathcal{B}$$

which is a bijection on the sets of objects.

(2) *If $\mathcal{E}$ is another additive ∞-category, then $\overline{(-)} : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ induces an equivalence between $\mathrm{Fun}^{\times}(\mathcal{A}/\mathcal{B}, \mathcal{E})$ and the full subcategory of $\mathrm{Fun}^{\times}(\mathcal{A}, \mathcal{E})$ consisting of those additive functors $\mathrm{G} : \mathcal{A} \to \mathcal{E}$ satisfying $\mathrm{G}|_{\mathcal{A}} \simeq 0$.*

(3) *We have the following commutative diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\quad \mathrm{can} \quad} & \mathcal{A}/\mathcal{B} \\ \downarrow & & \downarrow \\ \mathrm{PSh}_{\Sigma}(\mathcal{A}) & \xrightarrow{\mathrm{L}} & \mathrm{PSh}_{\Sigma}(\mathcal{A}/\mathcal{B}), \end{array}$$

where the bottom horizontal functor is left adjoint to the functor

$$\mathrm{res}_{\mathrm{can}} : \mathrm{PSh}_{\Sigma}(\mathcal{A}/\mathcal{B}) \to \mathrm{PSh}_{\Sigma}(\mathcal{A}) \qquad \mathrm{F} \mapsto \mathrm{F} \circ \overline{(-)}.$$

(4) *If $\mathcal{B}$ is generated under sums and retracts by a single object z. Then for any $\mathrm{F} \in \mathrm{PSh}_{\Sigma}(\mathcal{A})$*

$$\mathrm{LF}(-) \simeq \mathrm{Cofib}(\mathrm{F}(z) \otimes_{\mathrm{End}(z)} \mathrm{Maps}_{\mathcal{A}}(-, z) \to \mathrm{F}(\mathrm{X})).$$

(5) *In general for any $\mathrm{F} \in \mathrm{PSh}_{\Sigma}(\mathcal{A})$ there is an equivalence*

$$\mathrm{LF}(-) \simeq \mathrm{Cofib}\Big(\operatorname*{colim}_{z_1, \cdots, z_n \in \mathcal{B}} \mathrm{F}\Big(\bigoplus_{i=1}^{n} z_i\Big) \otimes_{\mathrm{End}(\bigoplus_{i=1}^{n} z_i)} \mathrm{Maps}_{\mathcal{A}}\Big(-, \bigoplus_{i=1}^{n} z_i\Big) \to \mathrm{F}(-)\Big),$$

where the colimit is taken over the poset of finite subsets $\{z_1, \cdots, z_n\}$ of objects of $\mathcal{B}$.

Proof. To prove assertion (1) it suffices to show that the the functor

$$h(\mathcal{A}) \to h(\mathcal{A}[\mathrm{W}^{-1}])$$

is an additive functor of additive categories by Proposition 2.1.2. This follows from Proposition 2.2.4 in [BS18b].

By the universal property of the localization, $\overline{(-)} : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ induces an equivalence between $\mathrm{Fun}^{\times}(\mathcal{A}/\mathcal{B}, \mathcal{E})$ and the full subcategory of $\mathrm{Fun}^{\times}(\mathcal{A}, \mathcal{E})$ consisting of those additive functors $\mathrm{G} : \mathcal{A} \to \mathcal{E}$ for which $\mathrm{G}(w)$ is an equivalence for all $w \in \mathrm{W}$. By definition of W and additivity of G the latter condition is equivalent to $\mathrm{G}(g)$ being an equivalence for any $g \colon b \to b' \in \mathcal{B}$ which is also equivalent to: $\mathrm{G}(b) \simeq 0$ for all $b \in \mathcal{B}$. This proves (2).

The third assertion follows automatically from [Lur17b, Proposition 5.3.6.2 and Proposition 5.3.5.13]. From this also follows the existence of the functor $i_{\mathcal{B}} : \mathrm{PSh}_{\Sigma}(\mathcal{B}) \to \mathrm{PSh}_{\Sigma}(\mathcal{A})$, which is right adjoint to $\mathrm{res}_{\mathcal{B}} : \mathrm{PSh}_{\Sigma}(\mathcal{A}) \to \mathrm{PSh}_{\Sigma}(\mathcal{B})$.

To prove the fourth claim denote by $\mathrm{L}'(-)$ the functor given by the formula on the right. There is also a natural transformation $\mathrm{id}_{\mathrm{PSh}_{\Sigma}(\mathcal{A})} \xrightarrow{\alpha} \mathrm{L}'$. It suffices to prove that L' is a localization onto the subcategory of all $\mathrm{G} \in \mathrm{PSh}_{\Sigma}(\mathcal{A})$ such that $\mathrm{G}|_{\mathcal{B}} = 0$. Note that the two maps

$$\mathrm{F}(z) \otimes_{\mathrm{End}(z)} \mathrm{Maps}_{\mathcal{A}}(z, z) \otimes_{\mathrm{End}(z)} \mathrm{Maps}_{\mathcal{A}}(-, z) \longrightarrow \mathrm{F}(z) \otimes_{\mathrm{End}(z)} \mathrm{Maps}_{\mathcal{A}}(-, z)$$

given by

$$f \otimes e \otimes e' \mapsto fe \otimes e' \text{ and } f \otimes e \otimes e' \mapsto f \otimes ee'$$

are equivalences for any F, so by construction we have a commutative diagram in $\mathrm{Fun}^{\times}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spt}_{\geqslant 0})$:

$$\begin{array}{ccccc} \mathrm{F}(z) \otimes_{\mathrm{End}(z)} \mathrm{Maps}_{\mathcal{A}}(z, z) \otimes_{\mathrm{End}(z)} \mathrm{Maps}_{\mathcal{A}}(-, z) & \xrightarrow{\simeq} & \mathrm{F}(z) \otimes_{\mathrm{End}(z)} \mathrm{Maps}_{\mathcal{A}}(-, z) & \longrightarrow & 0 \\ \downarrow \simeq & & \downarrow & & \downarrow \\ \mathrm{F}(z) \otimes_{\mathrm{End}(z)} \mathrm{Maps}_{\mathcal{A}}(-, z) & \longrightarrow & \mathrm{F}(-) & \xrightarrow{\alpha_{\mathrm{F}}} & \mathrm{LF}(-) \\ \downarrow & & \downarrow \alpha_{\mathrm{F}} & & \downarrow \alpha_{\mathrm{L}'(\mathrm{F})} \\ 0 & \longrightarrow & \mathrm{L}'\mathrm{F}(-) & \xrightarrow{\mathrm{L}'(\alpha)} & \mathrm{L}'\mathrm{L}'\mathrm{F}(-) \end{array}$$

whose rows and arrows are fiber sequences. Hence $\mathrm{L}'(\alpha)(\mathrm{F})$ and $\alpha_{\mathrm{L}'(\mathrm{F})}$ are equivalences. Now by [Lur17b, Proposition 5.2.7.4] L' is a localization onto its image. The map

$$\mathrm{F}(z) \otimes_{\mathrm{End}(z)} \mathrm{Maps}_{\mathcal{A}}(z,z) \longrightarrow \mathrm{F}(z)$$

is an equivalence, so $\mathrm{L}'\mathrm{F}(z) = 0$ and hence the image of L' is conatined in the image of L. Moreover, for any $\mathrm{G} \in \mathrm{PSh}_\Sigma(\mathcal{A})$ such that $\mathrm{G}|_{\mathcal{B}} = 0$

$$\mathrm{G}(z) \otimes_{\mathrm{End}(z)} \mathrm{Maps}_{\mathcal{A}}(z,z) = 0,$$

so $\mathrm{G} \simeq \mathrm{L}'(\mathrm{G})$ is in the image of L'.

The last claim follows from the fact that the functor

$$\mathrm{res}_{\mathrm{can}} \circ \mathrm{L} : \mathrm{PSh}_\Sigma(\mathcal{A}) \to \mathrm{PSh}_\Sigma(\mathcal{A}/\mathcal{B}) \to \mathrm{PSh}_\Sigma(\mathcal{A})$$

is a colimit of functors

$$\mathrm{PSh}_\Sigma(\mathcal{A}) \to \mathrm{PSh}_\Sigma(\mathcal{A}/\langle z_1, \cdots, z_n\rangle) \to \mathrm{PSh}_\Sigma(\mathcal{A})$$

over all finite subsets $\{z_1, \cdots, z_n\} \subset \mathcal{B}$. □

Corollary 2.2.3. *Let $\mathcal{A}$ be an additive ∞-category and $\mathcal{B} \subset \mathcal{A}$ a full subcategory. For any $x, y \in \mathcal{A}$*

(1) $\mathrm{Maps}_{\mathcal{A}/\mathcal{B}}(x,y)$ *can be computed using the formula*

$$\mathrm{Cofib}(\operatorname*{colim}_{z_1,\cdots,z_n \in \mathcal{B}} \mathrm{Maps}_{\mathcal{A}}(\bigoplus_{i=1}^{n} z_i, y) \otimes_{\mathrm{End}(\bigoplus_{i=1}^{n} z_i)} \mathrm{Maps}_{\mathcal{A}}(x, \bigoplus_{i=1}^{n} z_i) \to \mathrm{Maps}_{\mathcal{A}/\mathcal{B}}(x,y)),$$

(2) $\pi_0 \mathrm{Maps}_{\mathcal{A}/\mathcal{B}}(x,y) = \mathrm{Coker}(\bigoplus_{z \in \mathcal{B}} \pi_0\mathcal{A}(z,y) \otimes \pi_0\mathcal{A}(x,z) \to \pi_0\mathcal{A}(x,y))$.

Proof. The first claim follows directly from Theorem 2.2.2(3 and 5). This claim in particular implies $\pi_0 \mathrm{Maps}_{\mathcal{A}/\mathcal{B}}(x,y)$ can be computed as the quotient of $\mathrm{Maps}_{\mathcal{A}}(x,y)$ modulo the subgroup of those morphisms that factor thtough z. This is exactly what formula (2) says. □

Remark 2.2.4. A very similar formula to the one in Corollary 2.2.3(1) was obtained in the setting of dg-quotients of dg-categories in [Dri02].

Definition 2.2.5. Let $\mathcal{A}$ by an additive ∞-category.

(1) We say that an idempotent $e : x \to x$ in $\mathcal{A}$ **splits** if it is equivalent to a morphism of the form

$$x_1 \oplus x_2 \xrightarrow{\begin{pmatrix} \mathrm{id}_{x_1} & 0 \\ 0 & 0 \end{pmatrix}} x_1 \oplus x_2.$$

(2) A full subcategory $\mathcal{A} \subset \mathcal{B}$ is said to be **retract-closed** if every idempotent of an object of $\mathcal{A}$ that splits in $\mathcal{B}$ also splits in $\mathcal{A}$.

(3) If $\mathcal{A}$ is an additive ∞-category, then we say that $\mathcal{A}$ is **idempotent complete** (or **absolutely Karoubi closed**) if each idempotent $e : x \to x$ in $\mathcal{A}$ splits.

Lemma 2.2.6. *Let $\mathcal{A}$ be an additive ∞-category. Then there exists an additive ∞-category* $\mathrm{Kar}(\mathcal{A})$ *and an additive functor $\mathcal{A} \to$* $\mathrm{Kar}(\mathcal{A})$ *which is universal with respect to idempotent complete additive ∞-categories receiving an additive functor from $\mathcal{A}$. We call this category the idempotent completion of $\mathcal{A}$.*

Proof. According to [Lur17b, Proposition 5.1.4.2], we can take the embedding

$$\mathcal{A} \to \mathrm{PSh}_\Sigma(\mathcal{A}),$$

and set $\mathrm{Kar}(\mathcal{A})$ to be the full subcategory of $\mathrm{PSh}_\Sigma(\mathcal{A})$ spanned by functors which are retracts of the Yoneda image. Note that this is also an additive ∞-category. □

If $f : \mathcal{A} \to \mathcal{B}$ is an additive functor, then we get a functor $\mathrm{Kar}(f) : \mathrm{Kar}(\mathcal{A}) \to \mathrm{Kar}(\mathcal{B})$. We say that that f is an **equivalence up to idempotent completion** if $\mathrm{Kar}(f)$ is an equivalence. We will also the subcategory of additive ∞-categories spanned by those which are idempotent complete as:

$$\mathbf{Cat}_{\infty}^{\mathrm{Kar}} \subset \mathbf{Cat}_{\infty}^{\mathrm{add}}.$$

Definition 2.2.7. Let $\mathcal{A}$ by a additive ∞-category and $\mathcal{B} \subset \mathcal{A}$ be an additive full subcategory. We say that the sequence

$$\mathcal{B} \to \mathcal{A} \to \mathcal{C}$$

is an **exact sequence of additive ∞-categories** if the induced map $\mathcal{A}/\mathcal{B} \to \mathcal{A}$ is an equivalence up to idempotent completion.

2.3. Nilpotent extensions of additive ∞-categories.

We now introduce the first main definition of this paper.

Definition 2.3.1. A **nilpotent extension** of additive ∞-categories is an additive functor

$$f : \mathcal{A} \to \mathcal{B}$$

such that:

(1) f is essentially surjective,
(2) for all objects $x, y \in \mathcal{A}$ the map

$$\pi_0 \operatorname{Maps}_{\mathcal{A}}(x, y) \to \pi_0 \operatorname{Maps}_{\mathcal{B}}(f(x), f(y))$$

is a surjection,
(3) there exists $n \in \mathbb{N}$ such that for any sequence of composable morphisms $f_1, \cdots, f_n \in h(\mathcal{A})$ for which each $h(f)(f_i)$ is trivial[1], the composition $f_1 \circ \cdots \circ f_n$ is trivial.

The third condition is equivalent to saying that the kernel ideal of the functor $h(\mathcal{A}) \to h(\mathcal{B})$ is nilpotent.

Remark 2.3.2. We note that the integer n that appears in part (3) of Definition 2.3.1 only depends the additive functor f.

Proposition 2.3.3. *Suppose that $f : \mathcal{A} \to \mathcal{B}$ is a nilpotent extension of additive ∞-categories. Then f is conservative.*

Proof. Let $g : x \to y$ be a morphism in $\mathcal{A}$. By hypothesis, $f(g)$ is invertible in $\mathcal{B}$, hence we may pick an inverse $h' : f(y) \to f(x)$. Using (2) of Definition 2.3.1, we may choose a lift of h' which we call h. We claim that h is an inverse of g. Indeed, it suffices to prove that gh and hg are invertible. Since the arguments are the same we do it for gh. First note that $f(\mathrm{id} - gh) = \mathrm{id} - f(g)f(h) = 0$ since f is an additive functor and $f(h)$ is an inverse of $f(g)$. Therefore, by assumption (3) of Definition 2.3.1, there exists an n such that $(\mathrm{id} - gh)^n = 0$. But now, we can furnish and inverse to gh given by

$$(gh)^{-1} = (\mathrm{id} - (\mathrm{id} - gh))^{-1} = \mathrm{id} + (\mathrm{id} - gh) + (\mathrm{id} - gh)(\mathrm{id} - gh) + \cdots + (\mathrm{id} - gh)^{n-1}.$$

□

After Proposition 2.3.3, we can replace part (1) of Definition 2.3.1 by saying that the induced map on equivalence classes of objects:

$$\pi_0(\mathcal{A}^{\simeq}) \to \pi_0(\mathcal{B}^{\simeq})$$

is a bijection.

As a first example, we can view an associative, unital ring as an additive category with one object, we have:

---

[1] We note that any additive ∞-category is pointed, hence we say that a map $f : x \to y$ is said to be **trivial** if it is homotopic to $f : x \to 0 \to y$.

Example 2.3.4. Suppose that R, S are connective $\mathbb{E}_1$-ring spectra. Then we can consider the categories $\mathbf{Free}_{\mathrm{R}}$, $\mathbf{Free}_{\mathrm{S}}$ of free modules over these rings. A morphism of $\mathbb{E}_1$-rings $f : \mathrm{A} \to \mathrm{B}$ then defines an exact functor $f : \mathbf{Free}_{\mathrm{R}} \to \mathbf{Free}_{\mathrm{S}}$. The requirement that the extension is nilpotent is then simply the map $\pi_0(\mathrm{A}) \to \pi_0(\mathrm{B})$ having a nilpotent kernel.

Example 2.3.5. What follows is arguably the most important example from the point of view of this paper in the same way that if A is an "derived ring" (an animated ring or a connective $\mathbb{E}_1$-algebra) then $\mathrm{A} \to \pi_0(\mathrm{A})$ is a nilpotent extension. Let $\mathcal{A}$ be an additive ∞-category and let $h(\mathcal{A})$ be its homotopy category which is a 1-category. Then the map $\mathcal{A} \to h(\mathcal{A})$ is evidently a nilpotent extension of additive ∞-categories.

The next section, gives a general construction of examples of nilpotent extensions. Later in 4.3 we will also furnish a collection of concrete examples via the theory of weight structures.

2.4. **Square zero extensions of additive ∞-categories.** In this section, we introduce the square zero extension of additive ∞-categories which furnishes plenty of interesting examples of nilpotent extensions of additive ∞-categories.

Definition 2.4.1. Let $\mathcal{A}$ be an additive ∞-category. A **$\mathcal{A}$-bimodule** is a biadditive[2] functor

$$\mathcal{M} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Spt}_{\geqslant 0}.$$

By adjunction the functor gives rise to an additive functor

$$\mathcal{M} : \mathcal{A} \to \mathrm{Fun}^{\times}(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt})$$

which we refer to by the same letter.

Remark 2.4.2. Unpacking the structure of an $\mathcal{A}$-bimodule, we get the following pieces of data

1. fixing an object $y \in \mathcal{A}$, for any morphism $g : c \to c'$ in $\mathcal{A}$ we get a map of connective spectra:
$$g^* : \mathcal{M}(c', y) \to \mathcal{M}(c, y),$$
as part of a functor
$$\mathcal{M}(-, y) : \mathcal{A}^{\mathrm{op}} \to \mathrm{Spt}_{\geqslant 0}.$$
In particular, this endows $\mathcal{M}(y, y) \in \mathrm{Spt}_{\geqslant 0}$ the structure of a right $\mathrm{end}_{\mathcal{A}}(y, y)$-module.
2. Fixing an object $z \in \mathcal{A}$, for any morphism $g : c \to c'$ in $\mathcal{A}$ we get a map of connective spectra:
$$g_* : \mathcal{M}(z, c) \to \mathcal{M}(z, c'),$$
as part of a functor
$$\mathcal{M}(z, -) : \mathcal{A} \to \mathrm{Spt}_{\geqslant 0}.$$
In particular, this endows $\mathcal{M}(z, z) \in \mathrm{Spt}_{\geqslant 0}$ the structure of a left $\mathrm{end}_{\mathcal{A}}(z, z)$-module.
3. Altogether, we can package the two structures above by saying that each for $z \in \mathcal{C}$, $\mathcal{M}(z, z)$ is an $\mathrm{end}_{\mathcal{A}}(z, z)$-bimodule.

We would like to make sense of the square zero extension of additive ∞-categories, which we denote as $\mathcal{A} \oplus \mathcal{M}$. To do so, we first recall the formalism of lax equalizers [NS17, Section II.1] which is a special case of lax pullbacks as in [Tam18, Section I].

Construction 2.4.3. Let $\mathcal{A}, \mathcal{B}$ be ∞-categories equipped with functors $\mathrm{F}, \mathrm{G} : \mathcal{A} \to \mathcal{B}$. The **lax equalizer** of F and G

$$\mathrm{LEq}(\mathrm{F}, \mathrm{G})$$

[2] By this we mean a functor which is additive in each variable. In the presence of tensor products of additive ∞-categories as in [Lur18, Sections 10.1.6, D.2.1.1], we can say that such a functor defines an additive functor $\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathrm{Spt}_{\geqslant 0}$

is the ∞-category given by the pullback

$$\begin{array}{ccc} \mathrm{LEq}(\mathrm{F},\mathrm{G}) & \longrightarrow & \mathrm{Fun}(\Delta^1,\mathcal{B}) \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{res}_{\delta_0},\mathrm{res}_{\delta_1})} \\ \mathcal{A} & \xrightarrow{(\mathrm{F},\mathrm{G})} & \mathcal{B}\times\mathcal{B}. \end{array}$$

The objects of $\mathrm{LEq}(\mathrm{F},\mathrm{G})$ are given by pairs (c,f) where $c\in\mathcal{A}$ and a morphism $f:\mathrm{F}(c)\to\mathrm{G}(c)$ in $\mathcal{B}$. The mapping spaces in $\mathrm{LEq}(\mathrm{F},\mathrm{G})$ are given by an equalizer diagram

$$\mathrm{Maps}_{\mathrm{LEq}(\mathrm{F},\mathrm{G})}((c,f),(c',f')))\to\mathrm{Maps}_{\mathcal{A}}(c,c')\rightrightarrows\mathrm{Maps}(\mathrm{F}(c),\mathrm{G}(c'))$$

where one of the right maps is given by

$$\mathrm{Maps}_{\mathcal{A}}(c,c')\to\mathrm{Maps}_{\mathcal{B}}(\mathrm{F}c,\mathrm{F}c')\xrightarrow{f'_*}\mathrm{Maps}_{\mathcal{B}}(\mathrm{F}c,\mathrm{G}c'),$$

and the other is

$$\mathrm{Maps}_{\mathcal{A}}(c,c')\to\mathrm{Maps}_{\mathcal{B}}(\mathrm{G}c,\mathrm{G}c')\xrightarrow{f^*}\mathrm{Maps}_{\mathcal{B}}(\mathrm{F}c,\mathrm{G}c').$$

Lemma 2.4.4. *Suppose that $\mathcal{A},\mathcal{B}$ are additive ∞-categories with additive functors*

$$\mathrm{F},\mathrm{G}:\mathcal{A}\to\mathcal{B}.$$

Then, $\mathrm{LEq}(\mathrm{F},\mathrm{G})$ *is an additive ∞-category and the functor* $\mathrm{LEq}(\mathrm{F},\mathrm{G})\to\mathcal{A}$ *is additive.*

Proof. $\mathrm{Fun}(\Delta^1,\mathcal{B})$ is additive since limits and colimits in a functor category are computed pointwise (see [Lur17b, Corollary 5.1.2.3]). It suffices to show that a pullback of a diagram of additive categories is also an additive category. This follows from the fact a pullback of group-like $\mathbb{E}_\infty$-spaces is a group-like $\mathbb{E}_\infty$-space and Proposition 2.1.2. □

Construction 2.4.5. Given an additive ∞-category $\mathcal{A}$ and a $\mathcal{A}$-bimodule $\mathcal{M}$, we construct the **square zero extension** $\mathcal{A}\oplus\mathcal{M}$ in the following manner: take the (pointwise) suspension of $\mathcal{M}$ and adjoint to get a functor

$$\mathcal{A}\xrightarrow{\Sigma\mathcal{M}}\mathrm{Fun}^\times(\mathcal{A}^{\mathrm{op}},\mathrm{Spt}).$$

The additive ∞-category $\mathcal{A}\oplus\mathcal{M}$ is then defined as the lax equalizer between the Yoneda functor $\mathcal{A}\xrightarrow{よ}\mathrm{Fun}^\times(\mathcal{A}^{\mathrm{op}},\mathrm{Spt})$ and $\Sigma\mathcal{M}$:

$$\mathcal{A}\oplus\mathcal{M}:=\mathrm{LEq}(よ,\Sigma\mathcal{M}).$$

In other words, it is the pullback

$$\begin{array}{ccc} \mathcal{A}\oplus\mathcal{M} & \longrightarrow & \mathrm{Fun}(\Delta^1,\mathrm{Fun}^\times(\mathcal{A}^{\mathrm{op}},\mathrm{Spt})) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle (\mathrm{res}_{\delta_0},\mathrm{res}_{\delta_1})} \\ \mathcal{A} & \xrightarrow{(よ,\Sigma\mathcal{M})} & \mathrm{Fun}^\times(\mathcal{A}^{\mathrm{op}},\mathrm{Spt})\times\mathrm{Fun}^\times(\mathcal{A}^{\mathrm{op}},\mathrm{Spt}). \end{array}$$

Lemma 2.4.6. *The ∞-category $\mathcal{A}\oplus\mathcal{M}$ is an additive ∞-category and the functor $p:\mathcal{A}\oplus\mathcal{M}\to\mathcal{A}$ is additive.*

Proof. This follows from Lemma 2.4.4. □

Remark 2.4.7. Let us unpack the ∞-category $\mathcal{A}\oplus\mathcal{M}$. Its objects are pairs

$$x=(c\in\mathcal{A},t:よ(c)\to\Sigma\mathcal{M}(-,c)).$$

Now, since $\mathcal{M}$ takes values in connective spectra, we have that $\pi_0(\mathcal{M}(x,x))=0$, hence the data of t is, in a sense made precise in Lemma 2.4.11, redundant. Moreover, we can compute

$$\mathrm{Maps}_{\mathcal{A}\oplus\mathcal{M}}((c,t),(c',t'))$$

as the equalizer of the maps (notation as in Remark 2.4.2)

$$\mathrm{Maps}_{\mathcal{A}}(c,c')\ \underset{f\mapsto f^*t}{\overset{f\mapsto f_*t'}{\rightrightarrows}}\ \Sigma\mathcal{M}(c,c').$$

This means that a morphism $x = (c,t) \to x' = (c',t')$ can be viewed as the data of a morphism $f : c \to c'$ in $\mathcal{A}$ and a "loop" in $\Sigma\mathcal{M}(x,x')$ identifying f^*t with f_*t', i.e., a "point" in $\mathcal{M}(x,x)$ or, more precisely an element $m \in \pi_0(\mathcal{M}(x,x'))$.

The fiber sequence defining the equalizer also gives rise to a fiber sequence

$$\mathcal{M}(p(x),p(x')) \overset{j_{x,x'}}{\to} \mathrm{Maps}_{\mathcal{A}\oplus\mathcal{M}}(x,x') \overset{p_{x,x'}}{\to} \mathrm{Maps}_{\mathcal{A}}(p(x),p(x')). \tag{2.4.8}$$

of functors

$$(\mathcal{A}\oplus \mathrm{M})^{\mathrm{op}} \times (\mathcal{A}\oplus\mathcal{M}) \to \mathrm{Spt}_{\geqslant 0}.$$

Lemma 2.4.9. *The map $\mathcal{A}\oplus\mathcal{M} \to \mathcal{A}$ admits a section*

$$i : \mathcal{A} \to \mathcal{A}\oplus\mathcal{M}$$

that sends an object c to $(c,0)$.

Proof. The right vertical map in the diagram (2.4.3) admits a section given by the functor $(b,b') \mapsto (b \overset{0}{\to} b')$. Formally, this is the direct sum of functors

$$\mathcal{B}\times\mathcal{B} \overset{p_1}{\to} \mathcal{B} = \mathrm{Fun}(pt,\mathcal{B}) \overset{\mathrm{Ran}_{\delta_0}}{\to} \mathrm{Fun}(\Delta^1,\mathcal{B}) \text{ and}$$
$$\mathcal{B}\times\mathcal{B} \overset{p_2}{\to} \mathcal{B} = \mathrm{Fun}(pt,\mathcal{B}) \overset{\mathrm{Lan}_{\delta_1}}{\to} \mathrm{Fun}(\Delta^1,\mathcal{B}),$$

where Ran (resp. Lan) is right Kan extension (resp. Left Kan extension). This induces a section of the left vertical map by functoriality of pullbacks. It sends c to $(c,0)$ by construction. □

Proposition 2.4.10. *The functor p defined above is a nilpotent extension of additive ∞-categories.*

Proof. Part (1) of Definition 2.3.1 follows from Lemma 2.4.9. Observing the fiber sequence (2.4.8) and using the fact that $\pi_0\Sigma\mathcal{M}(c,c') = 0$ we also see that Part (2) is satisfied. Let us prove (3); we claim that the n appearing (3) is just 2. Indeed, suppose that $f : x \to x', g : x' \to x''$ are composable morphisms in $\mathcal{A}\oplus\mathcal{M}$ such that $p(f) \simeq p(g) \simeq 0$. We claim that $g \circ f = 0$. Indeed, by the bifunctoriality of the fiber sequence (2.4.8) we have the following commutative diagram where the rows are fiber sequences:

$$\begin{array}{ccccc}
\mathcal{M}(p(x'),p(x'')) & \xrightarrow{j_{x',x''}} & \mathrm{Maps}_{\mathcal{A}\oplus\mathcal{M}}(x',x'') & \longrightarrow & \mathrm{Maps}_{\mathcal{A}}(p(x'),p(x'')) \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
\mathcal{M}(p(x),p(x'')) & \xrightarrow{j_{x,x''}} & \mathrm{Maps}_{\mathcal{A}\oplus\mathcal{M}}(x,x'') & \longrightarrow & \mathrm{Maps}_{\mathcal{A}}(p(x),p(x'')).
\end{array}$$

Now since $g \in \mathrm{Maps}_{\mathcal{A}\oplus\mathcal{M}}(x',x'')$ is such that $p(g) \simeq 0$, we can find $m \in \mathcal{M}(p(x'),p(x''))$ such that $j_{x',x''}(m) = g$. On the other hand, since $p(f) \simeq 0$, we get that the left vertical f^* is nullhomotopic. Therefore, $g \circ f \simeq f^*(g) \simeq j_{x,x''} \circ f^*(m) \simeq 0$. □

Lemma 2.4.11. *Consider the subcategory of $\mathcal{A}\oplus\mathcal{M}^0 \hookrightarrow \mathcal{A}\oplus\mathcal{M}$ spanned by those objects of the form $(x \in \mathcal{A}, 0)$. Then the map $(\mathcal{A}\oplus\mathcal{M})^0 \hookrightarrow \mathcal{A}\oplus\mathcal{M}$ is an equivalence of categories.*

Proof. The functor in question is fully faithful, so it suffices to show essential surjectivity, i.e. that any object of $\mathcal{A}\oplus\mathcal{M}$ is equivalent to an object of the form $(c,0)$. By Proposition 2.4.10 $\mathcal{A}\oplus\mathcal{M} \overset{p}{\to} \mathcal{A}$ is a nilpotent extension, in particular, it is surjective on homotopy classes of morphisms. Hence we can find a map $(c,t) \overset{f}{\to} (c,0)$ such that $p(f) \simeq \mathrm{id}_c$ for any object $(c,t) \in \mathcal{A}\oplus\mathcal{M}$. By Proposition 2.3.3 p is conservative, so f is an equivalence. □

The next proposition gives us a way to compute composition in $\mathcal{A}\oplus\mathcal{M}$. We note that via the section $i : \mathcal{A} \to \mathcal{A}\oplus\mathcal{M}$ in Lemma 2.4.11, we get an additive functor

$$\mathrm{Maps}_{\mathcal{A}\oplus\mathcal{M}}(i(-),i(-)) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Spt}_{\geqslant 0}.$$

Furthermore, by Lemma 2.4.11 again, any object in $\mathcal{A}\oplus\mathcal{M}$ is equivalent to one which is of the form $i(c)$.

Proposition 2.4.12. *There is a* $\mathcal{A}^{\mathrm{op}} \times \mathcal{A}$*-canonical equivalence*

$$\mathrm{Maps}_{\mathcal{A}\oplus\mathcal{M}}(i(c), i(c')) \cong \mathrm{Maps}_{\mathcal{A}}(c, c') \oplus \mathcal{M}(c, c').$$

The composition

$$i(c) \overset{(f_1,m_1)}{\to} i(c') \overset{(f_2,m_2)}{\to} i(c'')$$

in $\mathcal{A} \ltimes \mathcal{M}$ *can be computed as* $(f_2 f_1, f_{2*}m_1 + f_1^* m_2)$.

Proof. Since the maps (2.4.7) are equal on objects of $\mathcal{A}$ the fiber sequence i_*(2.4.8) of functors on $\mathcal{A}^{\mathrm{op}} \times \mathcal{A}$ splits. This proves the first part of the statement.

To compute the composition, we note that since the composition operation is linear with respect to the additive $\mathbb{E}_\infty$-structure, we have that

$$\begin{aligned}(f_2, m_2) \circ (f_1, m_1) &\simeq ((f_2, 0) + (0, m_2)) \circ (f_1, m_1)\\ &\simeq (f_2, 0) \circ (f_1, m_1) + (0, m_2) \circ (f_1, m_1)\\ &\simeq (f_2, 0) \circ (f_1, 0) + (f_2, 0) \circ (0, m_1) + (0, m_2) \circ (f_1, 0) + (0, m_2) \circ (0, m_1).\end{aligned}$$

Hence the claim follows from the following sequence of equivalences:

(1) $(f_2, 0) \circ (f_1, m) \simeq (f_2 \circ f_1, f_{2*}m)$,
(2) $(f_2, m) \circ (f_1, 0) \simeq (f_2 \circ f_1, f_1^* m)$, and
(3) $(0, m) \circ (0, m') \simeq 0$.

We note that (3) was shown in the course of proving Proposition 2.4.10. To prove the first claim, because of the $\mathcal{A}^{\mathrm{op}} \times \mathcal{A}$-linearity of the decomposition, we get the commutative diagram

$$\begin{array}{ccc}\mathrm{Maps}_{\mathcal{A}\oplus\mathcal{M}}(i(c'), i(c'')) & \xrightarrow{\simeq} & \mathrm{Maps}_{\mathcal{A}}(c', c'') \oplus \mathcal{M}(c', c'')\\ {\scriptstyle (f_1,0)^*}\downarrow & & \downarrow{\scriptstyle f_1^* \oplus f_1^*}\\ \mathrm{Maps}_{\mathcal{A}\oplus\mathcal{M}}(i(c), i(c'')) & \xrightarrow{\simeq} & \mathrm{Maps}_{\mathcal{A}}(c, c'') \oplus \mathcal{M}(c, c'').\end{array}$$

This proves (1), while (2) follows from the commutativity of

$$\begin{array}{ccc}\mathrm{Maps}_{\mathcal{A}\oplus\mathcal{M}}(i(c), i(c')) & \xrightarrow{\simeq} & \mathrm{Maps}_{\mathcal{A}}(c, c') \oplus \mathcal{M}(c, c')\\ {\scriptstyle (f_2,0)_*}\downarrow & & \downarrow{\scriptstyle f_{2*} \oplus f_{2*}}\\ \mathrm{Maps}_{\mathcal{A}\oplus\mathcal{M}}(i(c), i(c'')) & \xrightarrow{\simeq} & \mathrm{Maps}_{\mathcal{A}}(c, c'') \oplus \mathcal{M}(c, c'').\end{array}$$

□

Remark 2.4.13. In [Dot18, Definition 1.3], Dotto defined the notion of square zero extension of Ab-enriched categories. Since $h(\mathcal{A})$ is, in particular, Ab-enriched our construction is an ∞-categorical generalization of his notion, at least when the category is additive. One advantage of our formulation is that the composition law is not *a priori* prescribed, but is a computation stemming from the presentation as a lax pullback. We can also view our definition as an ∞-categorical/spectral version of Tabuada's definition for dg categories [Tab09, Section 4].

Example 2.4.14. Let R be a ring and M be an R-bimodule. We can build the ring $\mathrm{R} \oplus \mathrm{M}$ with multiplication defined by the formula

$$(r_1, m_1), (r_2, m_2) \mapsto (r_1 r_2, r_1 m_2 + m_1 r_2).$$

Denote by $\mathcal{M}$ the functor

$$\mathbf{Perf}_{\mathrm{R}} \longrightarrow \mathrm{Fun}^{\times}(\mathbf{Perf}_{\mathrm{R}}^{\mathrm{op}}, \mathrm{Spt})$$

$$\mathrm{P}_1 \mapsto (\mathrm{P}_2 \mapsto \mathrm{P}_1 \otimes_{\mathrm{R}} \mathrm{M} \otimes_{\mathrm{R}} \mathrm{P}_2).$$

Then Proposition 2.4.12 and Lemma 2.4.11 yield an equivalence $\mathbf{Perf}_{\mathrm{R}\oplus\mathrm{M}} \simeq \mathbf{Perf}_{\mathrm{R}} \oplus \mathcal{M}$.

3. The DGM theorem for nilpotent extensions of additive ∞-categories

We begin with the additive version of the main result. Suppose that $\mathcal{A}$ is a small additive ∞-category then we can associate to it several other ∞-categories:

(1) We have the stable ∞-category of **additive spectral presheaves**

$$\mathrm{Fun}^{\times}(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt}) := \widehat{\mathcal{A}},$$

spanned by presheaves on $\mathcal{A}^{\mathrm{op}}$ that preserves finite sums. This is the stabilization of the prestable ∞-category $\mathrm{PSh}_{\Sigma}(\mathcal{A})$ so that the spectral Yoneda functor $よ : \mathcal{A} \to \widehat{\mathcal{A}}$ is fully faithful.

(2) We have the small stable ∞-category of **finite cell $\mathcal{A}$-modules** which is the smallest stable subcategory of $\mathrm{Fun}^{\times}(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt})$ containing the Yoneda image:

$$\mathcal{A}^{\mathrm{fin}} \subset \widehat{\mathcal{A}}.$$

(3) We have the subcategory of **ind-objects** of $\mathcal{A}$ defined as the subcategory of $\mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt})$ which preserves all finite limits:

$$\mathrm{Ind}(\mathcal{A}) \subset \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt}),$$

and the Yoneda embedding factors through this subcategory as $y : \mathcal{A} \to \mathrm{Ind}(\mathcal{A})$.

(4) We have the additive ∞-category of **big and flat cell $\mathcal{A}$-modules**

$$\mathcal{A}^{\mathrm{big}} \subset \mathrm{Ind}(\mathcal{A})$$

which is the additive subcategory of $\mathrm{Ind}(\mathcal{A})$ generated by the object $\oplus_{x_i \in \mathrm{Ob}\mathcal{A}} x_i$ and retracts thereof.

Remark 3.0.1. The ∞-category $\mathcal{A}^{\mathrm{fin}}$ is equivalent to the Spanier-Whitehead stabilization of the ∞-category $\mathrm{PSh}^{\mathrm{fin}}_{\Sigma}(\mathcal{A})$ — the smallest subcategory of $\mathrm{PSh}_{\Sigma}(\mathcal{A})$ containing all representable presheaves and closed under finite colimits. Indeed, $\mathrm{SW}(\mathrm{PSh}_{\Sigma}(\mathcal{A}))$ is a full subcategory of $\widehat{\mathcal{A}}$ (this follows from [Lur18, Remark C.1.1.6]) and the image coincides with the smallest subcategory of $\widehat{\mathcal{A}}$ containing representable presheaves and closed under taking shifts and finite colimits.

Now combining [Lur17b, Proposition 5.3.6.2] and [Lur18, Proposition C.1.1.7] we get a convenient universal property for $\mathcal{A}^{\mathrm{fin}}$:

$$\mathrm{Fun}^{\times}(\mathcal{A}, \mathcal{C}) \cong \mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}^{\mathrm{fin}}, \mathcal{C})$$

for any stable ∞-category $\mathcal{C}$. In other words, the functor

$$\mathbf{Cat}^{\mathrm{add}}_{\infty} \to \mathbf{Cat}^{\mathrm{st}}_{\infty}$$

given by applying $(-)^{\mathrm{fin}}$ is left adjoint to the forgetful functor. Hence we should regard $\mathcal{A}^{\mathrm{fin}}$ as the "free stable ∞-category" generated by $\mathcal{A}$. This features into the proof of the next result.

Lemma 3.0.2. *The functor $\mathbf{Cat}^{\mathrm{add}}_{\infty} \to \mathbf{Cat}^{\mathrm{st}}_{\infty}$ given by sending $\mathcal{A}$ to $\mathcal{A}^{\mathrm{fin}}$ sends exact sequences of ∞-categories into localization sequences of stable ∞-categories.*

Proof. Given an exact sequence of additive ∞-categories

$$\mathcal{B} \to \mathcal{A} \to \mathcal{C}$$

it suffices to show that $\mathcal{B}^{\mathrm{fin}} \to \mathcal{A}^{\mathrm{fin}}$ is fully faithful and that

$$\mathcal{A}^{\mathrm{fin}}/\mathcal{B}^{\mathrm{fin}} \to \mathcal{C}^{\mathrm{fin}}$$

is an equivalence up to idempotent completion.

Fully faithfulness follows, for example, from [Lur17b, Proposition 5.3.5.11]. It follows from Theorem 2.2.2(2) that $\mathcal{A} \to \mathcal{A}/\mathcal{B}$ can be identified with cofiber of the map $\mathcal{A} \to \mathcal{B}$, while the corresponding cofiber in $\mathbf{Cat}^{\mathrm{st}}_{\infty}$ is given by localization. As discussed in Remark 3.0.1 $(-)^{\mathrm{fin}}$ is a left adjoint, so it preserves cofibers. Lastly, note that

$$((-)^{\mathrm{fin}})^{\mathrm{Kar}} \simeq ((-)^{\mathrm{Kar}})^{\mathrm{fin}},$$

so

$$(\mathcal{A}^{\text{fin}}/\mathcal{B}^{\text{fin}})^{\text{Kar}} \simeq ((\mathcal{A}/\mathcal{B})^{\text{fin}})^{\text{Kar}} \to (\mathcal{C}^{\text{fin}})^{\text{Kar}}$$

is indeed an equivalence. □

With the above notation, the algebraic K-theory spectrum of $\mathcal{A}$ is defined to be the (non-connective) algebraic K-theory of stable ∞-categories in the sense of Blumberg, Gepner and Tabuada [BGT13]:

$$\mathrm{K}(\mathcal{A}) := \mathrm{K}(\mathcal{A}^{\text{fin}}).$$

The next Lemma follows from Lemma 3.0.2, and from definitions. We refer the reader to Theorem A.0.1 in the appendix to see why the K-theory of additive ∞-categories is, in some sense, "simpler" than the K-theory of stable ∞-categories.

Lemma 3.0.3. *Suppose that*

$$\mathcal{B} \to \mathcal{A} \to \mathcal{C}$$

is an exact sequence of additive ∞-categories. Then for any localizing invariant E,

$$\mathrm{E}(\mathcal{B}) \to \mathrm{E}(\mathcal{A}) \to \mathrm{E}(\mathcal{C})$$

is a cofiber sequence of spectra.

Here's our main result in the case of additive ∞-categories.

Theorem 3.0.4. *Let $f : \mathcal{A} \to \mathcal{B}$ be a nilpotent extension of small additive ∞-categories. Then the diagram*

$$\begin{array}{ccc} \mathrm{K}(\mathcal{A}) & \longrightarrow & \mathrm{TC}(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathrm{K}(\mathcal{B}) & \longrightarrow & \mathrm{TC}(\mathcal{B}). \end{array}$$

is a cartesian square.

Proof. We regard the trace map as a transformation (see [BGT13, Section 10.3], [AMGR17, Section 6] or [Gep] for a construction in the language of this paper)

$$\mathrm{K} \to \mathrm{TC} : \mathbf{Cat}_\infty^{\text{st}} \to \mathrm{Spt},$$

and we precompose with the functor $\mathrm{A} \mapsto \mathrm{A}^{\text{fin}}$

$$(\mathbf{Cat}_\infty^{\text{Kar}}) \to \mathbf{Cat}_\infty^{\text{st}},$$

so that by Lemma 3.0.3 we have a transformation between two functors that converts exact sequences of additive ∞-categories to cofiber sequences. For better bookkeeping, we consider the fiber of the trace map

$$\mathrm{K}^{\text{inv}} : (\mathbf{Cat}_\infty^{\text{Kar}})^{\mathbf{Perf}} \to \mathrm{Spt},$$

which is also a localizing invariant.

We will explain the following commutative diagram of additive ∞-categories

$$\begin{array}{ccccc} \mathcal{A} & \longrightarrow & \mathcal{A}^{\text{big}} & \longrightarrow & (\mathcal{A}^{\text{big}}/\mathcal{A})^{\text{Kar}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B} & \longrightarrow & \mathcal{B}^{\text{big}} & \longrightarrow & (\mathcal{B}^{\text{big}}/\mathcal{B})^{\text{Kar}}. \end{array} \tag{3.0.5}$$

For a small additive ∞-category $\mathcal{E}$, we can consider

$$\mathcal{E}^{\text{big}} \subset \mathrm{Ind}(\mathcal{E}),$$

which is the additive subcategory generated under retracts by

$$\mathrm{G} = \bigoplus_{x \in \mathrm{Obj}(\mathcal{E})} x_i.$$

The Yoneda functor $\mathcal{E} \to \mathrm{Ind}(\mathcal{A})$ factors through $\mathcal{E}^{\mathrm{big}}$ since the latter category is closed under retracts, whence contains the Yoneda image. Note that this construction is functorial in small additive ∞-categories and additive functors. This explains the left square. The right square is just the induced map on cofibers, i.e., on Verdier quotients in the sense explained in Section 2.1.

We note the following facts:

(1) we have an equivalence of additive ∞-categories
$$\mathcal{A}^{\mathrm{big}} \simeq \mathbf{Mod}(\mathrm{end}_{\mathcal{A}^{\mathrm{big}}}(\mathrm{G}))^c,$$
and

(2) the additive ∞-category $\mathcal{A}^{\mathrm{big}}/\mathcal{A}$ is generated by the image of $\bigoplus_{x\in\mathrm{Obj}(\mathcal{A})} x_i$.

Therefore:
$$(\mathcal{A}^{\mathrm{big}}/\mathcal{A})^{\mathrm{Kar}} \simeq \mathbf{Mod}(\mathrm{end}_{\mathcal{A}^{\mathrm{big}}/\mathcal{A}}(\mathrm{G}))^c,$$
The analogous equivalences also hold for $f^*\mathrm{G}$ in $\mathcal{B}^{\mathrm{big}}$ and in $(\mathcal{B}^{\mathrm{big}}/\mathcal{B})^{\mathrm{Kar}}$. Under these equivalences the middle and the right map correspond to base change functors along maps of $\mathbb{E}_1$-rings.

Now, the map
$$\pi_0\mathrm{end}_{\mathcal{A}^{\mathrm{big}}}(\mathrm{G}) \to \pi_0\mathrm{end}_{\mathcal{B}^{\mathrm{big}}}(f^*\mathrm{G})$$
is a nilpotent extension of rings, whose kernel is the ideal consists of infinite matrices with entries in the kernel of $\mathcal{A} \to \mathcal{B}$. Moreover, the same is true for the map
$$\pi_0\mathrm{end}_{\mathcal{A}^{\mathrm{big}}/\mathcal{A}}(\mathrm{G}) \to \pi_0\mathrm{end}_{\mathcal{B}^{\mathrm{big}}/\mathcal{B}}(f^*\mathrm{G})$$
by Corollary 2.2.3(2). Therefore, these ideals are nilpotent.

Now $\mathrm{K}^{\mathrm{inv}}$ applied to the middle or the right vertical functor of (3.0.5) is an equivalence by Theorem 1.0.1. Since it's a localizing invariant it's also an equivalence after applying to $\mathcal{A} \to \mathcal{B}$ which yields the claim. □

The last part of the proof only uses that $\mathrm{K}^{\mathrm{inv}}$ is nil-invariant on rings and is a localizing invariant. Thus in fact we have the following theorem.

Theorem 3.0.6. *Let* E *be a localizing invariant which is nil-invariant (for example, a truncating invariant in the sense of* [LT19]*). Then* E *sends any nilpotent extension of additive* ∞*-categories into an equivalence.*

Here is a couple of corollaries. The next one is an ∞-categorical version of Dotto's theorem [Dot18].

Corollary 3.0.7. *Suppose that* $\mathcal{A}$ *is an additive* ∞*-category and* $\mathcal{M}$ *and* $\mathcal{A}$*-bimodule. Then we have the following cartesian diagram:*
$$\begin{array}{ccc} \mathrm{K}(\mathcal{A}\oplus\mathcal{M}) & \longrightarrow & \mathrm{TC}(\mathcal{A}\oplus\mathcal{M}) \\ \downarrow & & \downarrow \\ \mathrm{K}(\mathcal{A}) & \longrightarrow & \mathrm{TC}(\mathcal{A}). \end{array}$$

The next corollary will be important for the sequel.

Corollary 3.0.8. *Let* $\mathcal{A}$ *be an addititive* ∞*-category. Then we have the following cartesian diagram*
$$\begin{array}{ccc} \mathrm{K}(\mathcal{A}) & \longrightarrow & \mathrm{TC}(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathrm{K}(h(\mathcal{A})) & \longrightarrow & \mathrm{TC}(h(\mathcal{A})). \end{array}$$

Remark 3.0.9. Let $\mathcal{A}$ be an additive ∞-category. Since $\mathcal{A}$ is the colimit of its subcategories generated by finitely many objects, we note that the K-theory spectrum of $\mathcal{A}$ can be computed as

$$\mathrm{K(A)} \cong \operatorname*{colim}_{\mathrm{X}\in\mathrm{Obj(A)}} \mathrm{K(End_A(X))}, \tag{3.0.10}$$

where the transition maps are given by inclusion of the smallest retract-closed subcategories containing X. Note that this result requires the fact that K-theory preserves filtered colimits. The formula (3.0.10) says that computing $\mathrm{K}(h(\mathcal{A}))$ can be reduced to computing K-theories of discrete rings which, at least in principle, is much more computable than the K-theory of non-discrete rings and of $\mathrm{K}(\mathcal{A})$.

Therefore, one can interpret Corollay 3.0.8 as saying that to understand $\mathrm{K}(\mathcal{A})$ one can understand $\mathrm{K}(h(\mathcal{A}))$ and the fiber of $\mathrm{TC}(\mathcal{A}) \to \mathrm{TC}(h(\mathcal{A}))$.

Remark 3.0.11. Unfortunately, TC is not a finitary invariant and the obvious analog of (3.0.10) for TC is probably false. Indeed, as in [LMT20, Section 3.1], [BCM20, Section 2.4], we can write the subcategory of p^∞-torsion objects in $\mathbf{Perf}_{\mathbf{Z}}$ as a colimit of categories of modules:

$$\operatorname{colim} \mathbf{Mod}_{\mathbf{Z}/p^n\mathbf{Z}}(\mathbf{Perf}_{\mathbf{Z}}).$$

However, TC does not preserve this colimits as explained in [LMT20, Remark 3.27]. Thus we cannot prove DGM for additive ∞- categories simply by applying the original DGM theorem to the maps

$$\mathrm{End_A(X)} \to \mathrm{End_B(X)}$$

and passing to the colimit.

Note that Theorem 3.0.4 and Remark 3.0.9 imply that at least $\mathrm{fib}(\mathrm{TC}(\mathcal{A}) \to \mathrm{TC}(h\mathcal{A}))$ commutes with filtered colimits.

Theorem 3.0.4 is a rather straightforward generalization of the original DGM theorem. The theory of weight structures for ∞-categories is the tool that allows us to bootstrap this rather easy theorem to obtain new and interesting cases of the DGM theorem.

4. The DGM theorem for weighted stable ∞-categories

So far we have only discussed additive ∞-categories and proved a DGM theorem in that context. However, the examples we are mostly interested in come from the context of stable ∞-categories, and their K-theories. The K-theory of $\mathcal{C}$ as a stable category is fundamentally different from its additive K-theory (in the sense of, say, [GGN15b, Section 8]), so our results do not directly apply. Moreover, we do not quite know how to define nilpotent extensions of general stable ∞-categories. Note that:

- Exact functors of stable ∞-categories are rarely nilpotent extensions of additive ∞-categories. For instance, we might think that $\mathbf{Perf}_{\mathbb{S}} \to \mathbf{Perf}_{\mathbb{Z}}$ should be a nilpotent extension, but the periodicity theorem in chromatic homotopy theory tells us that for any finite CW-complex X with torsion homology there are positive degree maps $\Sigma^\infty \mathrm{X} \xrightarrow{f} \Sigma^\infty \mathrm{X}[d]$ that are trivial in homology such that f^n is non-trivial for any $n > 0$.
- If we try to check the nilpotence condtition on the endomorphism rings of the generators, it will give us a wrong notion. For example, the functor $\mathbf{Perf}_{ku} \to \mathbf{Perf}_{\mathrm{KU}}$ is not expected to be a nilpotent extension, but the map of rings

  $$ku = \mathrm{End}_{ku}(ku) \to \mathrm{End_{KU}(KU)} = \tau_{\geqslant 0}\mathrm{KU}$$

  is an equivalence.

Of course, our result is applicable to those to those functors of stable ∞-categories that are equivalent to a functor of the form

$$\mathcal{A}^{\mathrm{fin}} \to \mathcal{B}^{\mathrm{fin}}$$

for some nilpotent extension of additive ∞-categories $\mathcal{A} \xrightarrow{f} \mathcal{B}$. So we can extend the DGM theorem to reasonably good stable ∞-categories if we had a "criterion" for a given stable ∞-category or a functor between those to be in the image of

$$(-)^{\mathrm{fin}}\colon \mathbf{Cat}_\infty^{\mathrm{add}} \to \mathbf{Cat}_\infty^{\mathrm{st}}.$$

We view this problem in analogy with describing loop spaces in topology in terms of spaces with some extra structure: we would like to have a *recognition principle* for finite cell modules of additive categories in terms of stable ∞-categories with some extra structure. This can be done using the notion of a boundedly weight structure and this allows us to formulate DGM for stable ∞-categories endowed with a bounded weight structure (weighted ∞-categories, for short).

We begin with a recollection of weighted ∞-categories for the reader's convenience.

Definition 4.0.1. A **weight structure** on a stable ∞-category $\mathcal{C}$ is the data of two retract-closed subcategories $(\mathcal{C}_{w\geqslant 0}, \mathcal{C}_{w\leqslant 0})$ such that:

(1) $\mathcal{C}_{w\geqslant 0}[1] \subset \mathcal{C}_{w\geqslant 0}$, $\mathcal{C}_{w\leqslant 0}[-1] \subset \mathcal{C}_{w\leqslant 0}$; write
$$\mathcal{C}_{w\geqslant n} := \mathcal{C}_{w\geqslant 0}[n] \qquad \mathcal{C}_{w\leqslant k} := \mathcal{C}_{w\leqslant 0}[k].$$
(2) if $x \in \mathcal{C}_{w\leqslant 0}, y \in \mathcal{C}_{w\geqslant 1}$ then
$$\mathrm{Maps}_{\mathcal{C}}(x, y) \simeq 0,$$
(3) for any object $x \in \mathcal{C}$ we have a cofiber sequence
$$x_{\leqslant 0} \to x \to x_{\geqslant 1}, \tag{4.0.2}$$
where $x_{\leqslant 0} \in \mathcal{C}_{\leqslant 0}$ and $x_{\geqslant 1} \in \mathcal{C}_{\geqslant 1}$. We call these **weight truncations** of x.

We say that the weight structure is **bounded** if $\mathcal{C} = \bigcup\limits_n (\mathcal{C}_{\geqslant -n} \cap \mathcal{C}_{\leqslant n})$. We call a pair of a stable ∞-category $\mathcal{C}$ equipped with a (bounded) weight structure a (**boundedly**) **weighted ∞-category**.

Remark 4.0.3. The axioms of a weight structure can be thought of as "dual" to those of a t-structure. However, weight structures differ from t-structures in at least two important aspects (and definitely more!):

(1) First, the cofiber sequence (4.0.2) is *not* functorial in x. Indeed, in the case of $\mathbf{K}(\mathcal{A})$, the ∞-category obtained from complexes in an additive category $\mathcal{A}$ by inverting homotopy equivalences, weight truncations correspond to the brutal/stupid truncation of complexes ([BS18a, Remark 1.2.3(1)]). Morally speaking, while the notion of a t-structure abstracts the Postnikov tower (say in the category of spectra of the derived category of a ring), the notion of a weight structure abstracts the cellular tower. The category $\mathcal{C}_{w\geqslant 0}$ should be thought of as the category of "cells of non-negative dimensions" and conversely, $\mathcal{C}_{w\leqslant 0}$ are "cells of non-positive dimensions". Axiom (2) abstracts the fact that there should be no maps from a lower-dimensional cell to a higher-dimensional cell, just as there is no map from a lower-dimensional sphere to a higher dimensional sphere. The non-functoriality of weight truncations is related to the fact that a cellular presentation of an object is a choice.
(2) Secondly, we define the **weight heart** of an weighted ∞-category as
$$\mathcal{C}^{\heartsuit_w} := \mathcal{C}_{\geqslant 0} \cap \mathcal{C}_{\leqslant 0}.$$
For t-structures, the heart is always an abelian discrete category while for weight structures, the heart is an additive ∞-category. The next few sections furnishes such examples.

Remark 4.0.4. We also note that beside being retract-closed, $\mathcal{C}_{w\geqslant 0}$ are also extension stable: if $x \to y \to z$ is a cofiber sequence in $\mathcal{C}$ where $x, z \in \mathcal{C}_{w\geqslant 0}$ then so is y. The same goes for $\mathcal{C}_{w\leqslant 0}$; we refer to [Bon10, Proposition 1.3.3(3)] for a proof.

Remark 4.0.5. Bounded weight structures are quite easy to construct. Assume we are given with a subcategory $\mathrm{N} \subset \mathcal{C}$ such that

1. N generates $\mathcal{C}$ under finite limits, colimits and retracts,
2. N is negatively self-orthogonal, that is, for any $x, y \in \mathrm{N}$ the mapping spectrum
$$\mathrm{maps}_{\mathcal{C}}(x, \Sigma^n y)$$
is connective. Equivalently, $\pi_0 \operatorname{Maps}_{\mathcal{C}}(x, \Sigma^n y) = 0$ for $n > 0$.

Then one can check that defining
$$\mathcal{C}_{w \geqslant 0} = \{\text{retracts of finite colimits of objects of N}\}$$
$$\mathcal{C}_{w \leqslant 0} = \{\text{retracts of finite limits of objects of N}\}$$
gives a weight structure on $\mathcal{C}$ whose heart is the minimal retract-closed additive subcategory containing N; see [BS18a, Corollary 2.1.2] for a reference.

The following is a result of the second author [Sos19, Corollary 3.4].

Theorem 4.0.6. *The functor*
$$\mathbf{WCat}_{\infty}^{\mathrm{st},b} \to \mathbf{Cat}_{\infty}^{\mathrm{add}} \qquad (\mathcal{A}, w) \mapsto \mathcal{A}^{\heartsuit_w}$$
is fully faithful. It is moreover an equivalence when restricted to the subcategories of idempotent complete ∞-categories on both sides.

Definition 4.0.7. The ∞-category of idempotent complete additive categories (as a full subcategory of $\mathbf{Cat}_{\infty}^{\mathrm{add}}$) will be denoted by $\mathbf{Cat}_{\infty}^{\mathrm{Kar}}$ and the ∞-category of idempotent complete weighted ∞-categories will be denoted by $\mathbf{WCat}_{\infty}^{\mathrm{st},b,\mathrm{Kar}}$.

4.1. **Motivic examples.** In this section, we discuss examples of boundedly weighted ∞-categories coming from the theory of motives.

Definition 4.1.1. Let k be a field. The additive ∞-category of **Chow motives**
$$\mathbf{Chow}_{\infty}(k) \subset \mathbf{DM}_{\mathrm{gm}}(k)$$
is the smallest additive ∞-category generated by
$$\{\mathrm{M}(\mathrm{X})(q)[2q] : q \in \mathbf{Z}, \mathrm{X} \text{ is smooth and projective over } k\},$$
and retracts thereof. If R is a coefficient ring, we also write
$$\mathbf{Chow}_{\infty}(k; \mathrm{R}) \subset \mathbf{DM}_{\mathrm{gm}}(k; \mathrm{R})$$
for the versions with coefficients.

We note that the mapping (connective) spectra in $\mathbf{Chow}_{\infty}$ is not discrete. Indeed, suppose that X, Y are smooth projective k-schemes and X is dimension d then for $j \geqslant 0$:

$$\begin{aligned}
\pi_j \operatorname{Maps}(\mathrm{M}(\mathrm{X})(n)[2n], \mathrm{M}(\mathrm{Y})(m)[2m]) &\cong \pi_0 \operatorname{Maps}(\mathrm{M}(\mathrm{X}), \mathrm{M}(\mathrm{Y})(m-n)[2m-2n-j]) \\
&\cong \pi_0 \operatorname{Maps}(\mathrm{M}(\mathrm{X} \times \mathrm{Y}), \mathrm{M}(k)(m-n+d)[2m-2n-j+2d]) \\
&\cong \mathrm{H}_{\mathrm{mot}}^{2(m+d-n)-j, m+d-n}(\mathrm{X} \times \mathrm{Y}) \\
&\cong \mathrm{CH}^{m+d-n}(\mathrm{X} \times \mathrm{Y}; j).
\end{aligned}$$

where we have used Friedlander-Voevodsky/Atiyah duality ([FV00, Rio05]) to conclude that:
$$\mathrm{M}(\mathrm{X})^{\vee} \cong \mathrm{M}(\mathrm{X})(-d)[-2d].$$

Lemma 4.1.2. *Let* X *be a smooth k-scheme of dimension d,* Y *a smooth projective k-scheme and $n, m \in \mathbf{Z}$. Then, for any coefficient ring* R*, the mapping spectrum*
$$\mathrm{maps}_{\mathbf{DM}_{\mathrm{gm}}(k;\mathrm{R})}(\mathrm{M}(\mathrm{X})(n)[2n], \mathrm{M}(\mathrm{Y})(m)[2m])$$
is connective.

Proof. By [Sus17] we can assume k is perfect. By the computation above, we have that

$$\pi_j \mathrm{maps}(\mathrm{M}(\mathrm{X})(n)[2n], \mathrm{M}(\mathrm{Y})(m)[2m]) \cong \mathrm{H}^{2(m+d-n)-j, m+d-n}_{\mathrm{mot}}(\mathrm{X} \times \mathrm{Y}; \mathrm{R}),$$

which is zero whenever $j < 0$ by the vanishing range of motivic cohomology [MVW06, Theorem 19.3]. □

The following is a result of Bondarko's which we give a proof for the reader's convenience.

Theorem 4.1.3 ([Bon10, Bon11])**.** *Let k be a field and suppose that e is the exponential characteristic of k. Assume that* R *is a ring where e is invertible. Then there exists a bounded weight structure on $\mathbf{DM}_{\mathrm{gm}}(k;\mathrm{R})$ such that $\mathbf{DM}_{\mathrm{gm}}(k,\mathrm{R})^{\heartsuit_w} \simeq \mathbf{Chow}_{\infty}(k,\mathrm{R})$.*

Proof. By [Sus17] we can assume k is perfect. By Remark 4.0.5 it suffices to check that the collection $\{\mathrm{M}(\mathrm{X})(q)[2q]\}$ where X is a smooth projective k-scheme and $q \in \mathbf{Z}$ is a collection of generators for $\mathbf{DM}_{\mathrm{gm}}(k,\mathrm{R})$ and that the mapping spectrum

$$\mathrm{maps}_{\mathbf{DM}_{\mathrm{gm}}(k;\mathrm{R})}(\mathrm{M}(\mathrm{X})(n)[2n], \mathrm{M}(\mathrm{Y})(m)[2m])$$

is connective for any smooth projective X, Y and any $n, m \in \mathbb{Z}$. The first fact follows from [Kel13, Proposition 5.5.3], the second fact is proved in Lemma 4.1.2.

□

Remark 4.1.4. The analogous weight structure can be constructed more generally on the ∞-categories $\mathbf{DM}_c(\mathrm{S};\mathbf{Q})$ (see [Bon14]) for quasi-excellent finite dimensional separated schemes S and even on some subcategories of $\mathbf{DM}_c(\mathrm{S};\mathbf{Z}[1/p])$, where S is a scheme of exponential characteristic p (see [BI15]). Moreover, similar weight structure can be constructed on the ∞-category of compact $\mathrm{KGL}_{\mathrm{S}}$-modules and the ∞-category of compact $\mathrm{MGL}_{\mathrm{S}}$-modules ([BS18a, 4]).

4.2. **Stacky examples.** Let k be a discrete commutative ring. We will need some basic notions about algebraic groups.

- A **group scheme** over k is a group object in the category of flat k-schemes.
- A group scheme G is called **embeddable** if there exists a homomorphism $\mathrm{G} \to \mathrm{GL}_n(k)$ which is a closed immersion.
- A group scheme G is called **linearly reductive** if the functor of taking invariants

$$\mathbf{QCoh}_{\mathrm{B}_k\mathrm{G}} \to \mathbf{QCoh}_k$$

  is t-exact.
- A group scheme G is called **nice** if it's linearly reductive and is an extension of a group scheme by a a group of multiplicative type.

Example 4.2.1. If k is characteristic zero, then reductive is equivalent to linearly reductive. If k is characteristic $p > 0$, then linearly reductive is equivalent to being nice and examples include tori, μ_{p^n} and constant group schemes of order prime to p.

The construction of weight structures on perfect complexes on stacks can be found in joint work of the second author with Bachmann, Khan and Ravi [BKRS].

Theorem 4.2.2 (Theorem 5.2.1, [BKRS])**.** *Let* G *be an embeddable linearly reductive group scheme over k.*

(1) *Let* R *be a $\mathbb{E}_\infty$-k-algebra endowed with an action of* G*. There is a bounded weight structure on $\mathbf{Perf}_{[\mathrm{Spec\,R/G}]}$ whose heart is the subcategory of vector bundles.*
(2) *Let $\mathrm{R} \xrightarrow{f} \mathrm{S}$ be a* G*-equivariant morphism of $\mathbb{E}_\infty$-k-algebras endowed with an action of* G*. Then the base change functor $\mathbf{Perf}_{[\mathrm{Spec\,R/G}]} \to \mathbf{Perf}_{[\mathrm{Spec\,S/G}]}$ is weight-exact.*

4.3. The DGM theorem for nilpotent extensions of weighted ∞-categories.

Definition 4.3.1. Let $(\mathcal{A}, w), (\mathcal{B}, w)$ be weighted ∞-categories. A weight exact-functor $f : \mathcal{A} \to \mathcal{B}$ is said to be a **nilpotent extension** if the functor of additive ∞-categories

$$f : \mathcal{A}^{\heartsuit_w} \to \mathcal{B}^{\heartsuit_w}$$

is a nilpotent extension in the sense of Definition 2.3.1.

Remark 4.3.2. Theorem 4.0.6 particular states that if $(\mathcal{A}, w)$ is an idempotent complete stable ∞-category with a bounded weight structure, then the functor

$$(\mathcal{A}^{\heartsuit_w})^{\mathrm{fin}} \to \mathcal{A}$$

is an equivalence.

In light of Theorem 4.0.6 we can trivially recast Definition 4.3.1 in the following way:

Lemma 4.3.3. *Let $(\mathcal{A}, w), (\mathcal{B}, w)$ be weighted ∞-categories. Let $f : \mathcal{A} \to \mathcal{B}$ be a weight exact functor. Then the following are equivalent*

1. *f is a nilpotent extension in the sense of Definition 4.3.1*
2. *$f^{\heartsuit_w}$ is a nilpotent extension in the sense of Definition 2.1.1*

Now Theorem 3.0.4 together with Remark 4.3.2 imply automatically

Theorem 4.3.4. *Let $f : \mathcal{A} \to \mathcal{B}$ be a nilpotent extension of boundedly weighted ∞-categories. Then the diagram*

$$\begin{array}{ccc} \mathrm{K}(\mathcal{A}) & \longrightarrow & \mathrm{TC}(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathrm{K}(\mathcal{B}) & \longrightarrow & \mathrm{TC}(\mathcal{B}). \end{array}$$

is a cartesian square.

Proof. This follows immediately from Theorem 3.0.4 and Theorem 4.0.6 which identifies a boundedly weighted ∞-category with $(-)^{\mathrm{fin}}$ of its weight heart.

□

Similarly we obtain a more general result from Theorem 3.0.6.

Theorem 4.3.5. *Let* E *be a localizing invariant which is nil-invariant (for example, a truncating invariant in the sense of* [LT19]*). Then* E *sends any nilpotent extension of boundedly weighted ∞-categories into an equivalence.*

Example 4.3.6. For any boundedly weighted ∞-category $\mathcal{C}$ one has a functor

$$\mathcal{C} \simeq (\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}} \to (h\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}}$$

which we call the **weight complex functor** (see [Sos19, Corollary 3.5]). This induces an equivalence on the homotopy categories of the heart, so it is a nilpotent extension of weighted categories. The ∞-category $(h\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}}$ can also be described as the *homotopy category of bounded complexes* in A, i.e. the localization of the 1-category of complexes by the set of homotopy equivalences.

Example 4.3.7. Since $h\mathbf{Chow}_\infty$ is the classical additive category of Chow motives, the previous example specializes to a functor

$$\mathbf{DM}_{\mathrm{gm}}(k; \mathrm{R}) \to (\mathbf{Chow}(k; \mathrm{R}))^{\mathrm{fin}}.$$

Applying Theorem 4.3.4 to this functor we obtain Corollary 1.0.4.

Example 4.3.8. In the context of Theorem 4.2.2(2) assume that $\mathrm{R} \xrightarrow{f} \mathrm{S}$ is a surjection with nilpotent kernel. Then $\mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{R}/\mathrm{G}]} \to \mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{S}/\mathrm{G}]}$ is a nilpotent extension.

Proof. By descent we may identify $\mathrm{Maps}_{[\mathrm{Spec}\,\mathrm{R}/\mathrm{G}]}(\mathrm{V},\mathrm{W})$ with $\pi_0\,\mathrm{Maps}_{\mathrm{R}}(\mathrm{V},\mathrm{W})^{\mathrm{G}}$. Note that the map

$$\pi_0\,\mathrm{Maps}_{\mathrm{R}}(\mathrm{V},\mathrm{W})^{\mathrm{G}} \to \pi_0\,\mathrm{Maps}_{\mathrm{S}}(\mathrm{V}_{\mathrm{S}},\mathrm{W}_{\mathrm{S}})^{\mathrm{G}}$$

is surjective since G is linearly reductive.

Let n be an integer such that $\mathrm{I}^n = 0$ where $\mathrm{I} = \mathrm{Ker}(\mathrm{R} \to \mathrm{S})$. Up to adding direct summands the morphisms in the kernel of

$$\pi_0\,\mathrm{Maps}_{\mathrm{R}}(\mathrm{V},\mathrm{W}) \to \pi_0\,\mathrm{Maps}_{\mathrm{S}}(\mathrm{V}_{\mathrm{S}},\mathrm{W}_{\mathrm{S}})$$

are defined by matrices with coefficients in I, so for any sequence of composable morphisms $f_1, \cdots, f_n$ in $\mathbf{Perf}_{\mathrm{R}}$ such that $f_{i,\mathrm{S}}$ is trivial, $f_1 \circ \cdots \circ f_n = 0$. Now since $\pi_0\,\mathrm{Maps}_{\mathrm{R}}(\mathrm{V},\mathrm{W})^{\mathrm{G}}$ is a subgroup in $\pi_0\,\mathrm{Maps}_{\mathrm{R}}(\mathrm{V},\mathrm{W})$, the same is true for composable sequences of morphisms in $\mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{R}/\mathrm{G}]}$ whose base change to S is trivial. So, the second and the third part of Definition 2.3.1 is satisfied.

Since $\overline{f}$ is a closed immersion, $\overline{f}_*$ is conservative. Consequently, $\overline{f}^*$ is essentially surjective up to retracts. Any idempotent $p \in \pi_0\,\mathrm{End}_{[\mathrm{Spec}\,\mathrm{S}/\mathrm{G}]}(\mathrm{V}_{\mathrm{S}})$ can be lift along a nilpotent extension of rings

$$\pi_0\,\mathrm{End}_{[\mathrm{Spec}\,\mathrm{R}/\mathrm{G}]}(\mathrm{V}) \to \pi_0\,\mathrm{End}_{[\mathrm{Spec}\,\mathrm{S}/\mathrm{G}]}(\mathrm{V}_{\mathrm{S}}),$$

so the first part of Definition 2.3.1 is also satisfied. □

Applying Theorem 4.3.4 to this functor we obtain Corollary 1.0.3.

4.4. **Applications to stacks.** We now discuss some applications of Corollary 1.0.3 to more general algebraic stacks. For this we use the Nisnevich topology on the category of stacks. We will not recall the definition here and refer the reader to [BKRS] instead. However, we point out that K-theory and TC satisfy descent with respect to the Nisnevich topology. This allows to directly deduce a DGM-type result for stacks that admit a Nisnevich cover by stacks of the form $[\mathrm{Spec}\,\mathrm{R}/\mathrm{G}]$. Such Nisnevich covers are available for a large class of stacks by results of [AHR19] and [AHHLR], for example for stacks with affine diagonal and nice stabilizers.

4.4.1. *Nil-invariance on stacks.*

Theorem 4.4.2. *Let* $\mathrm{X} \to \mathrm{Y}$ *be a nilpotent extension of algebraic stacks, i.e. a closed immersion whose ideals sheaf is nilpotent. Assume that* Y *admits an affine Nisnevich cover* $\{[\mathrm{Spec}\,\mathrm{R}_i/\mathrm{G}] \to \mathrm{Y}\}$ *where* G *is embeddable and nice.*

Then $\mathrm{E}(\mathrm{Y}) \to \mathrm{E}(\mathrm{X})$ *is an equivalence for any truncating localizing invariant. In particular,* $\mathrm{K}^{\mathrm{inv}}(\mathrm{Y}) \to \mathrm{K}^{\mathrm{inv}}(\mathrm{X})$ *is an equivalence.*

Proof. By Nisnevich descent it suffices to show that maps

$$[\mathrm{Spec}\,\mathrm{S}_i/\mathrm{G}] = [\mathrm{Spec}\,\mathrm{R}_i/\mathrm{G}] \times_{\mathrm{Y}} \mathrm{X} \to [\mathrm{Spec}\,\mathrm{R}_i/\mathrm{G}] \text{ and}$$

$$[\mathrm{Spec}\,\mathrm{S}_i \times_{\mathrm{X}} \mathrm{S}_j/\mathrm{G}] = [\mathrm{Spec}\,\mathrm{R}_i \times_{\mathrm{X}} \mathrm{R}_j/\mathrm{G}] \times_{\mathrm{Y}} \mathrm{X} \to [\mathrm{Spec}\,\mathrm{R}_i/\mathrm{G}]$$

induce equivalences in E. Then applying $\mathbf{Perf}_{(-)}$ to these maps gives rise to nilpotent extensions as described in Example 4.3.8. So it remains to apply Theorem 4.3.5 to the nilpotent extensions. □

4.4.3. *Cdh-excision for truncating invariants.* Recall that an abstract blow-up square of *Noetherian* stacks

$$\begin{array}{ccc} \mathrm{E} & \longrightarrow & \tilde{\mathrm{X}} \\ \downarrow & & \downarrow p \\ \mathrm{Y} & \xrightarrow{i} & \mathrm{X}, \end{array}$$

is a commutative square with p proper representable and i closed immersion such that p induces an isomorphism $\tilde{\mathrm{X}} - \mathrm{E} \to \mathrm{X} - \mathrm{Y}$. Define Y_n to be the n-th infinitesimal thickening of Y in X and E_n to be the n-th infinitesimal thickening of E in $\tilde{\mathrm{X}}$. We assume that X has affine diagonal and nice stabilizers. One of the main results of [BKRS] yields

Theorem 4.4.4. [BKRS] *Then for any localizing invariant* E *the square*

$$\begin{array}{ccc} \mathrm{E}(\mathrm{X}) & \longrightarrow & \text{“}\lim\text{”}\, \mathrm{E}(\mathrm{Y}_n) \\ \downarrow & & \downarrow \\ \mathrm{E}(\tilde{\mathrm{X}}) & \longrightarrow & \text{“}\lim\text{”}\, \mathrm{E}(\mathrm{E}_n) \end{array}$$

is a weak pullback of pro-spectra.

Now applying Theorem 4.4.2 to maps $\mathrm{E} \to \mathrm{E}_n$ and $\mathrm{Y} \to \mathrm{Y}_n$ we see that the pro-systems in question trivialize. In particular we obtain:

Corollary 4.4.5. *For any truncating localizing invariant* E *the square*

$$\begin{array}{ccc} \mathrm{E}(\mathrm{X}) & \longrightarrow & \mathrm{E}(\mathrm{Y}) \\ \downarrow & & \downarrow \\ \mathrm{E}(\tilde{\mathrm{X}}) & \longrightarrow & \mathrm{E}(\mathrm{E}) \end{array}$$

is a pullback of spectra.

Example 4.4.6. According to [LT19, Proposition 3.14], the homotopy K-theory functor is a truncating invariant. Setting E = KH in Corollary 4.4.5 reproves the main result of Hoyois and Krishna [HK19]. Of course, Corollary 4.4.5 also proves cdh descent on stacks for other truncating invariants discussed in [LT19] such as $\mathrm{K}^{\mathrm{inf}}_{\mathbf{Q}}$ [LT19, Corollary 3.9] and periodic cyclic homology HP [LT19, Corollary 3.11] on stacks over characteristic zero rings.

Appendix A. K-theory of additive ∞-categories

Let $\mathcal{A}$ be an additive ∞-category. In this section, we clarify what it means to take $\mathrm{K}(\mathcal{A})$ and prove that in the situations we are interested in, all notions of K-theory agree. To $\mathcal{A}$ we can attach the following spectra:

(1) the **direct sum** K-theory in the style of Segal [Seg74] and revisited by Gepner-Groth-Nikolaus [GGN15b, Section 8]
$$\mathrm{K}^{\oplus}(\mathcal{A}).$$
It is obtained by first taking the core of $\mathcal{A}$ to get an $\mathbb{E}_\infty$-monoid in spaces, $\mathcal{A}^{\simeq}$, whose operation is induced by direct sum. Then $\mathrm{K}^{\oplus}(\mathcal{A})$ is the *connective* spectrum obtained by taking group completion and invoking the identification between group-complete $\mathbb{E}_\infty$-monoids in spaces with connective spectra.

(2) As we have done for most of this paper we can apply the K-theory functor as in [BGT13], characterized as the universal localizing invariant, to $\mathcal{A}^{\mathrm{fin}}$; we denoted this by
$$\mathrm{K}(\mathcal{A}).$$

(3) Suppose that $\mathcal{A} \subset \mathcal{C}$ is an additive subcategory of a stable ∞-category (for example, the weight-heart of a weight structure on $\mathcal{C}$). Then we can equip $\mathcal{C}$ with the structure of a **Waldhausen ∞-category** in the sense of [Bar16] by the **maximal pair structure** of [Bar16, Example 2.11] insisting that all maps are ingressives. We can then induce the structure of a Waldhausen ∞-category on $\mathcal{A}$ where the ingressives are those morphisms with cofibers in $\mathcal{A}$. To this, we can attach the **Waldhausen-Barwick** K**-theory**:
$$\mathrm{K}^{\mathrm{WB}}(\mathcal{A}),$$
as constructed in [Bar16, Part 3].

Theorem A.0.1. *Let $\mathcal{C}$ be a boundedly weighted ∞-category and let $\mathcal{C}^{\heartsuit_w} \subset \mathcal{C}$ be its weight heart. Then:*

(1) There are canonical equivalences of connective spectra
$$\mathrm{K}^{\oplus}(\mathcal{C}^{\heartsuit_w}) \simeq \mathrm{K}^{\mathrm{WB}}(\mathcal{C}^{\heartsuit_w}) \simeq \mathrm{K}^{\mathrm{WB}}(\mathcal{C})$$

(2) There is a canonical morphism

$$\mathrm{K}^{\mathrm{WB}}(\mathcal{C}^{\heartsuit_w}) \to \mathrm{K}((\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}})$$

which identifies as a connective cover.

Proof. Using [Fon19, Theorem 5.1] or [Hel19, Theorem A.15], we have an equivalence of connective spectra.

$$\mathrm{K}^{\mathrm{WB}}(\mathcal{C}) \simeq \mathrm{K}^{\mathrm{WB}}(\mathcal{C}^{\heartsuit_w}).$$

Using [Hel19, Proposition A.19], using that ingressives are split in $\mathcal{C}^{\heartsuit_w}$, we a further identification:

$$\mathrm{K}^{\oplus}(\mathcal{C}^{\heartsuit_w}) \simeq \mathrm{K}^{\mathrm{WB}}(\mathcal{C}^{\heartsuit_w}).$$

For the second statement we note that, by Theorem 4.0.6, we have an equivalence of nonconnective spectra:

$$\mathrm{K}(\mathcal{C}) \simeq \mathrm{K}((\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}}).$$

On the other hand, the connective version of K-theory of the stable ∞-category $\mathcal{C}$ in the sense of [BGT13] is, by construction, the same as $\mathrm{K}^{\mathrm{WB}}(\mathcal{C})$ with the maximal pair structure. Therefore, we have a natural map to the nonconnective K-theory of [BGT13]

$$\mathrm{K}^{\mathrm{WB}}(\mathcal{C}) \to \mathrm{K}(\mathcal{C})$$

which witnesses a connective cover. Therefore, the map induced by the equivalence of the first part:

$$\mathrm{K}^{\mathrm{WB}}(\mathcal{C}^{\heartsuit_w}) \simeq \mathrm{K}^{\mathrm{WB}}(\mathcal{C}) \to \mathrm{K}((\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}}),$$

is indeed a connective cover. □

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Department of Mathematics, Harvard University, 1 Oxford St. Cambridge, MA 02138, USA
E-mail address: elmanto@math.harvard.edu
URL: https://www.eldenelmanto.com/

Laboratory "Modern Algebra and Applications", St. Petersburg State University, 14th line, 29B, 199178 Saint Petersburg, Russia
E-mail address: vsosnilo@gmail.com