MATH 232: ALGEBRAIC GEOMETRY I

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Contents

0.1. **Books.** In principle, this class is about Grothendieck's [EGA1] which signals the birth of modern algebraic geometry. It is an extremely technical document on its own and shows one of the many ways mathematics was developed organically. The French is not too hard and I recommend that you look through the book before the start of class — I might also assign readings from here occasionally with the promise that the French (and some Google translate) will not hinder your mathematical understanding.

Here are some textbooks in algebraic geometry.

([Sha13]) Shafarevich's book is a little more old school than the others in this list, but is valuable in the **examples** it gives.

([Har77]) Hartshorne's book has long been the "gold-standard" for algebraic geometry textbook. I learned the subject from this book first. It is **terse** and has plenty of **good exercises and problems**. However, the point of view that this book takes will be substantially different from one we will take in this class, though I will most definitely steal problems from here.

([GW10]) This is essentially a translation of Grothendieck's EGA (plus more) and is closer to the point of view of this class. Just like the original text, it is **relentlessly general** and very **lucid** in its exposition.

([Vak]) Arguably the most **inviting** book in this list, and modern in its outlook.

([DG80]) As far as I know this is still the only textbook reference to the **functor-of-points** point of view to algebraic geometry.

0.2. **Lecture Notes.** There are also several class notes online in algebraic geometry. I will add on to this list as the class progresses.

([Ras]) This is the closest document to our approach to this class. In fact, I will often present directly from these notes.

([Gat]) This is a "varieties" class, so the approach is very different, but I find it very helpful for lots of examples.

0.3. **Online textbooks.** There has been an explosion of online textbooks for algebraic geometry recently, though they are perhaps they are more like "encyclopedias."

([Stacks]) Johan de Jong at Columbia was the trailblazer in this industry and most, if not all, facts about algebraic geometry that will be taught will appear here, with proofs.

([cri]) Similar but for commutative algebra. Much more incomplete.

([fpp]) A translation project for EGA.

1. Lecture 1: What is algebraic geometry?

In its essence, algebraic geometry is the study of solutions to polynomial equations. What one means by "polynomial equations," however, has changed drastically throughout the latter part of the 20th century. To meet the demands in making constructions, ideas and theorems in classical algebraic geometry rigorous has given birth to a slew of techniques and ideas which are applicable to a much, much broader range of mathematical situations.

To begin with, let us recall the famous Fermat problem:

Theorem 1.0.1 (Taylor-Wiles)**.** *Let $n \geqslant 3$, then $x^n + y^n = 1$ has no solutions over $\mathbf{Q}$ when $x, y \neq 0$.*

This is a problem in algebraic geometry. In the language that we will learn in this class, we will be able to associate a **smooth, projective scheme** Fer_n which is, informally, given by a homogeneous polynomial equation $x^n + y^n = z^n$, equipped with a canonical **morphism**

$$\begin{array}{c}\mathrm{Fer}_n\\ \downarrow\\ \mathrm{Spec}\,\mathbf{Z},\end{array}$$

such that its set of sections

$$\begin{array}{c}\mathrm{Fer}_n\\ \uparrow\downarrow\\ \mathrm{Spec}\,\mathbf{Z},\end{array}$$

correspond to potential solutions to the Fermat equation. It is in this language that the Fermat problem was eventually solved.

The point-of-view we wish to adopt in this class, however, is one that goes by **functor-of-points**. In this highly abstract, but more flexible, approach schemes appear as what they are *supposed to be* which is often easier to think about. For us, the basic definition is:

Definition 1.0.2. A **prestack** is a functor from the category of commutative rings to sets:

$$\mathcal{F} : \mathrm{CAlg} \to \mathrm{Set}.$$

Remark 1.0.3. A note on terminology: this is non-standard. What should be called (and was called by Grothendieck) a prestack is a functor

$$\mathcal{F} : \mathrm{CAlg} \to \mathrm{Cat},$$

where Cat is the (large, $(2,1)$-)category of small categories. If we think of a set as a category with no non-trivial morphisms between the objects, then the above definition is a special case of this Grothendieck definition of a prestack. We will not consider functors into categories in this class so we will reserve the term prestack for such a functor above (as opposed to something like "a prestack in sets"). Perhaps it should be called a presheaf, but a presheaf should really just be an arbitrary functor

$$\mathcal{F} : \mathcal{D}^{\mathrm{op}} \to \mathrm{Set}.$$

To make this definition jibe with the Fermat scheme above, let us note that the equation $x^n + y^n = 1$ is defined for *any* ring. Therefore we can define

$$\widetilde{\mathrm{Fer}}_n(\mathrm{R}) = \{(a, b, c) : a^n + b^n = 1\} \subset \mathrm{R}^{\times 2}.$$

The theorem of Taylor and Wiles can then be restated as the fact that

$$\widetilde{\mathrm{Fer}}_n(\mathbf{Z}) = \emptyset \qquad n \geqslant 3;$$

However we caution that this is not the same as the scheme Fer_n that we have alluded to above since it is not projective — something that we will address in the class.

Another key idea in algebraic geometry is the question of parametrizing solutions of polynomial equations in a reasonable way. Let us consider $\widetilde{\mathrm{Fer}}_2$, which is the set of solutions to $x^2 + y^2 = 1$. We have a canonical equality (the first one is more or less the same as the above):

$$\widetilde{\mathrm{Fer}}_2(\mathbf{R}) = \mathrm{S}^1,$$

as we all know. Here are three other possible answers:

(1) $\widetilde{\mathrm{Fer}}_2(\mathbf{R}) = (\cos\theta, \sin\theta) \qquad 0 \leqslant \theta < 2\pi$,
(2) $\widetilde{\mathrm{Fer}}_2(\mathbf{R}) = (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}) \qquad t \in \mathbf{R}$,
(3) $\widetilde{\mathrm{Fer}}_2(\mathbf{R})$ is the set of all triangles with hypotenuse 1 up to congruence.

The first answer *does not* belong to the realm of (conventional) algebraic geometry which, by its very nature, concerns only polynomial functions. In other words, we dismiss transcendentals like exp and $\cos, \sin$. However the language of this class is actually powerful enough to capture transcendentals and reconstruct the familiar theory of differential geometry. The third answer will turn out to belong to the realm of algebraic geometry as well, but that will be reserved for a second course. The second answer does belong to the realm of algebraic geometry that we will study in this class: we can use **rational functions** of one variable in order to describe $\mathrm{Fer}_2(\mathbf{R})$. In fact, this parametrization proves

Theorem 1.0.4. *A quadric hypersurface in $\mathbf{P}^2$ with a rational point is rational. In fact, any quadric hypersurface with a rational point is rational.*

The proof of this result is "basically known" to pre-Grothendieck algebraic geometers: we pick the rational point and stereographcially project into a hyperplane. Since a quadric means that it is cut out by a degree two polynomial, it must hit one other point. This defines a **rational map** — one that is defined "almost everywhere" which is evidently an "isomorphism." One of the major thread of investigation in algebraic geometry and comes under the name of **birational geometry** and the above result belongs to this area. An example of a beautiful result that belongs to modern birational geometry is:

Theorem 1.0.5 (Clemens and Griffiths)**.** *A nonsingular cubic threefold over* $\mathbf{C}$ *is* not *rational.*

Theorem 1.0.5 is a non-existence proof — it says that there is no way to "rationally parametrize" the cubic threefold. If you have been trained in algebraic topology, you will feel like some kind of cohomological methods would be needed. The words that you should look for are "intermediate Jacobians," an object whose real birthplace is Hodge theory.

One of the major, open problems in the subject is:

Question 1.0.6. Is a generic cubic fourfold over $\mathbf{C}$ rational?

Recently, Katzarkov, Kontesvich and Pantev claimed to have made substantial progress towards this problem, but a write-up is yet to appear. More generally, a central question in algebraic geometry is:

Question 1.0.7. How does one classify algebraic varieties up to birational equivalence?

In topology, recall that a topological (closed) surface can be classified by genus or, better, Euler charateristic:

(1) if $\chi(\Sigma) < 0$, then Σ must be the Riemann sphere,
(2) if $\chi(\Sigma) = 0$ then Σ must be a torus — in the terminology of this class it is an **elliptic curve**,
(3) most surfaces have $\chi(\Sigma) > 0$ and they are, in some sense, the "generic situation."

This kind of trichotomy can be extended to higher dimensional varieties (topological surfaces being a 1-dimensional algebraic variety over $\mathbf{C}$). The **minimal model program** seeks to find "preferred" representatives in each class.

1.1. Algebraic geometry beyond algebraic geometry.

The field of birational geometry is extremely large and remains an active area of research. But classifying algebraic varieties is not the only thing that algebraic geometry is good for. We have seen how it can be used to phrase the Fermat problem and eventually hosts its solution. There are other areas where algebraic geometry has proven to be the optimal "hosts" for problems.

One of the most prominent areas is representation theory where the central definition is very simple a group homomorphism

$$\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}).$$

If we are interested in representations valued in k-vector spaces, then the collection of all G-representations form a category called $\mathbf{Rep}_k(\mathrm{G})$. This category has an algebro-geometric incarnation: it is the category of **quasicoherent sheaves** over the an algebro-geometric gadget called a **algebraic stack** (in this case, denoted by BG) which is a special, more manageable class of prestacks but are slightly more mysterious gadgets than just algebraic varieties. Quasi-coherent sheaves are fancy versions of vector bundles — they include gadgets whose fibers "can jump" although we will study restrictions on how exactly they jump. In any case, the field of **geometric representation theory** takes as starting point that representation theory is "just" the study of the geometric object BG and brings to bear the tools of algebraic geometry onto representation theory.

We have seen that algebraic geometry hosts number theory through the problem of the existence of rational points on a variety. Another deep problem of number theory that lives within modern algebraic geometry is the **Riemann hypothesis**. In algebro-geometric terms it can be viewed as a way to assemble solutions of an equation over fields of different characteristics.

Soon we will learn what it means for a morphism of schemes $f : \mathrm{X} \to \operatorname{Spec} \mathbf{Z}$ to be **proper** and for X to be **regular, geometrically connected** and **dimension d**. To this set-up we can associate the **Hasse-Weil zeta function**:

$$\zeta_{\mathrm{X}}(s) := \prod_{x \in |\mathrm{X}|} (1 - \sharp(\kappa(x))^{-s})^{-1}.$$

where:

(1) the set $|\mathrm{X}|$ is the set of **closed points** of X,

(2) $\kappa(x)$ is the **residue field** of x which is a finite extension of $\mathbf{F}_p$ for some prime $p > 0$.

This function is expected to be extending to all of the complex numbers (as a meromorphic function). There is a version $\zeta_{\overline{\mathrm{X}}}(s)$ which takes into account the "analytic part" of X as well:

Conjecture 1.1.1 (Generalized Riemann hypothesis)**.** *If $s \in \mathbf{C}$ is a zero of $\zeta_{\overline{\mathrm{X}}}(s)$ then:*

$$2\mathrm{Re}(s) = \nu,$$

where $\nu \in [0, 2d]$.

One of the more viable approaches to verifying the generalized Riemann hypothesis is via **cohomological methods** — one would like to find a cohomology theory for schemes to which one can "extract" in a natural way the Hasse-Weil zeta function. One reason why one might expect this is the (also conjectured) functional equation

$$\zeta_{\overline{\mathrm{X}}}(s) \sim \zeta_{\overline{\mathrm{X}}}(\dim(\mathrm{X}) - s)$$

where $\sim$ indicates "up to some constant." This is a manifestation of a certain Poincaré duality in this cohomology theory which witnesses a certain symmetry between the cohomology groups and governed by the dimension of X. If X is concentrated at a single prime, then the Riemann hypothesis was proved by Deligne using **étale cohomology**. Recent work of Hesselholt, Bhatt, Morrow and Scholze have made some breakthrough towards setting up this cohomology theory but the Riemann hypothesis is, to the instructor's knowledge, still out of reach.

1.2. **Exercises 1: categorical preliminaries.** Here is a standard definition. We assume that every category in sight is **locally small** so that $\mathrm{Hom}(x, y)$ is a set, while the set of objects, $\mathrm{Obj}(\mathcal{C})$, is not necessarily a set (so only a proper class).

Definition 1.2.1. A functor $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ is **fully faithful** if for all $x, y \in \mathcal{C}$, the canonical map

$$\mathrm{Hom}(x, y) \to \mathrm{H}om(\mathrm{F}x, \mathrm{F}y)$$

is an isomorphism. We say that it is **conservative** if it reflects isomorphisms: an arrow $f : c \to c'$ in $\mathcal{C}$ is an isomorphism if and onnly if $\mathrm{F}(f) : \mathrm{F}(c) \to \mathrm{F}(c')$ is.

Exercise 1.2.2. *Let*

$$\mathrm{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathrm{G}$$

be an adjunction (in these notes we always write the left adjoint on the left). Prove

(1) F *preserves all colimits,*

(2) G *preserves all limits,*

(3) *The functor* F *is fully faithful if and only if the unit transformation*

$$\mathrm{id} \to \mathrm{G} \circ \mathrm{F}$$

is an isomorphism.

(4) $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ *is an equivalence of categories if and only if* F *is fully faithful and* G *is conservative.*

Exercise 1.2.3. *Prove that* $\mathcal{C}$ *admits all colimits if and only if it admits coproducts and coequalizers. What kind of colimits do the following categories have (you do not have to justify your answer):*

(1) *the category of finite sets,*
(2) *the category of sets,*
(3) *the category of finitely generated free abelian groups,*
(4) *the category of abelian groups,*
(5) *the category of finite dimensional vector spaces,*
(6) *the category of all vector spaces,*
(7) *the category of finitely generated free modules over a commutative ring* R*,*
(8) *the category of finitely generated projective modules over a ring* R*,*
(9) *the category of all projective modules over a ring* R*.*

Exercise 1.2.4. *Give a very short proof (no more than one line) of the dual assertion:* $\mathcal{C}$ *admits all limits if and only if it admits products and equalizers.*

Exercise 1.2.5. *Prove that the limit over the empty diagram gives terminal object, while the colimit over the empty diagram gives the initial object.*

Exercise 1.2.6. *For any small category* $\mathcal{C}$*, we can form the presheaf category*

$$\mathrm{PSh}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}).$$

Prove:

(1) *If*

$$\mathrm{F} : \mathrm{I} \to \mathrm{PSh}(\mathcal{C}) \qquad i \mapsto \mathrm{F}_i$$

is a functor and I *is a small diagram, then for any* $c \in \mathcal{C}$ *the canonical map*

$$(\operatorname*{colim}_{\mathrm{I}} \mathrm{F}_i)(c) \to \operatorname*{colim}_{\mathrm{I}}(\mathrm{F}_i(c))$$

is an isomorphism.

(2) *Formulate and prove a similar statement for limits.*
(3) *Conclude that* $\mathrm{PSh}(\mathcal{C})$ *admits all limits and colimits.*

Exercise 1.2.7. *Prove the Yoneda lemma in the following form: the functor*

$$y : \mathcal{C} \to \mathrm{PSh}(\mathcal{C}) \qquad c \mapsto y(c)(x) = \mathrm{Hom}(x, c).$$

is fully faithful. Any functor in the image of y *is called* **representable**

Exercise 1.2.8. *Prove that any functor* $\mathrm{F} \in \mathrm{PSh}(\mathcal{C})$ *is a colimit of representable functors. This entails constructing a natural transformation*

$$\operatorname{colim} y(c) \to \mathrm{F}$$

where the domain is a colimit of a diagram of functors where each functor is representable, and proving that this natural transformation is an isomorphism when evaluated at each object of $\mathcal{C}$*.*

Exercise 1.2.9. *We say that a category* $\mathcal{C}$ *is* **essentially small** *if it is equivalent to small category. Let* R *be a commutative ring and consider* $\mathrm{CAlg}_{\mathrm{R}}$ *to be the category of commutative* R*-algebras. We say that an* R*-algebra* S *is* **finite type** *if it admits an* R*-linear* surjective *ring homomorphism*

$$\mathrm{R}[x_1, \cdots x_n] \to \mathrm{S}.$$

Consider the full subcategory $\mathrm{CAlg}^{\mathrm{ft}}_{\mathrm{R}} \subset \mathrm{CAlg}_{\mathrm{R}}$ *of finite type* R*-algebras. Prove that:*

(1) *The collection of* R*-algebras of the form*

$$\{\mathrm{R}[x_1, \cdots, x_n]/\mathrm{I} : \mathrm{I} \text{ is an ideal}\}$$

forms a set (this is not meant to be hard and does not require knowledge of "set theory").

(2) *Prove that the category of finite type* R*-algebras are equivalent to the subcategory of* R*-algebras of the form* $\mathrm{R}[x_1, \cdots, x_n]/\mathrm{I}$ *(this is not meant to be hard and does not require knowledge of "set theory").*

(3) *Conclude from this that* $\mathrm{CAlg}^{\mathrm{ft}}_{\mathrm{R}}$ *is an essentially small category.*

Exercise 1.2.10. *We define the subcategory of* **left exact functors**

$$\mathrm{PSh}_{\mathrm{lex}}(\mathcal{C}) \subset \mathrm{PSh}(\mathcal{C})$$

to be the subcategory of those functors which preserves finite limits. These are functors F *such that for any finite diagram*[1] $\alpha : \mathrm{I} \to \mathcal{C}$*, the canonical map*

$$\mathrm{F}(\operatorname*{colim}_{\mathrm{I}} \alpha) \to \lim_{\mathrm{I}} \mathrm{F}(\alpha)$$

is an isomorphism. Prove:

(1) *a category* $\mathcal{C}$ *admits all finite limits if and only if it admits final objects and pullbacks;*

(2) *for a functor* F *to be left exact, it is necessary and sufficient that* F *preserves final objects and pullbacks.*

(3) *Prove that the yoneda functor factors as* $y : \mathcal{C} \to \mathrm{PSh}_{\mathrm{lex}}(\mathcal{C})$.

(4) *If* $f : \mathcal{C} \to \mathcal{D}$ *is a functor, we define*

$$f^* : \mathrm{PSh}(\mathcal{D}) \to \mathrm{PSh}(\mathcal{C}) \qquad f^*\mathrm{F} = \mathrm{F} \circ f.$$

Prove that if f *preserves finite colimits, then we have an induced functor*

$$f^* : \mathrm{PSh}_{\mathrm{lex}}(\mathcal{D}) \to \mathrm{PSh}_{\mathrm{lex}}(\mathcal{C}).$$

In the third problem set, we will use this to prove the adjoint functor theorem and construct the sheafification functor.

Exercise 1.2.11. *We say that* $\mathcal{C}$ *is* **locally presentable** *if there exists a subcategory* $i : \mathcal{C}^c \subset \mathcal{C}$ *(called the category of* **compact objects***) which is essentially small and is closed under finite colimits such that the functor*

$$\mathcal{C} \to \mathrm{PSh}_{\mathrm{lex}}(\mathcal{C}) \xrightarrow{i^*} \mathrm{PSh}_{\mathrm{lex}}(\mathcal{C}^c)$$

is an equivalence of categories. Prove

(1) *the category* Sets *is locally presentable with* Sets^c *being the subcategory of finite sets,*

(2) *the category* Vect_k *is locally presentable with* Vect^c *being the subcategory of finite-dimensional vector spaces.*

2. Lecture 2: Prestacks

Throughout the course we will denote by CAlg the category of commutative rings.

Definition 2.0.1. A **prestack** is a functor

$$\mathrm{X} : \mathrm{CAlg} \to \mathrm{Set}.$$

This means that to each commutative ring R, we assign the set X(R) and for each morphism of commutative rings $f : \mathrm{R} \to \mathrm{S}$ we have a morphism of sets

$$f^* : \mathrm{X}(\mathrm{R}) \to \mathrm{X}(\mathrm{S}).$$

Furthermore, these satisfy the obvious compatibilities to be a functor.

[1] This just means that I is a category with finitely many objects and finitely many morphisms.

Definition 2.0.2. A **morphism of prestacks** is a natural transformation $g : \mathrm{X} \to \mathrm{Y}$ of functors. This means that for each morphism of commutative rings $f : \mathrm{R} \to \mathrm{S}$ we have a commuting diagram

$$\begin{array}{ccc} \mathrm{X}(\mathrm{R}) & \xrightarrow{f^*} & \mathrm{X}(\mathrm{S}) \\ {\scriptstyle g_{\mathrm{R}}}\downarrow & & \downarrow{\scriptstyle g_{\mathrm{S}}} \\ \mathrm{Y}(\mathrm{R}) & \xrightarrow{f^*} & \mathrm{X}(\mathrm{S}). \end{array}$$

An R-**point** of a prestack is point $x \in \mathrm{X}(\mathrm{R})$; this is the same thing as a morphism of prestacks $\operatorname{Spec}\mathrm{R} \to \mathrm{X}$ by the next

Lemma 2.0.3 (Yoneda)**.** *For all prestack* X *and all* $\mathrm{R} \in \mathrm{CAlg}$*, we have a canonical isomorphism*

$$\operatorname{Hom}(\operatorname{Spec}\mathrm{R}, \mathrm{X}) \cong \mathrm{X}(\mathrm{R}).$$

In particular we have that

$$\operatorname{Hom}(\operatorname{Spec}\mathrm{R}, \operatorname{Spec}\mathrm{S}) \cong \operatorname{Spec}\mathrm{S}(\mathrm{R}) = \operatorname{Hom}(\mathrm{S}, \mathrm{R}).$$

Note the reversal of directions.

We denote by **PStk** the category of prestacks. We already have a wealth of examples:

Definition 2.0.4. Let R be a commutative ring, We define

$$\operatorname{Spec}\mathrm{R} : \mathrm{CAlg} \to \mathrm{Set} \qquad \mathrm{S} \mapsto \operatorname{Hom}_{\mathrm{CAlg}}(\mathrm{R}, \mathrm{S}).$$

An **affine scheme** is a prestack of this form.

Remark 2.0.5. If a prestack is representable, then the ring **representing** it is unique up to unique isomorphism. This is a consequence of the Yoneda lemma. In more detail, the Yoneda functor takes the form

$$\operatorname{Spec} : \mathrm{CAlg}^{\mathrm{op}} \to \mathbf{PStk} = \operatorname{Fun}(\mathrm{CAlg}, \mathrm{Set}).$$

This functor is fully faithful so we may (somewhat abusively) identify $\mathrm{CAlg}^{\mathrm{op}}$ with its image in **PStk**. The category of affine schemes is then taken to be the opposite category of commutative rings.

Example 2.0.6. Let $n \geqslant 0$. Then we define the prestack of **affine space of dimension** n as

$$\mathbf{A}^n_{\mathbf{Z}} : \mathrm{CAlg} \to \mathrm{Set} \qquad \mathrm{R} \mapsto \mathrm{R}^{\times n}.$$

In the homeworks, you will be asked to prove that this prestack is an affine scheme, represented by $\mathbf{Z}[x_1, \cdots, x_n]$.

Example 2.0.7. Suppose that $f(x) \in \mathbf{Z}[x, y, z]$ is a polynomial in three variables; for a famous example this could be $f(x, y, z) = x^n + y^n - z^n$. For each ring A, we define

$$\mathrm{V}(f)(\mathrm{R}) := \{(a, b, c) : f(a, b, c) = 0\} \subset \mathrm{R}^{\times 3}.$$

Note that this indeed defines a prestack: given a morphism of rings $\varphi : \mathrm{R} \to \mathrm{S}$, we have a morphism of sets

$$\mathrm{V}(f)(\mathrm{R}) \to \mathrm{V}(f)(\mathrm{S})$$

since $f(\varphi(a), \varphi(b), \varphi(c)) = \varphi(f(a, b, c)) = \varphi(0) = 0$. In fact, we have a morphism of prestacks (in the sense of the next definition)

$$\mathrm{V}(f) \to \mathbf{A}^{\times 3}_{\mathbf{Z}},$$

where $\mathbf{A}^{\times 3}_{\mathbf{Z}}(\mathrm{R}) = \mathrm{R}^{\times 3}$.

2.1. **Operation on prestacks I: fibered products.** One of the key ideas behind algebraic geometry is to restrict ourselves to objects which are defined by polynomial functions. More abstractly we want to restrict ourselves to objects which arise from other objects in a *constructive manner*. This is both a blessing and a curse — on the one hand it makes objects in algebraic geometry rather rigid but, on the other, it gives objects in algebraic geometry a "tame" structure.

Example 2.1.1. As a warm-up, consider n-dimensional complex space $\mathbf{C}^n$ and suppose that we have a polynomial function $\mathbf{C}^n \to \mathbf{C}$. Then the **zero set** of f is defined via the pullback

$$\begin{array}{ccc} \mathrm{Z}(f) & \longrightarrow & \mathbf{C}^n \\ \downarrow & & \downarrow f \\ \{0\} & \longrightarrow & \mathbf{C}. \end{array}$$

We want to say that $\mathrm{Z}(f)$ has the structure of a prestack or, later, a scheme. Of course the above diagram presents $\mathrm{Z}(f)$ as a set but we can also take the pullback in, say, the category of $\mathbf{C}$-analytic spaces so that $\mathrm{Z}(f)$ inherits such a structure (if a pullback exists! and it does).

Example 2.1.2. Another important construction in algebraic geometry is the notion of the **graph**. Suppose that $f : \mathrm{X} \to \mathrm{Y}$, then its graph is the set

$$\Gamma_f := \{(x, y) : f(x) = y\} \subset \mathrm{X} \times \mathrm{Y}.$$

Suppose that X, Y have the structure of a scheme, or an $\mathbf{C}$-analytic spaces or a manifold etc., then we want to say that Γ_f does inherit this structure. To do so we note that we can present Γ_f in the following manner:

$$\begin{array}{ccc} \Gamma_f & \longrightarrow & \mathrm{Y} \\ \downarrow & & \downarrow \Delta \\ \mathrm{X} \times \mathrm{Y} & \xrightarrow{f \times \mathrm{id}} & \mathrm{Y} \times \mathrm{Y} \end{array}$$

Definition 2.1.3. Suppose that $\mathrm{X} \to \mathrm{Y} \leftarrow \mathrm{Z}$ is a **cospan** of prestacks, then the **fiber product of** X **and** Y **over** Z is defined as

$$(\mathrm{X} \times_{\mathrm{Y}} \mathrm{Z})(\mathrm{R}) := \mathrm{X}(\mathrm{R}) \times_{\mathrm{Y}(\mathrm{R})} \mathrm{Z}(\mathrm{R}).$$

It will be an exercise to verify the universal property of this construction.

Example 2.1.4. Suppose that we have a **span** of rings $\mathrm{R} \leftarrow \mathrm{S} \to \mathrm{T}$ so that we have a cospan of prestacks $\operatorname{Spec}\mathrm{R} \to \operatorname{Spec}\mathrm{S} \leftarrow \operatorname{Spec}\mathrm{T}$. Then (exercise) we have a natural isomorphism

$$\operatorname{Spec}\mathrm{R} \times_{\operatorname{Spec}\mathrm{S}} \operatorname{Spec}\mathrm{T} \cong \operatorname{Spec}(\mathrm{R} \otimes_{\mathrm{S}} \mathrm{T}).$$

Example 2.1.5. A **regular function on a prestack** is a morphism of prestacks $\mathrm{X} \to \mathbf{A}^1$. If $\mathrm{X} = \operatorname{Spec}\mathrm{R}$, then this classifies a map of commutative rings $\mathbf{Z}[x] \to \mathrm{R}$ which is equivalent to picking out a single element $f \in \mathrm{R}$. The **zero locus** of f is the prestack

$$\mathrm{Z}(f) := \mathrm{X} \times_{\mathbf{A}^1} \{0\}$$

where $\{0\} \to \mathbf{A}^1$ is the map corresponding to $\mathbf{Z}[x] \to \mathbf{Z}, x \mapsto 0$.

2.2. **Closed immersions.** A closed immersion is a special case of a **subprestack**

Definition 2.2.1. A **subprestack** of a prestack $\mathcal{F}$ is a prestack $\mathcal{G}$ equipped with a natural transformation $\mathcal{G} \to \mathcal{F}$ such that for any $\mathrm{R} \in \mathrm{CAlg}$, the map

$$\mathcal{G}(\mathrm{R}) \to \mathcal{F}(\mathrm{R})$$

is an injection. We will often write $\mathcal{F} \subset \mathcal{G}$ for subprestacks.

Remark 2.2.2. This is equivalent to saying that $\mathcal{G} \to \mathcal{F}$ is a monomorphism in the category of prestacks.

The important thing about a subprestack is that for any morphism $\mathrm{R} \to \mathrm{R}'$, the requirement that $\mathcal{G} \to \mathcal{F}$ is a natural transformation enforces the commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{G}(\mathrm{R}) & \longrightarrow & \mathcal{F}(\mathrm{R}) \\ \downarrow & & \downarrow \\ \mathcal{G}(\mathrm{R}') & \longrightarrow & \mathcal{F}(\mathrm{R}'). \end{array}$$

which should be read as: "the map $\mathcal{G}(\mathrm{R}) \to \mathcal{F}(\mathrm{R}')$ factors through the subset $\mathcal{G}(\mathrm{R}')$."

Definition 2.2.3. A **closed immersion** of affine schemes is a morphism $\operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{S}$ such that the induced map of rings $\mathrm{S} \to \mathrm{R}$ is surjective. In this case we say that $\operatorname{Spec} \mathrm{R}$ is a **closed subscheme** of $\operatorname{Spec} \mathrm{S}$.

Let us try to understand what this means. the map $\varphi : \mathrm{S} \to \mathrm{R}$, if surjective, is equivalent to the data of an ideal $\mathrm{I} = \ker(\varphi)$. As stated before, we should think of S as the ring of functions on a "space" $\operatorname{Spec} \mathrm{S}$ and so an ideal of S is a collection of functions which are closed under the action of S. Now, the "space" $\operatorname{Spec} \mathrm{R}$ should be thought of as the space on which the functions that belong to I vanish. In other words a closed immersion is one of the form $\operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{R}/\mathrm{I}$.

Exercise 2.2.4. *Suppose that $f : \mathbf{A}^n \to \mathbf{A}^1$ is regular function. Prove that the $\mathrm{Z}(f) \to \mathbf{A}^n$ is a closed immersion corresponding to a map of rings $\mathbf{Z}[x] \to \mathbf{Z}[x]/(f)$.*

Here is how one can globalize this definition:

Definition 2.2.5. A morphism of prestacks $\mathrm{X} \to \mathrm{Y}$ is a **closed immersion** or a **closed subprestack** if for any morphism $\operatorname{Spec} \mathrm{R} \to \mathrm{Y}$ then:

(1) the prestack $\operatorname{Spec} \mathrm{R} \times_{\mathrm{Y}} \mathrm{X}$ is representable and,
(2) the morphism
$$\operatorname{Spec} \mathrm{R} \times_{\mathrm{Y}} \mathrm{X} \to \operatorname{Spec} \mathrm{R}$$
is a closed immersion.

2.3. Exercises 2.

Exercise 2.3.1. *What does* $\operatorname{Spec}(0)$ *represent?*

Exercise 2.3.2. *Prove that the category of prestacks admit all limits and all colimits.*

Exercise 2.3.3. *Prove that the prestack $\mathbf{A}^n_{\mathbf{Z}}$ is representable by $\mathbf{Z}[x_1, \cdots, x_n]$.*

Exercise 2.3.4. *Consider the prestack*

$$\mathbf{G}_m : \mathrm{R} \mapsto \mathrm{R}^\times.$$

Here $\mathrm{R}^\times$ is the multiplicative group of unit elements in R. Prove that $\mathbf{G}_m$ is representable. What ring is it representable by?

Exercise 2.3.5. *Suppose that $f : \mathbf{A}^n \to \mathbf{A}^1$ is regular function. Prove that the $\mathrm{Z}(f) \to \mathbf{A}^n$ is a closed immersion corresponding to a map of rings $\mathbf{Z}[x] \to \mathbf{Z}[x]/(f)$.*

Exercise 2.3.6. *Consider the prestack*

$$\mathrm{GL}_n : \mathrm{R} \mapsto \mathrm{GL}_n(\mathrm{R}).$$

Prove that it is representable. What ring is it representable by?

If R is a ring we write $\mathrm{R}_{\mathfrak{p}}$ to be the localization of R at $\mathfrak{p}$. We write $\mathfrak{m}_{\mathfrak{p}}$ to be the maximal ideal of said local ring and write

$$\kappa(\mathfrak{p}) := \mathrm{R}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}.$$

Exercise 2.3.7. *Let* $\mathrm{R} \in \mathrm{CAlg}$ *and* K *a field. Prove that there is a natural bijection between*

$$\{\operatorname{Spec} \mathrm{K} \to \operatorname{Spec} \mathrm{R}\}$$

with

$$\{\textit{prime ideals } \mathfrak{p} \subset \mathrm{R} \textit{ with an inclusion } \kappa(\mathfrak{p}) \hookrightarrow \mathrm{K}\}.$$

The **Zariski tangent space** of $\operatorname{Spec} \mathrm{R}$ **at a prime ideal** $\mathfrak{p}$ is the $\kappa(\mathfrak{p})$-vector space given by

$$\mathrm{T}_{\mathfrak{p}} \operatorname{Spec} \mathrm{R} := (\mathfrak{m}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}^2)^{\vee}.$$

Exercise 2.3.8. *Let* $\mathrm{R} \in \mathrm{CAlg}$ *and* k *a field and suppose that* R *is a* k*-algebra. Prove that there is a bijection between*

$$\{k\textit{-morphisms of rings}\, \mathrm{R} \to \kappa[x]/(x^2)\}$$

with

$$\{\textit{prime ideals } \mathfrak{p} \textit{ of } \mathrm{R} \textit{ with residue field } k \textit{ and an element of the Zariski tangent space at } \mathfrak{p}\}.$$

Exercise 2.3.9. *Prove that the functor*

$$\operatorname{Spec} : \mathrm{CAlg} \to \mathbf{PStk}$$

1. *is fully faithful,*
2. *preserves all colimits in the sense that if* $\{\mathrm{R}_\alpha\}_{\alpha \in \mathrm{A}}$ *is a diagram of commutative rings then for all* $\mathrm{S} \in \mathrm{CAlg}$*, the canonical map*
$$\operatorname*{colim}_{\alpha}(\operatorname{Spec} \mathrm{R}_\alpha)(\mathrm{S}) \to \lim(\operatorname{Spec} \mathrm{R}_\alpha)(\mathrm{S})$$
is an isomorphism. Deduce, in particular, that Spec *converts tensor products of commutative rings to pullback.*
3. *Show, by example, that* Spec *does not preserve limits.*

Exercise 2.3.10. *Prove that a closed immersion of schemes is a subprestack.*

For the next exercise, recall that if $\mathcal{C}$ is a category with products and $\mathrm{X} \in \mathcal{C}$, then the identity morphism $\mathrm{id} : \mathrm{X} \to \mathrm{X}$ induces the **diagonal** map

$$\Delta : \mathrm{X} \to \mathrm{X} \times \mathrm{X}.$$

Exercise 2.3.11. *Let* $\mathrm{R} \in \mathrm{CAlg}$ *and consider the multiplication map* $\mathrm{R} \otimes_{\mathbf{Z}} \mathrm{R} \to \mathrm{R}$*. Prove:*

1. *the corresponding map* $\operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{R} \times \operatorname{Spec} \mathrm{R}$ *is given by the diagonal morphism,*
2. *prove that* $\Delta : \operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{R} \times \operatorname{Spec} \mathrm{R}$ *is a closed immersion of prestacks.*

Exercise 2.3.12. *Let* $\mathcal{C}$ *be a category with all limits and suppose that we have a diagram*

$$\begin{array}{ccccc} \mathrm{X} & & & & \\ & \searrow & & & \\ & & \mathrm{S} & \longrightarrow & \mathrm{T} \\ & \nearrow & & & \\ \mathrm{Y} & & & & \end{array}$$

Prove that the diagram (be sure to write down carefully how the maps are induced!)

$$\begin{array}{ccc} \mathrm{X} \times_{\mathrm{S}} \mathrm{Y} & \longrightarrow & \mathrm{X} \times_{\mathrm{T}} \mathrm{Y} \\ \downarrow & & \downarrow \\ \mathrm{S} & \xrightarrow{\Delta} & \mathrm{S} \times_{\mathrm{T}} \mathrm{S} \end{array}$$

is cartesian (hint: if you are unable to prove this result in full generality, feel free to assume that $\mathcal{C} = \mathbf{PStk}$*).*

Exercise 2.3.13. *A morphism of prestacks* $\mathrm{G} \to \mathrm{F}$ *is said to be* **representable** *if for any* $\operatorname{Spec} \mathrm{R} \to \mathrm{F}$*, the prestack* $\mathrm{G} \times_{\mathrm{F}} \operatorname{Spec} \mathrm{R}$ *is representable. Prove that the following are equivalent:*

(1) the diagonal $\Delta : \mathrm{F} \to \mathrm{F} \times \mathrm{F}$ *is representable;*
(2) any map $\operatorname{Spec} \mathrm{S} \to \mathrm{F}$ *(in other words any map from an affine scheme to* $\mathcal{F}$*) is representable.*

3. Lecture 3: Descent

We are about to define schemes. But first we define stacks[2]. In order to define the notion of stacks, we need the notion of open immersions, which are complementary to closed immersions.

Remark 3.0.1. Thinking about closed immersions of schemes is easier than thinking about open immersions, at least to the instructor. Indeed, every closed immersion of $\operatorname{Spec} \mathrm{R}$ corresponds to the set of all ideals of R. We can think of a poset of ideals $\{\mathrm{I} \subset \mathrm{J}\}$ which corresponds to a poset of closed subschemes $\{\operatorname{Spec} \mathrm{R}/\mathrm{I} \to \operatorname{Spec} \mathrm{R}/\mathrm{J}\}$. Of course, we want to say that open subschemes of $\operatorname{Spec} \mathrm{R}$ should be those of the form

$$\mathrm{D}(\mathrm{J}) := \operatorname{Spec} \mathrm{R} \smallsetminus \operatorname{Spec} \mathrm{R}/\mathrm{J}.$$

But now $\mathrm{D}(\mathrm{J})$ is in fact **not** representable — we will soon be able to prove this. In fact these $\mathrm{D}(\mathrm{J})$'s are the first examples of non-affine schemes. In particular $\mathrm{D}(\mathrm{J})$ is not the form Spec of a ring. In order to fomulate descent in a more digestible manner, we will restrict ourselves to open subschemes of $\operatorname{Spec} \mathrm{R}$ which are actually affine.

Suppose that I is an ideal of a ring R, let us recall that the **radical of** I, denoted usually by $\sqrt{\mathrm{I}}$ is defined as

$$\sqrt{\mathrm{I}} = \{x : x^{\mathrm{N}} \in \mathrm{I} \text{ for some } \mathrm{N} \gg 0\}.$$

Example 3.0.2. Let $\mathrm{I} = (0)$ be the zero ideal. Then the **nilradical** is $\sqrt{(0)} = \{f : f^{\mathrm{N}} = 0 \text{ for some } \mathrm{N} \gg 0\}$. We say that a ring is **reduced** if $\sqrt{(0)} = 0$.

Definition 3.0.3. Let R be a ring and $f \in \mathrm{R}$. A **basic Zariski cover of the ring** R_f consists of a set I and a collection $\mathcal{U} := \{f_i : f_i \in \mathrm{R}\}_{i \in \mathrm{I}}$ such that

$$f \in \sqrt{\Sigma(f_i)}.$$

In particular, a **basic Zariski cover** of a ring R consists of a set I and a collection $\mathcal{U} := \{f_i : f_i \in \mathrm{R}\}_{i \in \mathrm{I}}$ subject to the following condition:

$$1 \in \sqrt{\Sigma(f_i)}.$$

We write

$$\{\operatorname{Spec} \mathrm{A}_{f_i} \to \operatorname{Spec} \mathrm{A}_f\}_{i \in \mathrm{I}}$$

to denote an arbitrary basic Zariski cover.

Remark 3.0.4. If f is a unit so that $\mathrm{R}_f = \mathrm{R}$, this is a very redundant definition. Indeed, any element $x \in \sqrt{\Sigma(f_i)}$ means that

$$x^{\mathrm{N}} \in \Sigma(f_i)$$

But the sum of ideals means we have a sum of elements in each ideal where all except finitely many elements are zero so:

$$x^{\mathrm{N}} = a_1 f_1 + \cdots + a_n f_n,$$

up to rearrangements. But now

$$1 = 1^{\mathrm{N}} = a_1 f_1 + \cdots + a_n f_n.$$

[2]This is where some heavy conflict with the literature will occur so be wary. In the literature, the notion of stacks differs from this one in two, crucial ways. First the descent condition is asked with respect to something called the étale topology (which we will cover later in class) and, secondly, the functor lies in the $(2,1)$-category of groupoids. Functors landing in said version of categories are not really functors in the sense we are used to in class.

Therefore we can find a subcover of $\mathcal{U}$ such that

$$1 \in \Sigma(f_i).$$

Of course this argument also does show that a basic Zariski cover of R_f can be refined by a finite subcover.

Example 3.0.5. Let p, q be distinct primes in $\mathbf{Z}$. This means, by Bézout's identity, we can write

$$1 = kp + rq,$$

for some $k, r \in \mathbf{Z}$. In the language above we find that

$$\{\operatorname{Spec}\mathbf{Z}[\tfrac{1}{p}], \operatorname{Spec}\mathbf{Z}[\tfrac{1}{q}] \hookrightarrow \operatorname{Spec}\mathbf{Z}\}$$

defines a basic Zariski cover of $\operatorname{Spec}\mathbf{Z}$.

Example 3.0.6. Let k be a field and consider $k[x]$. Suppose that $p(x), q(x)$ are polynomials which are irreducible and are coprime. Then Bézout's identity again works in this situation:

$$1 = k(x)p(x) + r(x)q(x).$$

In this language we find that

$$\{\operatorname{Spec} k[x]_{p(x)}, \operatorname{Spec} k[x]_{q(x)} \hookrightarrow \operatorname{Spec} k[x]\}$$

defines a basic Zariski cover of $\operatorname{Spec} k[x]$.

Definition 3.0.7. A prestack $\mathcal{F} : \mathrm{CAlg} \to \mathrm{Set}$ is a **(Zariski) stack** if for any $\mathrm{A} \in \mathrm{CAlg}$ and for all basic Zariski cover $\{\operatorname{Spec}\mathrm{A}_{f_i} \to \operatorname{Spec}\mathrm{A}\}_{i \in \mathrm{I}}$ the diagram

$$\mathcal{F}(\mathrm{A}) \to \prod_i \mathcal{F}(\mathrm{A}_{f_i}) \rightrightarrows \prod_{i_0, i_1} \mathcal{F}(\mathrm{A}_{f_{i_0}} \otimes_{\mathrm{A}} \mathrm{A}_{f_{i_1}})$$

is an equalizer diagram where the maps are induced by

$$\prod \mathrm{A}_{f_i} \to \prod_{i_0, i_1} \mathrm{A}_{f_{i_0}} \otimes_{\mathrm{A}} \mathrm{A}_{f_{i_1}} \qquad (g_i) \mapsto (g_{i_0}|_{\mathrm{A}_{f_{i_0} \cdot f_{i_1}}}).$$

and

$$\prod \mathrm{A}_{f_i} \to \prod_{i_0, i_1} \mathrm{A}_{f_{i_0}} \otimes_{\mathrm{A}} \mathrm{A}_{f_{i_1}} \qquad (g_i) \mapsto (g_{i_1}|_{\mathrm{A}_{f_{i_0} \cdot f_{i_1}}}).$$

3.1. **Unpacking the descent condition and Serre's lemma.** Let us note a couple of easy properties about localizations

Lemma 3.1.1. *Let $f_1, f_2 \in \mathrm{R}$ then*

$$(\mathrm{R}_{f_1})_{f_2} \cong \mathrm{R}_{f_1 \cdot f_2} \cong \mathrm{R}_{f_1} \otimes_{\mathrm{R}} \mathrm{R}_{f_2}.$$

This will be homework. For the rest of this section, we will seek be taking a map $f : \mathrm{R} \to \mathrm{A}$ and then postcomposing then along some localization of A, say A'; for this it is convenient to use the notation

$$f|_{\mathrm{A}'}$$

and think about "restriction."

Let us fix a ring R and suppose that A is a "test-ring" and we are interested in the set

$$\operatorname{Hom}(\operatorname{Spec}\mathrm{A}, \operatorname{Spec}\mathrm{R}),$$

and we would like to recover this set in terms of a given basic Zariski cover of A. As we had discussed, this latter object is given by the data of elements $g_1, \cdots g_n \in \mathrm{A}$ such that $1 = \sum_{i=1}^n g_i$. Let us consider the following set

$$\operatorname{Glue}(\mathrm{R}, \mathrm{A}, \{g_i\}) \subset \prod_{i=1}^n \operatorname{Hom}(\operatorname{Spec}\mathrm{A}_{g_i}, \operatorname{Spec}\mathrm{R}),$$

consisting of the n-tuples $\{f_i : \mathrm{R} \to \mathrm{A}_{g_i}\}$ subject to the following condition (cocycle) $f_i|_{\mathrm{A}_{g_i \cdot g_j}} = f_j|_{\mathrm{A}_{g_j \cdot g_i}}$,

called the **cocycle condition**.

Lemma 3.1.2. *As above we have an isomorphism*

$$\mathrm{Glue}(\mathrm{R}, \mathrm{A}, \{g_i\}) \cong \mathrm{Eq}(\prod_i \mathrm{Hom}(\mathrm{R}, \mathrm{A}_{g_i}) \rightrightarrows \prod_{i_0, i_1} \mathrm{Hom}(\mathrm{R}, \mathrm{A}_{g_{i_0} \cdot g_{i_1}}))$$

This is an exercise in unpacking definitions. Even though Glue is more explicit, in order to prove actual results, we will work with the equalizer formulation. Our main theorem is as follows:

Theorem 3.1.3. *The map*

$$\mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{Spec}\,\mathrm{R}) \to \prod_{i=1}^n \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}_{g_i}, \mathrm{Spec}\,\mathrm{R}) \qquad f : \mathrm{R} \to \mathrm{A} \mapsto (f|_{\mathrm{A}_{g_i}})_{i \in \mathrm{I}}$$

factors as

$$\mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{Spec}\,\mathrm{R}) \to \mathrm{Glue}(\mathrm{R}, \mathrm{A}, \{g_i\})$$

and induces an isomorphism. Equivalently, $\mathrm{Spec}\,\mathrm{R}$ *is a Zariski stack.*

We will prove Theorem 3.1.3 in the right level of generality. The next lemma is called "Serre's lemma for modules."

Lemma 3.1.4. *Let* A *be a ring and* $f_1, \cdots, f_n$ *elements such that* $\sum_{i=1}^n f_i = 1$*. Then*

$$\mathrm{M} \to \prod \mathrm{M}_{f_i} \rightrightarrows \prod \mathrm{M}_{f_i \cdot f_j}$$

is an equalizer diagram.

Proof. We first assume

(1) there exists an element f_i, say f_1, which is invertible. So that $\mathrm{M} \cong \mathrm{M}_{f_1}$, via a map φ_1. Then we prove the result: indeed, denote the equalizer by Eq. Indeed, the map $\mathrm{M} \to \prod \mathrm{M}_{f_i}$ factors through the equalizer since this is just the map

$$m \mapsto (m|_{\mathrm{A}_{f_i}}),$$

and the cocycle condition is satisfied. We construct a map

$$\mathrm{Eq} \to \mathrm{M}$$

given by

$$(m_i)_{i \in \mathrm{I}} \mapsto \varphi_1^{-1}(m_1).$$

From this, we check the two composites. First, consider:

$$\mathrm{Eq} \to \mathrm{M} \to \mathrm{Eq} \qquad (m_i) \mapsto \varphi^{-1}(m_1) \mapsto (m_1, m_1|_{\mathrm{A}_{f_2}}, \cdots, m_1|_{\mathrm{A}_{f_n}}).$$

But now, for $j > 1$ we have that

$$m_1|_{\mathrm{A}_{f_j}} = \varphi^{-1}(m_1)|_{\mathrm{A}_{f_1 \cdot f_j}} = m_j|_{\mathrm{A}_{f_j \cdot f_1}} = m_j|_{\mathrm{A}_{f_1 \cdot f_j}} = m_j,$$

where the last equality comes from the assumption that $\mathrm{M} = \mathrm{M}_{f_1}$. This proves that the composite is the identity.

Second, note that:

$$\mathrm{M} \to \mathrm{Eq} \to \mathrm{M} \qquad m \mapsto (m|_{\mathrm{A}_{f_j}}) \mapsto \varphi^{-1}(m_{\mathrm{A}_{f_j}}).$$

is easily seen to be the identity.

Now, to prove the desired claim: take kernels and cokernels:

$$0 \to \mathrm{K} \to \mathrm{M} \to \mathrm{Eq} \to \mathrm{C} \to 0$$

The claim is that $\mathrm{K}, \mathrm{C} = 0$. They are 0 after inverting each f_i by the previous claim. The proof the finishes by the next claim.

∎

Lemma 3.1.5. *Let* A *be a ring and* $f_1, \cdots, f_n$ *elements such that* $\sum_{i=1}^n f_i = 1$. *Suppose that* $\mathrm{M}_{f_i} = 0$ *for all* $i = 1, \cdots, n$. *Then* $\mathrm{M} = 0$.

Proof. The condition means we can find an N such that $f_i^{\mathrm{N}} m = 0$. But then there is an even larger M:

$$m = 1^{\mathrm{M}} \cdot m = (\sum f_i)^{\mathrm{M}} \cdot m = 0.$$

□

Corollary 3.1.6. *Let* A *be a ring and* $f_1, \cdots, f_n$ *elements such that* $\sum_{i=1}^n f_i = 1$. *Then*

$$\mathrm{A} \to \prod \mathrm{A}_{f_i} \rightrightarrows \prod \mathrm{A}_{f_i \cdot f_j}$$

is an equalizer diagram of rings. In other words, we have proved that $\mathbf{A}^1$ *is a Zariski stack.*

Proof. This follows from what we have proved. The "in other words" part follows from the fact that

$$\mathbf{A}^1(\mathrm{A}) = \mathrm{A}.$$

□

Now let us prove

Proof of Theorem 3.1.3. We have proved that $\mathbf{A}^1$ is a Zariski stack. Any affine scheme can be written as a pullback

$$\begin{array}{ccc} \operatorname{Spec}\mathrm{R} & \longrightarrow & \mathbf{A}^{\mathrm{I}} \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbf{Z} & \xrightarrow{0} & \mathbf{A}^{\mathrm{J}}, \end{array}$$

where I, J are sets. The result then follows from the next lemmas (one of which is homework):

□

Lemma 3.1.7. *Zariski stacks are preserved under limits.*

Lemma 3.1.8. $\operatorname{Spec}\mathbf{Z}$ *is a Zariski stack.*

Proof. For any ring R, $\operatorname{Spec}\mathbf{Z}(\mathrm{R}) = *$. The claim then follows from the observation that the diagram

$$* \to * \rightrightarrows *$$

is an equalizer. □

3.2. **Exercises 3.**

Lemma 3.2.1. *Let* R *be a ring and consider the functor from* R*-algebras to* R*-modules*

$$\mathrm{U} : \mathrm{CAlg}_{\mathrm{R}} \to \mathrm{Mod}_{\mathrm{R}}.$$

Prove:

(1) this functor preserves final objects,
(2) this functor creates pullbacks: a diagram of rings

$$\begin{array}{ccc} \mathrm{A} & \longrightarrow & \mathrm{B} \\ \downarrow & & \downarrow \\ \mathrm{C} & \longrightarrow & \mathrm{D}, \end{array}$$

is a pullback diagram if and only if the corresponding diagram of modules is,
(3) if you are feeling up to it: prove that in fact U *creates all limits.*

Exercise 3.2.2. *Here is a formula for* R_f. *We will work in the generality of the category* $\mathrm{Mod}_{\mathrm{R}}$.

(1) *Let* $f \in \mathrm{R}$ *and consider the following* $\mathbf{N}$*-indexed diagram in the category of* R*-modules*

$$\mathrm{M} \xrightarrow{f\cdot} \mathrm{M} \xrightarrow{f\cdot} \mathrm{M} \xrightarrow{f\cdot} \cdots .$$

Define the colimit to be the R*-module* M_f*. Prove that we have a natural isomorphism: for any* $\mathrm{N} \in \mathrm{Mod}_{\mathrm{R}}$ *such that the map*

$$f\cdot : \mathrm{N} \to \mathrm{N},$$

is an isomorphism then:

$$\mathrm{Hom}_{\mathrm{Mod}_{\mathrm{R}}}(\mathrm{M}_f, \mathrm{N}) \cong \mathrm{Hom}_{\mathrm{Mod}_{\mathrm{R}}}(\mathrm{M}, \mathrm{N}).$$

(2) *Construct explicitly a multiplication on* R_f *and a compatible ring homomorphism* $\mathrm{R} \to \mathrm{R}_f$*.*

(3) *Consider the functor*

$$j_* : \mathrm{Mod}_{\mathrm{R}_f} \to \mathrm{Mod}_{\mathrm{R}}$$

given by restriction of scalars. Prove that this functor admits a left adjoint given by $j^* : \mathrm{M} \mapsto \mathrm{M}_f$*; part of the task is to explain why* M_f *is naturally an* R_f*-module.*

(4) *Use the formula from* 1 *to prove that* j_* *is fully faithful and show that the essential image identifies with the subcategory of* R*-modules where* $f\cdot$ *acts by an isomorphism.*

Lemma 3.2.3. *Let* $f_1, f_2 \in \mathrm{R}$ *then*

$$(\mathrm{R}_{f_1})_{f_2} \cong \mathrm{R}_{f_1 \cdot f_2} \cong \mathrm{R}_{f_1} \otimes_{\mathrm{R}} \mathrm{R}_{f_2}.$$

Exercise 3.2.4. *In this exercise, we will give a proof of a basic, but very clear formulation of descent. Let* A *be a ring and suppose that* $f, g \in \mathrm{A}$ *are elements*

(1) *Consider the square*

$$\begin{array}{ccc} \mathrm{A} & \longrightarrow & \mathrm{A}_f \\ \downarrow & & \downarrow \\ \mathrm{A}_g & \longrightarrow & \mathrm{A}_{fg}. \end{array}$$

Prove that it the top and bottom arrows are isomorphisms after inverting f*; conclude that the resulting square is cartesian.*

(2) *Prove that the left vertical and the right vertical arrows are isomorphisms after inverting* g*; conclude that the resulting square is cartesian.*

(3) *Now assume:*

$$1 \in (f) + (g).$$

Conclude that the square is cartesian.

Exercise 3.2.5. *Prove that Zariski stacks are preserved under limits: suppose that we have a diagram* $\mathrm{I} \to \mathbf{PStk}$*, then the functor*

$$\mathrm{R} \mapsto \lim_{\mathrm{I}} \mathcal{F}_i(\mathrm{R}),$$

defines a Zariski stack.

Exercise 3.2.6. *Prove that any affine scheme* $\mathrm{Spec}\,\mathrm{R}$ *can be written as a pullback*

$$\begin{array}{ccc} \mathrm{Spec}\,\mathrm{R} & \longrightarrow & \mathbf{A}^{\mathrm{I}} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,\mathbf{Z} & \xrightarrow{0} & \mathbf{A}^{\mathrm{J}}. \end{array}$$

4. Lecture 4: Onto schemes

We have done a bunch of abstract stuff. I would like to tell you how to say something concrete using abstract stuff.

4.1. **Diversion: multiplicative groups and graded rings.** Let us, for this section, consider what structure what one can endow on $\mathbf{G}_m = \operatorname{Spec} \mathbf{Z}[t, t^{-1}]$. One suggestive way to think about $\mathbf{Z}[t, t^{-1}]$ is as

$$\mathbf{Z}[t, t^{-1}] \cong \mathbf{Z}[\mathbf{Z}] \cong \bigoplus_{j \in \mathbf{Z}} \mathbf{Z}(j).$$

This is also called the **group algebra** on the (commutative) group $\mathbf{Z}$; we will say why this is a interesting at all later on. We want to say that $\mathbf{G}_m$ is a group object in the category of affine schemes. Unwinding definitions, we need to provide three pieces of data

(Mult.) The multiplication:

$$\mu : \mathbf{Z}[t, t^{-1}] \to \mathbf{Z}[t, t^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[t, t^{-1}] \qquad t \mapsto t \otimes t,$$

(Id.) The identity

$$\epsilon : \mathbf{Z}[t, t^{-1}] \to \mathbf{Z} \qquad t \mapsto 1$$

(Inv.) The inverse

$$\iota : \mathbf{Z}[t, t^{-1}] \to \mathbf{Z}[t, t^{-1}] \qquad t \mapsto t^{-1}.$$

These data (or, more precisely, the opposites thereof) are subject to the compatibilities that prescribe $\mathbf{G}_m$ as a group object in **PStk**.

Definition 4.1.1. An **affine group scheme** is an affine scheme $\mathrm{G} = \operatorname{Spec} \mathrm{R}$ with maps $\mu : \mathrm{G} \times \mathrm{G} \to \mathrm{G}, \epsilon : \operatorname{Spec} \mathbf{Z} \to \mathrm{G}, \iota : \mathrm{G} \to \mathrm{G}$ which endows it with the structure of a group object in prestacks.

When we speak of groups, we always want to speak about group actions. If G is an affine group scheme and $\mathcal{F}$ is a prestack then a **(left) action** is given by a morphism of prestacks

$$a : \mathrm{G} \times \mathcal{F} \to \mathcal{F},$$

satisfying the obvious compatibilities:

$$\begin{array}{ccc} \mathrm{G} \times \mathrm{G} \times \mathcal{F} & \xrightarrow{\mu \times \mathrm{id}} & \mathrm{G} \times \mathcal{F} \\ {\scriptstyle \mathrm{id} \times a} \downarrow & & \downarrow {\scriptstyle a} \\ \mathrm{G} \times \mathcal{F} & \xrightarrow{a} & \mathcal{F}. \end{array} \qquad \begin{array}{ccc} \mathcal{F} & \xrightarrow{\epsilon} & \mathrm{G} \times \mathcal{F} \\ & {\scriptstyle \mathrm{id}} \searrow & \downarrow {\scriptstyle a} \\ & & \mathcal{F}. \end{array}$$

If we restrict ourselves to $\mathbf{G}_m$ acting on affine schemes, we actually obtain the next result whose standard reference is [DG70, Exposé 1, 4.7.3]. Let us denote by $\mathrm{Aff}^{\mathrm{B}\mathbf{G}_m}$ the category of affine schemes equipped with a $\mathbf{G}_m$-action and $\mathbf{G}_m$-equivariant morphisms. This is not a subcategory of prestacks, but admits a forgetful functor

$$\mathrm{Aff}^{\mathrm{B}\mathbf{G}_m} \to \mathbf{PStk}.$$

On the other hand a **Z-graded ring** is a ring R equipped with a decomposition:

$$\mathrm{R} = \bigoplus_{i \in \mathbf{Z}} \mathrm{R}_i$$

such that:

1. each R_j is an additive subgroup of R (in other words, the direct sum above is taken in the category of abelian groups) and,
2. the multiplication induces $\mathrm{R}_i \mathrm{R}_j \subset \mathrm{R}_{i+j}$.

We say that an element $f \in \mathrm{R}$ is a **homogeneous element of degree** n if $f \in \mathrm{R}_n$. A **graded morphism** of graded rings is just a ring homomorphism $\varphi : \mathrm{R} \to \mathrm{S}$ such that $\varphi(\mathrm{R}_j) \subset \mathrm{S}_j$. We denote by grCAlg the category of $\mathbf{Z}$-graded rings.

Theorem 4.1.2. *There is an equivalence of categories*

$$\mathrm{Aff}^{\mathrm{B}\mathbf{G}_m} \simeq (\mathrm{grCAlg})^{\mathrm{op}}.$$

Remark 4.1.3. One of the main points of Theorem 4.1.2 is that it is interesting to read from left to right and right to left. One the one hand one can use the geometric language of groups acting on a scheme/variety to encode a combinatorial/algebraic structure. On the other hand, it gives a purely combinatorial/algebraic description of a geometric idea.

Proof. First we construct a functor:

$$\mathrm{Aff}^{\mathrm{B}\mathbf{G}_m} \to (\mathrm{grCAlg})^{\mathrm{op}}.$$

Recall that the tensor product of rings is computed as the tensor product of underlying modules. Therefore we can write isomorphisms of $\mathbf{Z}$-modules:

$$\mathrm{R} \otimes_{\mathbf{Z}} \mathbf{Z}[t, t^{-1}] \cong \mathrm{R} \otimes_{\mathbf{Z}} \mathbf{Z}[\mathbf{Z}] \cong \bigoplus_{j \in \mathbf{Z}} \mathrm{R}(j).$$

Hence a $\mathbf{G}_m$-action on $\operatorname{Spec} \mathrm{R}$ is the same data as giving a map

$$\varphi : \mathrm{R} \to \bigoplus_{j \in \mathbf{Z}} \mathrm{R}(j) \qquad f \mapsto (\varphi_j(f) \in \mathrm{R}(j)),$$

satisfying certain compatibilities. We note that the direct sum indicates that the components of $(\varphi_j(f))$ is finitely supported.

Using the identity axiom we see that the composite

$$\mathrm{R} \to \bigoplus_{j \in \mathbf{Z}} \mathrm{R}(j) \xrightarrow{t \mapsto 1} \mathrm{R}$$

must be the identity. Therefore, in coordinates, we get that for any $f \in \mathrm{R}$, we get that

$$f = \sum_{j \in \mathbf{Z}} \varphi_j(f),$$

so that any f can be uniquely written as a finite sum of the $\varphi_j(f)$'s. To conclude that this defines a grading on R we need to prove that each φ_j is an idempotent. If this was proved, then the grading would be such that $f \in \mathrm{R}$ is of homogeneous degree j whenever $\varphi(f) = ft^j$.

However, this is the case by associativity of the action:

$$\begin{array}{ccc} \mathrm{R} & \xrightarrow{\quad\varphi\quad} & \bigoplus_{j \in \mathbf{Z}} \mathrm{R}(j) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ \bigoplus_{j \in \mathbf{Z}} \mathrm{R}(j) & \xrightarrow{\mu} & \bigoplus_{j \in \mathbf{Z}} \bigoplus_{k \in \mathbf{Z}} \mathrm{R}(jk). \end{array}$$

Therefore we conclude that R splits, as a $\mathbf{Z}$-module (aka abelian group) as $\mathrm{R} \cong \bigoplus_j \varphi_j \mathrm{R} (:= \mathrm{R}_j)$ and one can check that this defines a graded ring structure on R where the compatibility of multiplication originates from the fact that φ is a ring map.

In more details. let $f \in \mathrm{R}$; generically we can write $\varphi(f) = \sum_i f_i t^i$ then going to the right and down gives:

$$\varphi(\varphi(f)) = \varphi(\sum_i f_i t^i) = \sum_i \varphi(f_i) t^i u^i,$$

while going down and then left gives

$$\mu(\varphi(f)) = \mu(\sum_i f_i t^i) = \sum_i f_i \mu(t^i) = \sum_i f_i t^i u^i;$$

so that $f_i = \varphi(f_i)$.

On the other hand, given a ring R equipped with the structure of a graded ring $\mathrm{R} = \bigoplus \mathrm{R}_j$ we define a map

$$\varphi : \mathrm{R} \to \bigoplus_{j \in \mathbf{Z}} \mathrm{R}(j),$$

on the level of abelian groups as

$$\mathrm{R} \to \pi_j \mathrm{R} \subset \mathrm{R}(j) \cong \mathrm{R},$$

where π_j is the projection map. This is checked easily to define a $\mathbf{G}_m$-action and the functors are mutually equivalent.

$\square$

Example 4.1.4. There is an action of $\mathbf{G}_m$ on $\mathbf{A}^1$ that "absorbs everything to the origin"; in coordinates this is written as $t \cdot x = tx$. An exercise in this week's homework will require you to translate this to a grading.

Example 4.1.5. The best way to define new graded rings is to mod out by homogenenous polynomial equations. Recall that a polynomial $f(x_1, \cdots, x_n) \in \mathrm{R}[x_1, \cdots, x_n]$ (over any ring R) is said to be **homogeneous of degree** d if for any $r \in \mathrm{R}$ $r^d f(x_1, \cdots, x_n) = f(rx_1, \cdots, rx_n)$. The instructor never found this a useful definition; we can equivalently define this to be a linear combination of monomials of degree d, i.e.,

$$ax_1^{r_1} \cdots x_k^{r_k} \qquad \sum_{j=1}^{k} r_j = d.$$

Here is a nice visual example: consider the **quadric cone**:

$$\operatorname{Spec} \mathbf{Z}[x, y, z]/(x^2 + y^2 - z^2).$$

Since $\mathbf{Z}[x, y, z]/(x^2 + y^2 - z^2)$ is the quotient of a graded ring by a homogeneous equation, it inherits a natural grading. This defines a $\mathbf{G}_m$-action. If we replace $\mathbf{Z}$ by a field, convince yourself that this is pictorially the "absorbing" action of the cone onto its cone point.

4.2. **Complementation and open subfunctors.** In the last class we defined the descent condition and also proved that Spec R satisfies this condition. This is like choosing a basis in a vector space — we could have two covers which are specified by $\{f_i\}$ or $\{g_j\}$ and we have to say something in order to prove that descend with respect to one cover implies descent for the other.

Let us try to characterize open immersions of affine schemes in terms of its functor of points. We know that open subschemes of Spec A should be one which is the complement of a closed subscheme where the latter is of the form Spec A/I. Furthermore we know the following example:

Remark 4.2.1. If $\mathrm{I} = (f)$, then $\operatorname{Spec} \mathrm{A}/f \hookrightarrow \operatorname{Spec} \mathrm{A}$ has a complement which is actually an affine scheme given by $\operatorname{Spec} \mathrm{A}_f$. Indeed, let us attempt to unpack this: suppose that $\operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{A}$ is a morphism of affine schemes corresponding to a map of rings $\mathrm{A} \to \mathrm{R}$. We want to say that Spec R lands in the open complement of $\operatorname{Spec} \mathrm{A}/f$ which translate algebraically to the following cartesian diagram

$$\begin{array}{ccc} \mathrm{A} & \longrightarrow & \mathrm{R} \\ \downarrow & & \downarrow \\ \mathrm{A}/f & \longrightarrow & 0. \end{array}$$

This means that the map $\varphi : \mathrm{A} \to \mathrm{R}$ must satisfy: $\mathrm{R}/f\mathrm{R} = 0$ and so $f\mathrm{R} = \mathrm{R}$ which exactly means that f acts invertibly on R and hence (by homework) defines uniquely a ring map

$$\mathrm{A}_f \to \mathrm{R}.$$

To summarize our discussion:

(1) intuitively (and actually!) the closed subscheme $\operatorname{Spec} \mathrm{A}/f$ is one which is cut out by f or, in other words, the locus where f vanishes. Its complement, which is an open subscheme (if you believe in topological spaces) is the locus where f is invertible so we should take something like $\operatorname{Spec} \mathrm{A}_f$.
(2) we need a new definition to make sense of complementation of prestacks.

Let us also note the following:

Lemma 4.2.2. *For any ring f, the following diagram is cartesian*

$$\begin{array}{ccc} \operatorname{Spec} \mathrm{R}_f & \longrightarrow & \operatorname{Spec} \mathrm{R} \\ \downarrow & & \downarrow f \\ \operatorname{Spec} \mathbf{Z}[t, t^{-1}] & \longrightarrow & \mathbf{A}^1 = \operatorname{Spec} \mathbf{Z}. \end{array}$$

Let us recall that a morphism of prestacks $\mathrm{X} \to \mathrm{Y}$ is a closed immersion if for any morphism $\operatorname{Spec} \mathrm{R} \to \mathrm{Y}$ then:

(1) the prestack $\operatorname{Spec} \mathrm{R} \times_{\mathrm{Y}} \mathrm{X}$ is representable and,
(2) the morphism
$$\operatorname{Spec} \mathrm{R} \times_{\mathrm{Y}} \mathrm{X} \to \operatorname{Spec} \mathrm{R}$$
is a closed immersion.

Definition 4.2.3. Let $\mathcal{G} \subset \mathcal{F}$ be a closed immersion of prestacks. The **complement of** $\mathcal{G}$, defined by $\mathcal{F} \setminus \mathcal{G}$ is the prestack given in the following manner: a morphism $x : \operatorname{Spec} \mathrm{R} \to \mathcal{F}$ is in $(\mathcal{F} \setminus \mathcal{G})(\mathrm{R})$ if and only if the following diagram is cartesian

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathcal{G} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathrm{R} & \xrightarrow{x} & \mathcal{F} \end{array}$$

Definition 4.2.4. A morphism of prestacks $\mathcal{G} \to \mathcal{F}$ is an **open immersions** if for each $\operatorname{Spec} \mathrm{R} \to \mathcal{F}$, the map $\operatorname{Spec} \mathrm{R} \times_{\mathcal{F}} \mathcal{G} \to \operatorname{Spec} \mathrm{R}$ is an open immersions. Equivalently, it is the complement of a closed immersion.

We note that $\mathcal{F} \setminus \mathcal{G}$ is indeed a prestack because the empty scheme pullsback

Lemma 4.2.5. *Let $\mathrm{R} \in \mathrm{CAlg}$ and $\mathrm{I} \subset \mathrm{R}$ an ideal. Then there is a natural bijection between*

(1) maps $\mathrm{R} \to \mathrm{A}$ such $\mathrm{IA} = \mathrm{A}$ and,
(2) morphisms $\operatorname{Spec} \mathrm{A} \to \operatorname{Spec} \mathrm{R}$ such that

$$\begin{array}{ccc} \emptyset & \longrightarrow & \operatorname{Spec} \mathrm{R}/\mathrm{I} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathrm{A} & \xrightarrow{x} & \operatorname{Spec} \mathrm{R}. \end{array}$$

Proof. The condition of the second item says that the following diagram is cocartesian:

$$\begin{array}{ccc} \mathrm{R} & \longrightarrow & \mathrm{A} \\ \downarrow & & \downarrow \\ \mathrm{R}/\mathrm{I} & \longrightarrow & 0. \end{array}$$

which means that $\mathrm{A} \otimes_{\mathrm{R}} \mathrm{R}/\mathrm{I} = \mathrm{A}/\mathrm{IA} = 0$ which exactly means that $\mathrm{A} = \mathrm{IA}$.

□

Definition 4.2.6. A subfunctor $\mathcal{F} \to \operatorname{Spec} \mathrm{R}$ of the form in Lemma 12.0.4 is called an **open subscheme**, while $\mathcal{F}$ is called a **quasi-affine** prestack. In this case we write

$$\mathcal{F} = \mathrm{D}(\mathrm{I}).$$

If $\mathcal{F} = \operatorname{Spec} \mathrm{R}_f$, we write $\mathrm{D}(f)$.

We will soon learn how to prove that not all quasi-affine prestacks are affine.

4.3. **Exercises 4.**

Exercise 4.3.1. *Consider the action of* $\mathbf{G}_m$ *on* $\mathbf{A}^n$ *given by*

$$\mathbf{Z}[x_1, \cdots, x_n] \to \mathbf{Z}[t, t^{-1}, x_1, \cdots, x_n] \qquad x_j \mapsto t^{-k_j} x_j.$$

Calculate the induced grading on $\mathbf{Z}[x_1, \cdots, x_n]$*.*

Exercise 4.3.2. *Let* $\mathrm{R}_\bullet$ *be a graded ring which is concentrated in* $\mathbf{Z}_{\geqslant 0}$*, i.e.,* $\mathrm{R}_{<0} = 0$*. Note that:*

(1) *each* R_j *is then canonically an* R_0*-module and, in fact,* $\mathrm{R}_\bullet$ *is an* R_0*-algebra;*
(2) *the subset* $\mathrm{R}_+ := \bigoplus_{i \geqslant 1} \mathrm{R}_i \subset \mathrm{R}_\bullet$ *is an ideal*

Prove that the following are equivalent:

(1) *the ideal* R_+ *is finitely generated as an* $\mathrm{R}_\bullet$*-ideal;*
(2) $\mathrm{R}_\bullet$ *is generated as an* R_0*-algebra by finitely many homogeneous elements of positive degree.*

In the above situation we say that $\mathrm{R}_\bullet$ *is a* **finitely generated graded ring**.

Exercise 4.3.3. *Prove the following locality properties for open immersions:*

(1) *Prove that the composite of open immersions of schemes is an open immersion.*
(2) *Suppose that* $\mathrm{X} \to \mathrm{Y}$ *is a morphism of schemes and suppose that* $\mathcal{V}$ *is an Zariski cover of* X *such that for each* $\mathrm{U} \to \mathrm{X}$ *in* $\mathcal{V}$ *the map*

$$\mathrm{U} \to \mathrm{Y}$$

is an open immersion. Prove that $\mathrm{X} \to \mathrm{Y}$ *is an open immersion.*

5. Lecture 5: Schemes, actually

5.1. **Quasi-affine prestacks are Zariski stacks.**

Proposition 5.1.1. *Any quasi-affine scheme is a prestack.*

Proof. Suppose that $\mathrm{U} = \mathrm{D}(\mathrm{I}) \subset \operatorname{Spec} \mathrm{A}$ is quasi-affine. We claim that it is a Zariski stack. Let R be a test ring and let $\{f_i \in \mathrm{R}\}$ determine a basic Zariski open cover. We first prove the following claim:

- a morphism $\varphi : \mathrm{A} \to \mathrm{R}$ satisfies $\mathrm{I} \cdot \mathrm{R} = \mathrm{R}$ if and only if $\mathrm{I} \cdot \mathrm{R}_{f_i} = \mathrm{R}_{f_i}$.

Indeed, we have a short exact sequence of modules:

$$0 \to \mathrm{I} \to \mathrm{A} \to \mathrm{A}/\mathrm{I} \to 0.$$

Applying $\mathrm{R} \otimes_{\mathrm{A}} -$, we get an exact sequence

$$\mathrm{I} \otimes_{\mathrm{A}} \mathrm{R} \to \mathrm{R} \to \mathrm{R} \otimes_{\mathrm{A}} \mathrm{A}/\mathrm{I}.$$

Since $\mathrm{I} \cdot \mathrm{R}$ is the image of the left-most map, we need only prove that $\mathrm{R} \otimes_{\mathrm{A}} \mathrm{A}/\mathrm{I}$ is zero as an R-module.

By Lemma 3.1.5, we need only check that for all f_i, tensoring the above further with $\otimes_{\mathrm{R}} \mathrm{R}_{f_i}$ is zero, But the map $\mathrm{R} \to \mathrm{R}_{f_i}$ is flat. Therefore the claim follows from:

$$\mathrm{R}_{f_i} \otimes_{\mathrm{R}} \mathrm{R} \otimes_{\mathrm{A}} \mathrm{A}/\mathrm{I} = \mathrm{R}_{f_i} \otimes_{\mathrm{A}} \mathrm{A}/\mathrm{I} = 0,$$

as was assumed.

I claim that we are now done. Indeed, shorthanding the relevant equalizers as $\mathrm{Eq}(\mathcal{F})(\mathrm{R})$ we get a diagram:

$$\begin{array}{ccc} \mathrm{D}(\mathrm{I})(\mathrm{R}) & \longrightarrow & \mathrm{Eq}(\mathrm{D}(\mathrm{I}))(\mathrm{R}) \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec} \mathrm{A}(\mathrm{R}) & \xrightarrow{\cong} & \mathrm{Eq}(\operatorname{Spec} \mathrm{A})(\mathrm{R}). \end{array}$$

Since the left vertical map is injective and the bottom horizontal map is an isomorphism, the top vertical map is injective. Now, the top vertical map is also surjective by what we have proved. □

5.2. **General open covers.** We formulated a coordinate-dependent way of phrasing descent because we have used the notion of a basic Zariski cover. This is like choosing a basis for a topology which is, in turn, like choosing a basis for a vector space in linear algebra. One should not do this ever or, at least, whenever possible. We will get rid of these choices now.

Recall from last class that if $\mathrm{Spec}\,\mathrm{R} = \mathrm{X}$ is an affine scheme, then an open immersion is a morphism of prestacks where the domain is of the form $\mathrm{D}(\mathrm{I})$ where I is an ideal of R. Recall also that closed immersions of prestacks are defined by appeal to the affine case: it is a morphism $\mathcal{G} \to \mathcal{F}$ such that for every morphism $\mathrm{Spec}\,\mathrm{R} \to \mathcal{F}$, the pullback $\mathrm{Spec}\,\mathrm{R} \times_{\mathcal{F}} \mathcal{G}$ is representable by a scheme and the morphism to $\mathrm{Spec}\,\mathrm{R}$ is indeed a closed immersion which is specified by an ideal.

Definition 5.2.1. Let $\mathcal{F}$ be a prestack. Then an **open subprestack** of $\mathcal{F}$ or an **open immersion of prestacks** is a morphism $\mathcal{G} \to \mathcal{F}$ such that for any morphism $\mathrm{Spec}\,\mathrm{R} \to \mathcal{F}$, the morphism $\mathrm{Spec}\,\mathrm{R} \times_{\mathrm{Y}} \mathrm{X} \to \mathrm{Spec}\,\mathrm{R}$ is an open subscheme. In other words, $\mathrm{Spec}\,\mathrm{R} \times_{\mathrm{Y}} \mathrm{X}$ is of the form $\mathrm{D}(\mathrm{I})$ where I is an ideal of R.

Since we have plenty of examples of open immersions of affine schemes which are not themselves affine, we do not want the representability condition which we saw was imposed in the closed immersion case. The next lemma is left as an exercise.

Lemma 5.2.2. *Let $\mathcal{Z} \hookrightarrow \mathcal{F}$ be an closed immersion, then the complement $\mathcal{F} \smallsetminus \mathcal{Z}$ is canonically an open immersion.*

Definition 5.2.3. A **Zariski cover** of an affine scheme $\mathrm{X} = \mathrm{Spec}\,\mathrm{A}$ is a collection of open embeddings of prestacks

$$\mathcal{U} = \{\mathrm{U} \hookrightarrow \mathrm{X}\},$$

such that for any nonzero ring R, $\mathrm{S} = \mathrm{Spec}\,\mathrm{R}$ with a map $\mathrm{S} \to \mathrm{X}$ there exists a $\mathrm{U} \hookrightarrow \mathrm{Spec}\,\mathrm{A} \in \mathcal{U}$ such that

$$\mathrm{U} \times_{\mathrm{X}} \mathrm{S} \neq \emptyset.$$

At this point nothing then stops us from defining Zariski covers of any prestack:

Definition 5.2.4. A **Zariski cover** of an prestack $\mathcal{F}$ is a collection of open embeddings of prestacks

$$\mathcal{U} = \{\mathrm{U} \hookrightarrow \mathcal{F}\},$$

such that for any nonzero ring R, $\mathrm{S} = \mathrm{Spec}\,\mathrm{R}$ with a map $\mathrm{S} \to \mathcal{F}$ there exists a $\mathrm{U} \hookrightarrow \mathcal{F} \in \mathcal{U}$ such that

$$\mathrm{U} \times_{\mathcal{F}} \mathrm{S} \neq \emptyset.$$

Definition 5.2.5. A prestack $\mathcal{F} : \mathrm{CAlg} \to \mathrm{Set}$ is a **(Zariski) stack** (resp. **(Zariski) stack for the Zariski cover** $\mathcal{U}$) if for all $\mathrm{A} \in \mathrm{CAlg}$ and all (resp. the) Zariski cover $\mathcal{U} := \{\mathrm{U}_i \to \mathrm{Spec}\,\mathrm{A}\}_{i \in \mathrm{I}}$ the diagram

$$\mathcal{F}(\mathrm{Spec}\,\mathrm{A}) \to \prod \mathrm{Hom}(\mathrm{U}_i, \mathcal{F}) \rightrightarrows \prod_{i_0, i_1} \mathrm{Hom}(\mathrm{U}_{i_0} \times_{\mathrm{Spec}\,\mathrm{A}} \mathrm{U}_{i_1}, \mathcal{F})$$

is an equalizer diagram where the maps are induced are the obvious ones.

Remark 5.2.6. It is a bit dangerous to write $\mathcal{F}(\mathrm{U}_i)$ since U_i is not known (and will be known not to be) an affine scheme. So we will stick with the notation $\mathrm{Hom}(\mathrm{U}_i, \mathcal{F})$ in these notes, though the instructor does lapse to writing $\mathcal{F}(\mathrm{U}_i)$. This notation will be justified later on.

From this definition, our previous definition of a scheme asks that $\mathcal{F}$ is a Zariski stack for covers $\mathcal{U}$ which are made of Zariski open covers. Another exercise in unpacking definitions:

Lemma 5.2.7. *A basic Zariski cover is a Zariski cover. In particular if $\mathcal{F}$ is a Zariski stack, then it is a Zariski stack with respect to basic open covers.*

This tells us that Definition 5.2.4 is stronger than Definition 3.0.7. In some sense Definition 5.2.4 is preferable — it affords the flexibility of working with covers where the opens are not necessarily affine. We will work towards proving the equivalence of these two definitions shortly. More precisely:

Theorem 5.2.8. *Suppose that $\mathcal{F} : \mathrm{CAlg} \to \mathrm{Set}$ is a functor. Then the following are equivalent:*

(1) $\mathcal{F}$ is a Zariski stack in the sense of Definition 5.2.4.
(2) $\mathcal{F}$ is a Zariski stack in the sense of Definition 3.0.7.

5.3. **Quasicompactness of affine schemes.** First let us consider the different ways we can think about Zariski covers.

Remark 5.3.1. One of the weird things about algebraic geometry is how large open sets are. For example, consider $\mathbf{R}$ with the usual topology. Then there exists many open covers of $\mathbf{R}$ with no finite subcovers. Now $\mathbf{R} = \mathrm{Spec}\,\mathbf{A}^1(\mathbf{R})$ but, as an affine scheme $\mathrm{Spec}\,\mathbf{A}^1$ is, in a precise way, compact.

We call the next lemma *quasicompactness of affine schemes*.

Lemma 5.3.2. *Let $\mathrm{X} = \mathrm{Spec}\,\mathrm{A}$, and $\mathcal{U} = \{\mathrm{U} \hookrightarrow \mathrm{Spec}\,\mathrm{A}\}$ is a collection of open immersions. then the following are equivalent:*

(1) $\mathcal{U}$ is a Zariski cover.
(2) $\mathcal{U}$ has a finite subset $\mathcal{V}$ which is also an open cover.
(3) for any field k and any map $x : \mathrm{Spec}\,k \to \mathrm{X}$ there exists an $\mathrm{U} \in \mathcal{U}$ such that x factors through U.

Proof. The implication (1) ⇒ (2) comes under the term "affine schemes are **quasicompact**" which is one way in which open subsets look very different in algebraic geometry than what you have experienced before. To prove this, we note that giving a Zariski open cover of an affine scheme is to give a collection of ideals $\{\mathrm{I}_j\}$ such that

$$1 \in \sqrt{\sum \mathrm{I}_j}. \tag{5.3.3}$$

Indeed, we know that each element of $\mathcal{U}$ is of the form $\mathrm{D}(\mathrm{I}_j) \hookrightarrow \mathrm{Spec}\,\mathrm{A}$. Suppose that there exists a cover $\mathcal{U}$ for which (5.3.3) is not satisfied. Then, consider the ring:

$$\mathrm{B} := \mathrm{A}/(\sqrt{\Sigma \mathrm{I}_j}),$$

which induces a morphism $\mathrm{Spec}\,\mathrm{B} \to \mathrm{Spec}\,\mathrm{A}$. By assumption $\mathrm{B} \neq 0$, however for no j do we have $\mathrm{Spec}\,\mathrm{B} \times_{\mathrm{Spec}\,\mathrm{A}} \mathrm{D}(\mathrm{I}_j) \neq \emptyset$, contradicting the condition to be an open cover.

Continuing with the proof, from the definition of sums of ideals, there exists a finite subcollection $i_0, \cdots, i_k$ such that

$$1 \in \sqrt{\sum_{0 \leqslant s \leqslant k} \mathrm{I}_{i_s}}.$$

This in turn means that we can refine the above open cover by

$$\{\mathrm{D}(\mathrm{I}_{i_s}) \to \mathrm{Spec}\,\mathrm{A}\}_{0 \leqslant s \leqslant k}.$$

Let us prove (2) ⇒ (3). Given a morphism into a field $\varphi : \mathrm{A} \to k$, we want to find $\mathrm{U} \hookrightarrow \mathrm{X} \in \mathcal{V}$ such that x factors through U. Suppose that there is none, then $\varphi(\Sigma \mathrm{I}_j) = 0$, which means that $\varphi(1) = 0$ but this is not possible.

Let us prove (3) ⇒ (1). Suppose that R is a nonzero ring with a map $\mathrm{Spec}\,\mathrm{R} \to \mathrm{Spec}\,\mathrm{A}$ so that we have a morphism $\varphi : \mathrm{A} \to \mathrm{R}$. Since R is nonzero, it has a maximal ideal $\mathfrak{m}$ so that $\mathrm{R}/\mathfrak{m} = \kappa$ a field. By hypothesis, there exists a $\mathrm{U} \in \mathcal{U}$ such that $\mathrm{U} \times_{\mathrm{X}} \mathrm{Spec}\,\kappa$ is nonempty, but this also means that $\mathrm{U} \times_{\mathrm{X}} \mathrm{Spec}\,\mathrm{R}$ is nonempty since there is a morphism $\mathrm{U} \times_{\mathrm{X}} \mathrm{Spec}\,\kappa \to \mathrm{U} \times_{\mathrm{X}} \mathrm{Spec}\,\mathrm{R}$. □

We make this a definition:

Definition 5.3.4. A prestack $\mathcal{F}$ is said to be **quasicompact** if there exists an Zariski cover $\mathcal{U}$ of $\mathcal{F}$ such that $\mathcal{U}$ consists of a *finite* collection of affine schemes.

This provides the first mechanism by which a scheme can be non-affine.

Definition 5.3.5. Suppose that R, S are rings, we define the **scheme-theoretic coproduct** or the **coproduct in schemes** as

$$\operatorname{Spec} \mathrm{R} \sqcup \operatorname{Spec} \mathrm{S} := \operatorname{Spec} \mathrm{R} \times \mathrm{S}.$$

This affine scheme equipped with maps

$$\operatorname{Spec} \mathrm{R} \hookrightarrow \operatorname{Spec} \mathrm{R} \sqcup \operatorname{Spec} \mathrm{S} \hookleftarrow \operatorname{Spec} \mathrm{S}.$$

There is an obvious guess for what a coproduct of schemes could be. However, this latter notion is bad as it does not have Zariski descent — you will be asked to address this in the exercises.

Lemma 5.3.6. *Suppose that we have a diagram $\mathbf{N} \to \mathbf{PStk}, i \mapsto \mathcal{F}_i$ such that for each i the prestack $\mathcal{F}_i$ is a Zariski stack. Then $\operatorname{colim}_i \mathcal{F}_i$ is a Zariski stack.*

Proof. This follows from the fact that colimits in **PStk** are computed pointwise (as was proved in the previous exercise) and $\mathbf{N}$-indexed colimits commutes with finite limits (or, more generally, filtered colimits commute with finite limits) which is in this week's problem set. ☐

Lemma 5.3.7. *Let $(\mathrm{A}_i)_{i \in \mathbf{N}}$ be a collection of nonzero rings and consider a countable product $\coprod_{\mathbf{N}} \operatorname{Spec} \mathrm{A}_i$. This prestack (which in fact a Zariski stack) is not an affine scheme.*

Proof. We first note that $\mathrm{X} := \coprod \operatorname{Spec} \mathrm{A}_i = \operatorname{colim}_{n \to \infty} \coprod_{i=1}^n \operatorname{Spec} \mathrm{A}_i$ so the previous lemma asserts that $\coprod \operatorname{Spec} \mathrm{A}_i$ is a Zariski stack. Consider the collection of maps

$$\{\operatorname{Spec} \mathrm{A}_i \to \coprod \mathrm{X}\}$$

I claim that $\operatorname{Spec} \mathrm{A}_j \to \mathrm{X}$ is an open immersion for each fixed j. Indeed, suppose that we have a map $\operatorname{Spec} \mathrm{R} \to \mathrm{X}$. In this case we note that:

$$\mathrm{X}(\mathrm{R}) = (\operatorname*{colim}_{n \to \infty} \coprod_{i=1}^n \operatorname{Spec} \mathrm{A}_i)(\mathrm{R}) = \operatorname*{colim}_{n \to \infty} (\coprod_{i=1}^n \operatorname{Spec} \mathrm{A}_i)(\mathrm{R}) = \operatorname*{colim}_{n \to \infty} \operatorname{Hom}(\prod_{i=1}^n \mathrm{A}_i, \mathrm{R}),$$

which means that the map $\operatorname{Spec} \mathrm{R} \to \mathrm{X}$ corresponds to a map $\prod_{i=1}^n \mathrm{A}_i \to \mathrm{R}$ or, equivalently, a map $\operatorname{Spec} \mathrm{R} \to \coprod_{i=1}^n \operatorname{Spec} \mathrm{A}_i$ or, in other words, the map $\operatorname{Spec} \mathrm{R} \to \mathrm{X}$ factors through a finite stage of the colimit. Therefore the pullback is computed as

$$\operatorname{Spec} \mathrm{A}_j \times_{\mathrm{X}} \operatorname{Spec} \mathrm{R} = \operatorname{Spec} \mathrm{A}_j \times_{\coprod_{i=1}^n \operatorname{Spec} \mathrm{A}_i} \operatorname{Spec} \mathrm{R}.$$

But, by this week's exercise we know that (1) $\operatorname{Spec} \mathrm{A}_j \to \coprod_{i=1}^n \operatorname{Spec} \mathrm{A}_i$ is an open immersion (since summand inclusions are open immersions) and (2) open immersions are stable under pullbacks and therefore, $\operatorname{Spec} \mathrm{A}_j \times_{\coprod_{i=1}^n \operatorname{Spec} \mathrm{A}_i} \operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{R}$ is an open immersion.

To conclude that the desired collection of maps are covers, we just need to check on k-points and invoke Lemma 5.3.2. To conclude, note that there is no refinement of this subcover.

☐

5.4. **The definition of a scheme.** Finally:

Definition 5.4.1. A prestack X is a **scheme** if:

(1) it is a Zariski stack, and
(2) there exists an Zariski cover $\mathcal{U}$ of X such that $\mathcal{U}$ consists of affine schemes.

So far we have seen that affine schemes are schemes; they are furthermore quasicompact schemes. We have also seen that infinite coproducts of schemes are schemes which are not quasicompact. We will prove that quasi-affine schemes are schemes in the next lecture.

5.5. **Exercises 5.** Let Eq be the category displayed as $\bullet \rightrightarrows \bullet$.

Exercise 5.5.1. *Suppose that we have a diagram* $\mathrm{F} : \mathbf{N} \times \mathrm{Eq} \to \mathrm{Set}$. *On the one hand we can view this as*

$$\mathbf{N} \to \mathrm{Fun}(\mathrm{Eq}, \mathrm{Set})$$

and take

$$\operatorname*{colim}_{\mathbf{N}} \lim_{\mathrm{Eq}} \mathrm{F}$$

or we can view this as

$$\mathrm{Eq} \to \mathrm{Fun}(\mathbf{N}, \mathrm{Set})$$

and take

$$\lim_{\mathrm{Eq}} \operatorname*{colim}_{\mathbf{N}} \mathrm{F}.$$

Construct a natural map between the two sets and prove that they are isomorphic. Conclude the same commutation property for **PStk** *in place of* Set. *This is the first instance of an unexpected commutation of limits versus a colimit that you have seen in this class.*

Exercise 5.5.2. *Prove that open immersions are stable under pullbacks: suppose that* $\mathcal{U} \to \mathcal{F}$ *is an open immersion of prestacks, then for each map* $\mathcal{G} \to \mathcal{F}$, *the induced map of prestacks* $\mathcal{G} \times_{\mathcal{F}} \mathcal{U} \to \mathcal{G}$ *is an open immersion. Prove the same property for closed immersions.*

Exercise 5.5.3. *Here is a simple algebra fact that you should know: let* R *be a ring and* e *is an idempotent. Then prove that we have a decomposition* $e\mathrm{R} \oplus (1-e)\mathrm{R} \cong \mathrm{R}$ *in the category of abelian groups, while neither* $e\mathrm{R}$ *nor* $(1-e)\mathrm{R}$ *are subrings unless* e *is zero or* 1. *However note that the projections* $\mathrm{R} \to e\mathrm{R}$ *and* $\mathrm{R} \to (1-e)\mathrm{R}$ *are ring homomorphisms.*

Exercise 5.5.4. *Let* R, S *be rings and consider a test ring* A, *then consider the map*

$$\mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{Spec}\,\mathrm{R}) \sqcup \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{Spec}\,\mathrm{S}) \to \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{Spec}\,\mathrm{R} \times \mathrm{S}),$$

given on one factor of the coproduct by

$$g : \mathrm{R} \to \mathrm{A} \mapsto \mathrm{R} \times \mathrm{S} \to \mathrm{R} \xrightarrow{g} \mathrm{A},$$

$$f : \mathrm{S} \to \mathrm{A} \mapsto \mathrm{R} \times \mathrm{S} \to \mathrm{S} \xrightarrow{f} \mathrm{A}.$$

Prove:

1. *if* A *has no nontrivial idempotent, then the map is an isomorphism.*
2. *if* A *has nontrivial idempotent, then the map is not an isomorphism in general.*
3. *Prove that, in this case, the functor*

$$\mathrm{X}(\mathrm{A}) = \begin{cases} \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{Spec}\,\mathrm{R}) \sqcup \mathrm{Hom}(\mathrm{Spec}\,\mathrm{A}, \mathrm{Spec}\,\mathrm{S}) & \mathrm{A} \neq 0 \\ * & \textit{else}, \end{cases}$$

   is not *a Zariski stack.*

Definition 5.5.5. We say that a scheme X is **connected** if it is not the empty scheme and for all 2-fold open covers $\mathcal{U} = \{\mathrm{U}, \mathrm{V} \hookrightarrow \mathrm{X}\}$ such that $\mathrm{U} \times_{\mathrm{X}} \mathrm{V} = \emptyset$, then $\mathrm{U} = \mathrm{X}$ or $\mathrm{V} = \mathrm{X}$.

Exercise 5.5.6. *Prove that any Zariski stack converts a product of rings to a product of sets, i.e., if* $\mathrm{A} \cong \mathrm{B} \times \mathrm{C}$ *then the map*

$$\mathcal{F}(\mathrm{B} \times \mathrm{C}) \to \mathcal{F}(\mathrm{B}) \times \mathcal{F}(\mathrm{C}),$$

is an isomorphism whenever $\mathcal{F}$ *is a Zariski stack. Conclude that Zariski stack converts finite coproducts of affine schemes to finite products.*

Exercise 5.5.7. *Prove that:*

1. *an affine scheme* $\mathrm{X} = \mathrm{Spec}\,\mathrm{R}$ *is connected if and only if* R *has no nontrivial idempotent,*
2. *if* e *is a nontrivial idempotent and* $\mathrm{R} = e\mathrm{R} \times (1-e)\mathrm{R}$ *prove that the map* $\mathrm{R} \to e\mathrm{R}$ *induces an map* $\mathrm{Spec}\, e\mathrm{R} \to \mathrm{Spec}\,\mathrm{R}$ *which is both an open and a closed immersion.*

(3) let R *be a noetherian ring. Prove that we can write*

$$\mathrm{R} \simeq \prod_{\mathrm{I}} \mathrm{R}_i$$

where each R_i *is nonzero and has no nontrivial idempotent and* I *is a finite set.*

Exercise 5.5.8. *The following is a generalization of Exercise 5.5.4. First we note that in point-set topology if* $\mathrm{U} \hookrightarrow \mathrm{X}$ *is a clopen subset of a topological space* X*, then* U *is a union of connected components. So we can, in topological spaces, write* $\mathrm{X} = \mathrm{U} \sqcup \mathrm{X} \smallsetminus \mathrm{U}$ *where* $\mathrm{X} \smallsetminus \mathrm{U}$ *is again clopen, whence also a union of its connected components. But we have seen that the operation of coproduct in schemes is not taken set-wise. The following exercise, however, still proves that we have a coproduct decomposition for clopen subfunctors:*

Let R *be a ring and suppose that*

$$\mathrm{U} \hookrightarrow \operatorname{Spec} \mathrm{R}$$

is both an open and closed subfunctor (in other words **clopen***). Note that this means that we also know that* $\operatorname{Spec}\mathrm{R} \smallsetminus \mathrm{U}$ *is quasi-affine.*

1. *suppose that* $\{\operatorname{Spec}\mathrm{R}_{f_i} \hookrightarrow \mathrm{U}\}, \{\operatorname{Spec}\mathrm{R}_{g_j} \hookrightarrow \operatorname{Spec}\mathrm{R} \smallsetminus \mathrm{U}\}$ *are open covers, which we can choose to be finite since we already know that quasi-affines are quasicompact. Prove that each* $f_i g_j$ *is nilpotent.*
2. *Prove that there exists an* N *such that* $\mathrm{I}^{\mathrm{N}} + \mathrm{J}^{\mathrm{N}} = \mathrm{R}$ *so that we can write* $1 = x + y$ *where* $x \in \mathrm{I}^{\mathrm{N}}, y = \mathrm{J}^{\mathrm{N}}$ *and prove that* x *and* y *are idempotents.*
3. *Let* $\mathrm{J} = (f_1, \cdots, f_n), \mathrm{I} = (g_1, \cdots, g_m)$*. Prove that* $\operatorname{Spec}\mathrm{R}/\mathrm{I}^{\mathrm{N}} \cong \mathrm{U}, \operatorname{Spec}\mathrm{R}/\mathrm{J}^{\mathrm{M}} \cong \operatorname{Spec}\mathrm{R} \smallsetminus \mathrm{U}$ *for* M, N *large enough. Conclude that* $\operatorname{Spec}\mathrm{R} = \operatorname{Spec}\mathrm{R}/\mathrm{I} \sqcup \operatorname{Spec}\mathrm{R}/\mathrm{J}$.
4. *Conclude that if* $\mathrm{U} \hookrightarrow \operatorname{Spec}\mathrm{R}$ *is an open and closed subfunctor then*

$$\operatorname{Spec}\mathrm{R} \cong \mathrm{U} \sqcup \mathrm{V},$$

where U, V *are affine schemes (remember what we mean by coproducts!).*

5. *Conclude the same for arbitrary schemes: if* $\mathrm{U} \hookrightarrow \mathrm{X}$ *is a clopen subscheme then*

$$\mathrm{X} \cong \mathrm{U} \sqcup \mathrm{V}.$$

6. *On the other hand, prove the following: if* e *is idempotent, then* $\operatorname{Spec}\mathrm{R}/e \hookrightarrow \operatorname{Spec}\mathrm{R}$ *is a clopen embedding.*

Exercise 5.5.9. *Let* $\mathbf{A}^{\infty} := \operatorname{Spec}\mathbf{Z}[x_1, x_2, \cdots, x_n, \cdots]$*. This an affine scheme hence quasicompact. Is the quasi-affine scheme* $\mathbf{A}^{\infty} \smallsetminus 0$ *quasicompact?*

6. Lecture 6: Quasi-affine schemes, a dévissage in action

Last time, we wanted to make more examples of schemes so we want to claim that quasi-affine schemes are schemes. We have already proved that quasi-affine schemes are Zariski stacks. It remains to produce a Zariski cover.

Lemma 6.0.1. *Let* A *be a ring and* I *and ideal of* A*. Then, the collection* $\{\operatorname{Spec}\mathrm{A}_{f_i} \to \mathrm{D(I)} : f_i \in \mathrm{I} \smallsetminus 0\}$ *is a Zariski open cover of the quasi-affine scheme* D(I)*. In particular* D(I) *is a scheme.*

Proof. This gives me an opportunity to compute pullbacks and show you how one can make computations of how some pullbacks look like. We want to compute

$$\operatorname{Spec}\mathrm{A}_{f_i} \times_{\operatorname{Spec}\mathrm{A}} \mathrm{D(I)}.$$

Recall that pullbacks are computed pointwise and therefore we want to understand explicitly how this prestack looks like when one maps into a ring R or, in other words, if one maps Spec R in. So let R be such a ring and, by definition, the following diagram is cartesian (in sets)

$$\begin{array}{ccc} (\operatorname{Spec} \mathrm{A}_{f_i} \times_{\operatorname{Spec} \mathrm{A}} \mathrm{D}(\mathrm{I}))(\mathrm{R}) & \longrightarrow & \{\mathrm{A} \to \mathrm{R} : \mathrm{IR} = \mathrm{R}\} \\ \downarrow & & \downarrow \\ \{\mathrm{A}_{f_i} \to \mathrm{R}\} & \longrightarrow & \{\mathrm{A} \to \mathrm{R}\}. \end{array}$$

Now we see that the pullback is given by the subset of maps $\varphi : \mathrm{A}_{f_i} \to \mathrm{R}$ such that if we precompose with $\mathrm{A} \to \mathrm{A}_{f_i}$ we have that $\mathrm{IR} = \mathrm{R}$. Now, assume that $f_i \in \mathrm{I} \smallsetminus 0$. I claim that this last condition is no condition at all. Indeed, suppose that $\varphi : \mathrm{A}_{f_i} \to \mathrm{R}$ is such a map, then given any $r \in \mathrm{R}$ we can write

$$r = \varphi(f_i)/\varphi(f_i)^{-1} r = \varphi(f_i) r'$$

which exactly means that $\mathrm{IR} = \mathrm{R}$. Therefore we conclude that

$$\operatorname{Spec} \mathrm{A}_{f_i} \times_{\operatorname{Spec} \mathrm{A}} \mathrm{D}(\mathrm{I}) \cong \operatorname{Spec}_{\mathrm{A}_{f_i}}.$$

We also know that open immersions are stable under pullbacks and therefore the map

$$\operatorname{Spec}_{\mathrm{A}_{f_i}} \hookrightarrow \mathrm{D}(\mathrm{I}).$$

is open since the map $\operatorname{Spec} \mathrm{A}_{f_i} \to \operatorname{Spec} \mathrm{A}$ is.

To prove the covering condition. Let $\operatorname{Spec} \mathrm{R} \to \mathrm{D}(\mathrm{I})$ be nonzero. We consider the diagram where each square is cartesian.

$$\begin{array}{ccc} \operatorname{Spec} \mathrm{R} \times_{\mathrm{D}(\mathrm{I})} \operatorname{Spec} \mathrm{A}_{f_i} & \longrightarrow & \operatorname{Spec} \mathrm{R} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathrm{A}_{f_i} & \longrightarrow & \mathrm{D}(\mathrm{I}) \\ \downarrow \cong & & \downarrow \\ \operatorname{Spec} \mathrm{A}_{f_i} & \longrightarrow & \operatorname{Spec} \mathrm{A}. \end{array}$$

Our goal is to find an f_i for which the top left corner is nonempty. But now, we note that the resulting rectangle is cartesian whenever $f_i \in \mathrm{I} \smallsetminus 0$ since the bottom square is from our previous computation. But by assumption on $\mathcal{U}$, we can indeed find such an f_i.

□

6.1. **Universality of descent and dévissage.** This is a technical section which proves the following result.

Theorem 6.1.1. *Suppose that $\mathcal{F} : \mathrm{CAlg} \to \mathrm{Set}$ is a functor. Then the following are equivalent:*

(1) $\mathcal{F}$ is a Zariski stack in the sense of Definition 5.2.4.
(2) $\mathcal{F}$ is a Zariski stack in the sense of Definition 3.0.7.

Of course the direction (1)⇒(2) is immediate. We note that Theorem 6.1.1 is akin to a maneuver in point-set topology where we say that everything we want to know about the topology of a space is basically determined by the basic opens. We will prove this in a rather fancy way, but the crux is the next lemma:

Lemma 6.1.2. *Any open cover $\mathcal{U}$ of an affine scheme $\operatorname{Spec} \mathrm{A}$ admits a refinement by basic Zariski covers. More precisely: given $\mathcal{U}$ a Zariski open cover of $\operatorname{Spec} \mathrm{A}$, there exists a basic Zariski open cover $\mathcal{V} := \{\mathrm{A} \to \mathrm{A}_{f_i}\}$ with the property that for each $\mathrm{U} \in \mathcal{U}$, the set*

$$\mathcal{V}_{\mathrm{U}} := \{\operatorname{Spec} \mathrm{A}_{f_i} \in \mathcal{V} : \operatorname{Spec} \mathrm{A}_{f_i} \hookrightarrow \mathrm{U}\}$$

is a Zariski open cover of U.

Proof. By Lemma 12.0.4, $\mathcal{U} = \{D(I_\alpha)\}_\alpha$. By the proof of Lemma 5.3.2 we have that

$$A = \sqrt{\sum I_\alpha}.$$

From this, extract the set

$$\{f : \exists \alpha, f \in I_\alpha \setminus 0\}$$

Then $\{\operatorname{Spec} A_f\}$ is the desired refinement after Lemma 6.0.1. □

We now begin the proof. Our first goal is to observe that if $\mathcal{F}$ is a Zariski stack in the sense of Definition 3.0.7, then there is an extension of $\mathcal{F}$ to the category of quasi-affine schemes:

Definition 6.1.3. Let QAff to be the full subcategory of **PStk** spanned by affine schemes and quasi-affine schemes. The objects of this category are prestacks of the form D(I) for some ideal $I \subset A$ is a ring A.

To see, this we make the following observation. Suppose that $D(I) \hookrightarrow \operatorname{Spec} A$ is a quasi-affine. Consider open immersions

$$\operatorname{Spec} B, \operatorname{Spec} C \hookrightarrow D(I).$$

Then we have a map

$$\operatorname{Spec} B \times_{D(I)} \operatorname{Spec} C \to \operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} C.$$

Lemma 6.1.4. *This is an isomorphism.*

Proof. According to Exercise 2.3.12, we have the following cartesian diagram

$$\begin{array}{ccc} \operatorname{Spec} B \times_{D(I)} \operatorname{Spec} C & \longrightarrow & \operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} C \\ \downarrow & & \downarrow \\ D(I) & \xrightarrow{\quad \Delta \quad} & D(I) \times_{\operatorname{Spec} A} D(I). \end{array}$$

Now, Δ is an isomorphism since D(I) is open (exercise!), hence the claim follows. □

In particular: $\operatorname{Spec} B \times_{D(I)} \operatorname{Spec} C$ turns out to be affine. Using this observation, we will extend $\mathcal{F} : \mathrm{CAlg} \to \mathrm{Set}$.

Construction 6.1.5. Suppose that $\mathcal{F}$ is a Zariski stack and D(I) is quasi-affine. For any basic open cover of $\mathcal{U} = \{\operatorname{Spec} A_i \hookrightarrow D(I)\}$ define:

$$\widetilde{\mathcal{F}}(D(I)) := \mathrm{Eq}(\prod_i \mathcal{F}(\operatorname{Spec} A_i) \rightrightarrows \prod_{i,j} \mathcal{F}(\operatorname{Spec} A_i \times_{D(I)} \operatorname{Spec} A_j)$$

Lemma 6.1.6. *The above construction gives a well-defined functor*

$$\widetilde{\mathcal{F}} : \mathrm{QAff}^{\mathrm{op}} \to \mathrm{Set}.$$

This is kind of tedious, but you can imagine its proof

Proof Sketch. Given two covers $\mathcal{U} = \{\operatorname{Spec} A_i \to D(I)\}$, $\mathcal{V} = \{\operatorname{Spec} B_j \to D(I)\}$ then

1. for each i we have that $\{\operatorname{Spec} B_j \times_{D(I)} \operatorname{Spec} A_i \to \operatorname{Spec} A_i\}$ is a cover of $\operatorname{Spec} A_i$ and
2. for each i we have that $\{\operatorname{Spec} A_i \times_{D(I)} \operatorname{Spec} A_i \to \operatorname{Spec} B_j\}$ is a cover of $\operatorname{Spec} B_j$.

With this we can rewrite

$$\mathrm{Eq}(\prod_i \mathcal{F}(\operatorname{Spec} A_i) \rightrightarrows \prod_{i,j} \mathcal{F}(\operatorname{Spec} A_i \times_{D(I)} \operatorname{Spec} A_j))$$

as products of equalizers involving B_j's and vice versa. The proof follows from the fact that equalizers and products are both limits and hence they commute. □

The next concept is the key to proving a result like Theorem 6.1.1

Definition 6.1.7. Fix a prestack

$$\mathcal{F} : \mathrm{QAff}^{\mathrm{op}} \to \mathrm{Set}.$$

We say that a morphism of $\mathrm{X} \to \mathrm{Y}$ in QAff is of **$\mathcal{F}$-descent** if

$$\mathcal{F}(\mathrm{Y}) \to \mathcal{F}(\mathrm{X}) \rightrightarrows \mathcal{F}(\mathrm{X} \times_{\mathrm{Y}} \mathrm{X}),$$

is an equalizer diagram. It is **of universal $\mathcal{F}$-descent** if for any map $\mathrm{T} \to \mathrm{Y}$,

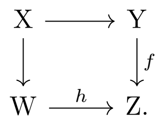
$$\mathcal{F}(\mathrm{T}) \to \mathcal{F}(\mathrm{T} \times_{\mathrm{Y}} \mathrm{X}) \rightrightarrows \mathcal{F}(\mathrm{T} \times_{\mathrm{Y}} (\mathrm{X} \times_{\mathrm{Y}} \mathrm{X})).$$

is an equalizer diagram, i.e., the map $\mathrm{X} \times_{\mathrm{Y}} \mathrm{T} \to \mathrm{T}$ is also of **$\mathcal{F}$-descent**.

The next lemma summarizes why this definition is a good one:

Lemma 6.1.8. *Let $f : \mathrm{Y} \to \mathrm{Z}, g : \mathrm{X} \to \mathrm{Y}$ be a morphism in* QAff*. Then*

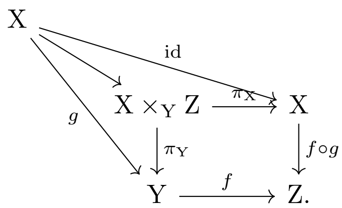
(1) morphisms of universal $\mathcal{F}$-descent are stable under base change.
(2) it f admits a section, then it is of universal $\mathcal{F}$-descent.
(3) suppose that we have a cartesian diagram



Suppose that (1) the base change of h to Y *and to $\mathrm{Y} \times_{\mathrm{Z}} \mathrm{Y}$ and (2) the base change of f to $\mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}$ are of universal $\mathcal{F}$-descent. Then f is of universal $\mathcal{F}$-descent.*

(4) If f, g are of universal $\mathcal{F}$-descent then so is their composite.
(5) If $f \circ g$ is universal $\mathcal{F}$-descent, then so is f.
(6) If $\mathcal{F}$ converts coproducts to products, morphisms of the form $\sqcup_{i=1}^{n} \mathrm{Spec}\, \mathrm{B}_i \to \mathrm{Spec}\, \mathrm{A}$ is of universal $\mathcal{F}$-descent.

Proof sketch. Let us prove (5), assuming (1-3). We have a diagram



This proves that π_{X} is a retraction and hence is of universal $\mathcal{F}$-descent by (2). In particular, the base change of f along $h := f \circ g$ is of universal $\mathcal{F}$-descent and so are its base changes along $\mathrm{X} \times_{\mathrm{Z}} \mathrm{X} \rightrightarrows \mathrm{X}$ using (2). Now, by assumption $h := f \circ g$ is of universal $\mathcal{F}$-descent so that its base change along f is again of universal $\mathcal{F}$-descent and hence its base changes along $\mathrm{Y} \times_{\mathrm{Z}} \mathrm{Y} \rightrightarrows \mathrm{Z}$, again using (2). Using (3), we conclude that f is universal $\mathcal{F}$-descent. □

Remark 6.1.9. Warning: this is not meant to be a technical remark. A lot of statements in algebraic geometry are proved by proving closure properties for that statement and then demonstrating a base case. This kind of argument style is called *dévissage* (which is the French word for "unscrewing"). Often this kind of argument can be replaced by a more ad hoc one which one proves by hand but we have chosen to give a demonstration of how this general principle can work.

Here is a sharper formulation of Theorem 6.1.1

Theorem 6.1.10. *Suppose that $\mathcal{F} : \mathrm{QAff}^{\mathrm{op}} \to \mathrm{Set}$ is a prestack. Assume that:*

(1) $\mathcal{F}$ converts finite coproducts to products,
(2) $\mathcal{F}|_{\mathrm{Aff}}$ is a Zariski stack.

Then $\mathcal{F}$ is a Zariski stack in the sense of Definition 5.2.4.

Proof. Suppose that $\mathcal{U}$ is a Zariski cover of $\operatorname{Spec} R$. By the quasicompactness of $\operatorname{Spec} R$, we can assume that $\mathcal{U}$ consists of a finite collection of morphisms. In this case, consider the map in QAff:

$$f : \coprod_{i=1}^{n} U_i \to \operatorname{Spec} R,$$

noting that quasi-affine schemes admit finite coproducts. We claim that f is of universal $\mathcal{F}$-descent if $\mathcal{F}$ satisfies (1) and (2). Now, by Lemma 6.1.2, we can find a (finite) refinement of the $\mathcal{U}$ consisting of affine schemes so that we have a sequence of composable morphisms

$$\coprod_{i,j} V_{ij} \to \coprod_{i=1}^{n} U_i \to \operatorname{Spec} R.$$

By Lemma 6.1.8.5, the composite is of universal $\mathcal{F}$-descent, which means we do know that the original map is too using Lemma 6.1.8.3. □

6.2. **Exercises 6.**

Definition 6.2.1. Recall that a morphism $f : X \to Y$ is **monic** if for any $g, h : Z \to X$ and $fg = fh$ implies that $g = h$.

Exercise 6.2.2. *Let $\mathcal{C}$ be a category with pullbacks. Then prove that the following are equivalent:*

(1) *$f : X \to Y$ is monic,*
(2) *$\Delta : Y \to Y \times_X Y$ is an isomorphism.*

Corollary 6.2.3. *Prove that Quasi-affine schemes are also quasi-compact.*

Definition 6.2.4. Let R be a ring. We define the **reduction of** R to be the ring $R_{\text{red}} := R/\sqrt{(0)}$, i.e., it is the ring R modulo its nilpotent elements.

Exercise 6.2.5. *Let $f : X \to Y$ be a morphism of schemes, prove that there exists a unique morphism $f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$ rendering the diagram*

$$\begin{array}{ccc} X_{\text{red}} & \xrightarrow{f_{\text{red}}} & Y_{\text{red}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

commutative. Furthermore:

(1) *prove that $(X \times_Z Y)_{\text{red}} \cong (X_{\text{red}} \times_{Z_{\text{red}}} Y_{\text{red}})_{\text{red}}$,*
(2) *but even if X, Y are reduced then $X \times_Z Y$ need not be.*

Exercise 6.2.6 (Optional)**.** *Prove Lemma 6.1.8 (Hint: you should only use formal properties of pullbacks).*

Definition 6.2.7. A scheme X is **noetherian** if it is (1) quasicompact, and (2) there exists a Zariski cover $\mathcal{U}$ of X consisting of affines schemes where each member is Spec of a noetherian ring.

Exercise 6.2.8. *Prove that a scheme X is noetherian if and only if any chain of closed subschemes*

$$\cdots Z^i \subset Z^{i-1} \subset \cdots Z^1 \subset Z^0 = X,$$

terminates.

Exercise 6.2.9. *Let $f : X \to Y$ be a morphism of schemes. Define the graph prestack as*

$$\Gamma_f : R \mapsto \{(x, y) : f(x) = y\} \subset (X \times Y)(R).$$

Prove that Γ_f is a scheme.

7. Lecture 7: Relative algebraic geometry and quasicoherent sheaves

We have defined schemes and proved an independence result on the condition of being a Zariski stack. Next, we will discuss relative algebraic geometry and the theory of quasicoherent sheaves.

7.1. **Relative algebraic geometry.** One of the key innovations of the Grothendieck school is the idea that one should be working with algebraic geometry over a base scheme; this is also called "*relative algebraic geometry.*" At heart, the following proposition is key:

Proposition 7.1.1. *Let* X, Y, Z *be schemes and suppose that we have cospan of prestacks*

$$\begin{array}{ccc} X & & Y \\ & \searrow \quad \swarrow & \\ & Z & \end{array},$$

then the fiber product in prestacks $X \times_Z Y$ *is, in fact, a scheme.*

Proof. It is a Zariski stack since limits commute with each other. The interesting part is to furnish an open cover of X, Y, Z.

The proof breaks down naturally in steps:

(1) If $X = \operatorname{Spec} A, Y = \operatorname{Spec} B, Z = \operatorname{Spec} C$, then $X \times_Y Z = \operatorname{Spec} A \otimes_C B$ as we have seen before.
(2) Assume that Y, Z are affine. Our goal is to furnish an open cover $X \times_Y Z$. So pick an open cover of X consisting of affines $\mathcal{U} = \{\operatorname{Spec} A_i \hookrightarrow X\}$. From the previous result, we do know that $\operatorname{Spec} A_i \times_Y Z$ is an (affine) scheme so taking
$$\mathcal{U}_X := \{\operatorname{Spec} A_i \times_Y Z \to X \times_Y Z\}$$
furnishes an open cover since we do know that open immersions are stable under base change. Similarly, we are okay if Z, Y are both affines.
(3) Now, let us assume that X, Y are both affines and suppose that Z is not. Take an affine open cover of Z $\mathcal{V} := \{\operatorname{Spec} C_i \hookrightarrow Z\}$. Now, the collection
$$\mathcal{V}_X := \{X \times_Z \operatorname{Spec} C_i \hookrightarrow X\},$$
is a collection of open immersions of X. We note that, however, $X \times_Z \operatorname{Spec} C_i$ is not necessarily affine — they are only quasi-affine. Similarly, we also have
$$\mathcal{V}_Y := \{Y \times_Z \operatorname{Spec} C_i \hookrightarrow Y\}.$$
Now consider
$$\mathcal{V}_X \times_Z \mathcal{V}_Y := \{(Y \times_Z \operatorname{Spec} C_i) \times_{\operatorname{Spec} C_i} (\operatorname{Spec} C_i \times_Z X) \hookrightarrow X \times_Z Y\}$$
This is an improvement of the situation as the terms in the cover are made out of *taking fiber products over an affine scheme* so that we can just arrange one of the other terms, say $(\operatorname{Spec} C_i \times_Z X)$ to be affine by taking a further cover (using Lemma 6.0.1, say). We then appeal to the previous case.
(4) Lastly, if none of them are affine, then we take open covers
$$\mathcal{U} := \{\operatorname{Spec} A_i \hookrightarrow X\} \qquad \mathcal{V} := \{\operatorname{Spec} B_j \hookrightarrow Y\},$$
and note that
$$\mathcal{U} \times_Z \mathcal{V} := \{\operatorname{Spec} A_i \times_Z \operatorname{Spec} B_i \hookrightarrow X \times_Z Y\}$$
is an open cover by the previous situation.

□

Here is a way to phrase what the above says:

Corollary 7.1.2. *The inclusion* $\mathrm{Sch} \subset \mathbf{PStk}$ *creates fiber products. In fact, it creates finite limits.*

Equipped with the above result, we define:

Definition 7.1.3. Let B be a scheme, the category of **B-schemes** is the slice category $\mathrm{Sch}_{\mathrm{B}} := \mathrm{Sch}_{/\mathrm{B}}$. In other words, its objects are given by morphisms of schemes

$$\mathrm{X} \to \mathrm{B},$$

while morphisms are

$$\begin{array}{ccc} \mathrm{X} & \longrightarrow & \mathrm{Y} \\ & \searrow \quad \swarrow & \\ & \mathrm{Z} & \end{array} .$$

The following lemma is a consequence of the existence of fiber products:

Lemma 7.1.4. *The category* $\mathrm{Sch}_{\mathrm{B}}$ *admits finite limits.*

We note that products in $\mathrm{Sch}_{\mathrm{B}}$ are given by fiber products.

Example 7.1.5. Let R be a fixed commutative ring. Then the category of Spec R-schemes (sometimes also called R-schemes) which are *also affine* (this is different from saying "affine R-schemes) is equivalent to the category R-algebras, i.e., a commutative ring admitting a ring morphism from R and those ring maps under R which renders the obvious diagram commutative.

Remark 7.1.6. Let $\mathrm{X} \to \mathrm{Y}$ be a morphism between schemes, then we have an adjunction

$$\mathrm{forget} : \mathrm{Sch}_{\mathrm{X}} \rightleftarrows \mathrm{Sch}_{\mathrm{Y}} : \times_{\mathrm{Y}} \mathrm{X},$$

where the left adjoint is given by sending an X-scheme $\mathrm{T} \to \mathrm{X}$ to $\mathrm{T} \to \mathrm{X} \to \mathrm{Y}$. Indeed, convince yourself that this is an adjunction. The right adjoint is often called the "base change functor."

Example 7.1.7. Here are some examples of difference with working with *absolute* algebraic geometry versus *relative* algebraic geometry.

(1) consider the map

$$\mathbf{C} \to \mathbf{C} \qquad z = a + ib \mapsto \overline{z} = a - ib.$$

This is is a **Z**-linear map. However, it is *not* a **C**-linear map:

$$(x+iy)\overline{ib} = (x+iy)(-ib) = -xib + yb,$$

but

$$\overline{(x+iy)(ib)} = \overline{(xib - yb)} = -xib - yb.$$

This means that the involution

$$\iota : \mathrm{Spec}\,\mathbf{C} \to \mathrm{Spec}\,\mathbf{C},$$

is *not* a morphism in **C**.

(2) Let R be a commutative ring which is also an algebra over $\mathbf{F}_p$. Then there is a map

$$\mathrm{F}_{\mathrm{R}} : \mathrm{R} \to \mathrm{R} \qquad x \mapsto x^p,$$

called the Frobenius. The Frobenius exhbits the following functoriality if $f : \mathrm{Spec}\,\mathrm{R} \to \mathrm{Spec}\,\mathrm{S}$ is a morphism then:

$$\begin{array}{ccc} \mathrm{Spec}\,\mathrm{R} & \xrightarrow{\mathrm{F}_{\mathrm{R}}} & \mathrm{Spec}\,\mathrm{R} \\ \downarrow f & & \downarrow f \\ \mathrm{Spec}\,\mathrm{S} & \xrightarrow{\mathrm{F}_{\mathrm{S}}} & \mathrm{Spec}\,\mathrm{S}. \end{array}$$

In particular, unless $\mathrm{F}_{\mathrm{S}} = \mathrm{id}$, then F_{R} is *not* a morphism of S-schemes.

Relative algebraic geometry prompts us to ask the following question:

Question 7.1.8. Is there a good theory of "algebra" over a base scheme B?

It turns out, as you will prove in the exercises, that repeating the theory of prestacks as functors

$$\mathcal{F} : \mathrm{CAlg}_{\mathrm{A}} \to \mathrm{Set}$$

does recover relative algebraic geometry over Spec A.

7.2. **Linear algebra over schemes.** There is a one-shot definition of quasicoherent sheaves. This is the best definition but maybe not the most workable since it involves 2-categories:

$$\mathbf{QCoh}(\mathrm{X}) := \mathrm{holim}_{\mathrm{Spec\,R}\to\mathrm{X}}\mathbf{Mod}_{\mathrm{R}}.$$

The the holim indicates a more sophisticated but correct notion of a limit in the context of the 2-category of 1-categories but I will spare everyone this formulation. If you have, however, worked with categories like this, I encourage you to think that way. The theory of quasicoherent sheaves can be quite difficult to stomach on first try. Here are some signposts in the wilderness:

(1) if $\mathrm{X} = \mathrm{Spec\,A}$, an affine scheme, then

$$\mathbf{QCoh}(\mathrm{X}) = \mathbf{Mod}_{\mathrm{A}}.$$

In other words, the theory of quasicoherent sheaves over an affine scheme is "just linear algebra."

(2) Insider $\mathbf{QCoh}(\mathrm{X})$ there is a distinguished (in the royal sense, but not in any precise sense):

$$\mathbf{Vect}(\mathrm{X}) \subset \mathbf{QCoh}(\mathrm{X})$$

which are much much more manageable and also interesting — they are called "vector bundles" and behave like objects under the same name that you might have encountered in differential geometry or other contexts. If $\mathrm{X} = \mathrm{Spec\,A}$ then

$$\mathbf{Vect}(\mathrm{X}) = \mathbf{Mod}_{\mathrm{A}}^{\mathrm{fgproj}},$$

the category of finitely generated projective A-modules. Here's one way to think about it from this point of view (which is not necessarily good since this is equivalent to picking a basis): to give finitely generated projective module M is equivalent to giving a free module $\mathrm{A}^{\oplus n}$ and an idempotent $e : \mathrm{A}^{\oplus n} \to \mathrm{A}^{\oplus n}$. In other words, these are just *(square) matrices.*

(3) Other than vector bundles, quasicohernet sheaves which are interesting are those "coming from closed subschemes." In the affine case: if $\mathrm{X} = \mathrm{Spec\,A}$, recall that a closed subscheme is given by a surjection $\mathrm{A} \to \mathrm{B}$. From this we can extract $\mathrm{I} := \ker(\mathrm{A} \to \mathrm{B})$ which is thus an A-module and is thus a quasicoherent sheaf on X.

(4) Some good categorical/homological properties: $\mathbf{QCoh}(\mathrm{X})$ is a symmetric monoidal Grothendieck abelian category with enough injectives. This is not an easy theorem and is due to Gabber (a name you will hear many times if you are an algebraic geometer) but, in particular, you can:
   - take kernels and cokernels;
   - take tensor products.
   - if $0 \to \mathcal{F}_i \to \mathcal{G}_i \to \mathcal{H}_i \to 0$ is an exact sequence, then the sum

   $$0 \to \bigoplus_i \mathcal{F}_i \to \bigoplus_i \mathcal{G}_i \to \bigoplus_i \mathcal{H}_i \to 0,$$

   remains exact;
   - and, most importantly, you can *take cohomology* which, by the end of this class, you will be addicted to doing.

Here comes the definition:

Definition 7.2.1. Let X be a scheme. A **quasicoherent sheaf** $\mathcal{F}$ on X is the data:

(1) for each map $x : \mathrm{Spec\,R} \to \mathrm{X}$ an A-module which we denote by $x^*\mathcal{F} \in \mathbf{Mod}_{\mathrm{A}}$,

(2) for each map $\mathrm{Spec}\,\mathrm{S} \xrightarrow{g} \mathrm{Spec}\,\mathrm{R} \xrightarrow{x} \mathrm{X}$ whose composite we denote by

$$y = x \circ g : \mathrm{Spec}\,\mathrm{S} \to \mathrm{X}.$$

we are given an isomorphism

$$\alpha_{x,g} : (x \circ g)^*\mathcal{F} \xrightarrow{\cong} g^*x^*\mathcal{F},$$

subject to the following condition: given

$$\mathrm{Spec}\,\mathrm{T} \xrightarrow{h} \mathrm{Spec}\,\mathrm{S} \xrightarrow{g} \mathrm{Spec}\,\mathrm{R} \xrightarrow{x} \mathrm{X}$$

so that the composite is denoted as

$$z : \mathrm{Spec}\,\mathrm{T} \to \mathrm{X}$$

we have an equality of maps

$$\alpha_{x,g\circ h} = \alpha_{x\circ g,h} : z^*\mathcal{F} \xrightarrow{\cong} h^*g^*f^*\mathcal{F}.$$

A **morphism of quasicoherent sheaves** $q : \mathcal{F} \to \mathcal{G}$ is the data: for each morphism $x : \mathrm{Spec}\,\mathrm{R} \to \mathrm{X}$ a morphism

$$x^*q : f^*\mathcal{F} \to x^*\mathcal{G}$$

of R-modules such that all the induced diagrams commute. We denote by **QCoh**(X) the category of quasicoherent sheaves on X.

Example 7.2.2. Let X be a scheme. Then the **structure sheaf** on X, which we denote by $\mathcal{O}_\mathrm{X}$ is the object $\mathcal{O}_\mathrm{X} \in \mathbf{QCoh}(\mathrm{X})$ given by

$$f^*\mathcal{O}_\mathrm{X} = \mathrm{R}$$

for any $f : \mathrm{Spec}\,\mathrm{R} \to \mathrm{X}$. Let us unpack the compatibility condition. Suppose that we have a composite

$$\mathrm{Spec}\,\mathrm{T} \xrightarrow{h} \mathrm{Spec}\,\mathrm{S} \xrightarrow{g} \mathrm{Spec}\,\mathrm{R} \xrightarrow{x} \mathrm{X},$$

so that, on the level of rings, we have morphisms

$$\mathrm{R} \to \mathrm{S} \to \mathrm{T}.$$

We are two different isomorphisms of

$$z^*\mathcal{O}_\mathrm{X} = \mathrm{T} \xrightarrow{\cong} h^*g^*x^*\mathcal{O}_\mathrm{X} = h^*g^*\mathrm{R}$$

given by

$$\mathrm{T} \cong \mathrm{R} \otimes_\mathrm{S} (\mathrm{S} \otimes_\mathrm{S} \mathrm{T})$$

and

$$\mathrm{T} \cong (\mathrm{R} \otimes_\mathrm{S} \mathrm{S}) \otimes_\mathrm{S} \mathrm{T}.$$

The claim that $\mathcal{O}_\mathrm{X}$ is a quasicoherent sheaf follows from the associativity of the tensor product.

7.3. **A word on: why quasicoherent sheaves?** It is algebro-geometric propaganda that quasicoherent sheaves are important and that they should be the next thing one introduces after introducing schemes. One motivation is that it that to have a mechanism of showing that some scheme is not affine, we will need to examine the global sections of the structure sheaf. Hence, the data of "global functions" or, more precisely, the global sections of the structure sheaf captures the entire affine scheme. On the other hand, the birational classification of algebraic varieties relies on studying invariants that one can extract out of quasicoherent sheaves.

In other words, to study an affine scheme it is not enough to look at global sections of $\mathcal{O}$, we must look at the global sections of $\mathbf{QCoh}(-)$. I hope this is enough motivation for us to spend sometime trying to study quasicoherent sheaves.

There is however a more primary motivation. In the 60's Gabriel and Rosenberg proved the following remarkable "reconstruction theorem."

Theorem 7.3.1 (Gabriel, Rosenberg)**.** *Let* X, Y *be schemes which are quasiseparated (to be defined later), then the following are equivalent:*

(1) *there is an equivalence of categories* $\mathbf{QCoh}(\mathrm{X}) \cong \mathbf{QCoh}(\mathrm{Y})$*;*
(2) X *and* Y *are isomorphic as schemes.*

7.4. **Exercises.**

Definition 7.4.1. A **closed point** of a scheme X is a closed immersion of the form

$$x : \operatorname{Spec} k \hookrightarrow \mathrm{X}.$$

Exercise 7.4.2. *Prove that under the correspondence of Exercise 2.3.7 a closed point of* $\mathrm{X} = \operatorname{Spec} \mathrm{A}$, $\operatorname{Spec} k \hookrightarrow \mathrm{X}$ *is the same thing as a maximal ideal of* A *with* $\mathrm{A}/\mathfrak{m} = k$.

Definition 7.4.3. Let R be a ring and $\mathrm{R} \to \mathrm{S}$ an R-algebra. Then S is said to be **of finite type** if it is of the form (as an R-algebra)

$$\mathrm{S} \cong \mathrm{R}[x_1, \cdots, x_n]/\mathrm{I},$$

where I is an ideal. In this case, we say that $\operatorname{Spec} \mathrm{S} \to \operatorname{Spec} \mathrm{R}$ is of **finite type**.

For the next exercise, you may, and should, invoke Hilbert's Nullstellensatz.

Exercise 7.4.4. *Suppose that* $\mathrm{X} = \operatorname{Spec} \mathrm{A} \to \operatorname{Spec} k$ *is of finite type. Then the following are equivalent:*

(1) *the* k*-morphism* $\operatorname{Spec} \mathrm{K} \to \operatorname{Spec} \mathrm{A}$ *is a closed point,*
(2) K *is a finite field extension of* k*,*
(3) K *is algebraic as an extension over* k*.*

Exercise 7.4.5. *Let* $\mathcal{O}$ *be a discrete valuation ring with fraction field* K*. Prove that the map*

$$\operatorname{Spec} \mathrm{K} \to \operatorname{Spec} \mathcal{O},$$

is an open immersion. Conclude that $\operatorname{Spec} \mathrm{K}$ *is not a closed point of* $\operatorname{Spec} \mathcal{O}$ *but has an open neighborhood in which it is closed.*

Exercise 7.4.6. *Let* $\mathcal{C}$ *be a category and* $\mathrm{X} \in \mathcal{C}$ *a fixed object. Consider* $\mathrm{Z} \xrightarrow{f} \mathrm{X}, \mathrm{Y} \to \mathrm{X}$ *Then prove that the following is a pullback diagram in sets*

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{C}_{/\mathrm{X}}}(\mathrm{Z}, \mathrm{Y}) & \longrightarrow & \operatorname{Hom}_{\mathcal{C}}(\mathrm{Z}, \mathrm{Y}) \\ \downarrow & & \downarrow \\ \{f\} & \longrightarrow & \operatorname{Hom}_{\mathcal{C}}(\mathrm{Z}, \mathrm{X}) \end{array}$$

where the right vertical map is postcomposition:

$$\mathrm{Z} \to \mathrm{Y} \mapsto \mathrm{Z} \to \mathrm{Y} \to \mathrm{X}.$$

Exercise 7.4.7. *Let* X *be a scheme and* R *a ring. Prove that the set* $\mathrm{X}(\mathrm{R})$ *is canonically isomorphic to sections of* $\mathrm{X} \times_{\operatorname{Spec} \mathbf{Z}} \operatorname{Spec} \mathrm{R}$*. Compute the following sets:*

(1) *sections of* $\mathbf{A}^1_{\mathbf{R}} \to \operatorname{Spec} \mathbf{R}$*,*
(2) *sections of* $\mathbf{A}^1_{\mathbf{C}} \to \operatorname{Spec} \mathbf{C}$*.*

This illustrates how rational points can differ over different base rings.

Exercise 7.4.8. *The following exercises corrects a mistake made in the formulation of a previous exercise. Let* B *be a base scheme and* K *a field. Consider the following commutative diagram*

$$\begin{array}{ccc} \operatorname{Spec} \mathrm{K} & \xrightarrow{\quad \xi \quad} & \mathrm{X} \\ & \searrow \quad \swarrow & \\ & \mathrm{B} & \end{array} .$$

Define

$$\mathrm{T}_\xi(\mathrm{X}/\mathrm{B}) := \operatorname{Hom}_{\mathrm{B}}(\operatorname{Spec} \mathrm{K}[x]/(x^2), \mathrm{X}) \times_{\operatorname{Hom}_{\mathrm{S}}(\operatorname{Spec} \mathrm{K}, \mathrm{X})} \{\xi\}.$$

Prove:

(1) we can endow $\mathrm{T}_\xi(\mathrm{X}/\mathrm{B})$ *with the structure of a* K*-vector space;*
(2) if $\mathrm{S} = \operatorname{Spec} k$ *where* k *is a field and* $\mathrm{K} = k$*, prove that* $\mathrm{T}_\xi(\mathrm{X}/\mathrm{B})$ *recovers the Zariski tangent space from Exercise 2.3.7.*
(3) Let S *remain the same but let* K *be a finite extension of* k*. Let* $\mathrm{X} = \operatorname{Spec}\mathrm{K}$ *and let* $x : \operatorname{Spec}\mathrm{K} \to \mathrm{X}$ *classify the identity. Prove that*

$$\mathrm{T}_x\mathrm{X} = 0,$$

but

$$\mathrm{T}_\xi(\mathrm{X}/\operatorname{Spec} k) = 0$$

if and only if K/k *is a separable extension.*

Exercise 7.4.9. *Let* $\mathcal{F} : \mathrm{CAlg}_\mathrm{A} \to \mathrm{Set}$ *be a functor where* A *is a nonzero ring.*
(1) prove that in $\mathrm{Fun}(\mathrm{CAlg}_\mathrm{A}, \mathrm{Set})$*, the Yoneda embedding exhibits* Spec A *as the final object by constructing a canonical map* $\mathcal{F} \to \operatorname{Spec}\mathrm{A}$.
(2) Prove that "being a scheme" is a local property by proving that the following are equivalent:
  (a) the prestack

$$\mathcal{F} : \mathrm{CAlg} = \mathrm{CAlg}_{\mathbf{Z}} \xrightarrow{\otimes \mathrm{A}} \mathrm{CAlg}_\mathrm{A} \xrightarrow{\mathcal{F}} \mathrm{Set}$$

  is a scheme,
  (b) for each $\operatorname{Spec}\mathrm{R} \to \operatorname{Spec}\mathrm{A}$*, the pullback* $\mathcal{F} \times_{\operatorname{Spec}\mathrm{R}} \operatorname{Spec}\mathrm{A}$ *is a scheme.*

Exercise 7.4.10. *Let* X *be a scheme. We say that a quasicoherent sheaf* $\mathcal{F}$ *is* **finite type** *if for all* $f : \operatorname{Spec}\mathrm{R} \to \mathrm{X}$*, the pullback* $f^*\mathcal{M} \in \mathbf{Mod}_\mathrm{R}$ *is a finitely generated* R*-module. Define the subprestack*

$$\mathrm{Supp}(\mathcal{F}) \hookrightarrow \mathrm{X}$$

as follows: a morphism $g : \operatorname{Spec}\mathrm{R} \to \mathrm{X}$ *factors through* $\mathrm{Supp}(\mathcal{F})$ *if and only if* $g^*\mathcal{F} \neq 0$*. Prove that* $\mathrm{Supp}(\mathcal{F})$ *is a closed subscheme (Hint: you may take for granted the following fact: if* M *is a module over a ring* R*, then the support of a module* M *(in the sense of commutative algebra) which is finite type is given by the quotient ring* $\mathrm{R}/\mathrm{Ann}_\mathrm{R}(\mathrm{M})$*; but be sure to know that this* not *true when* M *is note finitely generated).*

Exercise 7.4.11. *Let* k *be an algebraically closed field. Consider the* $\mathbf{A}^1$*-scheme* $\operatorname{Spec} k[x,y,t]/(ty - x^2) \to \operatorname{Spec} k[t]$ *induced by the obvious morphism.*
(1) Prove that this morphism is surjective on all R*-points where* R *is a* k*-algebra.*
(2) For each $a \in k$ *which is not zero consider the map* $\operatorname{Spec} k \to \operatorname{Spec} k[t]$ *induced by the map* $k[t] \to k$ *sending* t *to* a*. Prove that the pullback*

$$\operatorname{Spec} k \times_{\operatorname{Spec} k[t]} \operatorname{Spec} k[x,y,t]/(ty - x^2),$$

  is a reduced scheme.
(3) If $\operatorname{Spec} k \to \operatorname{Spec} k[t]$ *is induced by the map* $k[t] \to k$ *sending* t *to zero, then prove that*

$$\operatorname{Spec} k \times_{\operatorname{Spec} k[t]} \operatorname{Spec} k[x,y,t]/(ty - x^2),$$

  is, however, not reduced.

8. Lecture 8: more quasicoherent sheaves

Last class, we defined the category $\mathbf{QCoh}(\mathrm{X})$. Let us see what happens in the affine situation. So suppose that $\mathrm{X} = \operatorname{Spec}\mathrm{A}$ is an affine scheme and say, $\mathcal{F}$ is a quasicoherent sheaf on X. Among other things, we have the identity morphism $\mathrm{id} : \operatorname{Spec}\mathrm{A} \to \operatorname{Spec}\mathrm{A}$ so that we have an A-module

$$\mathrm{id}^*\mathcal{F} =: \mathrm{M}.$$

Furthermore, we also note that $\mathrm{id}^*\mathcal{O}_\mathrm{X} = \mathrm{A}$.

On the other hand, if M is an A-module, and we are given maps $\operatorname{Spec} S \xrightarrow{g} \operatorname{Spec} R \xrightarrow{f} X$ which means maps of rings

$$A \to R \to S,$$

then we have a canonical isomorphism:

$$M \otimes_A R \otimes_R S \cong M \otimes_A S.$$

The canoncity of this isomorphism implies that the cocycle condition is satisfied. We denote this quasicoherent sheaf by $\widetilde{M}$ and this assembles into a functor

$$\mathbf{Mod}_A \to \mathbf{QCoh}(X) \qquad M \mapsto \widetilde{M}.$$

Proposition 8.0.1. *Let* $X = \operatorname{Spec} A$*. Consider the functor*

$$\Gamma : \mathbf{QCoh}(X) \to \mathbf{Mod}_A,$$

given by

$$\mathcal{F} \mapsto \mathrm{id}^* \mathcal{F}.$$

This functor is an equivalence of categories.

Proof. We have a functor $\mathbf{Mod}_A \to \mathbf{QCoh}(X)$ given by sending an A-module M to $\widetilde{M}$. But now we note that

$$\Gamma(\widetilde{M}) = \mathrm{id}^* \widetilde{M} = M,$$

by construction. On the other hand, for any $x : \operatorname{Spec} R \to X$, we have that

$$x^* \widetilde{\Gamma(\mathcal{F})} = x^* \Gamma(\mathcal{F}) = x^* \mathrm{id}^* \mathcal{F} = x^* \mathcal{F}.$$

Hence we have that $\widetilde{\Gamma(\mathcal{F})} \cong \mathcal{F}$. □

More generally, we define:

Definition 8.0.2. Let $\mathcal{F}$ be a quasicoherent sheaf on X, then the set of **global sections** of $\mathcal{F}$ is defined to be

$$\Gamma(X, \mathcal{F}) := \mathrm{Hom}_{\mathbf{QCoh}(X)}(\mathcal{O}_X, \mathcal{F})$$

Let us attempt to unpack Definition 8.0.2. Suppose that we have a morphism $f : \operatorname{Spec} R \to X$, then to an element of $\Gamma(X, \mathcal{F})$ gives us a map of R-modules:

$$f^* \mathcal{O}_X = R \to f^* \mathcal{F},$$

which just means that we are picking out an element of $f^*\mathcal{F}$ since this map is R-linear. So if we think of $f^*\mathcal{F}$ as a sheaf over the space X, this picks out a local section of $\mathcal{F}$. As R varies, these maps should then be compatible in a suitable way. In particular we note that if $X = \operatorname{Spec} A$ then

$$\Gamma(X, \widetilde{M}) = \mathrm{id}^* \widetilde{M} = M,$$

and is thus naturally an A-module. We will later say what extract structure $\Gamma(X, \mathcal{F})$ is.

8.1. **A(nother) result of Serre's.**

In the next topic, we will "calculate" what the category $\mathbf{QCoh}(X)$ looks like, i.e., produce a smaller amount of data that determines the whole category of quasicoherent sheaves. We have two motivations for this:

8.1.1. *Ideal sheaves.* Let X be a scheme. If X is affine, we note that an closed subscheme $\mathrm{Z} \hookrightarrow \mathrm{X}$ determines and is determined by an ideal I, i.e., an algebraic data. We wish to prove that there is a global version of this phenomenon. We would like to define the *ideal sheaf* of a closed immersion $\mathrm{Z} \hookrightarrow \mathrm{X}$ which is often denoted by $\mathcal{I}_{\mathrm{Z}}$. Here's one way to begin doing so: suppose that $x : \operatorname{Spec} \mathrm{R} \to \mathrm{X}$ is a morphism, then we know that

$$\operatorname{Spec} \mathrm{R} \times_{\mathrm{X}} \mathrm{Z} \cong \operatorname{Spec} \mathrm{R}/\mathrm{I}.$$

We would like to set

$$x^*\mathcal{I}_{\mathrm{Z}} := \mathrm{I} \in \mathbf{Mod}_{\mathrm{R}}.$$

However we note that if $\operatorname{Spec} \mathrm{S} \xrightarrow{f} \operatorname{Spec} \mathrm{R}$ and

$$\operatorname{Spec} \mathrm{S} \times_{\mathrm{X}} \mathrm{Z} \cong \operatorname{Spec} \mathrm{S}/\mathrm{J},$$

then the ideal J need not be equal to $f^*\mathrm{I}$. Indeed, we have an exact sequence of R-modules

$$0 \to \mathrm{I} \to \mathrm{R} \to \mathrm{R}/\mathrm{I} \to 0.$$

If we apply f^* we get a sequence which may not be left exact

$$f^*\mathrm{I} \to \mathrm{S} \to \mathrm{S}/\mathrm{J} = \mathrm{S} \otimes_{\mathrm{R}} \mathrm{R}/\mathrm{I} \to 0,$$

though there is a comparison map

$$f^*\mathrm{I} \to \mathrm{J}.$$

To fix this problem, we will have an excuse to discuss the first basic functoriality of quasicoherent sheaves.

Construction 8.1.2. Let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism of schemes. Then the **pullback functor**

$$f^* : \mathbf{QCoh}(\mathrm{Y}) \to \mathbf{QCoh}(\mathrm{X})$$

is constructed as follows: for $\mathcal{F} \in \mathbf{QCoh}(\mathrm{Y})$, then for any $x : \operatorname{Spec} \mathrm{R} \to \mathrm{X}$, $x^*(f^*\mathcal{F})$ is defined as

$$x^* f^* \mathcal{F}.$$

The compatibility condiitions for $f^*\mathcal{F}$ is inherited by those of $\mathcal{F}$'s.

We observe

Proposition 8.1.3. *Let* $\mathrm{X} = \operatorname{Spec} \mathrm{A}$ *be an affine scheme and suppose that* $j : \mathrm{U} \hookrightarrow \mathrm{X}$ *is an open immersions. Then given an injection of* A*-modules*

$$0 \to \mathrm{M}' \to \mathrm{M},$$

we have an injection

$$0 \to j^*\mathrm{M}' \to j^*\mathrm{M}$$

Proof. We will soon develop the full-blown theory of exact sequences in quasicoherent sheaves. For now, we only note that in order to check that the map $j^*\mathrm{M}' \to j^*\mathrm{M}$ is injective, it suffices to prove injectivity on a cover of U: for an open affine cover $\{j_i : \operatorname{Spec} \mathrm{R}_i \to \mathrm{U}\}$, the map

$$j_i^* j^* \mathrm{M}' \to j_i^* j^* \mathrm{M}$$

is injective. It suffices to then check for any open immersion of affine schemes $j : \operatorname{Spec} \mathrm{S} \to \operatorname{Spec} \mathrm{R}$, the functor $j^* = - \otimes_{\mathrm{R}} \mathrm{S}$ is flat (which is homework). ☐

8.1.4. *Non-affineness results.* Suppose that we want to prove the following result:

Theorem 8.1.5. *The scheme* $\mathbf{A}^2 \setminus 0$ *is not an affine scheme.*

For this we need to find a property of affine schemes which makes them "affine." One of the things that makes a scheme is affine is the fact that $\mathbf{QCoh}(X) \simeq \mathbf{Mod}_A$. In fact,

$$A \cong \Gamma(\mathcal{O}_X),$$

as A-modules. But in fact, this equivalence is an equivalence of algebras. To make sense of this we need to talk about the symmetric monoidal structure on $\mathbf{QCoh}(X)$.

Proposition 8.1.6. *The category of quasicoherent sheaves is a symmetric monoidal category with unit* $\mathcal{O}_X$*. Furthermore, if* $f : X \to Y$ *is a morphism of schemes, then the induced functor*

$$\mathbf{QCoh}(Y) \xrightarrow{f^*} \mathbf{QCoh}(X)$$

is strongly symmetric monoidal.

Proof. For a pair $\mathcal{F}, \mathcal{G}$ of quasicoherent sheaves, we define

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \in \mathbf{QCoh}(X)$$

to be the quasicoherent sheaf such that for each $f : \operatorname{Spec} R \to X$, we get

$$f^*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) := f^*\mathcal{F} \otimes_R f^*\mathcal{G}.$$

With this note that

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{F},$$

since for all $f : \operatorname{Spec} R \to X$ we have that

$$f^*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X) = f^*\mathcal{F} \otimes_R R = f^*\mathcal{F}.$$

Checking the axioms of being a symmetric monoidal category is a standard check.

For the second claim, this follows from the fact that if $f : \operatorname{Spec} R \to \operatorname{Spec} S$ is a morphism of affine schemes, then

$$f^*(M \otimes_S N) = R \otimes_S (M \otimes_S N) \cong (M \otimes_S R) \otimes_R (R \otimes_S N) = f^*(M) \otimes_S f^*(N),$$

and

$$f^*(S) = R.$$

$\square$

By construction, we see that if X is affine, then the equivalence $\mathbf{QCoh}(X) \cong \mathbf{Mod}_A$ respects symmetric monoidal structures. In particular we can recover A as a ring from this equivalence. To make this more precise, we introduce the main focus of next week's lectures:

Definition 8.1.7. A **quasicoherent algebra** or a **quasicoherent sheaves of algebras** is a quasicoherent sheaf $\mathcal{F}$ together with maps

$$m : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F} \qquad \epsilon : \mathcal{O}_X \to \mathcal{F},$$

which makes $\mathcal{F}$ as a commutative algebra object in the symmetric monoidal category $\mathbf{QCoh}(X)$.

Corollary 8.1.8. *Let* $X = \operatorname{Spec} A$*. The equivalence of Proposition 8.0.1 descends to a compatible equivalence:*

$$\mathrm{CAlg}_A \rightleftarrows \mathrm{CAlg}(\mathbf{QCoh}(X)).$$

In particular:

$$\Gamma(\mathcal{O}_X) = A$$

as ring and thus $\operatorname{Spec} \Gamma(\mathcal{O}_X) \cong X$.

Now, let X be a scheme in general; we do not expect $\Gamma(\mathcal{O}_X)$ to recover X in general. But we can still ask what kind of structure $\Gamma(\mathcal{F})$ has. This will sharpen our picture a little bit more:

(1) if $s, t : \mathcal{O}_X \to \mathcal{F}$ are two sections, then we can add sections

$$s + t : \mathcal{O}_X \to \mathcal{F},$$

which is locally given by

$$s + t : R \to x^*\mathcal{F} \qquad (s+t)(r) = s(r) + t(r),$$

using the module addition. Therefore $\Gamma(\mathcal{F})$ is an abelian group.

(2) Furthermore, if $\mathcal{A}$ is a quasicoherent sheaf of algebras, then we can multiply sections:

$$s \cdot t : \mathcal{O}_X \to \mathcal{A},$$

which is locally given by

$$s \cdot t : R \to x^*\mathcal{A} \qquad s \cdot t(r) = s(t)r(t).$$

From these observations, we conclude that

$$\Gamma(\mathcal{O}_X)$$

is an algebra.

As a result, Γ assembles into a functor

$$\Gamma : \mathbf{QCoh}(X) \to \mathbf{Mod}_{\mathbf{Z}} = \mathbf{QCoh}(\operatorname{Spec} \mathbf{Z}).$$

Soon, will recast the above discussion in a more general context later and discover Γ as the pushforward map

$$\pi_{X*} : \mathbf{QCoh}(X) \to \mathbf{QCoh}(\operatorname{Spec} \mathbf{Z}),$$

where $\pi_X : X \to \operatorname{Spec} \mathbf{Z}$ is the canonical map.

Anyway, we now have a clear strategy to prove that $\mathbf{A}^2 \smallsetminus 0$ is not affine: we just prove that

$$\operatorname{Spec} \Gamma(\mathcal{O}_{\mathbf{A}^2 \smallsetminus 0}) \ncong \mathbf{A}^2 \smallsetminus 0.$$

In fact we know exactly that $\Gamma(\mathcal{O}_{\mathbf{A}^2 \smallsetminus 0})$ is:

Lemma 8.1.9. *The ring* $\Gamma(\mathcal{O}_{\mathbf{A}^2 \smallsetminus 0}) \cong \mathbf{Z}[x, y]$.

This begs the question:

Question 8.1.10. How does one compute $\Gamma(\mathcal{O}_X)$?

8.2. **Formulation of Serre's theorem.** Suppose that $j : U \hookrightarrow V$ be an open immersion of schemes and $\mathcal{F} \in \mathbf{QCoh}(V)$. We will sometimes abusively write

$$j^*\mathcal{F} = \mathcal{F}|_V.$$

If $W \hookrightarrow V$ is another open immersion we also abusively write

$$W \times_V U =: W \cap U$$

With this:

Definition 8.2.1. Let X be a scheme and $\mathcal{U}$ an open cover of X. We define

$$\mathbf{QCoh}(X, \mathcal{U})$$

to be the category consisting of $\mathcal{F}_U \in \mathbf{QCoh}(U)$ and isomorphisms

$$\alpha_{UV} : \mathcal{F}_U|_{U \cap V} \cong \mathcal{F}_V|_{U \cap V},$$

satisfying the cocycle condition: for triple intersections U, V, W the composite of isomorphisms

$$\begin{aligned} \mathcal{F}_U|_{U\cap V\cap W} &\simeq (\mathcal{F}_U|_{U\cap V})|_{U\cap V\cap W} \\ &\xrightarrow{\alpha_{UV}} (\mathcal{F}_V|_{U\cap V})|_{U\cap V\cap W} \\ &\xrightarrow{\alpha_{VW}} (\mathcal{F}_W|_{V\cap W})|_{U\cap V\cap W} \\ &\simeq \mathcal{F}_W|_{U\cap V\cap W}. \end{aligned}$$

must be equal to

$$\mathcal{F}_{\mathrm{U}}|_{\mathrm{U}\cap\mathrm{V}\cap\mathrm{W}} \simeq (\mathcal{F}_{\mathrm{U}}|_{\mathrm{U}\cap\mathrm{W}})|_{\mathrm{U}\cap\mathrm{V}\cap\mathrm{W}} \xrightarrow{\alpha_{\mathrm{UW}}} \mathcal{F}_{\mathrm{W}}|_{\mathrm{U}\cap\mathrm{V}\cap\mathrm{W}}.$$

Remark 8.2.2. The cocycle condition appearing in the above definition makes it clearer how a quasicoherent sheaf is actually a "sheaf" in some sense.

Now, we note that we can indeed define the ideal sheaf $\mathcal{I}_{\mathrm{Z}}$ of a closed immersion as an object of $\mathbf{QCoh}(\mathrm{X}, \mathcal{U})$. We have a functor

$$\text{forget} : \mathbf{QCoh}(\mathrm{X}) \to \mathbf{QCoh}(\mathrm{X}, \mathcal{U})$$

which *a priori* seems to forget a lot of information. This is not the case.

8.3. **Exercises.** We begin to elucidate the structure of $\mathbf{QCoh}(\mathrm{X})$ in the exercises; fix a Zariski cover of X consisting of affine schemes $\mathcal{U}$. We say that a sequence of quasicoherent sheaves

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

is a **exact** if, under the identification of Serre's theorem

$$\mathbf{QCoh}(\mathrm{X}, \mathcal{U}) \simeq \mathbf{QCoh}(\mathrm{X})$$

the sequence

$$\mathcal{F}'|_{\mathrm{U}} \to \mathcal{F}|_{\mathrm{U}} \to \mathcal{F}''|_{\mathrm{U}}$$

is exact for all $\mathrm{U} \hookrightarrow \mathrm{X}$.

Exercise 8.3.1. *Prove that for any open immersion* $j : \mathrm{U}' \hookrightarrow \mathrm{X}$*, the functor*

$$j^* : \mathbf{QCoh}(\mathrm{X}, \mathcal{U}) \to \mathbf{QCoh}(\mathrm{U}')$$

is exact.

Exercise 8.3.2. *Prove that a sequence*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

is exact if and only if for all $\mathcal{G} \in \mathbf{QCoh}(\mathrm{U})$*, the sequence of* $\Gamma(\mathrm{U})$*-modules*

$$0 \to \mathrm{Hom}_{\mathbf{QCoh}(\mathrm{U})}(\mathcal{G}, \mathcal{F}'|_{\mathrm{U}}) \to \mathrm{Hom}_{\mathbf{QCoh}(\mathrm{U})}(\mathcal{G}, \mathcal{F}|_{\mathrm{U}}) \to \mathrm{Hom}_{\mathbf{QCoh}(\mathrm{U})}(\mathcal{G}, \mathcal{F}''|_{\mathrm{U}}).$$

Prove that

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is exact if and only if for all $\mathcal{G} \in \mathbf{QCoh}(\mathrm{U})$*, the sequence of* $\Gamma(\mathrm{U})$*-modules*

$$0 \to \mathrm{Hom}_{\mathbf{QCoh}(\mathrm{U})}(\mathcal{F}''|_{\mathrm{U}}, \mathcal{G}) \to \mathrm{H}om_{\mathbf{QCoh}(\mathrm{U})}(\mathcal{F}|_{\mathrm{U}}, \mathcal{G})) \to \mathrm{Hom}_{\mathbf{QCoh}(\mathrm{U})}(\mathcal{F}'|_{\mathrm{U}}, \mathcal{G}).$$

is exact.

Exercise 8.3.3. *Let* X *be a scheme and* $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathbf{QCoh}(\mathrm{X})$*. Prove that there exists*

$$\mathcal{H}\mathrm{om}(\mathcal{G}, \mathcal{H}) \in \mathbf{QCoh}(\mathrm{X})$$

such that we have a natural isomorphism of abelian groups

$$\mathrm{Hom}_{\mathbf{QCoh}(\mathrm{X})}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \cong \mathrm{Hom}_{\mathbf{QCoh}(\mathrm{X})}(\mathcal{F}, \mathcal{H}\mathrm{om}(\mathcal{G}, \mathcal{H})).$$

Exercise 8.3.4. *Consider the scheme* $\mathrm{X} = \mathbf{A}^2 \smallsetminus 0 \subset \mathrm{Spec}\, \mathbf{Z}[x_1, x_2]$*. Consider the quadric*

$$\mathrm{Q}_3 := \mathrm{Spec}\, \mathbf{Z}[x_1, x_2, y_1, y_2]/(x_1y_1 + x_2y_2 = 1) \subset \mathbf{A}^4.$$

Consider the map $p : \mathrm{Q}_3 \to \mathbf{A}^2$ *given by*

$$\mathbf{Z}[x_1, x_2] \to \mathbf{Z}[x_1, x_2, y_1, y_2]/(x_1y_1 + x_2y_2 = 1) \qquad x_1 \mapsto x_1, x_2 \mapsto x_2.$$

Prove that

1. *the map* p *factors through* $\mathbf{A}^2 \smallsetminus 0$*,*
2. *over any field point* $\mathrm{Spec}\, k \to \mathbf{A}^2 \smallsetminus 0$*, the pullback* $\mathrm{Spec}\, k \times_{\mathbf{A}^2 \smallsetminus 0} \mathrm{Q}_3 \cong \mathbf{A}^1_k$*.*
3. *Furnish an open cover of* $\mathbf{A}^2 \smallsetminus 0$ *by affine schemes,* $\{\mathrm{U} \hookrightarrow \mathbf{A}^2 \smallsetminus 0\}$ *such that*

$$\mathrm{U} \times_{\mathbf{A}^2 \smallsetminus 0} \mathrm{Q}_3 \cong \mathbf{A}^1_{\mathrm{U}}.$$

The scheme Q_3 *is called the* **Jouanalou-Thomason device** *of* $\mathbf{A}^2 \setminus 0$*: it is a "bundle" (which we have not yet defined in this class) of affine spaces over a non-affine scheme whose "total space" is an affine scheme. We will later see that most reasonable schemes admit Jouanalou devices.*

Exercise 8.3.5. *Generalize Exercise 8.3.4 to*

$$Q_{2n} := \operatorname{Spec} \mathbf{Z}[x_i, \cdots, x_n, y_1, \cdots, y_n]/(x_1y_1 + \cdots + x_ny_n = 1) \to \mathbf{A}^n \setminus 0.$$

9. Lecture 9: vector and line bundles

Here is Serre's theorem:

Theorem 9.0.1. *The functor*

$$\text{forget} : \mathbf{QCoh}(X) \to \mathbf{QCoh}(X, \mathcal{U})$$

is an equivalence of categories.

Proof sketch. This proof is tedious, so let me indicate the main ideas:

(Reduce to affines) By definition, a quasicoherent sheaf is a compatible collection of $x^*\mathcal{F}$ for each $x : \operatorname{Spec} R \to X$ and isomorphisms as $\operatorname{Spec} R$ varies. For each point $x : \operatorname{Spec} R \to X$, we can consider the cover of $\operatorname{Spec} R$ given by $\mathcal{U}_R := \{\operatorname{Spec} R \times_X U_i \to \operatorname{Spec} R\}$. Suppose that we have proved the claim for affine schemes, then an object of $\mathbf{QCoh}(X, \mathcal{U})$ determines an object of $\mathbf{QCoh}(\operatorname{Spec} R, \mathcal{U}_R)$ which then defines a quasicoherent sheaf on $\mathbf{QCoh}(\operatorname{Spec} R) = \mathbf{Mod}_R$. Doing this as x varies, we procure an R-module with the desired compatibilities and isomorphisms.

(Reduce to basic opens) This is left to the reader and formulated precisely in the exercises.

(Case of basic opens) We are now left with the case of $X = \operatorname{Spec} R$ and $\mathcal{U} = \{R \to R_{f_i}\}$ a Zariski cover. An object of $\mathbf{QCoh}(X, \mathcal{U})$ is the data of R_{f_i}-modules M_i with isomorphisms

$$\alpha_{ij} : M_i[f_j^{-1}] \cong M_j[f_i^{-1}].$$

I claim that we can produce an R-module. There is only one possible way to do this of course. Form the diagram in $\mathbf{Mod}_R$

$$\prod_i M_i \rightrightarrows \prod_{i,j} M_{ij},$$

where $M_{ij} := M_i[f_j^{-1}] \cong M_j[f_i^{-1}]$. Then taking equalizers, we produce $M \in \mathbf{Mod}_R$.

(Assembling the proof) We we have done is to construct a functor

$$\text{glue} : \mathbf{QCoh}(X, \mathcal{U}) \to \mathbf{QCoh}(X).$$

I claim that the composite

$$\mathbf{QCoh}(X) \to \mathbf{QCoh}(X, \mathcal{U}) \to \mathbf{QCoh}(X).$$

Indeed this composite is equivalent to the identity from our old friend Lemma 3.1.4. In particular, this prove that the forgetful functor is fully faithful. Now, we note that forget and glue are adjoint (the counit is tautologically id = glue ∘ forget). To finish off, prove that the other composite is equivalent to the identity (another exercise, using an old trick).

□

We can now conclude:

Proof of Lemma 8.1.9. We have a morphism $\mathbf{A}^2 \setminus 0 \to \mathbf{A}^2$. I claim that the map

$$\Gamma(\mathcal{O}_{\mathbf{A}^2}) \cong \Gamma(\mathcal{O}_{\mathbf{A}^2 \setminus 0})$$

is an isomorphism. Consider the following commutative diagram

$$\begin{array}{ccc} \mathbf{A}^1 \smallsetminus 0 \times \mathbf{A}^1 \smallsetminus 0 & \longrightarrow & \mathbf{A}^2 \smallsetminus \mathbf{A}^1_v \\ \downarrow & & \downarrow \\ \mathbf{A}^2 \smallsetminus \mathbf{A}^1_h & \longrightarrow & \mathbf{A}^2 \smallsetminus 0. \end{array}$$

Here $\mathbf{A}^1_h$ (resp. $\mathbf{A}^1_z$) is the horizontal $\mathbf{A}^1$ (resp. vertical $\mathbf{A}^1$). We note that

$$\{\mathbf{A}^2 \smallsetminus \mathbf{A}^1_v, \mathbf{A}^2 \smallsetminus \mathbf{A}^1_h \hookrightarrow \mathbf{A}^2 \smallsetminus 0\}$$

is an open cover of $\mathbf{A}^2 \smallsetminus 0$ and furthermore we have:

$$\mathbf{A}^2 \smallsetminus \mathbf{A}^1_h = \operatorname{Spec} \mathbf{Z}[x, y, x^{-1}] \qquad \mathbf{A}^2 \smallsetminus \mathbf{A}^1_v = \operatorname{Spec} \mathbf{Z}[x, y, y^{-1}].$$

Using Serre's theorem, a quasicoherent sheaf on $\mathbf{A}^2 \smallsetminus 0$ is given by:

(1) a $\mathbf{Z}[x, y, x^{-1}]$-module M,
(2) a $\mathbf{Z}[x, y, y^{-1}]$-module N,
(3) an isomorphism $\alpha : \mathrm{M}[y^{-1}] \cong \mathrm{M}[x^{-1}]$.

And so $\Gamma(\mathcal{O}_\mathrm{X})$ is exactly (by an exercise in this week's homework):

$$f \in \mathbf{Z}[x, y, x^{-1}] \qquad g \in \mathbf{Z}[x, y, y^{-1}],$$

such that $f = g$ as elements in $\mathbf{Z}[x, y, x^{-1}, y^{-1}]$. But this means that $f = g \in \mathbf{Z}[x, y]$. □

Along the way we have used:

Lemma 9.0.2. *Let* $\{j_i : \mathrm{U}_i \hookrightarrow \mathrm{X}\}$ *be a cover of* X *by affine schemes. Under the identification of Theorem 9.0.1, the functor*

$$\Gamma : \mathbf{QCoh}(\mathrm{X}, \mathcal{U}) \to \mathbf{QCoh}(\mathbf{Z})$$

is given by

$$(\mathcal{F}_{\mathrm{U}_i}, \alpha_{ij}) \mapsto \{f_i \in \mathcal{F}(\mathrm{U}_i, \mathcal{F}_{\mathrm{U}_i}) \in \mathbf{QCoh}(\mathrm{U}_i) : \alpha_{ij}(f_i) = f_j|_{\mathrm{U}_j}\}.$$

9.1. **Vector bundles.** One of the reasons to develop the theory of quasicoherent sheaves over a scheme was to do algebra over a scheme. We are on our way to doing linear algebra. But to actually do linear algebra we need a good theory of projective modules.

Here's a motivation:

Lemma 9.1.1. *Let* R *be a ring and* M *an* R*-module. Then the following are equivalent:*

(1) M *is in the smallest idempotent complete additive subcategory of* $\mathbf{Mod}_\mathrm{R}$ *containing* R.
(2) M *is finitely generated and projective,*
(3) M *is a direct summand of a finitely generated free module,*
(4) M *is finite and locally free in the sense that for any prime ideal* $\mathfrak{p}$ *of* R*, the module*

$$\mathrm{M}_\mathfrak{p} \in \mathbf{Mod}_{\mathrm{R}_\mathfrak{p}}$$

is a free module of finite rank,
(5) for any open cover $\{j_\mathrm{U} : \mathrm{U} \hookrightarrow \operatorname{Spec} \mathrm{R}\}$ *of affine schemes,* $j_\mathrm{U}^*\widetilde{\mathrm{M}}$ *is projective and finitely generated.*
(6) there exists an open cover $\{j_\mathrm{U} : \mathrm{U} \hookrightarrow \operatorname{Spec} \mathrm{R}\}$ *of affine schemes such that* $j_\mathrm{U}^*\widetilde{\mathrm{M}}$ *is finitely generated and free.*

The first definition gives finitely generated projective modules a kind of universal property. The third definition gives it an "equational property" — it is basically the same thing as an idempotent square matrix with entries in R. The fourth definition gives it a "local description".

I would like to prove the equivalence between the last two definitions which are geometric in nature. Having this we define

Definition 9.1.2. Let X be a scheme and $\mathcal{E} \in \mathbf{QCoh}(\mathrm{X})$ is a quasicoherent sheaf. Then $\mathcal{E}$ is a **vector bundle on** X (also often called **locally free sheaf of finite rank** if it is finite type and locally projective, i.e., there exists an cover of X of affines $\{j : \mathrm{U}_\alpha = \operatorname{Spec} \mathrm{A}_\alpha \hookrightarrow \mathrm{X}\}$ such that $j^*\mathcal{F} \in \mathbf{Mod}_{\mathrm{A}_\alpha}$ is finitely generated and projective.

We have the subcategory
$$\mathbf{Vect}(\mathrm{X}) \subset \mathbf{QCoh}(\mathrm{X})$$
spanned by those quasicoherent sheaves which are vector bundles.

In order to prove the equivalence of parts (4) and (5) of Lemma 9.1.1, let us formulate a version of Nakayama's lemma which is geometric in nature

9.2. **Exercises.**

Exercise 9.2.1. *Complete the proof of "reduction to basic opens" in Theorem 9.0.1.*

Exercise 9.2.2. *Prove Lemma 9.0.2.*

Exercise 9.2.3. *Construct a natural map in* $\mathbf{QCoh}(\mathrm{X})$
$$\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G}) \otimes \mathcal{H} \to \mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G} \otimes \mathcal{H})$$
such that if $\mathcal{F}$ *or* $\mathcal{H}$ *are vector bundles, then this map is an isomorphism.*

Exercise 9.2.4. *If* $\mathcal{E}$ *is a vector bundle, then denote its* **dual** *by*
$$\mathcal{E}^\vee := \mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{O}_\mathrm{X})$$
Prove:

(1) $(\mathcal{E}^\vee)^\vee \cong \mathcal{E}$,
(2) for any quasicoherent sheaf $\mathcal{F}$*, then* $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^\vee \otimes \mathcal{F}$.

Exercise 9.2.5. *Let* X *be a scheme. Let* $\mathcal{F} \in \mathbf{QCoh}(\mathrm{X}) \cong \mathbf{QCoh}(\mathrm{X}, \mathcal{U})$ *under Serre's theorem. Then* $\mathcal{I} \to \mathcal{O}_\mathrm{X}$ *is a* **quasicoherent ideal** *or also called an* **ideal sheaf** *if for each* $\mathrm{U} = \operatorname{Spec} \mathrm{A} \in \mathcal{U}$
$$\mathcal{I}_\mathrm{U} \subset \mathcal{O}_\mathrm{U}$$
is the inclusion of an ideal in $\mathbf{Mod}_\mathrm{A}$*. Let* $\mathcal{I}$ *be a quasicoherent ideal and define the following data: on* $j : \mathrm{U} \hookrightarrow \mathrm{X}$ *where* $\mathrm{U} \in \mathcal{U}$ *consider the closed subscheme of* U *defined via* $\mathcal{I}_\mathrm{U}$*, i.e.,*
$$\mathrm{Z}_\mathrm{U} := \operatorname{Spec} \Gamma(\mathcal{O}_\mathrm{U})/\mathcal{I}_\mathrm{U}.$$
Show that this defines subprestack $\mathrm{Z} \hookrightarrow \mathrm{X}$ *which is a scheme and is, furthermore, a closed subscheme.*

Prove that there is a canonical bijection between quasicoherent ideals of $\mathcal{O}_\mathrm{X}$ *and closed subschemes of* X.

Exercise 9.2.6. *Let* $f : \mathrm{Y} \to \mathrm{X}$ *be a morphism of schemes and* $\mathcal{F}, \mathcal{G} \in \mathbf{QCoh}(\mathrm{X})$.

(1) construct a natural morphism
$$\alpha_{\mathcal{F},\mathcal{G}} : f^*\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}\mathrm{om}(f^*\mathcal{F}, f^*\mathcal{G}),$$
which, if $f : \operatorname{Spec} \mathrm{A} \to \operatorname{Spec} \mathrm{B}$ *is a morphism of affine schemes, then the morphism is given by a morphism of* A*-modules*
$$\mathrm{Hom}_\mathrm{B}(\mathrm{M}, \mathrm{N}) \otimes_\mathrm{B} \mathrm{A} \to \mathrm{Hom}_\mathrm{A}(\mathrm{M} \otimes_\mathrm{B} \mathrm{A}, \mathrm{N} \otimes_\mathrm{B} \mathrm{A}),$$
adjoint to a map of B*-modules*
$$\mathrm{Hom}_\mathrm{B}(\mathrm{M}, \mathrm{N}) \to \mathrm{Hom}_\mathrm{A}(\mathrm{M} \otimes_\mathrm{B} \mathrm{A}, \mathrm{N} \otimes_\mathrm{B} \mathrm{A}) \qquad f \mapsto f \otimes \mathrm{id}_\mathrm{A}.$$
where the right hand side is given the structure of a B*-module by forgetting strucutre.*
(2) Show by example that such a morphism is not, in general, an isomorphism.
(3) Prove that if $\mathcal{F}$ *is a vector bundle, then the map is always an isomorphism.*

Exercise 9.2.7. *Prove all equivalences of (1)-(4) of Lemma 9.1.1*

10. Lecture 10: Nakayama's lemma, leftover on vector bundles

10.1. Nakayama's lemma revisited.

We all have seen many versions of Nakayama's lemma. Here's another one:

Lemma 10.1.1. *Let $\mathcal{F}$ be a finite type quasicoherent sheaf on* X. *Let k be a field and $x : \operatorname{Spec} k \to \mathrm{X}$ be a k-point of* X *such that $x^*\mathcal{F} = 0$, then there exists an open immersion $\mathrm{U} \hookrightarrow \mathrm{X}$ containing x such that $j^*\mathcal{F} = 0$.*

This is a geometric statement: if $\mathcal{F}$ vanishes on a point, it vanishes in a neighborhood.

Remark 10.1.2. Consider $\mathbf{Q}$ as a $\mathbf{Z}$-module. Then for any prime p consider the $\mathbf{F}_p$-point of $\mathbf{Z}$:

$$i : \operatorname{Spec} \mathbf{F}_p \to \operatorname{Spec} \mathbf{Z}.$$

Then $i^*\mathbf{Q} = \mathbf{F}_p \otimes_{\mathbf{Z}} \mathbf{Q} = 0$. However we note that $\mathbf{Z}_{(p)} \otimes_{\mathbf{Z}} \mathbf{Q} \neq 0$ and $\operatorname{Spec} \mathbf{Z}_{(p)}$ is the "smallest open" containing $\operatorname{Spec} \mathbf{F}_p$ in $\operatorname{Spec} \mathbf{Z}$. In particular $\mathbf{Q}$ does not vanish on any open of $\operatorname{Spec} \mathbf{Z}$. This shows that we really do need the finite generation of $\mathbf{Z}$.

Remark 10.1.3. In contrast, the easiest finitely generated $\mathbf{Z}$-module imaginable is of the form $\mathbf{Z}/q$. Say q is a prime which is not p. Then we note that

$$\mathbf{F}_p \otimes_{\mathbf{Z}} \mathbf{Z}/q = 0,$$

and furthermore

$$\mathbf{Z}_{(p)} \otimes_{\mathbf{Z}} \mathbf{Z}/q = 0,$$

exactly since q is invertible in $\mathbf{Z}_{(p)}$ so that

$$1 \otimes 1 = q/q \otimes 1 = 1/q \otimes 0 = 0.$$

There isn't really anything special about this argument, $\mathbf{Z}$ or $\mathbf{Z}/q$ and other friends. Say A is a local ring with maximal ideal $\mathfrak{m}$. Suppose that M is an A-module which is generated by a single generator; this means that we have a surjection of A-modules

$$\mathrm{A} \to \mathrm{M} = \mathrm{A}/\mathrm{I} \to 0$$

If we do know that $\mathrm{A}/\mathfrak{m} \otimes_{\mathrm{A}} \mathrm{M} = 0$ then for any $\overline{x}$

$$\overline{x} = \tfrac{y}{y}\overline{x} = \tfrac{\overline{x}}{y} y = 0 \cdot y = 0.$$

This simple algebraic observation is the basis of Nakayama's lemma.

Proof. We again break the proof down into several steps:

(Affine case) Let $\mathrm{X} = \operatorname{Spec} \mathrm{A}$ so that $\mathcal{F}$ corresponds to a finitely generated A-module M. In this case a k-point of X is the same thing as a map $\mathrm{A} \to k$. By hypothesis, $\mathrm{M} \otimes_{\mathrm{A}} k = 0$. We induct on the number of generators of M. If M is generated by no elements (so the zero module), then we are done. Now, say M is generated by n-elements. Therefore we can write M as an extension

$$0 \to \mathrm{N} \to \mathrm{M} \to \mathrm{A}/\mathrm{I} \to 0,$$

where N is generated by $n-1$-elements. We do know that $\mathrm{M} \otimes_{\mathrm{A}} k = 0$. The tensor product is right exact so that we have a surjection

$$0 \to \mathrm{A}/\mathrm{I} \otimes_{\mathrm{A}} k \to 0,$$

hence $\mathrm{A}/\mathrm{I} \otimes_{\mathrm{A}} k = 0$ and thus $k/\mathrm{I}k = 0$ and thus

$$k = \mathrm{I}k.$$

Since k is not the zero ring, this means that we can find an $f \in \mathrm{A}$ such that its image in k is nonzero and therefore the map $\mathrm{A} \to k$ factors as

$$\mathrm{A}[f^{-1}] \to k.$$

Localizing is exact and thus we have an exact sequence

$$0 \to \mathrm{N}[f^{-1}] \to \mathrm{M}[f^{-1}] \to \mathrm{A}/\mathrm{I}[f^{-1}] \to 0,$$

But now $\mathrm{A}/\mathrm{I}[f^{-1}] = 0$ and thus $\mathrm{N}[f^{-1}] \to \mathrm{M}[f^{-1}]$ is an isomorphism. As A_f-modules, these are generated by $n-1$-elements and so the inductive hypothesis applies.

(Assembling) We have the open immersion $j : \operatorname{Spec} \mathrm{A}_f \hookrightarrow \operatorname{Spec} \mathrm{A}$. We note that $j^*\mathrm{N} \cong j^*\mathrm{M}$ and are modules over A_f generated by $n-1$-elements. We further note that the map $x : \operatorname{Spec} k \to \operatorname{Spec} \mathrm{A}$ factors through $\operatorname{Spec} \mathrm{A}_f$ as noted above so that $x^*j^*\mathrm{N} \cong x^*j^*\mathrm{M} = 0$. Therefore we can find an open U

$$x \in \mathrm{U} \subset \operatorname{Spec} \mathrm{A}_f \subset \operatorname{Spec} \mathrm{A}$$

such that $\mathrm{M}|_{\mathrm{U}} = 0$. This is the desired open.

(Globalizing) By the definition of a scheme, there exists an affine scheme Spec A such that $\operatorname{Spec} \mathrm{A} \times_{\mathrm{X}} \operatorname{Spec} k \neq \emptyset$. We can then replace X by Spec A.

□

We now prove

Proof of (some parts of) Lemma 9.1.1. Everything in sight is local so we assume that $\mathrm{X} = \operatorname{Spec} \mathrm{A}$.

First, let us assume that $\mathcal{E}$ is finitely generated and projective. Our goal is to produce a cover of X on which $\mathcal{E}$ is finitely generated and free. Let $\mathfrak{m}$ be a maximal ideal of A so that we obtain a field point

$$i_\kappa : \operatorname{Spec} \mathrm{A}/\mathfrak{m} = \kappa \to \operatorname{Spec} \mathrm{A}$$

of the scheme Spec A. Then we note that $i_\kappa^*\mathcal{E} = \mathcal{E}/\mathfrak{m}$ is a finitely generated $\mathrm{A}/\mathfrak{m}$-module, i.e., a vector space over κ of finite dimension. Let's say the dimension is n we can choose a basis

$$\overline{v}_1, \cdots \overline{v}_n \in \mathcal{E}/\mathfrak{m},$$

which we can then lift to elements of the A-module

$$v_1, \cdots, v_n \in \mathcal{E}.$$

In particular we obtain a morphism

$$v : \mathrm{A}^{\oplus} \to \mathcal{E}.$$

I claim:

- there exists an open affine $\mathrm{U} \hookrightarrow \operatorname{Spec} \mathrm{A}$ such that i_κ factors through it:

$$\operatorname{Spec} \kappa \xrightarrow{i_\kappa} \mathrm{U} \hookrightarrow \operatorname{Spec} \mathrm{A}$$

  such that the map $v|_{\mathrm{U}} : \widetilde{\mathrm{A}^{\oplus n}}|_{\mathrm{U}} \to \mathcal{E}|_{\mathrm{U}}$ is an isomorphism.

From this we are done: extract the open cover of affines

$$\{\mathrm{U}_\kappa \to \operatorname{Spec} \mathrm{A}\},$$

and furthermore can be refined to a finite collection.

The claim breaks down into injectivity and surjectivity (of course we need to find such a U):

(Surj.) We have the exact sequence of A-modules (equivalently an exact sequence in $\mathbf{QCoh}(\operatorname{Spec} \mathrm{A})$):

$$\mathrm{A}^{\oplus n} \to \mathcal{E} \to \operatorname{coker}(v) \to 0.$$

Observe that $\operatorname{coker}(v)$ is finitely generated as well since it is the quotient of a map between finitely generated modules. Furthermore we know that $\operatorname{coker}(v)|_{\operatorname{Spec}\kappa} = 0$ and therefore Nakayama's lemma furnishes an open $\mathrm{U} \hookrightarrow \operatorname{Spec} \mathrm{A}$ such that $\operatorname{coker}(v)|_{\mathrm{U}} = 0$. By shrinking U further we may assume that U is actually affine.

(Inj.) Here we need to use exactness: the map v splits since $\mathcal{E}$ is projective and therefore $\ker(v)$ is finitely generated. Furthermore $\ker(v)|_{\operatorname{Spec}\kappa} = 0$ and therefore, by Nakayama again, we can find an affine open V of Spec A through which i_κ factors and v is injective. Taking $\mathrm{U} \times_{\operatorname{Spec} \mathrm{A}} \mathrm{V}$ we are done.

The converse trickier. We first leave an exercise the following assertion:

- for a ring A and $\{A \to A_{f_i}\}$ a Zariski open cover. Let M be an A-module and assume that each M_{f_i} is finitely generated (resp. finitely presented) as an A_{f_i}-module, then M is a finitely generated (finitely presented) A-module.

Assuming this exercise, let's prove the result. To prove projectivity take

$$0 \to N' \to N \to N'' \to 0$$

a short exact sequence of A-modules. We want to prove that

$$0 \to \mathrm{Hom}_A(M, N') \to \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N'') \to 0$$

is a short exact sequence of R-modules. It suffices, from a previous lemma, to prove that

$$0 \to \mathrm{Hom}_A(M, N')[f_i^{-1}] \to \mathrm{Hom}_A(M, N)[f_i^{-1}] \to \mathrm{Hom}_A(M, N'')[f_i^{-1}] \to 0$$

is exact for all f_i's such that $\{A \to A_{f_i}\}$ an Zariski open cover. The next lemma finishes the proof.

□

The next lemma is kind of underrated.

Lemma 10.1.4. *Let* A *be a ring,* M *finitely presented* A*-module and* N *an* R*-module. Then for any* $f \in A$

$$\mathrm{Hom}_A(M, N)[f^{-1}] \cong \mathrm{Hom}_A(M[f^{-1}], N[f^{-1}]).$$

Proof. If M is finitely presented, then we can present M as

$$\bigoplus_{i=1}^{n} A \to \bigoplus_{j=1}^{m} A \to M \to 0.$$

In particular we must prove this in the case that M is just finitely generated and free. For this, we have

$$\begin{aligned} \mathrm{Hom}_A(A^{\oplus n}, N)[f^{-1}] &\cong N^{\oplus n}[f^{-1}] \\ &\cong N[f^{-1}]^{\oplus n} \\ &\cong \mathrm{Hom}_{A_f}(A[f^{-1}]^{\oplus n}, N[f^{-1}]). \end{aligned}$$

The proof then finishes off by examining the following diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_A(M, N)[f^{-1}] & \longrightarrow & \mathrm{Hom}_A(\bigoplus_{j=1}^{m} A, N)[f^{-1}] & \longrightarrow & \mathrm{Hom}_A(\bigoplus_{j=1}^{n} A, N)[f^{-1}] \\ & \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 \longrightarrow & \mathrm{Hom}_A(M[f^{-1}], N[f^{-1}]) & \longrightarrow & \mathrm{Hom}_A(\bigoplus_{j=1}^{m} A[f^{-1}], N[f^{-1}]) & \longrightarrow & \mathrm{Hom}_A(\bigoplus_{j=1}^{n} A[f^{-1}], N[f^{-1}]) \end{array}$$

□

10.2. **Line bundles and examples.** One of the things algebraic geometry is obsessed about (for good reasons) is the idea of defining *numerical invariants*: we associate to an algebro-geometric object a number which varies well (to varying degrees) in families. This last requirement is more difficult to satisfy than you think in the sense that invariants are usually not easy to define globally but easy to define locally. The task is then to give a local definition then prove that it globalizes well.

Definition 10.2.1. Let X be a scheme and suppose that $\mathcal{E}$ is a vector bundle on X. Let $x : \mathrm{Spec}\, k \to X$ be a field point, then the **rank of** $\mathcal{E}$ **at x** is defined as follows: by Lemma 9.1.1 there exists an affine open containing x,

$$U \hookrightarrow \mathrm{Spec}\, X$$

such that $\mathcal{E}|_U \simeq \mathcal{O}_U^{\oplus n}$. We define the the rank to be n:

$$\mathrm{rank}_x(\mathcal{E}) := n.$$

We say that a vector bundle $\mathcal{E}$ on X is **of constant rank** n or, if the context is clear, is **rank** n if for all $x : \operatorname{Spec} k \to \mathrm{X}$, it is indeed of rank n.

It is left an exercises to check that the notion of rank at a point is well-defined.

Remark 10.2.2. Suppose that $\mathrm{X} = \operatorname{Spec} \mathrm{A}$ and we have a decomposition

$$\operatorname{Spec} \mathrm{A} \cong \operatorname{Spec} \mathrm{B} \sqcup \operatorname{Spec} \mathrm{C}.$$

Then we see that a vector bundle on X is the same thing as a vector bundle on Spec B and another on Spec C. They do not have to have same rank. For now, rank is an intrinsically local notion.

Definition 10.2.3. A **line bundle** on a scheme X is a vector bundle of rank 1.

Example 10.2.4. We would like to define a vector bundle on $\mathbf{A}^2 \setminus 0$. Here is one way to do it.

- set M on $\mathbf{Z}[x, x^{-1}, y]$ to be $\mathcal{O}$ and N to be $\mathcal{O}$ on $\mathbf{Z}[x, y, y^{-1}]$.
- Now on $\mathbf{Z}[x, x^{-1}, y, y^{-1}]$ we need to specify an isomorphism; we can take any automorphism
$$\mathcal{O} \to \mathcal{O};$$
we note that this automorphism must be $\mathbf{Z}[x, x^{-1}, y, y^{-1}]$-linear and hence is determined by an element of
$$(\mathbf{Z}[x, x^{-1}, y, y^{-1}])^{\times} = \{\pm x^{\pm k}, \pm y^{\pm j}, \}$$
and this will specify the desired line bundle. Weirdly, any such line bundle is trivial (exercise).

10.3. Exercises.

Exercise 10.3.1. *Let* A *be a ring and* $\{\mathrm{A} \to \mathrm{A}_{f_i}\}$ *a Zariski open cover. Let* M *be an* A*-module and assume that each* M_{f_i} *is finitely generated (resp. finitely presented) as an* A_{f_i}*-module, then* M *is a finitely generated (finitely presented)* A*-module.*

Exercise 10.3.2. *Prove that the notion of rank is independent of choices.*

Exercise 10.3.3. *Prove that the line bundles defined in Example 10.2.4 is actually trivial.*

Exercise 10.3.4. *Let* X *be a scheme. Prove that the following are equivalent:*

(1) $\mathcal{L}$ *is a line bundle on* X*,*
(2) the evaluation map
$$\mathcal{L}^{\vee} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{L} \to \mathcal{O}_{\mathrm{X}}$$
is an isomorphism.

Exercise 10.3.5. *Let* A *be a ring. Prove that* $\mathcal{L}$ *is a line bundle if and only if the modules associated to* $\mathcal{L}$*, call it* L*, is projective and for any maximal ideal* $\mathfrak{m}$ *of* A*,* $\mathrm{L}_{\mathfrak{m}}$ *is rank* 1*. Conclude that the following affine schemes admits no nontrivial line bundles:*

(1) A *is a local ring,*
(2) A *is factorial (all prime ideals of height* 1 *are principal).*

Exercise 10.3.6. *Let* R *be a ring of characteristic* p *and* $\mathcal{L}$ *a line bundle on* Spec R*. Prove that the Frobenius map*

$$\mathrm{F} : \operatorname{Spec} \mathrm{R} \to \operatorname{Spec} \mathrm{R}$$

induces an isomorphism, for any line bundle $\mathcal{L}$*,*

$$\mathrm{F}^*(\mathcal{L}) \cong \mathcal{L}^{\otimes p}.$$

Conclude the same result for any scheme over $\mathbf{F}_p$*.*

11. Lecture 11: the projective space

We continue trying to give examples of line bundles.

Example 11.0.1. Let k be a field of characteristic zero. We will consider the following affine scheme: let R be

$$\mathrm{R} = k[x,y]/(y^2 - x^3 - 1).$$

In other words, it is the closed subscheme of $\mathbf{A}^2$ cut out by the equation $y^2 = x^3 + 1$. This is an example of a **punctured elliptic curve** but more on that later. Consider

$$\mathrm{X} = \operatorname{Spec}\mathrm{R}.$$

Our goal is to produce a nontrivial vector bundle on R. In order to do so, let us pick a Zariski open cover of $\operatorname{Spec}\mathrm{R}$; so we give two elements of R, g_1, g_2 such that the ideal

$$(g_1, g_2) = \mathrm{R}.$$

Consider $g_1 = y - 1, g_2 = y + 1$. In this case, we have that

$$g_1 - g_2 = y - 1 - (y+1) = -2$$

so that indeed $(g_1, g_2) = \mathrm{R}$.

Consider the ideal

$$\mathcal{L} = (x, y+1).$$

Geometrically, this coincides with the point $(0, -1) \in \mathrm{X}(k)$. Now, I claim that $\mathcal{L}$ is free on the charts $\operatorname{Spec}\mathrm{R}_{g_1}, \operatorname{Spec}\mathrm{R}_{g_2}$. Indeed

$$\mathcal{L}|_{\mathrm{R}_{g_2}} = (x, y+1)[\tfrac{1}{y+1}] = (y+1),$$

while

$$\mathcal{L}|_{\mathrm{R}_{g_1}} = (x, y+1)[\tfrac{1}{y-1}] = (x)$$

since

$$x^3 = 1 - y^2 = (1-y)(1+y) \Rightarrow \tfrac{x^3}{1-y} = 1 + y.$$

Therefore we see that $\mathcal{L}$ is a locally free sheaf.

11.1. An attempted definition.

For this discussion, we suppose that we are working in Sch_k where k is a field. Our goal is to define $\mathbf{P}^n_k$, the projective space over k. We have an idea of what this is: its k-points should be should be the set of lines in the $n+1$-dimensional vector space k^{n+1} or, equivalently,

$$\mathbf{P}^n_k(k) = (k^{n+1} \smallsetminus 0)/k^{\times}$$

where $k^{\times}$ act on the vector space k^{n+1} by scaling. In topology, we can make this set and declare a topology on it by giving it the quotient topology. This misses so much of the point of algebraic geometry of course.

Let us attempt to define the $\operatorname{Spec}\mathrm{A}$-points of $\mathbf{P}^n$ in a functorial way: given a k-morphism $\mathrm{A} \to \mathrm{B}$, we need a map

$$\mathbf{P}^n_k(\mathrm{A}) \to \mathbf{P}^n_k(\mathrm{B});$$

in particular given a map $\mathrm{A} \to k$ (thought of as $\operatorname{Spec} k \to \operatorname{Spec}\mathrm{A}$) we need to get

$$\mathbf{P}^n_k(\mathrm{A}) \to \mathbf{P}^n_k(k).$$

If we had rolled with something like $\mathbf{A} \smallsetminus 0/\mathrm{A}^{\times}$ then note that we might accidentally "pull points" back to zero: indeed say $\mathrm{A} = k[\epsilon]$ and we consider the map $k[\epsilon] \to k, \epsilon \mapsto 0$, then the class of ϵ will be sent to zero.

This says that the naive formulation of projective space is not quite right. Here's another thing we can do:

Definition 11.1.1. Define the prestack

$$(\mathbf{P}^n_k)^{\mathrm{naive}} : \mathrm{CAlg}_k \to \mathrm{Set}$$

$$\mathrm{A} \mapsto \{v \in \mathrm{A}^{n+1} : \forall x : \operatorname{Spec}\kappa \to \mathrm{A}, x^*(v) \in k^{n+1} \neq 0\}/\{v = aw : a \in \mathrm{A}^{\times}\}.$$

This is not such a bad definition but this does not satisfy Zariski descent.

11.2. **Line bundles as a solution.** Here is the problem: nontrivial line bundles exist on affine schemes! Indeed the equivalence class of v above is basically given by a map

$$\mathcal{O}_{\mathrm{Spec\,A}} \to \mathrm{A}^{\oplus n+1}.$$

Now, a line bundle is, in particular, a quasicoherent sheaf which is locally of the form $\mathcal{O}$ but *not* globally so. In other words, as A varies, there is no assurance that we can glue together the maps $\mathcal{O}_{\mathrm{A}} \to \mathrm{A}^{n+1}$ as the $\mathcal{O}$'s need not glue. To make this precise let us go right into the solution. First we need to define the above "nondegeneracy condition" in a more elegant manner.

11.3. **Nondegeneracy conditions and the definition of projective space.**

Lemma 11.3.1. *Let* A *be a ring and* M, N *be* A*-modules which are locally free and* $\varphi : \mathrm{M} \to \mathrm{N}$ *be a morphism. The following are equivalent:*

(1) for each maximal ideal $\mathfrak{m} \subset \mathrm{A}$*, the induced map*

$$\mathrm{M}/\mathfrak{m}\mathrm{M} \to \mathrm{N}/\mathfrak{m}\mathrm{N}$$

is injective,

(2) the dual map

$$\varphi^\vee : \mathrm{M}^\vee \to \mathrm{N}^\vee$$

surjects,

(3) the image of M *in* N *can be split off* M *as a summand,*

(4) the map φ *is injective with locally free cokernel.*

This lemma is an exercise.

Lemma 11.3.2. *Let* $\varphi : \mathcal{L} \to \mathcal{E}$ *be a morphism of vector bundles over* X *where* $\mathcal{L}$ *is a line bundle and* $\mathcal{E}$ *is of constant rank* n*. The following are equivalent:*

(1) for each point $\mathrm{Spec}\, k \to \mathrm{X}$*, the map*

$$x^*\varphi : x^*\mathcal{L} \to k^{\oplus n}$$

is injective,

(2) the dual map

$$\varphi^\vee : \mathcal{E}^\vee \to \mathcal{L}^\vee$$

is surjective,

(3) the image of $\mathcal{L}$ *under* φ *can be split off* $\mathcal{E}$ *as a summand,*

(4) the map φ *is injective and cokernel is locally free,*

(5) there exists a Zariski open cover $\{\mathrm{U} \hookrightarrow \mathrm{X}\}$ *such that on each* $j : \mathrm{U} \hookrightarrow \mathrm{X}$*,* $j^*\mathcal{E}$ *is trivial and*

$$\mathcal{L}|_{\mathrm{U}} \to \mathcal{E}|_{\mathrm{U}} \cong \mathcal{O}^{\oplus n} \to \mathcal{O}_{\mathrm{U}}$$

is an isomorphism.

Proof. We assume (1). Let us prove that for each $\mathrm{Spec}\, k \to \mathrm{X}$, there exists an affine open $\mathrm{U} \hookrightarrow \mathrm{X}$ over which $\mathcal{E}$ is trivial and such such that

$$\mathcal{L}|_{\mathrm{U}} \to \mathcal{E}|_{\mathrm{U}} \to \mathcal{O}_{\mathrm{U}}$$

is an isomorphism. Indeed, since the map $x^*\mathcal{L} \to x^*\mathcal{E} \cong k^{\oplus n}$ is an isomorphism, Nakayama's lemma furnishes an open affine containing x such that

$$\mathcal{L}|_{\mathrm{U}} \cong \mathcal{O}_{\mathrm{U}}$$

is an isomorphism; we can of course arrange this map compatibly with the trivialization of $\mathcal{E}$ by letting U be small enough. □

Any morphism $\mathcal{L} \to \mathcal{E}$ of the above form will be called a **bundle injection**. Finally:

Construction 11.3.3. Let $n \geqslant 0$. We define

$$\mathbf{P}^n : \mathrm{CAlg} \to \mathrm{Set}$$

as sending A to

$$\{(\mathcal{L}, \varphi) : \varphi : \mathcal{L} \to \mathrm{A}^{\oplus n} \text{ a bundle injection}\}$$

Theorem 11.3.4. *The prestack* $\mathbf{P}^n$ *is a scheme.*

11.4. **Exercises.**

Exercise 11.4.1. *Prove Lemma 11.3.1.*

Exercise 11.4.2. *Let* X *be a scheme. By Serre's theorem an* **ideal sheaf** *on a* X *is an object* $\mathcal{I} \in \mathbf{QCoh}(\mathrm{X}, \mathcal{U})$ *such that for each* $j_{\mathrm{U}} : \mathrm{U} = \operatorname{Spec} \mathrm{A} \hookrightarrow \mathrm{X}$, $j_{\mathrm{U}}^*(\mathcal{I})$ *is an ideal of* A. *Prove that there is a natural bijection between*

$$\{\mathrm{Z} \subset \mathrm{X} : \mathrm{Z} \textit{ is a closed immersion}\}$$

and

$$\{\mathcal{I} \subset \mathcal{O}_{\mathrm{X}} : \mathcal{I} \textit{ is an ideal sheaf of } \mathcal{O}_{\mathrm{X}}\}$$

Exercise 11.4.3. *Let* $\mathcal{E}$ *be a locall free sheaf on* X. *Consider the functor*

$$\mathrm{End}(\mathcal{E}) : \mathrm{CAlg} \to \mathrm{Set}$$

be the functor that sends $\mathrm{A} \in \mathrm{CAlg}$ *to a pair*

$$(x, \varphi)$$

where $x : \operatorname{Spec} \mathrm{A} \to \mathrm{X}$ *is a map and* $\varphi : x^*\mathcal{E} \to x^*\mathcal{E}$ *is an endomorphism. Prove that* $\mathrm{End}(\mathcal{E})$ *is representable by a scheme (hint: if* R *is a ring, why is* $\mathrm{M}_n(\mathrm{R})$ *a scheme?).*

Exercise 11.4.4. *Let* $\mathrm{Aut}(\mathcal{E}) \hookrightarrow \mathrm{End}(\mathcal{E})$ *be the subprestack defined by those endomorphisms which are automorphisms. Prove that* $\mathrm{Aut}(\mathcal{E})$ *is a scheme and the map above is an open immersion.*

12. Lecture 12: vector bundles, affine morphisms and projective bundles

Last time, we defined the projective n-space as a prestack. It assigns to a commutative ring A the set of isomorphism classes of "pictures"

$$\{0 \to \mathcal{L} \to \mathcal{O}_{\mathrm{A}}^{\oplus n+1}\},$$

which are subject to the nondegeneracy conditions spelled out in Lemma 11.3.2. One of the most useful formulation of this condition is as follows: (1) we have an injection:

$$0 \to \mathcal{L} \to \mathcal{O}_{\mathrm{A}}^{\oplus n+1},$$

such that the (2) cokernel is a locally free A-module or, equivalently, a vector bundle on $\mathcal{O}_{\mathrm{A}}^{\oplus n+1}$. Necessarily the cokernel is $\mathcal{E}$, a vector bundle of rank $n-1$ though it does not have to be globally free. We seek to prove that $\mathbf{P}^n$ is indeed a scheme, but this turns out to push us towards understanding better the nature of quasicoherent sheaves and that will be a digression that takes precedence this week. I will first define the relative version of $\mathbf{P}^n$. By this, I mean that we replace the trivial vector bundle of rank n, $\mathcal{O}^{\oplus n}$, with a vector bundle on a scheme $\mathcal{E}$.

Definition 12.0.1. Let X be a scheme and $\mathcal{E} \in \mathrm{Vect}(\mathrm{X})$. Then $\mathbf{P}_{\mathrm{X}}(\mathcal{E})$ (or just $\mathbf{P}(\mathcal{E})$ when the context is clear), the **projectivization of** $\mathcal{E}$ or the **projective bundle associated to** $\mathcal{E}$ is the functor

$$\mathbf{P}(\mathcal{E}) : \mathrm{CAlg} \to \mathrm{Set},$$

defined as follows for a commutative ring A, the set $\mathbf{P}(\mathcal{E})(\mathrm{A})$ are isomorphism classes of

$$(\mathcal{L}, x, i),$$

where $x : \operatorname{Spec} \mathrm{A} \to \mathrm{X}$ is a morphism, $\mathcal{L}$ is a line bundle on X and $i : \mathcal{L} \to x^*(\mathcal{E})$ is a bundle injection.

Theorem 12.0.2. *The prestack* $\mathbf{P}_{\mathrm{X}}(\mathcal{E})$ *is a scheme. It comes equipped with a canonical morphism* $\mathbf{P}_{\mathrm{X}}(\mathcal{E}) \to \mathrm{X}$ *and there exists a Zariski cover* $\mathcal{U}$ *of* X *such that for each* $\mathrm{U}_i \hookrightarrow \mathrm{X} \in \mathcal{U}$*, we have that*

$$\mathbf{P}_{\mathrm{X}}(\mathcal{E}) \times_{\mathrm{X}} \mathrm{U}_i \cong \mathbf{P}^n \times \mathrm{U}_i$$

for some $n \geqslant 0$.

We now work towards proving this result. To this end, we will introduce a very important technique in algebraic geometry: specifying the nonvanishing of an equation by the nonvanishing of a section.

Remark 12.0.3. Let $\mathrm{X} = \operatorname{Spec} \mathrm{A}$ be an affine scheme. Throughout this class, we seem to like those affine opens in Spec A which are defined as complement of the vanishing of an $f \in \mathrm{A}$: this is usually written as $\operatorname{Spec} \mathrm{A}_f \hookrightarrow \mathrm{X}$. Under the equivalence $\mathbf{QCoh}(\mathrm{X}) \simeq \mathbf{Mod}_{\mathrm{A}}$ recall that we can regard f as an element of $\Gamma(\mathcal{O}_{\mathrm{X}})$, i.e., a map $f : \mathcal{O}_{\mathrm{X}} \to \mathcal{O}_{\mathrm{X}}$. The open subscheme $\operatorname{Spec} \mathrm{A}_f$ is, in this way, a special case of the **nonvanishing locus** of a section of a vector bundle.

To formalize this, we have the following lemma which invites a rather lengthy but motivated digression.

Lemma 12.0.4. *Let* X *be a scheme and* $\mathcal{E} \in \mathbf{QCoh}(\mathrm{X})$ *and* $f \in \Gamma(\mathcal{E})$ *is a global section. Consider the subprestack of* X*:*

$$\mathrm{X}_f \subset \mathrm{X}$$

defined as follows: a morphism $x : \operatorname{Spec} \mathrm{A} \to \mathrm{X}$ *factors through* x_f *if and only if*

$$f_{x,\mathrm{A}} : \mathrm{A} \to x^*\mathcal{E}$$

is nonzero. Then X_f *is an open subscheme of* X.

Let us see why this is useful

Start proof of Theorem 11.3.4. We will repeat this proof again, so do not worry but I just want to give a taster about the above lemma. Suppose that we want to produce an open affine cover of $\mathbf{P}^n$. We define a $\mathbf{P}^n_{s_i \neq 0} \hookrightarrow \mathbf{P}^n$ as follows: a morphism $x : \operatorname{Spec} \mathrm{A} \to \mathbf{P}^n$ factors through $\mathbf{P}^n_{s_i \neq 0}$ if and only if the map

$$(s_0, \cdots, s_n) : \mathcal{L} \to \mathcal{O}_{\mathrm{A}}^{n+1}$$

satisfies $s_i \neq 0$. Indeed, let us note that, collectively, $(s_0, \cdots, s_n)$ is nonvanishing but we are imposing a stronger condition.

To prove that $\mathbf{P}^n_{s_i \neq 0}$ is open, we take an affine scheme $x : \operatorname{Spec} \mathrm{A} \to \mathbf{P}^n$ so that this corresponds to a picture $0 \to \mathcal{L} \to \mathcal{O}_{\mathrm{A}}^{n+1}$. We look at the i-th coordinate $\mathcal{L} \to \mathcal{O}_{\mathrm{A}}$ and also its dual $f : \mathcal{O}_{\mathrm{A}} \to \mathcal{L}^{\vee}$ which is a global section of the line bundle $\mathcal{L}^{\vee}$. Now, we have an isomorphism:

$$\operatorname{Spec} \mathrm{R} \times_{\mathbf{P}^n} \mathbf{P}^n_{s_i \neq 0} \cong (\operatorname{Spec} \mathrm{A})_f,$$

and hence Lemma 12.0.4 tells us that we have an open subscheme of Spec R. □

I would really like to emphasize that X_f is nothing but a fancy version of $\operatorname{Spec} \mathrm{A}_f$. On the other hand, since affine schemes can have trivial line bundles this construction is strictly more general than just $\operatorname{Spec} \mathrm{A}_f$. In order to prove this result, we would like to express X_f as a pullback of an open immersion. As I have alluded to throughout these lectures, we should really think of quasicoherent sheaves as bundles. Suppose that we can do this, then we should functorially associate to every quasicoherent sheaf $\mathcal{F}$ on X a scheme $\mathbf{V}(\mathcal{F})$, equipped with a canonical map

$$\mathbf{V}(\mathcal{F}) \to \mathrm{X}.$$

Furthermore, we should be able to make sense of a "zero section":

$$\mathbf{V}(\mathcal{F}) \leftarrow \mathrm{X} : z,$$

whose image is a closed subscheme of $\mathbf{V}(\mathcal{F})$ so that its complement is an open subscheme. Therefore X_f is the pullback

$$X_f = X \times_{\mathbf{V}(\mathcal{F})} \mathbf{V}(\mathcal{F}) \setminus z(X),$$

where the first map is defined by the section defined by f. Let us make these ideas precise.

12.1. **Total space.** Here is the much anticipated geometric interpretation of quasicoherent sheaves.

Definition 12.1.1. Let $\mathcal{F} \in \mathbf{QCoh}(X)$, then its **total space** is the prestack

$$\mathbf{V}_X(\mathcal{F}) : \mathrm{CAlg} \to \mathrm{Set},$$

which sends A to the isomorphism classes of data:

$$(x, \varphi),$$

where $x : \operatorname{Spec} A \to X$ is a morphism and φ is a map in $\mathbf{QCoh}(\operatorname{Spec} A)$ given by

$$\varphi : x^*\mathcal{F} \to \mathcal{O}_A.$$

By definition $\mathbf{V}_X(\mathcal{F})$ comes equipped with a map $\mathbf{V}_X(\mathcal{F}) \to X$, endowing it with the structure of an X-scheme.

Example 12.1.2. Let $\mathcal{F} = \mathcal{O}_X^{\oplus n}$, then we see that $\mathbf{V}(\mathcal{O}_X^{\oplus n})$ is isomorphic as an X-scheme to $\mathbf{A}^n \times X =: \mathbf{A}^n_X$. Indeed this can be checked by just looking at A-points.

Remark 12.1.3. The intuition behind $\mathbf{V}_X(\mathcal{F})$ is that it is the "classifying scheme" for sections of $\mathcal{F}$ or, rather, its dual. This interpretation, however, is only completely valid when $\mathcal{F}$ is a vector bundle; the general situation only lets us think about "cosections." Indeed, A-points of $\mathbf{V}_X(\mathcal{F})$ is given by maps

$$\varphi : x^*\mathcal{F} \to \mathcal{O}_A,$$

which is, equivalently, a section:

$$s : \mathcal{O}_A \to (x^*\mathcal{F})^\vee.$$

Note that other textbooks might employ the opposite conventions though. One reason to adopt our convention is the following one: let $X = \operatorname{Spec} k$, then a vector bundle on X is just a finite dimensional vector space V. Consider points of

$$\mathbf{V}_k(V),$$

which we abstractly know is isomorphic to affine n-space. A k-point of this scheme then classifies a k-linear map

$$k \to V^\vee,$$

which picks up some element in the dual space. In particular, we can choose a dual basis of V this way which we call $x_1, \cdots, x_n$ and eventually write

$$\mathbf{V}_k(V) \cong \operatorname{Spec} k[x_1, \cdots, x_n].$$

This makes sense: the coordinates x_i are the equations that cut out various hyperplanes in the vector space V and hence are naturally elements of the dual space.

We would like to prove the following result.

Theorem 12.1.4. *The total space $\mathbf{V}(\mathcal{F})$ is a scheme equipped with a canonical morphism to X.*

In order to prove Theorem 12.1.4, we will capture it as an instance of a more general construction.

Construction 12.1.5. Let $\mathrm{E} \in \mathbf{Mod}_{\mathrm{A}}$ be an A-module. Then its **tensor algebra** is the associative A-algebra given by

$$\mathrm{Tens}(\mathrm{E}) = \mathrm{Tens}_{\mathrm{A}}(\mathrm{E})^3 := \bigoplus_{n \geqslant 0} \mathrm{T}_{n,\mathrm{A}}(\mathrm{E}),$$

where each $\mathrm{T}_{n,\mathrm{A}}(\mathrm{E}) := \mathrm{E}^{\otimes_{\mathrm{A}} n}$ and the multiplication is given by concantenating of tensors. The **symmetric algebra** is commutative, graded A-algebra

$$\mathrm{Sym}(\mathrm{E}) = \mathrm{Sym}_{\mathrm{A}}(\mathrm{E}) := \mathrm{Tens}(\mathrm{E})/\mathrm{I} \cong \bigoplus_{n \geqslant 0} \mathrm{S}_{n,\mathrm{A}}(\mathrm{E})$$

where I is the two-sided ideal generated by

$$\{m \otimes n - n \otimes m\}.$$

We note that:

(1) $\mathrm{S}_{0,\mathrm{A}}(\mathrm{E}) = \mathrm{A}$,
(2) $\mathrm{S}_{1,\mathrm{A}}(\mathrm{E}) = \mathrm{E}$.

Example 12.1.6. Let $\mathrm{E} = \mathrm{A}^{\oplus n}$, then

$$\mathrm{Sym}(\mathrm{E}) = \mathrm{A}[x_1, \cdots, x_n].$$

In particular, when presenting $\mathrm{E} = \mathrm{A}^{\oplus n}$ we have implicitly chosen a basis for the free A-module of rank n and each of these x_i's then correspond to the "pure tensor" on a basis element.

Lemma 12.1.7. *There is a canonical morphism of* A*-modules* $\mathrm{E} \to \mathrm{Sym}(\mathrm{E})$ *which satisfies the following universal property: for any (commutative)* A*-algebra* B*, the canonical map above indues a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{CAlg}_{\mathrm{A}}}(\mathrm{Sym}(\mathrm{E}), \mathrm{B}) \cong \mathrm{Hom}_{\mathbf{Mod}_{\mathrm{A}}}(\mathrm{E}, \mathrm{B}).$$

Corollary 12.1.8. *Let* $\varphi : \mathrm{A} \to \mathrm{B}$ *be a ring map, then for any* $\mathrm{E} \in \mathbf{Mod}_{\mathrm{A}}$*, we have a canonical isomorphism*

$$\mathrm{Sym}(\mathrm{E}) \otimes_{\mathrm{A}} \mathrm{B} \xrightarrow{\cong} \mathrm{Sym}(\mathrm{E} \otimes_{\mathrm{A}} \mathrm{B}).$$

Proof. This is easy using Lemma 12.1.7 and is probably one of your first times checking that two objects are isomorphic using universal properties so let's do it briefly. The claim is that to given a B-algebra map $\mathrm{Sym}(\mathrm{E}) \otimes_{\mathrm{A}} \mathrm{B} \to \mathrm{C}$ it is the same as giving a B-module map $\mathrm{E} \otimes_{\mathrm{A}} \mathrm{B} \to \mathrm{C}$. However, by the universal property of the tensor product, the former map is the same data as an A-algebra map $\mathrm{Sym}(\mathrm{E}) \to \mathrm{C}$, which is the same as an A-module map $\mathrm{E} \to \mathrm{C}$ by Lemma 12.1.7 and, hence the same as a B-module map $\mathrm{E} \otimes_{\mathrm{A}} \mathrm{B} \to \mathrm{C}$ by the universal properties of the tensor product. □

We now globalize the construction above

Construction 12.1.9. Suppose that $\mathcal{F} \in \mathbf{QCoh}(\mathrm{X})$, we construct the quasicoherent sheaf on X:

$$\mathrm{Sym}_{\mathcal{O}_{\mathrm{X}}}(\mathcal{F})$$

in the following manner: given

$$x : \operatorname{Spec} \mathrm{A} \to \mathrm{X}$$

we set

$$x^*\mathrm{Sym}_{\mathcal{O}_{\mathrm{X}}}(\mathcal{F}) := \mathrm{Sym}_{\mathrm{A}}(x^*\mathcal{F}).$$

By Corollary 12.1.8 we see that this defines a quasicoherent sheaf on X. We again drop the "$\mathcal{O}_{\mathrm{X}}$" decoration whenever the context is clear.

The next definition has been a long time coming: it captures the appropriate basic notion for "commutative algebra over a scheme X."

[3] The "A" decoration will be omitted if the context is clear".

Definition 12.1.10. A **quasicoherent $\mathcal{O}_X$-algebra** is an algebra object in **QCoh**(X). In other words, it is a quasicoherent sheaf $\mathcal{A}$ equipped with multiplication and unit maps:

$$m : \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{A} \qquad \epsilon : \mathcal{O}_X \to \mathcal{A},$$

satisfying the obvious compatibility conditions to make it into a commutative algebra object in **QCoh**(X). We denote the category of quasicoherent $\mathcal{O}_X$-algebras and algebra morphisms by

$$\mathbf{CAlg}(\mathbf{QCoh}(X)),$$

which comes with a canonical forgetful functor:

$$\mathbf{CAlg}(\mathbf{QCoh}(X)) \to \mathbf{QCoh}(X).$$

It is easy to check that $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{F})$ is indeed a quasicoherent $\mathcal{O}_X$-algebra. It also enjoys the following universal property inherited by its restriction to affine schemes: in particular it says that the functor

$$\mathcal{F} \mapsto \mathrm{Sym}(\mathcal{F})$$

is a left adjoint to the forgetful functor $\mathbf{CAlg}(\mathbf{QCoh}(X)) \to \mathbf{QCoh}(X)$.

Lemma 12.1.11. *There is a canonical morphism of quasicoherent sheaves on* X*:* $\mathcal{O}_X \to \mathrm{Sym}(\mathcal{F})$ *which satisfies the following universal property: for any quasicoherent* $\mathcal{O}_X$*-algebra* $\mathcal{A}$ *we have a canonical isomorphism*

$$\mathrm{Hom}_{\mathbf{CAlg}(\mathbf{QCoh}(X))}(\mathrm{Sym}(\mathcal{F}), \mathcal{A}) \cong \mathrm{Hom}_{\mathbf{QCoh}(X)}(\mathcal{F}, \mathcal{A}).$$

Construction 12.1.12. Let $\mathcal{A} \in \mathbf{CAlg}(\mathbf{QCoh}(X))$. Then the **relative Spectrum** of $\mathcal{A}$ is the prestack $\mathrm{Spec}_X(\mathcal{A})$ defined as follows: given $R \in \mathrm{CAlg}$,

$$\mathrm{Spec}_X(\mathcal{A})(R)$$

is the set of isomorphism classes of pairs

$$(x, f),$$

where $x : \mathrm{Spec}\, R \to X$ and f is a morphism of schemes over $\mathrm{Spec}\, R$:

$$f : \mathrm{Spec}\, R \to \mathrm{Spec}\, x^*(\mathcal{A}).$$

12.2. Exercises.

Exercise 12.2.1. *Prove that the relative spectrum construction yields a scheme.*

13. Lecture 13: Total spaces, and affine morphisms

Let us recall what we are trying to do. Let $\mathcal{F}$ be a quasicoherent sheaf. We have defined the prestack $\mathbf{V}_X(\mathcal{F})$ such that anytime we have a map $x : \mathrm{Spec}\, A \to \mathbf{V}_X(\mathcal{F})$, we get a "co-section": a map

$$x^*(\mathcal{F}) \to \mathcal{O}_A.$$

The first important feature of this construction is that it admits a canonical morphism, the **projection morphism**,

$$\mathbf{V}_X(\mathcal{F}) \to X.$$

Now suppose that $f \in \Gamma(\mathcal{F})$, a global section of $\mathcal{F}$. Then, f is a morphism of quasicoherent sheaves on X

$$f : \mathcal{O}_X \to \mathcal{F}.$$

We can apply $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$ which was featured in the exercise before this week to get a map

$$f^\vee : \mathcal{F}^\vee \to \mathcal{O}_X$$

This lets us define a morphism $X \to \mathbf{V}_X(\mathcal{F}^\vee)$ *as soon as we know that the target is at least a Zariski stack*. If so, such a map corresponds to a cosection of $\mathcal{F}^\vee$ on X and thus, we can speak of a correspondence between

$$f \in \Gamma(\mathcal{F}),$$

and morphisms

$$f : \mathrm{X} \to \mathbf{V}_{\mathrm{X}}(\mathcal{F}) \qquad \text{such that } p \circ f = \mathrm{id}.$$

Why were we trying to do this? Well we wanted to take the following pullback

$$\mathrm{X} \times_{\mathbf{V}(\mathcal{E})} \mathbf{V}(\mathcal{E}) \smallsetminus z(\mathrm{X}) =: \mathrm{X}_f,$$

where the first map classifies the section f of a vector bundle, while the other map is the open immersion, complementary to the zero section. We want to think of this open subscheme as the global version of $\operatorname{Spec} \mathrm{A}_f \subset \operatorname{Spec} \mathrm{A}$ and we want to specify open subschemes of $\mathbf{P}^n$ in these terms.

To achieve our goals, we introduced the notion of a relative spectrum. Let us unpack this notion of a relative spectrum. Suppose that $x : \operatorname{Spec} \mathrm{R} \to \mathrm{X}$ is a morphism, i.e., an R-point. Then we can pull back $\mathcal{A}$ to $\operatorname{Spec} \mathrm{R}$ to obtain

$$x^*(\mathcal{A}) \in \mathrm{CAlg}_{\mathrm{R}};$$

in particular it has a ring morphism $\mathrm{R} \to x^*(\mathcal{A})$. On the other hand, the map $f : \operatorname{Spec} \mathrm{R} \to \operatorname{Spec} x^*(\mathcal{A})$, which we insisted is a $\operatorname{Spec} \mathrm{R}$-morphism, is the same thing as a morphism of R-algebras:

$$x^*(\mathcal{A}) \to \mathrm{R},$$

so that it is a *section* of the R-algebra structure on $x^*(\mathcal{A})$.

Theorem 13.0.1. *The prestack* $\operatorname{Spec}_{\mathrm{X}}(\mathcal{A})$ *is a scheme, equipped with a canonical morphism* $\operatorname{Spec}_{\mathrm{X}}(\mathcal{A}) \to \mathrm{X}$

This will be exercise this week; the proof proceeds via an analogous fashion to the proof that the Spec of a ring is indeed a scheme. We also note that there is another way to talk about "morphisms which are of the form:

$$\operatorname{Spec}_{\mathrm{X}}(\mathcal{A}) \to \mathrm{X}.$$

"

Definition 13.0.2. An **affine morphism** of schemes $\mathrm{Y} \to \mathrm{X}$ is one such that for all $\operatorname{Spec} \mathrm{R} \to \mathrm{X}$, the pullback $\operatorname{Spec} \mathrm{R} \times_{\mathrm{X}} \mathrm{Y}$ is an affine scheme.

Example 13.0.3. Here are examples of affine morphisms:

(1) any morphism between affine schemes is an affine morphism;
(2) closed immersions are affine morphisms.

We denote by $\mathrm{Aff}_{\mathrm{X}}$ the category of affine morphsims over X; this is to be regarded as a subcategory of the slice category $\mathrm{Sch}_{\mathrm{X}}$. The next proposition generalizes the fact that the opposite category of affine schemes is equivalent to CAlg.

Proposition 13.0.4. *Suppose that* $\mathrm{X} \to \mathrm{S}$ *is a morphism of schemes. Then the followign are equivalent:*

(1) f *is an affine morphism,*
(2) there exists a quasicoherent S*-algebra* $\mathcal{A}$ *such that*

$$\mathrm{X} \cong \operatorname{Spec}_{\mathrm{S}}(\mathcal{A}),$$

over S.

Proof. We first claim that the functor

$$\mathcal{A} \mapsto (\operatorname{Spec}_{\mathrm{X}}(\mathcal{A}) \to \mathrm{X})$$

yields an affine morphism over X. Indeed, if $x : \operatorname{Spec} \mathrm{R} \to \mathrm{X}$ is a map then I claim that

$$\operatorname{Spec}_{\mathrm{X}}(\mathcal{A}) \times_{\mathrm{X}} \operatorname{Spec} \mathrm{R} \cong \operatorname{Spec} x^*(\mathcal{A}).$$

Note that since $x^*(\mathcal{A})$ is an R-algebra, we get a ring map

$$\mathrm{R} \to x^*(\mathcal{A}),$$

hence a canonical map of schemes

$$q : \operatorname{Spec} x^*(\mathcal{A}) \to \operatorname{Spec} \mathrm{R}.$$

Next, we note that to give a morphism $\operatorname{Spec} x^*(\mathcal{A}) \to \operatorname{Spec}_{\mathrm{X}}(\mathcal{A})$ is to give a map to X (which we can through $\operatorname{Spec} x^*(\mathcal{A}) \xrightarrow{q} \operatorname{Spec} \mathrm{R} \to \mathrm{X}$ and a section

$$q^*(x^*(\mathcal{A})) \to x^*(\mathcal{A});$$

but the domain unpacks as

$$x^*(\mathcal{A}) \otimes_{\mathrm{R}} x^*(\mathcal{A}) \to x^*(\mathcal{A})$$

so we can let this map be the multiplication map, which is a section of the unit map $x^*(\mathcal{A}) \to x^*(\mathcal{A}) \otimes_{\mathrm{R}} x^*(\mathcal{A})$ adjoint to the unit map $\mathrm{R} \to x^*(\mathcal{A})$.

To prove the claim, let $\mathrm{B} \in \mathrm{CAlg}$, then I claim that the square below (constructed by the discussion in the previous paragraph) is a cartesian square:

$$\begin{array}{ccc} \operatorname{Spec} x^*(\mathcal{A})(\mathrm{B}) & \longrightarrow & \operatorname{Spec}_{\mathrm{X}}(\mathcal{A})(\mathrm{B}) \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathrm{R}(\mathrm{B}) & \longrightarrow & \mathrm{X}(\mathrm{B}). \end{array}$$

Indeed, the pullback unpacks to $y : \operatorname{Spec} \mathrm{B} \to \operatorname{Spec} \mathrm{R}$ and a map of B-algebras

$$y^*(x^*\mathcal{A}) \cong x^*\mathcal{A} \otimes_{\mathrm{R}} \mathrm{B} \to \mathrm{B};$$

which is the same thing as an R-algebra map

$$x^*(\mathcal{A}) \to \mathrm{B},$$

as desired.

To prove the claim we need to construct the functor the other way. We need to introduce pushforwards.

□

Example 13.0.5. Let $\mathrm{A} \to \mathrm{B}$ be a morphism of rings. We have been using the tensor product $\mathrm{M} \mapsto \mathrm{M} \otimes_{\mathrm{A}} \mathrm{B}$. This is right adjoint to the functor

$$\mathbf{Mod}_{\mathrm{B}} \to \mathbf{Mod}_{\mathrm{A}},$$

which simply forgets that M is a B-module and regard it as an A-module. On the level of quasicoherent sheaves this will be written as

$$\pi_* : \mathbf{QCoh}(\operatorname{Spec} \mathrm{B}) \to \mathbf{QCoh}(\operatorname{Spec} \mathrm{A}).$$

It is not easy to define the pushforward *even* using Serre's lemma. Indeed, suppose that $\mathrm{Y} \to \mathrm{X}$ is a morphism of schemes. To globalize Example 13.0.5 we would choose a Zariski open cover of Y to get maps $\operatorname{Spec} \mathrm{B}_i \to \mathrm{X}$. There is, however, no guarantee that this morphism will factor through some open affine of X (unless, say B_i is a field). We can also try to pick a Zariski open cover of X and we are left with trying to define π_* for morphisms of the form

$$\operatorname{Spec} \mathrm{R} \times_{\mathrm{X}} \mathrm{Y} \to \operatorname{Spec} \mathrm{R}.$$

This is a much more promising approach: we can try to pick an open cover of $\operatorname{Spec} \mathrm{R} \times_{\mathrm{X}} \mathrm{Y}$ and glue the pushforward functor as morphsims of the form

$$\operatorname{Spec} \mathrm{B}_i \hookrightarrow \operatorname{Spec} \mathrm{R} \times_{\mathrm{X}} \mathrm{Y} \to \operatorname{Spec} \mathrm{R}$$

varies. One motivation for the definition of an affine morphism is that no further glueing is needed.

Construction 13.0.6. Let $f : \mathrm{Y} \to \mathrm{X}$ be an affine morphism of schemes. We construct a functor

$$f_* : \mathbf{QCoh}(\mathrm{Y}) \to \mathbf{QCoh}(\mathrm{X})$$

as follows: given $\mathcal{F} \in \mathbf{QCoh}(\mathrm{Y})$, then

$$f_*(\mathcal{F}) \in \mathbf{QCoh}(\mathrm{X}),$$

is the quasicoherent sheaf on Y defined by giving, for each $x : \operatorname{Spec} \mathrm{A} \to \mathrm{X}$, the A-module

$$(f')_*(x')^*(\mathcal{F}),$$

where the maps are:

$$\begin{array}{ccc} \operatorname{Spec} \mathrm{A} \times_{\mathrm{X}} \mathrm{Y} = \operatorname{Spec} \mathrm{C} & \xrightarrow{x'} & \mathrm{Y} \\ \downarrow f' & & \downarrow f \\ \operatorname{Spec} \mathrm{A} & \xrightarrow{x} & \mathrm{X}. \end{array}$$

Note how we have used the affineness property here.

Lemma 13.0.7. *The functor f_* preserves quasicoherent algebras,*

Proof. We need to construct a map

$$f_*(\mathcal{A}) \otimes_{\mathcal{O}_{\mathrm{X}}} f_*(\mathcal{A}) \to f_*(\mathcal{A}).$$

By adjunction, this is the same thing as a map

$$f^*(f_*(\mathcal{A}) \otimes_{\mathcal{O}_{\mathrm{X}}} f_*(\mathcal{A})) \to \mathcal{A},$$

which, by symmetric monoidality, is the same as

$$f^*f_*(\mathcal{A}) \otimes_{\mathcal{O}_{\mathrm{Y}}} f^*f_*(\mathcal{A}) \to \mathcal{A}.$$

But we have the counit $f^*f_* \to \mathrm{id}$ so we get map

$$f^*f_*(\mathcal{A}) \otimes_{\mathcal{O}_{\mathrm{Y}}} f^*f_*(\mathcal{A}) \to \mathcal{A} \otimes_{\mathcal{O}_{\mathrm{Y}}} \mathcal{A} \to \mathcal{A}.$$

I will leave you to check that you do indeed get an algebra. □

Proof of Proposition 13.0.4 continued. Now, given an affine morphsim $f : \mathrm{X} \to \mathrm{S}$, we get that $f_*(\mathcal{O}_{\mathrm{X}})$ as a quasicoherent $\mathcal{O}_{\mathrm{S}}$-algebra. We need to prove that

$$\operatorname{Spec}_{\mathrm{S}}(f_*(\mathcal{O}_{\mathrm{X}})) \cong \mathrm{X}.$$

To prove this, we first provide a map

$$\mathrm{X} \to \operatorname{Spec}_{\mathrm{S}}(f_*(\mathcal{O}_{\mathrm{X}})).$$

Pick an Zariski cover of S made up of affines $x_j : \operatorname{Spec} \mathrm{A}_j \to \mathrm{S}$. Since f is an affine morphism we get that

$$x'_j : \operatorname{Spec} \mathrm{B}_j := \operatorname{Spec} \mathrm{A}_j \times_{\mathrm{S}} \mathrm{X} \to \mathrm{X}$$

also makes up an Zariski covering of X made up of affines. Now the map $x'_j : \operatorname{Spec} \mathrm{B}_j \to \mathrm{X}$ defines a map

$$(x'_j)^*(\mathcal{O}_{\mathrm{X}}) \to \mathrm{B}_j$$

But now, by definition of f_*, this is the same thing as a map

$$(f \circ x')^*(f_*\mathcal{O}_{\mathrm{X}}) \to \mathrm{B}_j,$$

which assembles into a map (using the Zariski stack property of $\operatorname{Spec}_{\mathrm{S}}(f_*(\mathcal{O}_{\mathrm{X}}))$) to a map

$$\mathrm{X} \to \operatorname{Spec}_{\mathrm{S}}(f_*(\mathcal{O}_{\mathrm{X}})).$$

We leave it to the reader to prove that this is an isomorphism. □

Proposition 13.0.8. *Let $\mathcal{F}$ be quasicoherent sheaf on* X. *Then we have an isomorphism*

$$\mathbf{V}_{\mathrm{X}}(\mathcal{F}) \cong \operatorname{Spec}_{\mathrm{X}}(\operatorname{Sym}(\mathcal{F})).$$

Proof. We have the string of isomorphisms for $\mathrm{T} = \operatorname{Spec} \mathrm{A}$ and a map $x : \operatorname{Spec} \mathrm{A} \to \mathrm{X}$

$$\begin{aligned} \mathrm{Hom}_{\mathrm{X}}(\mathrm{T}, \mathbf{V}_{\mathrm{X}}(\mathcal{F})) &\cong \mathrm{Hom}_{\mathcal{O}_{\mathrm{T}}}(x^*\mathcal{F}, \mathcal{O}_{\mathrm{T}}) \\ &\cong \mathrm{Hom}_{\mathrm{CAlg}(\mathbf{QCoh}(\mathrm{T}))}(x^*\mathrm{Sym}(\mathcal{F}), \mathcal{O}_{\mathrm{T}})). \end{aligned}$$

□

13.1. **Exercises.**

Exercise 13.1.1. *Finish the proof of Proposition 13.0.4.*

Exercise 13.1.2. *Let $\mathcal{C}, \mathcal{D}$ be symmetric monoidal categories. Suppose that*

$$\mathrm{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathrm{G}$$

is an adjunction such that F *is a symmetric monoidal functor. Prove that* G *takes commutative algebra objects in $\mathcal{D}$ to commutative algebra objects in $\mathcal{C}$.*

Exercise 13.1.3. *Let $f : \operatorname{Spec} \mathrm{A} \to \mathrm{X}$ be a morphism from an affine scheme to a scheme* X*. Prove or give a counterexample: f is an affine morphism.*

Exercise 13.1.4. *Prove that a morphism $f : \mathrm{X} \to \mathrm{Y}$ is an affine morphism if and only if for any open immersion* $\operatorname{Spec} \mathrm{A} \hookrightarrow \mathrm{Y}$*, the pullback* $\operatorname{Spec} \mathrm{A} \times_{\mathrm{Y}} \mathrm{X}$ *is an affine scheme.*

Exercise 13.1.5. *Let $\mathcal{F}$ be quasicoherent sheaf on* X *and $f \in \Gamma(\mathcal{F})$. Prove that the morphism $\mathrm{X}_f \hookrightarrow \mathrm{X}$ is affine.*

Exercise 13.1.6. *Let* X *be a scheme of characteristic p (so it is a scheme over* $\operatorname{Spec} \mathbf{F}_p$*). We have the Frobenius morphism*

$$\mathrm{Frob} : \mathrm{X} \to \mathrm{X}$$

defined on affine opens by

$$\operatorname{Spec} \mathrm{A} \to \operatorname{Spec} \mathrm{A} \qquad \mathrm{A} \to \mathrm{A}, x \mapsto x^p.$$

(1) is the Frobenius morphism affine?
(2) let $\mathcal{L}$ be a line bundle on X*, prove that*

$$\mathrm{Frob}^*(\mathcal{L}) \cong \mathcal{L}^{\otimes p}.$$

14. Lecture 14: projective space is a scheme

Let us recall that we have done. Let $\mathcal{F}$ be a quasicoherent sheaf on $\mathcal{F}$. Then $f \in \Gamma(\mathcal{F})$ defines for us a map of quasicoherent sheaves

$$\mathcal{O}_{\mathrm{X}} \to \mathcal{F};$$

and hence a dual map

$$f^{\vee} : \mathcal{F}^{\vee} \to \mathcal{O}_{\mathrm{X}}.$$

Such a map defines a section

$$f : \mathrm{X} \to \mathbf{V}(\mathcal{F}^{\vee}),$$

so that the composite

$$\mathrm{X} \to \mathbf{V}(\mathcal{F}^{\vee}) \xrightarrow{f} \mathbf{V}(\mathcal{F}^{\vee}) \to \mathrm{X}$$

is the identity. There is also the zero section

$$z : \mathrm{X} \to \mathbf{V}(\mathcal{F}^{\vee})$$

which is a closed immersion (in fact, any section defines a closed immersion; this is this week's homework) and so

$$\mathbf{V}(\mathcal{F}^{\vee}) \smallsetminus z(\mathrm{X}) \subset \mathbf{V}(\mathcal{F}^{\vee})$$

is an open subscheme. In Lemma 12.0.4, we introduced $\mathrm{X}_f \subset \mathrm{X}$ which we can recast as

$$\mathrm{X}_f \cong \mathrm{X} \times_{\mathbf{V}(\mathcal{F}^{\vee})} \mathbf{V}(\mathcal{F}^{\vee}) \smallsetminus z(\mathrm{X}),$$

manifestly presenting it as an open subscheme of X; this is the subscheme of X where f is nonvanishing. We can use this to prove that $\mathbf{P}^n$ is indeed a scheme:

Proof. This week's homework is to prove that $\mathbf{P}^n$ is a Zariski stack. The basic idea is that line bundles glue. In any case, let us accept that for a moment.

Let us define
$$\mathrm{U}_i \hookrightarrow \mathbf{P}^n$$
in the following way: a morphism $x : \mathrm{S} = \operatorname{Spec} \mathrm{A} \to \mathbf{P}^n$ factors through U_i if and only if
$$s = (s_0, \cdots, s_{n+1}) : \mathcal{L} \to \mathcal{O}_{\mathrm{S}}^{\oplus n+1}$$
has the property that $s_i \neq 0$. We claim that U_i is an open immersion. Indeed, consider the cartesian square
$$\begin{array}{ccc} \mathrm{S} \times_{\mathbf{P}^n} \mathrm{U}_i & \longrightarrow & \mathrm{U}_i \\ \downarrow & & \downarrow \\ \mathrm{S} & \xrightarrow{\quad x \quad} & \mathbf{P}^n. \end{array}$$
Noting that the map x defines the cosection
$$s_i : \mathcal{L} \to \mathcal{O}_{\mathrm{S}}^{n+1} \xrightarrow{\pi_i} \mathcal{O}_{\mathrm{S}}$$
where the second map is given by projection to the i-coordinate, we see that
$$\mathrm{S} \times_{\mathbf{P}^n} \mathrm{U}_i \cong \mathrm{S} \times_{\mathbf{V}(\mathcal{L}^\vee)} \mathbf{V}(\mathcal{L}^\vee) \smallsetminus 0 \cong \mathrm{S}_{s_i},$$
which we saw was an open subscheme of S.

Next we claim that $\mathrm{U}_i \cong \mathbf{A}^n$. Indeed, it suffices to prove that we have an S-isomorphism
$$\mathrm{S}_s \cong \mathbf{A}^n_{\mathrm{S}}.$$
But now, the data of a map $\mathcal{L} \to \mathcal{O}_{\mathrm{S}}^{n+1}$ which is nonzero on the i-coordinate, lets us pick the other n-coordinates freely and so we are looking only at n-elements of A without any further constrain.

Let us now prove that $\{\mathrm{U}_i\}_{i=0,\cdots,n}$ is an cover of $\mathbf{P}^n$. Let $\operatorname{Spec} k \to \mathbf{P}^n$ be a morphism; this classifies for a us a 1-dimensional vector space L and a map
$$s : \mathrm{L} \to k^{\oplus n}$$
with the nondegeneracy conditions above. In this case, by linear algebra, we see that there must be at least a single coordinate where s picks out a nonzero element. For that coordinate, $\operatorname{Spec} k \to \mathbf{P}^n$ factors through U_i. □

Proof that $\mathbf{P}_{\mathrm{X}}(\mathcal{E})$ *is a scheme.* For a cover $\{\mathrm{U}_i \hookrightarrow \mathrm{X}\}$ such that $\mathcal{E}|_{\mathrm{U}_i}$ is trivial, one can check that
$$\mathbf{P}_{\mathrm{X}}(\mathcal{E}) \times_{\mathrm{X}} \mathrm{U}_i \cong \mathbf{P}^n \times \mathrm{U}_i.$$
Therefore, this furnishes the cover of $\mathbf{P}_{\mathrm{X}}(\mathcal{E})$ as a scheme. □

Here's an extremely important object in algebraic geometry.

Definition 14.0.1. The **tautological bundle** is the line bundle $\mathcal{O}_{\mathbf{P}^n}(1)$ given as follows: a map
$$x : \operatorname{Spec} \mathrm{A} \to \mathbf{P}^n$$
defines a bundle injection
$$0 \to \mathcal{L} \to \mathcal{O}_{\mathrm{A}}^{\oplus n+1}.$$
We set $x^*\mathcal{O}_{\mathbf{P}^n}(1) := \mathcal{L}^\vee$.

If the context is clear we write $\mathcal{O}(1)$. Furthermore we also have:

Construction 14.0.2. We have three cases:

$(m > 0)$ We set
$$\mathcal{O}(m) := \mathcal{O}(1)^{\otimes m};$$
$(m = 0)$ we set
$$\mathcal{O}(0) = \mathcal{O};$$

$(m < 0)$ we set

$$\mathcal{O}(m) := (\mathcal{O}(1)^{\vee})^{\otimes m},$$

In particular $\mathcal{O}(-1) := \mathcal{O}(1)^{\vee}$ and

$$\mathcal{O}(m) \otimes_{\mathcal{O}} \mathcal{O}(-m) \cong \mathcal{O}.$$

Remark 14.0.3. Note that if $x : \operatorname{Spec} \mathrm{A} \to \mathbf{P}^n$ is a morphism classifying $\mathcal{L} \to \mathcal{O}_{\mathrm{A}}^{\oplus n+1}$, then

$$x^*\mathcal{O}(-1) \cong \mathcal{L}$$

by definition.

Remark 14.0.4. Let X be a scheme. Under the interpretation above, given a map $x : \mathrm{X} \to \mathbf{P}^n_{\mathrm{A}}$, is the same thing as giving a map

$$0 \to \mathcal{L} \to \mathcal{O}_{\mathrm{X}}^{n+1}.$$

By convention, $\mathcal{L}^{\vee} = x^*(\mathcal{O}(1))$, hence we can rewrite the above as $n+1$-tuple of sections

$$s_0, \cdots, s_n : \mathcal{O}_{\mathrm{X}} \to x^*(\mathcal{O}(1)).$$

This is the "usual interpretation" of maps into projective space: to give a morphism from a scheme X to $\mathbf{P}^n_{\mathrm{A}}$ is to give $n+1$ global sections of a line bundle $\mathcal{L}$ up to nondegeneracy conditions. Soon, we will see how to read off properties of this morphism in terms of the datum of these line bundles.

Remark 14.0.5. Suppose that $\mathcal{L}$ is a line bundle on X. Then $n+1$-sections of $\mathcal{L}$, denoted by $s_0, \cdots, s_n$ lets us "pick a framing" on X in the following way: suppose that we find an affine open subscheme $\operatorname{Spec} \mathrm{A} \hookrightarrow \mathrm{X}$ such that $\mathcal{L}$ trivializes on $\operatorname{Spec} \mathrm{A}$. Then these sections define $s_0|_{\operatorname{Spec} \mathrm{A}}, \cdots s_n|_{\operatorname{Spec} \mathrm{A}}$ which, in turns, defines maps

$$\operatorname{Spec} \mathrm{A} \to \mathbf{V}((\mathcal{L}^{\vee})^{\oplus n+1}) \cong \operatorname{Spec} \mathrm{A} \times \mathbf{A}^{n+1}.$$

Now there is no reason for this map to be a closed immersion but in the best possible situations, Spec A is defined as the simulatneous vanishing locus of these equations. However, even in this situation, there is no reason for these framings to glue. The discrepancy, however, turns out to be in the automorphism of the line bundles, i.e., units. This brings in the relevance of projective space.

We will soon see that the correspondence between projective space and line bundles lets us pick a framing on X: a nice line bundle will define a closed immersion

$$\mathrm{X} \hookrightarrow \mathbf{P}^n,$$

endowing X with a set of projective coordinates.

14.1. **Playing with projective space.** Let us now play with projective space; we fix a base ring A. We first note that we can construct the A-algebra

$$\bigoplus_{j \geqslant 0} \Gamma(\mathbf{P}^n_{\mathrm{A}}, \mathcal{O}(j)),$$

which encodes all the global sections of the various twists $\mathcal{O}(j)$ at once. We have a map of A-modules (the presence of the dual will be clear in a moment):

$$(\mathrm{A}^{\oplus n+1})^{\vee} \to \Gamma(\mathbf{P}^n_{\mathrm{A}}, \mathcal{O}(1))$$

defined in the following manner: such a map is the same data as $n+1$-sections

$$s_0, \cdots, s_j : \mathcal{O}_{\mathbf{P}^n_{\mathrm{A}}} \to \mathcal{O}(1).$$

To describe these sections: given $x : \operatorname{Spec} \mathrm{B} \to \mathbf{P}^n_{\mathrm{A}}$ corresponding to a map

$$0 \to \mathcal{L} \to \mathcal{O}_{\operatorname{Spec} \mathrm{B}}^{\oplus n+1} \tag{14.1.1}$$

But now, the restriction of the section s_i to Spec B is a map

$$\mathcal{O}_{\operatorname{Spec} \mathrm{B}} \to \mathcal{L}^{\vee},$$

i.e., an element of the dual space. And so, we declare this to be the i-th component of the map in (14.1.1). Since this is a morphism of A-modules and the target is an A-algebra we get a graded map of graded rings:

$$\bigoplus_{m\geqslant 0} \mathrm{Sym}_{\mathrm{A}}((\mathrm{A}^{\oplus n+1})^{\vee}) \to \bigoplus_{m\geqslant 0} \Gamma(\mathbf{P}^n_{\mathrm{Spec\,A}}, \mathcal{O}(r)).$$

Now note that the right hand side is nothing but

$$\bigoplus_{m\geqslant 0} \mathrm{A}[\mathrm{T}_0, \cdots, \mathrm{T}_n]_m,$$

the graded ring of homogeneous polynomials with coefficients in A. The next lemma is left as an exercise for this week:

Lemma 14.1.2. *Let $r \geqslant 0$, then*

$$\bigoplus_{m\geqslant 0} \mathrm{Sym}_{\mathrm{A}}((\mathrm{A}^{\oplus n+1})^{\vee}) \to \bigoplus_{m\geqslant 0} \Gamma(\mathbf{P}^n_{\mathrm{Spec\,A}}, \mathcal{O}(m)),$$

is a graded isomorphism of graded rings.

Since the above isomorphism respects grading, we know exactly how to interpret global sections of $\mathcal{O}(r)$ on $\mathbf{P}^n$: they are given by

$$f(x_0, \cdots, x_n) \in \mathrm{A}[x_0, \cdots, x_n]_r,$$

a degree n polynomial with coefficients in A and of homogeneous degree r.

Example 14.1.3. Let A be a ring and consider the line bundle $\mathcal{O}(d)$ on $\mathbf{P}^n_{\mathrm{A}}$. The global sections of $\mathcal{O}(d)$ is then a free A-module of rank

$$\binom{n+d}{d.}$$

For example when $n = 2$ and $d = 2$, an A-basis is given by

$$\{x^2, y^2, z^2, xy, xz, yz\}.$$

Now, let $\{f_j\}$ denote a basis for $\mathrm{A}[x_0, \cdots, x_n]_r$. Then we obtain a morphism

$$\mathcal{O}_{\mathbf{P}^n} \to \bigoplus_{\binom{n+d}{d}} \mathcal{O}_{\mathbf{P}^n}(d)$$

which is dual to

$$\mathcal{O}_{\mathbf{P}^n}(-d) \to \bigoplus_{\binom{n+d}{d}} \mathcal{O}_{\mathbf{P}^n};$$

therefore we obtain a morphism

$$\mathbf{P}^n \to \mathbf{P}^{\binom{n+d}{d}-1}.$$

We will soon see that this is a closed immersion and is the famous **Veronese embedding**. Two examples of this are significant:

(d=2,n=2) This is the so-called Veronese surface; in projective coordinates that you might be more familiar with:

$$[x : y : z] \mapsto [x^2 : y^2 : z^2 : xy : xz : yz].$$

(n=1) This is a morphism

$$\mathbf{P}^1 \to \mathbf{P}^{\mathrm{N}}$$

and is called the **rational normal curve of degree** N.

Example 14.1.4. We can also consider products. Let $\mathcal{L}$ be a line bundle on X and $\mathcal{M}$ a line bundle on Y. We denote the line bundle on the product by

$$\mathcal{L} \boxtimes \mathcal{M} := \pi_{\mathrm{X}}^*\mathcal{L} \otimes \pi_{\mathrm{Y}}^*\mathcal{M}.$$

Then consider the line bundle

$$\mathcal{O}(d,e) := \mathcal{O}(d) \boxtimes \mathcal{O}(e)$$

on $\mathbf{P}^n \times \mathbf{P}^m$. One checks that sections of $\mathcal{O}(1,1)$ are exactly given by products of the various linear monomials in n and m-variables; there are $n+m+nm$ of them and hence defines a morphism

$$\mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^{n+m+mn}.$$

This will also turn out to be a closed embedding and is called the **Segre embedding**. There is actually a Segre embedding here hidden somewhere:

$$\mathbf{P}^n \to \mathbf{P}^n \times \mathbf{P}^n \to \mathbf{P}^{n^2+2n}.$$

Example 14.1.5. Let k be a field. The above is an instance of a **linear series**. If $\mathcal{L}$ is a line bundle on $\mathbf{P}^n$, we say that a **linear series** is a k-subspace:

$$\mathrm{V} \subset \Gamma(\mathbf{P}^n_k, \mathcal{L}).$$

A linear series is **complete** if the above inclusion is an equality. Hence the vector space of degree d homogeneous polynomials in n-variables determines a complete linear series. We will soon define what a projective scheme is and a linear series makes sense for projective schemes over a field as a subspace of

$$\mathrm{V} \subset \Gamma(\mathrm{X}, \mathcal{L}).$$

We are now changing topic to discuss finiteness conditions on schemes.

14.2. **Exercises.**

Exercise 14.2.1. *Prove Lemma 14.1.2. Here's one way to proceed: we have already constructed a morphism of* A*-algebras.*

$$t_n^{\mathrm{A}} : \bigoplus_{m\geqslant 0} \mathrm{Sym}_{\mathrm{A}}((\mathrm{A}^{\oplus n+1})^\vee) \to \bigoplus_{m\geqslant 0} \Gamma(\mathbf{P}^n_{\mathrm{Spec\,A}}, \mathcal{O}(m)).$$

Prove that it suffices to know that:

1. *for any localization* $j : \mathrm{Spec\,A}_{\mathfrak{p}} \hookrightarrow \mathrm{Spec\,A}$, $j^*t_n^{\mathrm{B}} = t_n^{\mathrm{B}}$*;*
2. *over any field* k*, the rank of* $\mathrm{Sym}_k((k^{\oplus n+1})^\vee)$ *and* $\Gamma(\mathbf{P}^n_{\mathrm{Spec\,A}}, \mathcal{O}(r)$ *are equal to*

$$\binom{n+d}{d.}$$

Exercise 14.2.2. *Let* X *be a fixed scheme and* $\mathcal{E}, \mathcal{F}$ *be vector bundles on* X*. What kind of morphisms* $\mathcal{E} \to \mathcal{F}$ *induces a map on* $\mathbf{P}$*? Use your answer to show that:*

1. *let* $\mathrm{X} = \mathrm{Spec\,A}$*, then there is a bijection between homogeneous ideals of* $\mathrm{A}[x_0, \cdots, x_n]$ *and closed subschemes of* $\mathbf{P}^n$*.*
2. *if* $\mathcal{E}, \mathcal{F}$ *are vector bundles, then the surjective morphism*

$$\mathrm{Sym}(\mathcal{E}^\vee \otimes \mathcal{F}^\vee) \to \bigoplus_{j=0}^{\infty} \mathrm{Sym}^j(\mathcal{E}^\vee) \otimes \mathrm{Sym}^j(\mathcal{F}^\vee),$$

induces the Segre embedding above; in particular it is a closed immersion by the first part.

15. Lecture 15: quasicompact and quasiseparated schemes

15.1. **Where do we go from here?** Now that we know what $\mathbf{P}^n$ is, we can speak about a collection of schemes which are of geometric interest in algebraic geometry. For maximal generality we work in the relative situation:

Definition 15.1.1. We say that a morphism $f : \mathrm{X} \to \mathrm{Y}$ is **projective** if it factors as

$$\begin{array}{ccc} \mathrm{X} & \longrightarrow & \mathbf{P}^n_{\mathrm{Y}} \\ & \overset{f}{\searrow} & \downarrow \\ & & \mathrm{Y}, \end{array}$$

where the top horizontal arrow is a *closed immersion.* If Y is affine so that $\mathrm{Y} = \operatorname{Spec} \mathrm{A}$ we also sometimes call $\mathrm{X} \to \operatorname{Spec} \mathrm{A}$ a **projective scheme over** A or, if $\mathrm{A} = k$ a field, then X is a **projective scheme**. A morphism $f : \mathrm{X} \to \mathrm{Y}$ is **quasiprojective** if factors as

$$\begin{array}{ccc} \mathrm{X} & \longrightarrow & \mathrm{X}' \\ & \overset{f}{\searrow} & \downarrow g \\ & & \mathrm{Y}, \end{array}$$

where g is a projective morphism and the top horizontal map is an *open immersion.*

We have spoken of an closed immersion into $\mathbf{P}^n$ as a kind of a "framing" or a generalized version of projective coordinates defining the scheme. Indeed if $\mathrm{A} = k$, we should think of a projective scheme as a geometric gadget we are interested in, such as a conic or an elliptic curve and so on. In fact, if we want to connect back to the italians, then we would go back in time and say the following:

Definition 15.1.2. A **variety** is an **integral, separated scheme** of finite type. A **projective variety** (resp. **quasiprojective variety**) is one such that X is also a projective scheme (resp. quasiprojective scheme).

We will make sense of all the words in bold. In any case, a projective morphism is then an *in-family* version of projective morphisms. Indeed: if $\operatorname{Spec} k \to \mathrm{Y}$ is a morphism, then $\mathrm{X} \times_{\mathrm{Y}} \operatorname{Spec} k$ is a projective scheme.

We now ask:

Question 15.1.3. What good properties do projective morphisms and projective schemes enjoy?

Question 15.1.4. How does one produce a projective morphism?

This last question is addressed by the theory of linear systems and divisors that we touched upon at the end of last class.

15.2. **Reduced schemes.** Before we speak of finiteness conditions on schemes, let us address a property of schemes which was in a previous iteration of the homeworks. Recall that there is a natural bijection between closed subschemes of a scheme and ideal sheaves.

Definition 15.2.1. The **reduced locus** of a scheme X is the closed subscheme $\mathrm{X}_{\mathrm{red}} \subset \mathrm{X}$ defined by the ideal sheaf:

$$\mathcal{I} =: \sqrt{(0} \subset \mathcal{O}_{\mathrm{X}}.$$

The ideal sheaf $\sqrt{0}$ takes the following form: given a morphism $x : \operatorname{Spec} \mathrm{R} \to \mathrm{X}$, we have htat $x^* \sqrt{(0)} = \sqrt{(0)}$ or the nilradical of R, i.e., the set of all elements x such that $x^{\mathrm{N}} = 0$ for $\mathrm{N} \gg 0$. You should convince yourself of the following:

Lemma 15.2.2. *The nilradical defines an ideal sheaf.*

The key is to use Serre's lemma and note that if $\operatorname{Spec} A \to \operatorname{Spec} B$ is an open immersion, then nilradicals pullback.

Definition 15.2.3. Let X be a scheme. We say that X is **reduced** if the closed immersion

$$X_{\mathrm{red}} \hookrightarrow X$$

is, in fact, an isomorphism.

Proposition 15.2.4. *Let $f : Y \to X$ be morphism of schemes where Y is reduced, then there exists a unique factorization of the following form*

$$\begin{array}{ccc} Y_{f_{\mathrm{red}}} & \dashrightarrow & X_{\mathrm{red}} \\ & \searrow & \downarrow \\ & & X. \end{array}$$

Proof. It is enough to verify this for $Y = \operatorname{Spec} R$ and $X = \operatorname{Spec} S$ (why?). In this case, we have a map $f : \operatorname{Spec} R \to \operatorname{Spec} S$ which we want to prove to factor through $\operatorname{Spec} S_{\mathrm{red}} = S/\sqrt{0}$. But this is the same thing as asking a filler for

$$\begin{array}{ccc} R & \dashleftarrow & S_{\mathrm{red}} \\ & \nwarrow^{\varphi} & \uparrow \\ & & S. \end{array}$$

But now, we note that $\varphi(\sqrt{0}) \subset 0 = \sqrt{0}$ since the nilradical of R is zero. □

Example 15.2.5. The easiest example of a non-reduced scheme is the so-called dual numbers $\operatorname{Spec} k[x]/(x^2)$ which already appears in the definition of the Zariski tangent space. While non-reduced schemes feel quite "geometric", the reduced ones play the role of determining "infinitesimal information" in algebraic geometry.

Example 15.2.6. It is quite easy to "degenerate" schemes to nilpotent ones; let k be a field for simplicity.

$$\operatorname{Spec} k[t,x]/(x^2 - t) \to \operatorname{Spec} k[t]$$

defines a family of schemes whose fiber over 0 is $\operatorname{Spec} k[x]/(x^2)$. For another, more interesting example consider

$$\operatorname{Spec} k[x,y,t]/(ty - x^2) \to \operatorname{Spec} k[t].$$

Then the fiber over a point classified by a $\operatorname{Spec} k \to \operatorname{Spec} k[t], t \mapsto a$ where $a \neq 0$ gives us the conic

$$\operatorname{Spec} k[x,y]/(ay - x^2),$$

which is a reduced ring.

However, if $a = 0$, then we get

$$\operatorname{Spec} k[x,y]/(x^2).$$

Clearly this is not a reduced scheme. How does one picture this? Note that we have a further quotient (computing its reduction) $k[x,y]/(x^2) \to k[x,y]/(x)$. The Spec of the target is visualized as the y-axis (we are setting $x = 0$ in $\mathbf{A}^2_k$). And so this is a closed subscheme of $\mathbf{A}^2$ which *contains* the y-axis! In this way we think of $\operatorname{Spec} k[x,y]/(x^2)$ as a thickened axis. More on this in a moment.

Example 15.2.7. We saw that degenerations is a way to produce non-reduced schemes. Let's attempt to understand collisions! Consider the ideal $(x) \subset k[x]$. This corresponds to the point $0 \in \mathbf{A}^1_k$. Suppose that $t \in k$ and consider $(x - t) \subset k[x]$ which corresponds to a point $t \in \mathbf{A}^1_k$. To collide them we consider

$$\mathbf{A}^2_k = \operatorname{Spec} k[x,t]$$

and take the intersection of ideals:

$$(x(x-t)).$$

This computes the scheme theoretic union: these are schemes cut out by vanishing of functions in *both* (x) and $(x-t)$. In any case, if we want collide the idea of (x) and the ideal of $(x-t)$ we set $t \mapsto 0$ to get

$$\operatorname{Spec} k[x]/(x^2),$$

which is again the double point. This justifies the idea that $k[x]/(x^2)$ is a point with a direction.

Example 15.2.8. Let us attempt to generalize the above example further: suppose that we have the double point inside $\mathbf{A}^2$ as a closed immersion:

$$\operatorname{Spec} k[\epsilon]/(\epsilon^2) \hookrightarrow \mathbf{A}^2_k;$$

and suppose further that the inverse image of the ideal ϵ under the corresponding surjection

$$\varphi : k[x,y] \to k[\epsilon]$$

is exactly the ideal $(x,y) \in k[x,y]$. Pictorially, this means that the map $\operatorname{Spec} k[\epsilon]/(\epsilon^2) \hookrightarrow \mathbf{A}^2_k$ has image the point at zero. Our goal is to express the image of this immersion as a closed subscheme of $\mathbf{A}^2_k$, i.e., find its defining ideal. This then classifies all possible double-point thickening of the origin (and, in fact, any point) of the $\mathbf{A}^2_k$. Equivalently, this is an attempt at computing the kernel of φ.

Now, we know that

$$\epsilon^2 = 0,$$

and thus the map φ factors through

$$k[x,y]/(x^2, xy, y^2) = (x,y)^2 \to k[\epsilon].$$

Therefore we conclude that the above morphism factors further as

$$\operatorname{Spec} k[\epsilon]/(\epsilon^2) \hookrightarrow \operatorname{Spec} k[x,y]/(x^2,xy,y^2) = (x,y)^2 \hookrightarrow \mathbf{A}^2_k$$

Now we might guess that the first map is an isomorphism. But this *not* the case since $k[\epsilon]/(\epsilon^2)$ as a k-vector space is 2-dimensiona, generated by $1, \epsilon$ while $k[x,y]/(x^2,xy,y^2)$ is generated by $1, x, y$ so it is 3-dimensional. This means that we need one more equation! Which means that there exists α, β one of which is nonzero such that

$$k[x,y]/(x^2,xy,y^2,\alpha x + \beta y) = k[\epsilon]/(\epsilon^2).$$

We see that the coordinates (α, β) specifies a direction, which is an infinitesimal thickening of the point.

15.3. **Why finiteness conditions?** In what is to come, we want to perform induction arguments on covers of schemes by affines. For this it will be useful to know when schemes can be covered by finitely many affines. That much is clear. However, if we cover X by $\operatorname{Spec}\mathrm{A}$ and $\operatorname{Spec}\mathrm{B}$, it may not be the case that $\operatorname{Spec}\mathrm{A} \times_{\mathrm{X}} \operatorname{Spec}\mathrm{B}$ is again affine. A natural way to repair this is to say that you can further cover intersections by finitely many affines. We will in fact introduce the relative variants of these notions.

Definition 15.3.1. Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of schemes. We say that f is a **quasicompact morphism** if for each $\operatorname{Spec}\mathrm{A} \to \mathrm{X}$, the pullback $\mathrm{X} \times_{\mathrm{Y}} \operatorname{Spec}\mathrm{A}$ is a quasicompact scheme.

Lemma 15.3.2. *A morphsim $f : \mathrm{X} \to \operatorname{Spec}\mathbf{Z}$ is a quasicompact morphism if and only if X is a quasicompact scheme.*

Example 15.3.3. Any closed immersion is quasicompact since the pullback along any morphism from an affine scheme is represented by an affine scheme, which is quasicompact.

Example 15.3.4. Open immersions do not have to be quasicompact in general: for example $\mathbf{A}^\infty \smallsetminus \{0\}$ which was an earlier example.

We now examine the notion of quasiseparatedness. To motivate this notion more geometrically, let us recall the following fact from point-set topology:

Lemma 15.3.5. *A topological space* X *is Hausdorff if and only if the diagonal morphism*

$$\mathrm{X} \to \mathrm{X} \times \mathrm{X}$$

is a closed immersion.

In fact, it is often useful to consider a more flexible notion than just asking for a closed diagonal. First, just like quasicompactness, we have the absolute notion:

Definition 15.3.6. We say that a scheme X is **quasiseparated** if the intersection of any two open affines in X is quasicompact.

In other words, being quasiseparated is the next best thing to ask for cartesian squares

$$\begin{array}{ccc} \operatorname{Spec} \mathrm{A} \times_{\mathrm{X}} \operatorname{Spec} \mathrm{B} & \longrightarrow & \operatorname{Spec} \mathrm{A} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathrm{B} & \longrightarrow & \mathrm{X} \end{array}$$

To massage our definition closer to this one we record the following lemma:

Lemma 15.3.7. *Let* X *be a scheme. Then the following are equivalent:*
1. X *is a quasiseparated scheme,*
2. *the diagonal*

$$\Delta_{\mathrm{X}} : \mathrm{X} \to \mathrm{X} \times \mathrm{X}$$

is quasicompact.

Proof. Indeed, we have the following pullback square

$$\begin{array}{ccc} \operatorname{Spec} \mathrm{A} \times_{\mathrm{X}} \operatorname{Spec} \mathrm{B} & \longrightarrow & \mathrm{X} \\ \downarrow & & \downarrow \Delta \\ \operatorname{Spec} \mathrm{A} \times \operatorname{Spec} \mathrm{B} & \xrightarrow{j \times j'} & \mathrm{X} \times \mathrm{X} \end{array}$$

.

Therefore Δ is quasicompact if and only if $\operatorname{Spec} \mathrm{A} \times_{\mathrm{X}} \operatorname{Spec} \mathrm{B}$ is quasicompact. □

Example 15.3.8. Here is an infamously freaky example in algebraic geometry. Consider the cospan

$$\begin{array}{ccc} \mathbf{A}^{\infty} \smallsetminus \{0\} & \hookrightarrow & \mathbf{A}^{\infty} \\ \downarrow & & \\ \mathbf{A}^{\infty} & & . \end{array}$$

In the homeworks, we will go through the formation of pushouts of schemes but take for granted that it exists and we call it $\mathbf{A}^{\infty}_{\pm}$ to indicate that this is the infinite dimensional affine space with the origin doubled (one + and one −). We have morphism including the infinite affine spaces:

$$\operatorname{Spec} \mathbf{A}^{\infty}_{+} \times \operatorname{Spec} \mathbf{A}^{\infty}_{-} \to \mathbf{A}^{\infty}_{\pm},$$

going through the " + " and the " − ". The intersection is then $\mathbf{A}^{\infty} \smallsetminus 0$ which we know to not be quasicompact.

Of course being quasicompact does not mean the diagonal is a closed immersion, although Example 15.3.3 does say that if the diagonal is indeed a closed immersion, then we have a quasicompact scheme.

Definition 15.3.9. Let $f : \mathrm{X} \to \mathrm{Y}$ be morphism of schemes:

(1) we say that f is **quasiseparated** if the diagonal

$$\Delta_{\mathrm{X/Y}} : \mathrm{X} \to \mathrm{X} \times_{\mathrm{X}} \mathrm{X}$$

is a quasicompact morphism;

(2) we say that f is **separated** if the diagonal morphism is a closed immersion;

(3) we say that X is **separated** if the structure morphism $\mathrm{X} \to \operatorname{Spec} \mathbf{Z}$ is separated.

Example 15.3.10. As a variant of Example 15.3.10 we have $\mathbf{A}^1_{\pm}$ the affine line with double origin. In this case, the diagonal morphism is actually quasicompact but is not a closed immersion as the intersection of $\mathbf{A}^1_{-}, \mathbf{A}^1_{+}$ is $\mathbf{A}^1 \smallsetminus 0$.

Example 15.3.11. An interesting example of a non-quasiseparated morphims is $\operatorname{Spec} \overline{\mathbf{Q}} \to \operatorname{Spec} \mathbf{Q}$. Indeed, note that

$$\operatorname{Spec} \overline{\mathbf{Q}} \times_{\operatorname{Spec} \mathbf{Q}} \operatorname{Spec} \overline{\mathbf{Q}} \cong \operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q});$$

as *schemes*; it is a little tricky to say how $\operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is a scheme but the point is that any profinite set is actually a scheme. Because of the profinite nature of $\operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, every point is closed but the complement is far from being quasicompact.

We will make good use of the following lemma:

Lemma 15.3.12. *Let $f : \mathrm{X} \to \mathrm{Y}$ be a quasicompact and quasiseparated (qcqs) morphsim. Then there exists a finite Zariski cover of* Y*,*

$$\mathcal{U} := \{\operatorname{Spec} \mathrm{A}_i \hookrightarrow \mathrm{Y}\}_{i=1,\cdots n}$$

and for each i a finite open covering

$$\mathcal{V}_i = \{\operatorname{Spec} \mathrm{B}_{ij} \hookrightarrow \operatorname{Spec} \mathrm{A}_i \times_{\mathrm{X}} \mathrm{Y}\}_{j \in \mathrm{I}_i},$$

such that $\operatorname{Spec} \mathrm{B}_{ij} \times_{\mathrm{X}} \operatorname{Spec} \mathrm{B}_{ij'}$ *is quasicompact.*

This is a mouthful! But this lemma will allow us to construct, for any f qcqs a pushforward map

$$f_* : \mathbf{QCoh}(\mathrm{X}) \to \mathbf{QCoh}(\mathrm{Y})$$

by applying Serre's lemma simultaneously.

To finish off, we also record:

Lemma 15.3.13. *Suppose that* X, Y *are schemes over* S *and*

$$f, g : \mathrm{X} \to \mathrm{Y}$$

be morphisms over S*. Then if $\pi_{\mathrm{Y}} : \mathrm{Y} \to \mathrm{S}$ is separated the largest subscheme $\mathrm{Z} \subset \mathrm{X}$ where f and g agrees is closed.*

Proof. Follows from the pullback

$$\begin{array}{ccc} \mathrm{Z} & \longrightarrow & \mathrm{X} \\ \downarrow & & \downarrow{\scriptstyle f \times g} \\ \mathrm{Y} & \xrightarrow{\Delta_{\mathrm{Y/S}}} & \mathrm{Y} \times_{\mathrm{S}} \mathrm{Y}, \end{array}$$

since the bottom morphism is a closed immersion. □

This is an extremely pleasant property of separated schemes. We will prove this in the next class once we make sense of the notion of density.

Corollary 15.3.14. *Suppose that there exist a dense open $\mathrm{W} \subset \mathrm{X}$ such that $f|_{\mathrm{W}} = g|_{\mathrm{W}}$ then $f = g$.*

15.4. Exercises.

Exercise 15.4.1 (Gluing schemes)**.** *Consider the following diagram of schemes*

$$\begin{array}{ccc} \mathrm{U} & \xrightarrow{j'} & \mathrm{V} \\ \big\downarrow{\scriptstyle j} & & \\ \mathrm{W} & & , \end{array}$$

where j, j' are open immersions. Prove that the pushout exists in the category of schemes by (1) taking the pushout in the category of prestacks and then applying the left adjoint to the inclusion of stacks into prestacks ("stackification") and (2) exhibiting a cover of the resulting object using a cover of V *and* W *(hint: we might as well assume that* U, V, W *are all affines to begin with). This construction is called* **gluing***.*

Exercise 15.4.2. *Consider the morphism of rings*

$$\begin{array}{ccc} \mathbf{Z}[t, t^{-1}] & \longleftarrow & \mathbf{Z}[t] \\ \big\uparrow & & \\ \mathbf{Z}[t^{-1}] & & . \end{array}$$

Prove that after taking Spec *and gluing, we get* $\mathbf{P}^1$.

Exercise 15.4.3 (Scheme-theoretic union)**.** *Let* $\mathrm{Z}, \mathrm{W} \subset \mathrm{X}$ *be closed subschemes of* X *with ideal sheaves* $\mathcal{I}_{\mathrm{Z}}, \mathcal{I}_{\mathrm{W}}$*. The* **scheme-theoretic union** $\mathrm{Z} \cup \mathrm{W}$ *is defined to be the closed subscheme defined by the ideal sheaf* $\mathcal{I}_{\mathrm{Z}} \cap \mathcal{I}_{\mathrm{W}}$ *(use Serre's theorem to make sense of this, as always). On the other hand, the* **scheme-theoretic intersection** *(which we have already seen many times) is defined to be the closed subscheme defined by* $\mathrm{Z} \times_{\mathrm{X}} \mathrm{W}$*. Prove*

1. *the ideal sheaf corresponding to the scheme-theoretic interseciton is given by* $\mathcal{I}_{\mathrm{Z}} + \mathcal{I}_{\mathrm{W}}$*;*
2. *the diagram*

$$\begin{array}{ccc} \mathrm{Z} \times_{\mathrm{X}} \mathrm{W} & \longrightarrow & \mathrm{W} \\ \big\downarrow & & \big\downarrow{\scriptstyle f \times g} \\ \mathrm{Z} & \xrightarrow{\Delta_{\mathrm{Y/S}}} & \mathrm{Z} \cup \mathrm{W}, \end{array}$$

is both cartesian and cocartesian.

16. Lecture 16: more geometric properties

Consider a morphism $f : \mathrm{X} \to \mathrm{Y}$. We have the pullback functor

$$f^* : \mathbf{QCoh}(\mathrm{Y}) \to \mathbf{QCoh}(\mathrm{X}).$$

Our first order of business is to construct its right adjoint.

Construction 16.0.1. We want to construct

$$f_* : \mathbf{QCoh}(\mathrm{X}) \to \mathbf{QCoh}(\mathrm{Y}),$$

right adjoint to f^* *under the assumption that the morphism f is quasicompact and quasiseparated (qcqs).*

1. First consider the case when $f : \mathrm{X} \to \operatorname{Spec} \mathrm{A}$ so a qcqs morphism to a base affine scheme A. Given $\mathcal{F} \in \mathbf{QCoh}(\mathrm{X})$ we want to produce $f_*\mathcal{F}$, which is an A-module. To do so, pick an open affine cover of X. We note that $\mathrm{X} \to \operatorname{Spec} \mathrm{A} \to \operatorname{Spec} \mathbf{Z}$ is a composite of quasicompact morphisms and therefore is quasicompact (an exercise this week). Therefore, we get that X is quasicompact as a scheme and we can let this cover be finite:
$$\mathcal{V} = \{\operatorname{Spec} \mathrm{B}_j \to \mathrm{X}\}_{j=1,\cdots n}.$$

Now, we also note that the following diagram is cartesian:

$$\begin{array}{ccc} \operatorname{Spec} \mathrm{B}_j \times_{\mathrm{X}} \operatorname{Spec} \mathrm{B}_k & \longrightarrow & \operatorname{Spec} \mathrm{B}_j \times_{\operatorname{Spec} \mathrm{A}} \operatorname{Spec} \mathrm{B}_k \\ \downarrow & & \downarrow \\ \mathrm{X} & \xrightarrow{\quad \Delta \quad} & \mathrm{X} \times_{\operatorname{Spec} \mathrm{A}} \mathrm{X} \end{array}$$

.

Since $f : \mathrm{X} \to \operatorname{Spec} \mathrm{A}$ is quasicompact we have that the intersections of the covers $\operatorname{Spec} \mathrm{B}_j \times_{\mathrm{X}} \operatorname{Spec} \mathrm{B}_k$ are all quasicompact. Hence enlarging $\mathcal{V}$ is necessary we can assume that $\mathcal{V}$ is a Zariski cover of X which is (1) finite and (2) the intersections are affine. By Serre's theorem $\mathcal{F}$ is then equivalent to the data of B_j-modules $\mathcal{F}|_{\mathrm{B}_j}$ equipped with isomorphisms of quasicoherent sheaves over affine schemes $\operatorname{Spec} \mathrm{B}_j \times_{\mathrm{X}} \operatorname{Spec} \mathrm{B}_k =: \operatorname{Spec} \mathrm{B}_{jk}$

$$(\mathcal{F}|_{\mathrm{B}_j})_{\mathrm{B}_k} \cong (\mathcal{F}|_{\mathrm{B}_k})_{\mathrm{B}_j}.$$

(2) Let $f_j : \operatorname{Spec} \mathrm{B}_j \to \operatorname{Spec} \mathrm{A}$ be restriction of f to $\operatorname{Spec} \mathrm{B}_j$ and define

$$\mathcal{F}_j := (f_j)_*(\mathcal{F}|_{\mathrm{B}_j}),$$

which is defined since we have a morphism of affine schemes. Furthermore we also have maps of B_j-modules

$$\mathcal{F}_{\mathrm{B}_j} \to \mathcal{F}_{\mathrm{B}_{jk}} (\cong \mathcal{F}_{\mathrm{B}_j} \otimes_{\mathrm{B}_j} \mathrm{B}_{jk});$$

and therefore morphisms

$$\mathcal{F}_j \to \bigoplus_k (f_j)_*(\mathcal{F}_{\mathrm{B}_{jk}}).$$

By the cocycle condition that $\mathcal{F}$ satisfies, we altogether get a diagram

$$\bigoplus \mathcal{F}_j \rightrightarrows \bigoplus (f_j)_*(\mathcal{F}_{\mathrm{B}_{jk}}) \cong (f_k)_*(\mathcal{F}_{\mathrm{B}_{kj}}),$$

and set $\mathcal{F}$ to be the equalizer of the above diagram (concrete, the kernel of the difference). I will leave it for the reader to check that this is the right adjoint.

(3) Finally, if $f : \mathrm{X} \to \mathrm{Y}$ is an arbitrary quasicompact, quasiseparated morphism then we do the following: pick an affine open cover of Y (which need not be finite) and $f_*\mathcal{F}$ is specified on $j : \operatorname{Spec} \mathrm{A} \to \mathrm{Y}$ by

$$j^*(f_*\mathcal{F}) := f'_* j'^*(\mathcal{F}),$$

where $j' : \operatorname{Spec} \mathrm{A} \times_{\mathrm{Y}} \mathrm{X} \hookrightarrow \mathrm{X}$ is the open immersion and $f' : \operatorname{Spec} \mathrm{A} \times_{\mathrm{Y}} \mathrm{X} \to \operatorname{Spec} \mathrm{A}$ is qcqs morphism, whence the previous construction does apply.

Altogether we obtain $f_* : \mathbf{QCoh}(\mathrm{X}) \to \mathbf{QCoh}(\mathrm{Y})$ whenever f is qcqs.

Lemma 16.0.2. *Let* X *be qcqs scheme and* $f : \mathrm{X} \to \operatorname{Spec} \mathbf{Z}$, *then* $f_*(\mathcal{F}) \cong \Gamma(\mathcal{F})$ *canonically.*

More generally, if $\mathrm{X} \to \operatorname{Spec} \mathrm{A}$ is a qcqs morphism over A, we set $\Gamma_{\mathrm{A}}(\mathcal{F}) := f_*(\mathcal{F})$, which picks up the structure of an A-module. Another point is that if $f : \mathrm{X} \to \mathrm{Y}$ is a qcqs morphism of schemes, then we have a canonical map

$$f^*\mathcal{O}_{\mathrm{Y}} \to \mathcal{O}_{\mathrm{X}},$$

and so we have its adjoint

$$\mathcal{O}_{\mathrm{Y}} \to f_*\mathcal{O}_{\mathrm{X}}.$$

Most textbooks prefer to consider this second data as we will see later and we write it as $f^\sharp : \mathcal{O}_{\mathrm{Y}} \to f_*\mathcal{O}_{\mathrm{X}}$.

16.1. **Some topological properties of schemes.** One of our first goals is to prove

Corollary 16.1.1. *Let* X *be reduced. Suppose that there exist a dense open* $W \subset X$ *such that* $f|_W = g|_W$ *then* $f = g$.

In order to speak of dense opens, we need to define the **scheme-theoretic image** of morphisms: if $f : X \to Y$ is defined as the closed subscheme of Y defined via the ideal sheaf

$$\mathcal{I}_{f(X)} := \ker(f^\sharp),$$

so that we have a short exact sequence in **QCoh**(Y)

$$0 \to \mathcal{I}_{f(X)} \to \mathcal{O}_Y \xrightarrow{f^\sharp} f_*\mathcal{O}_X.$$

We now that this ideal shea fcorresponds to a closed subscheme which we write as $\overline{f(X)} \hookrightarrow Y$. Furthermore, the morphism $f : X \to Y$ factors as

$$X \to \overline{f(X)} \to Y.$$

Example 16.1.2. If $f : X = \operatorname{Spec} B \to Y = \operatorname{Spec} A$. Then we get a morphism of A-algebras $\varphi : A \to B$. We then have the kernel I and the map φ factors as

$$A \to A/I \to B.$$

The scheme-theoretic image of f is then given by $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} B$ and we have a factorization

$$\operatorname{Spec} B \to \operatorname{Spec} A/I \to \operatorname{Spec} A.$$

Let us have a look at two examples:

(1) let f be a nonzero divisor in B; therefore we have an injection

$$B \hookrightarrow B[\tfrac{1}{f}],$$

so that $I = 0$ and $B/0 = B$. Therefore, scheme-theoretic image of the open immersion $\operatorname{Spec} B[\frac{1}{f}] \hookrightarrow \operatorname{Spec} B$ is all of B.

(2) Consider the morphism

$$j : \mathbf{A}^1 \hookrightarrow \mathbf{P}^1,$$

which makes up one of the affine open covers of $\mathbf{P}^1$. From this we get a morphism

$$j^\sharp : \mathcal{O}_{\mathbf{P}^1} \to j_*\mathcal{O}_{\mathbf{A}^1}.$$

We claim that this morphism is injective. Indeed, if we restrict $j^\sharp$ along j itself, then we get the identity morphism. On the other hand, if $j' : \mathbf{A}^1 \hookrightarrow \mathbf{P}^1$ is the other affine line, then we are looking at

$$j \times_{\mathbf{P}^1} \mathbf{A}^1 : \mathbf{A}^1 \setminus 0 \hookrightarrow \mathbf{A}^1,$$

and so this is a morphsim between affine schemes and

$$j'^* j^\sharp : \mathbf{Z}[t] \to \mathbf{Z}[t, t^{-1}].$$

As a result, $\mathcal{I} = 0$ and the scheme theoretic image of j is all of $\mathbf{P}^1$.

These two are examples of the next definition.

Definition 16.1.3. A quasicompact (any open immersion is, in fact, quasiseparated) open immersion $U \hookrightarrow X$ is **dense** if $\overline{U} = X$.

16.2. **Irreducibility and generic points.** Before we prove our result about maps on dense opens, we need to speak about a generalization of connected components of schemes.

Definition 16.2.1. An affine scheme $X = \operatorname{Spec} A$ is **irreducible** if the nilradical of A is a prime ideal.

Remark 16.2.2. There was a previous false assertion that primality of the nilradical is the same thing as every zerodivisor being nilpotent. This is false: consider $k[x, y]/(x^2, xy)$; we note that the nilradical is given by (x) and hence is a prime ideal, but y is a zero divisor which is evidently not nilpotent. On the other hand, if every zero divisor is nilpotent, then I claim that $\sqrt{(0)}$ is a prime ideal. Indeed, let $xy \in \sqrt{(0)}$, then we see that $(xy)^n = 0$ and hence we may assume that x is a zero divisor. This means that x is nilpotent as desired.

Remark 16.2.3. Recall that if X is a topological space, then it is **irreducible** whenever it cannot be written as a union of two proper closed subsets. Then, Definition 16.2.1 is equivalent to saying that the Zariski spectrum (discussed in problem set) is irreducible.

Example 16.2.4. Any irreducible scheme is connected. This is in the correct exercises.

Example 16.2.5. Consider $\operatorname{Spec} k[x, y]/(xy) = X$. Then X is connected but it is not irreducible. Indeed, x is a zero divisor which is not nilpotent.

Example 16.2.6. Let $\mathcal{O}$ be a local ring. Then $\operatorname{Spec} \mathcal{O}$ need not be irreducible: for example $k[x, y]/(xy)_{(x,y)}$. Here, the nilradical is (0) but then $xy = 0$ and neither is zero.

The next Lemma is one of the key innovations of the Grothendieck set-up for algebraic geometry: one can index irreducible components (to be defined soon) in terms of **generic points**. This is a supremely helpful concept all over algebraic geometry: for example it lets us speak of "the generic cubic 3-fold" say within the moduli of cubic 3-folds while being precise.

Definition 16.2.7. Let X be a scheme. We say that a morphism $\eta : \operatorname{Spec} K \to X$, where K is a field, is a **generic point** of X if it factors through $U \hookrightarrow X$ for any nonempty affine open U of X.

Lemma 16.2.8. *Let* $X = \operatorname{Spec} A$ *be an affine scheme which is irreducible. Then it admits a unique generic point.*

Proof. Let $X = \operatorname{Spec} A$ then $\sqrt{(0)}$ is a prime ideal by definition. From this we get that $A/\sqrt{(0)}$ is an integral domain and set $K := \operatorname{Frac}(A/\sqrt{(0)})$ so that we have a map $A \to \operatorname{Frac}(A/\sqrt{(0)}$. It suffices to prove that $\eta : \operatorname{Spec} K \to X$ factors through $\operatorname{Spec} A_f$ for any $f \in A$. On the level of algebra we want to construct the filler

$$\begin{array}{ccc} A & \xrightarrow{\eta^\sharp} & K \\ \downarrow & \nearrow & \\ A_f & & \end{array} .$$

There are two cases:

(1) if f is nilpotent element then $A_f = 0$ so that the existence is filler is assured since f is mapped to the zero element in K;
(2) otherwise it is not nilpotent and f is mapped to a nonzero, hence invertible, element in K.

We leave uniqueness to the reader.

□

Definition 16.2.9. Let X be a scheme. We say that it is **irreducible** if X is nonempty and for every nonempty affine open $U \hookrightarrow X$, then U is irreducible.

One can globalize Lemma 16.2.8 to prove that:

Lemma 16.2.10. *Any irreducible scheme* X *has a unique generic point.*

Example 16.2.11. The easiest irreducible scheme is probably Spec $\mathbf{Z}$. We note that its generic point is Spec $\mathbf{Q} \to$ Spec $\mathbf{Z}$. For contrast: let k be a field and $\mathbf{A}^1_k$ be the affine line over k. Then Spec $k(t) \to \mathbf{A}^1_k$ is the generic point. This also extends to non-affine schemes, Spec $k(t) \to \mathbf{P}^1$ is the generic point.

Definition 16.2.12. An **integral scheme** is a non-empty scheme which is reduced and irreducible.

The following falls out from the definitions above.

Lemma 16.2.13. *The following are equivalent:*

(1) $\mathrm{X} = \operatorname{Spec} \mathrm{A}$ *is integral.*
(2) A *is an integral domain.*

16.3. **Exercises.**

Exercise 16.3.1. *Prove that irreducible schemes are connected. Prove that* Spec *of local rings are connected.*

Exercise 16.3.2. *Prove that any irreducible scheme* X *has a unique generic point.*

Exercise 16.3.3. *A prime ideal* $\mathfrak{p} \subset \mathrm{A}$ *is* **minimal** *if for any other prime ideal* $\mathfrak{p}'$ *such that* $\mathfrak{p}' \subset \mathfrak{p}$ *then* $\mathfrak{p}' \subset \mathfrak{p}$. *Let* A *be ring such that* Spec A *is irreducible, prove that* $\sqrt{(0)}$ *is a minimal prime.*

Let X be a scheme. We will denote by $\mathrm{X}^{(0)}$ the set of all irreducible components of X.

Exercise 16.3.4. *Let* A *be a ring and let* $\mathrm{X} = \operatorname{Spec} \mathrm{A}$. *Consider the set* $|\operatorname{Spec} \mathrm{A}|$ *of all prime ideals of* A. *Consider*

$$|\operatorname{Spec} \mathrm{A}| \to \mathrm{X}^{(0)} \qquad \mathfrak{p} \mapsto \operatorname{Spec} \mathrm{A}/\mathfrak{p}.$$

Prove that this defines a bijection from the minimal primes of A *to the set of all irreducible components of* X.

Remark 16.3.5. The point of the above exercises is that irreducible components of X are indexed by minimal primes.

17. Lecture 17: Integral and normal schemes

Let us first get the following result out of the way:

Corollary 17.0.1. *Let* S *be a base scheme and let* X, Y *be* S*-schemes. Assume that* $f, g : \mathrm{X} \to \mathrm{Y}$ *is a morphsim of* S*-schemes such that there exists a dense open* $\mathrm{U} \subset \mathrm{X}$ *such that* $f|_{\mathrm{U}} = g|_{\mathrm{U}}$. *Then if* Y *is separated, we have that* $f = g$.

Proof. By Lemma 15.3.13, the subscheme of X where f agrees with g is a closed subscheme $\mathrm{Z} \hookrightarrow \mathrm{X}$. Now, we have that $\mathrm{U} \hookrightarrow \mathrm{Z}$ (the open immersion $\mathrm{U} \hookrightarrow \mathrm{X}$ factors through Z) by assumption. But now U is dense in X which means that $\overline{\mathrm{U}} = \mathrm{X}$. But now, we invoke the universal property of the scheme-theoretic image from Lemma 17.0.2 to conclude that $\overline{\mathrm{U}} \subset \mathrm{Z}$, and thus $\mathrm{X} = \mathrm{Z}$. □

The universal property of the scheme-theoretic image roughly states that the scheme-theoretic image of the morphism f is the *smallest* closed subscheme of Y for which f factors through.

Lemma 17.0.2. *Let* $f : \mathrm{X} \to \mathrm{Y}$ *be a qcqs morphism of schemes. Then the scheme-theoretic image* $\overline{f(\mathrm{X})} \hookrightarrow \mathrm{Y}$ *has the following universal property: given another closed subscheme* $\mathrm{Z}' \hookrightarrow \mathrm{Y}$ *such that* f *factors through as* $\mathrm{X} \to \mathrm{Z}' \to \mathrm{Y}$, *then we have a closed immersion*

$$\overline{f(\mathrm{X})} \hookrightarrow \mathrm{Z}'.$$

Proof. Recall that the scheme-theoretic image of f is given by the ideal $\mathcal{I} := \ker(f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X)$. We note that f factors through Z' if and only if the ideal sheaf corresponding to Z', denoted by $\mathcal{I}'$ is *contained in* $\mathcal{I}$:

$$\mathcal{I}' \subset \mathcal{I};$$

which is equivalent to saying that $\overline{f(X)} \subset Z'$. □

Remark 17.0.3. If there were topological spaces involved in the definition of schemes, there is a meaningful notion of image and a corresponding notion being dense topologically and our notion corresponds to being scheme-theoretically dense. We will not have to discuss this topological notion, but it agrees with our notion whenever the scheme is reduced [Stacks, Tag 056E].

Remark 17.0.4. If we drop the assumption that Y is separated, the result is false. For example let $\mathbf{A}^1_{\pm}$ be the affine line with doubled origin, then we have morphisms

$$+, - : \mathbf{A}^1 \to \mathbf{A}^1_{\pm};$$

one going to the "+"-part of $\mathbf{A}^1$ and the other going to the "−"-part. Then these morphims agree on $\mathbf{A}^1 \smallsetminus 0$ but evidently do not agree on all of $\mathbf{A}^1$.

17.1. **Integral schemes.** Last time, we discussed the existence of a generic point η on an irreducible scheme X: it is a point that exists in all the nonempty open subschemes of X. Since generic points are unique, we can index irreducible components of X using generic points. Just as we can decompose a scheme into its connected components, we also want to decompose a scheme into its irreducible components. The correct definition involves looking only at the underlying "subset" without any "scheminess."

Definition 17.1.1. We say that a scheme is **integral** if it is reduced and irreducible.

Lemma 17.1.2. *Let* X *be a scheme. Then the following are equivalent:*

(1) X *is integral;*
(2) for every $U \hookrightarrow X$ *an open immersion where* U *is affine, then* U *is* Spec *of a reduced ring;*
(3) there exists an open affine cover of X *made up of* Spec *of integral domains.*

Proof. It suffices to prove the following statement:

- if A is a ring and $X = \operatorname{Spec} A$, then X is integral if and only if A is an integral domain.

Indeed, this is equivalent to saying that $\sqrt{(0)} = (0)$ and is a prime ideal. □

Example 17.1.3. The scheme $\operatorname{Spec} k[x]/(x^2)$ is not reduced. But is irreducible: its nilradical is generated is the prime ideal x. For an example of a ring A which is reduced but not irreducible consider $\mathbf{Z}[X]/(2X)$ — this has no nilpotents but its 2 is a zero divisor which is clearly not nilpotent.

Definition 17.1.4. Let X be a scheme, then $Z \subset X$ is an **irreducible component** if it is a closed subscheme of X which is 1) integral and 2) is maximal: if $Z' \hookrightarrow X$ is a closed immersion which is also integral such that it contains Z then $Z' = Z$.

Remark 17.1.5. Let Z be a closed subscheme of X. Then we can consider the poset of closed subschemes

$$\{Z \subset X : Z'_{\text{red}} = Z_{\text{red}}\};$$

we say that this is the posed of closed subschemes of X whose **support** is Z_{red}. Since the reduction was proved to be a scheme, the closed subscheme Z_{red} is a minimal element of this poset. Hence any irreducible (but not necessarily reduced) closed subscheme of X does determine uniquely an irreducible component of X. We insist on integrality to ensure that we are dealing with the "underlying topological space" of Z and ignoring the thickenings, making sure that the above notion is manifestly topological.

Example 17.1.6. The scheme $\operatorname{Spec} k[x,y]/(xy)$ which is the coordinate axis is not irreducible. It has two components $\operatorname{Spec} k[x,y]/(xy,y), \operatorname{Spec} k[x,y]/(xy,x)$.

We briefly speak about decomposing schemes in terms of irreducible components with details left as exercises. The next two lemmas a purely "topological" statement.

Lemma 17.1.7. *Let* X *be a reduced scheme. Then the following are equivalent:*

(1) X *is integral;*
(2) X *is its own irreducible component;*
(3) X *cannot be written as a union of two proper (in the sense that it is not the whole of* X*), integral closed subschemes;*
(4) for every proper closed subscheme $\mathrm{Z} \subset \mathrm{X}$*, then* $\mathrm{X} \smallsetminus \mathrm{Z}$ *is dense.*

Via a Zorn's lemma argument we can then prove the following decomposition:

Lemma 17.1.8. *Let* X *be a scheme. Then we can write*

$$\mathrm{X}_{\mathrm{red}} = \bigcup_i \mathrm{X}_i,$$

where each X_i *is an irreducible component of* X.

The most useful version of Lemma 17.1.8, however, will be its noetherian version where we can write X as a union of its irreducible components.

17.2. **Normal schemes.** We have seen some topological properties of schemes. We will finish off today's lecture with the first *regularity* property that we can impose on an integral scheme (so topologically as nice as you can get) but one that relies on some exclusively codimension one phenomenon (though we have yet to introduce dimension theory). This notion is called **normality** It has two meanings which you should bear in mind:

(1) for a "curve" (this will be defined in some generality next class), it is normal if and only if it is a smooth curve. So we are avoiding nodes: something modeled by the xy-axis and, more precisely, $\operatorname{Spec} k[x,y]/(y^2 = x^3+x^2)$ or the cusp $\operatorname{Spec} k[x,y]/(x^2 = y^3)$. Even if we had not defined the notion of smooth scheme yet, clearly these schemes cannot ever be called smooth.
(2) Hartog's lemma in complex geometry states that a holomorphic function defined away from a codimension two subset (this is also "measure zero"!) can be extended over that locus.

We want to expand on this second point. Let us recall some concepts from commutative algebra: let A be a domain so that we have an injection $\mathrm{A} \subset \mathrm{K}$ into the fraction field. Recall the notion of integral closure if $\mathrm{A} \subset \mathrm{B}$ is an inclusion of rings, then $b \in \mathrm{B}$ is an integral element over A if any of the following conditions are satisfied

(1) there exists $n \geqslant 1$ and a monic polynomial $p(x) \in \mathrm{A}[x]$ such that $p(b) = 0$;
(2) the subring $\mathrm{A}[b] \subset \mathrm{B}$ is a finitely generated A-module;
(3) there exists $\mathrm{C} \subset \mathrm{B}$ which is a subring and a finitely generated A-module that contains $\mathrm{A}[b]$;

We say that $\mathrm{A} \subset \mathrm{B}$ is an **integral extension** or that B **is integral over** A if every element of B is integral over A. If $\mathrm{A} \subset \mathrm{B}$ is an arbitrary extension, the **integral closure** is the subset of B which are integrally closed over A; fact: this is a ring. We say that A is **integrally closed** in B if the integral closure of A in B is itself. In other words any element $b \in \mathrm{B}$ which is integral over A is, in fact, in A. Here comes the key definition:

Definition 17.2.1. A domain A is **integrally closed** if A is integrally closed in its fraction field. We say a ring A (not necessarily a domain) is **normal** if the localization at each prime $\mathrm{A}_{\mathfrak{p}}$ is a domain which is integrally closed in its fraction field.

We say that an affine scheme is a **normal scheme** if Spec of a normal ring. We say that X is a **normal scheme** if every open affine $\mathrm{U} \subset \mathrm{X}$ is, in fact, a normal scheme.

Remark 17.2.2. Here is a quick warning: being integral is not a local condition, hence a ring whose localizations are all integral domains need not be an integral domain. For example take $k[x]/(x(x-1))$. Then there are two prime ideals, (x) and $(x-1)$. The localization $k[x]/(x(x-1))_{(x)} \cong k[x]/(x) = k$ and, similarly, $k[x]/(x(x-1))_{(x-1)} \cong k$. These are both domains but $k[x]/(x(x-1))$ itself is not a domain. Geometrically, this scheme corresponds to two copies of $\operatorname{Spec} k$; from this point of view it is obvious that a scheme which is "locally irreducible" need not be globally irreducible.

Remark 17.2.3. Because of the above remark, it is often not a good idea to speak about normal rings which are not domains. If we further impose the condition that A is a domain, we have the following equivalences:

(1) A is integrally closed,
(2) $A_{\mathfrak{p}}$ is integrally closed for all prime ideals $\mathfrak{p}$,
(3) $A_{\mathfrak{m}}$ is integrally closed for all maximal ideals $\mathfrak{m}$.

First, we note an unconditional lemma which we refer to [Stacks, Tag 034M] for a proof; we will give a proof of this in the case of A is a domain below which is only linguistically simple as we can use the field of fractions as opposed to the total ring of fractions.

Lemma 17.2.4. *If* A *is a normal ring (so not necessarily a domain), then* A *is integrally closed in* Q(A).

Hence what is at stake is the statement that: if A is integrally closed in Q(A) (but not a domain), then $A_{\mathfrak{p}}$ is a domain which integrally closed in its fraction field. This is *not going to be true*. What is true if when A is furthermore asked to be a domain:

Lemma 17.2.5. *Let* A *be a domain. Then the following are equivalent:*

(1) A *is integrally closed,*
(2) $A_{\mathfrak{p}}$ *is integrally closed for all prime ideals* $\mathfrak{p}$*,*
(3) $A_{\mathfrak{m}}$ *is integrally closed for all maximal ideals* $\mathfrak{m}$*.*

Proof. I claim that the property of being integrally closed is stable under localizations at least when A is a domain. Indeed, let S be a multiplicatively closed subset of A, then I claim that $S^{-1}A$ is a integrally closed. To see this, consider a monic polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in S^{-1}A[x]$ and assume that there exists a root $x_0 \in \operatorname{Frac}(S^{-1}A)$. We need to prove that $x_0 \in S^{-1}A$. Indeed, we may clear denominators in the coefficients appearing in $p(x)$ by multiplying with the product appearing in the denominators of the coefficients of $p(x)$; call this polynomial $p'(x) = gp(x) \in A[x]$, $g \in A$. Now, $\operatorname{Frac}(S^{-1}A) = \operatorname{Frac}(A)$, since A is a domain, and we further have that $p'(x_0) = gp(x_0) = g \cdot 0 = 0$. This shows that $x_0 \in A$ since A was assumed to be integrally closed and thus in $S^{-1}A$. This proves that A is an integrally closed domain, then $A_{\mathfrak{p}}$ is integrally closed for any prime ideal $\mathfrak{p}$ and hence (1) ⇒ (2). Clearly, (2) ⇒ (1). Now I claim that (3) ⇒ (1). As in the proof of Hartog, let $x \in \operatorname{Frac}(A)$ and assumed to be integral. Let $I := \{f : fx \in A\}$ so that $I \neq A$ if and only if $x \notin A$. Assume that A is not integrally closed so that $x \notin A$. Then I is a proper ideal and thus contained in some maximal ideal $\mathfrak{m}$. But this means that $x \notin A_{\mathfrak{m}}$ but x is integral over $A_{\mathfrak{m}}$ and since $A_{\mathfrak{m}}$ is integrally closed we achieve a contradiction. □

The converse

Example 17.2.6. Right off the bat let us look at $\operatorname{Spec} k[x,y]/(y^2 - x^3 - x^2)$ or the nodal cubic. Consider the element $t = y/x$ in the function field; we see that

$$t^2 - (x+1) = y^2/x^2 - (x+1) = (x^2 + x^3/x^2) - (x+1) = (x+1) - (x+1) = 0;$$

but clearly y/x is not in the original ring. This proves that the nodal cubic is not a normal scheme.

Example 17.2.7. Consider the scheme $\operatorname{Spec} k[x,y,z]/(x^2 + y^2 - z^2)$. This is a conic in 3-space and is a normal scheme; pictorially, however, this better not be smooth.

18. Lecture 18: Normality continued and finiteness conditions

Let us recall that a ring is said to be normal if it is a domain and is integrally closed in its fraction field. We saw that the node and the cusp are not normal. Here's a large number of examples of schemes which are normal:

Example 18.0.1. A **Dedekind domain** R is an integral domain which is not a field such that R is noetherian and the localization at every prime ideal of R is a discrete valuation ring. Examples: $\mathbf{Z}, k[x], \mathbf{Z}_p$ and various other examples from number theory that one might like. These are all normal domains.

One useful way to determine if a scheme is normal is the next lemma; it is an exercise in going from local to global statements.

Lemma 18.0.2. *Let* X *be an integral scheme. Then* $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$ *is a normal ring.*

Proof. It is easy to see that $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$ is a domain. Now suppose that we have an equation

$$f^d + \alpha_{d-1}f^{d-1} + \cdots + \alpha_0 = 0,$$

where $f = \frac{x}{y}$ is an element of the fraction field and $\alpha_j \in \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$. I claim that, in fact, $f \in \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$. Using Serre's theorem the $\alpha_n' s, x, y$ and f are all determined by their restrictions to open affines. Now our goal is to concretely prove that $\frac{x}{y} \in \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$. But now, we have

$$\frac{x|_{\mathrm{U}}}{y|_{\mathrm{U}}} = \frac{x}{y}|_{\mathrm{U}} = (f^d + \alpha_{d-1}f^{d-1} + \cdots + \alpha_0)|_{\mathrm{U}} = 0,$$

for all open affines in U. Since U is the spectrum of an integral scheme, we must have that $\frac{x}{y}|_{\mathrm{U}} \in \Gamma(\mathrm{U}, \mathcal{O}_{\mathrm{U}})$ and so $\frac{x}{y}$ is in $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$ as desired. □

We mentioned that our main interest behind normality is the following theorem:

Theorem 18.0.3 (Algebraic Hartog's Lemma)**.** *Let* A *be a noetherian domain which is also normal. Then*

$$\mathrm{A} = \cap_{\mathfrak{p}, \mathrm{cod}_{\mathfrak{p}}\mathrm{A}=1} \mathrm{A}_{\mathfrak{p}},$$

within K.

Before we define the terms involved in algebraic Hartog's let's try to understand what this says: an element $t \in \mathrm{Frac}(\mathrm{A})$ in its fraction field should be thought of as rational function on Spec A: indeed it is of the form $t = \frac{f}{g}$ where $f, g \in \mathrm{A}$, which determine maps

$$f, g : \mathrm{Spec}\,\mathrm{A} \to \mathbf{A}^1_{\mathbf{Z}}.$$

Therefore, we can think of t as having poles being the zero's of g. Now, the intersection above is taken in K so being regular means that t must live in $\mathrm{A}_{\mathfrak{p}}$ for each $\mathfrak{p} \in \mathrm{A}$ which is a prime of codimension 1. Intuitively, a prime of codimension 1 is a hypersurface; this is true for example in a unique factorization domain. What does it mean for $t \in \mathrm{Frac}(\mathrm{A})$ to be in $\mathrm{A}_{\mathfrak{p}=(h)}$ in this case? Well it is actually easier to say what it means for t to not be in $\mathrm{A}_{(h)}$: it means that t has a pole along A/h. From this point-of-view we see that an element of fraction field is regular if and only if it has no pole along each codimension one subscheme. We will use this result a lot when we discuss the theory of divisors.

This gives us a leeway to discuss **numerical invariants** of schemes such as dimension, codimension etc. These invariants are terribly behaved unless the scheme at hand is noetherian. We have already seen the noetherian condition come up in the problem sets.

18.1. Noetherian schemes.

One of the main motivation to begin imposing noetherianity conditions is to be able to decompose a scheme easily into irreducible components. Recall that a ring is noetherian if it has the ascending chain condition for ideals: given a sequence of strictly increasing inclusions of ideals

$$\emptyset = \mathrm{I}_{-1} \subset \mathrm{I}_0 \subset \mathrm{I}_1 \cdots \mathrm{I}_k \subset \mathrm{I}_{k+1} \subset \cdots,$$

there exists a $j \gg 0$ such that $\mathrm{I}_j = \mathrm{I}_{j+1}$. This is also equivalent to more "global conditions":

(1) every ideal is finitely generated;
(2) every submodule of a finitely generated module is also finitely generated.

and a geometric condition that for any flag of strictly decreasing closed subschemes

$$\cdots \hookrightarrow Z_k \cdots \hookrightarrow Z_1 \hookrightarrow Z_0 = X,$$

the chain terminates.

Example 18.1.1. If R is a noetherian ring, then the **Hilbert basis theorem** asserts that R[T] is also noetherian. Furthermore the quotient of a noetherian ring is also noetherian. As a result this gives a way to generate lots of examples of noetherian rings: a morphism of rings $S \to R$, equivalently a morphism of affine schemes $\operatorname{Spec} R \to \operatorname{Spec} S$ is said to be **finite type** or S **is a finitely generated** R**-algebra** if there exists $n \in \mathbf{Z}$ and a surjection

$$S[T_1, \cdots, T_n] \twoheadrightarrow R.$$

We then record:

Lemma 18.1.2. *If* R *is a noetherian ring, then if we have a finite type morphism* $\operatorname{Spec} S \to \operatorname{Spec} R$*,* S *is noetherian.*

To globalize we adopt the following perspective:

Definition 18.1.3. A scheme X is **locally noetherian** if it admits an Zariski open cover by Spec of Noetherian rings. It is **noetherian** if it is locally noetherian and quasicompact.

Of course we have

Lemma 18.1.4. *Let* X *be a scheme. The following are equivalent:*

(1) X *is noetherian;*
(2) X *has the descending chain condition with respect to closed subschemes: if*

$$\cdots \hookrightarrow Z_k \cdots \hookrightarrow Z_1 \hookrightarrow Z_0 = X,$$

is a flag of closed subschemes, then this chain terminates.

Proof. The equivalence has been discussed when X is affine. Now, if X is noetherian, then the chain terminates at the maximal index for which the restriction of this chain to an affine open terminates. But X is quasicompact so this number is finite. On the other hand, if $\mathcal{U}$ is a cover of X we can pick an ordering on this cover and construct an ascending chain of opens covering X, its complement is a descending chain of closed subschemes which must terminate by assumption hence X is quasicompact; the local noetherianness of X is immediate. □

Definition 18.1.5. A morphism of schemes $f : X \to Y$ is **locally finite type** if for each affine open $U = \operatorname{Spec} A \subset Y$ there exists an affine open cover $\mathcal{V}_U$ of $U \times_Y X$ such that for each $\operatorname{Spec} B_j \hookrightarrow U \times_Y X$, the map $\operatorname{Spec} B_j \to \operatorname{Spec} A$ is of finite type. We say that f is finite type if it is furthermore quasicompact.

Lemma 18.1.6. *Let* $f : X \to Y$ *be a morphism and let* Y *be a (locally) noetherian scheme. Then if* f *is (locally) of finite type, then* X *is (locally) noetherian as well.*

Example 18.1.7. Our favorite example $\mathbf{P}^n$ is a noetherian scheme: it is locally noetherian since it admits a Zariski open cover by $\mathbf{A}^n$ and since the number of $\mathbf{A}^n$'s needed to cover $\mathbf{P}^n$ is finite, it is also quasicompact.

Example 18.1.8. It is easy to see that closed subschemes of noetherian schemes are noetherian. The next example is slightly more surprising.

Example 18.1.9. Let $U \hookrightarrow X$ be an open immersion and let X be noetherian. I claim that U is, in fact, noetherian. It is not hard to check that U is locally noetherian, what is interesting is that the morphism $U \hookrightarrow X$ is quasicompact: yet another demonstration that the notion of quasicompactness for schemes should not be compared to the notion of compactness for manifolds. To see this, we note that a morphism is quasicompact if and only if the inverse image of every open affine must be quasicompact and so we need to know that any open subscheme of a notherian affine scheme is quasicompact (we know this is false without the noetherian hypothesis), which we do.

The next lemma is left to the reader, but it's actually easier to prove using the descending chain condition definition.

Lemma 18.1.10. *Any noetherian scheme has finitely many irreducible components.*

In view of dimension theory, we have

Definition 18.1.11. A point $x : \operatorname{Spec} k \to \mathrm{X}$ is a **closed point** if it is a closed immersion

Lemma 18.1.12. *Let* X *be a scheme and* $x : \operatorname{Spec} k \to \mathrm{X}$ *is a morphism. Then the following are equivalent:*

(1) x *is a closed point,*
(2) x *factors through an affine open* $\mathrm{U} = \operatorname{Spec} \mathrm{A} \subset \mathrm{X}$ *such that* x *corresponds to the residue field of a maximal ideal.*

Example 18.1.13. Let $\mathcal{O}$ be a local ring, then $\operatorname{Spec} \mathcal{O}$ has a unique closed point given by $\operatorname{Spec} \mathcal{O}/\mathfrak{m} \hookrightarrow \operatorname{Spec} \mathcal{O}$.

Proposition 18.1.14. *Every non-empty closed subscheme of a quasicompact scheme contains a closed point.*

Proof. By Zorn's lemma, any closed subscheme Z has a minimal closed subscheme $\mathrm{Z}_0 \subset \mathrm{Z}$ (a closed subscheme with no nonempty proper closed subscheme). We may further assume that Z_0 is reduced. Now choose a nonempty open affine $\mathrm{U} \subset \mathrm{Z}_0$, then the complement $\mathrm{Z}_0 \smallsetminus \mathrm{U}$ is a proper closed subscheme of Z_0 so that $\mathrm{Z}_0 = \mathrm{U}$, hence Z_0 must be affine. But by minimality of $\mathrm{Z}_0 = \operatorname{Spec} \mathrm{A}$, we conclude that A is a ring whose only proper ideal is a field. Therefore $\mathrm{Z}_0 \hookrightarrow \mathrm{X}$ is of the form $\operatorname{Spec} k \hookrightarrow \mathrm{X}$, which is a closed immersion. □

19. Lecture 19: Dimension theory; proof of Hartgog

Last time, we introduced noetherian schemes: they are schemes which are locally noetherian and quasicompact. There are plenty of examples of locally noetherian schemes coming from the Hilbert basis theorem: if R is a noetherian ring and $\operatorname{Spec} \mathrm{S} \to \operatorname{Spec} \mathrm{R}$ is of finite type, we get that $\operatorname{Spec} \mathrm{S}$ is a noetherian scheme. We introduced also the global notion of being finite type which is a morphism of schemes $f : \mathrm{Y} \to \mathrm{X}$ which is locally of finite type and quasicompact. In particular if f is of finite type and X is a noetherian scheme, then Y is of finite type. Lastly, we note that closed subschemes of noetherian schemes are noetherian since closed immersions are quasicompact morphisms and it is locally of finite type in a tautologoical way. We thus get that if A is a noetherian ring, then any closed subscheme

$$\mathrm{X} \hookrightarrow \mathbf{P}^n_{\mathrm{A}}$$

is a noetherian scheme over $\operatorname{Spec} \mathrm{A}$; a little more surprisingly is the fact that open subschemes of noetherian schemes are also noetherian since open immersions into a noetherian scheme is, in fact, a quasicompact morphism. It is worth introducing the following:

Definition 19.0.1. A morphism of schemes $f : \mathrm{X} \to \mathrm{Y}$ is said to be a **locally closed immersion** if it can be factored as

$$\mathrm{X} \xrightarrow{i} \overline{\mathrm{X}} \xrightarrow{j} \mathrm{Y},$$

where i is an open immersion and j is an open immersion.

Let us get back to our regularly scheduled program: dimension theory.

19.1. **Dimenson theory I: local dimension theory.** Dimension theory is only reasonable for noetherian schemes so let us now assume this. In any case dimension theory for schemes is actually pretty tricky business. Here are things to keep in mind to never be lost:

- Dimension is an intrinsically local notion: given any k-point $x : \operatorname{Spec} k \to \mathrm{X}$, then we should be able to make sense of $\dim_x(\mathrm{X})$.
- Never compute the dimension of a scheme without going into irreducible components.
- Never try to compute the codimension of a non-reduced scheme (the dimension is a number which only depends on the reduced scheme).
- The codimension of a point is always the dimension of its local ring and *this is often easier to compute.*

I would like to expand on this last notion because that's all we need to speak about the algebraic Hartog's theorem. We first review quickly the notion of the dimension of a ring.

Definition 19.1.1. The **Krull dimension** of a ring R is

$$\sup\{n : \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n\};$$

each of the inclusions above are assumed to be proper. The **height** of a prime ideal is the dimension of the localization $\mathrm{R}_{\mathfrak{p}}$.

This is all fine and good and you probably have your own way of thinking about dimension, but I want to empahsize that dimension is "algorithmic" in the local noetherian case. I would like to offer one way of thinking about this, premised upon the following result from commutative algebra:

Theorem 19.1.2. *Let* R *be a noetherian ring. Then the following are equivalent:*

(1) $\dim(\mathrm{R}) = 0$*;*

(2) R *is an Artinian ring: it satisfies the descending chain condition for ideals.*

Furthermore we have a complete classification of Artinian rings: R *is Artinian if and only if*

$$\mathrm{R} \cong \prod_{i=1}^{n} \mathrm{R}_i,$$

where each R_i *is an Artin local ring. In this case, we have that there exists an* $n \gg 0$ *such that*

$$\mathfrak{m}^n = 0.$$

In fact, a noetherian local ring is precisely those whose maximal ideal is nilpotent (in other words if $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ *for all* n*, then* R *is not Artinian).*

The above result tells us that, from the point-of-view of dimension theory, we should treat Artin local rings as the most "atomic" unit of rings. They are furthermore easy to think about: they are exactly those noetherian local rings with nilpotent maximal ideals.

Example 19.1.3. Fields are the simplest examples of artinian local rings. Conversely, if A is a *reduced* noetherian local ring then $\mathfrak{m}^n = (0) \Rightarrow \mathfrak{m} = (0)$ and hence A is a field. Therefore we conclude:

- any reduced Artinian ring must be a product of fields.

In scheme theoretic terms, any dimension zero, noetherian, reduced local scheme must be a finite coproduct of Spec of fields.

Example 19.1.4. There are plenty of Artin local rings which are not reduced: like our good friend $\operatorname{Spec} k[x]/(x^2)$, note that this also dimension zero.

Knowing that Artin local rings are exactly those rings which are dimension zero is incredibly helpful: say we wish to measure the dimension of the local ring R which is assumed to be noetherian. How would we do this? For a moment, suppose that $\mathrm{R} = k[x_1, \cdots, x_n]_{(x_1,\cdots,x_n)}$ whose Spec is the localization affine space at the point zero. How do we know that it is dimension n?

Well we feel that this must be dimension n because we can "slash down" the dimension of $\operatorname{Spec} R$ all the way down to the Spec of an Artin local ring by looking the intesection of all the standard hyperplanes: there are n of them and we know that the point must be the smallest dimensional subscheme of $\mathbf{A}^n_k$. We can read this the other way around: it is the closed subscheme of the *largest codimension* so that we might expect $\mathbf{A}^n$ to have the exactly this dimension as if it had a higher dimension, then there would be a higher codimension subscheme — so an even smaller point. However, such a definition should not make you happy as there are various ways of going from $\mathbf{A}^n_k$ to a point: for example we can cut down by hypersurfaces and such: take for example

$$(x_1^2, x_2^2, \cdots, x_n^2) \subset k[x_1, \cdots, x_n]_{(x_1, \cdots x_n)};$$

this also lets us slash down the (localized) affine space to the point. On the other hand, the ideal

$$(x_1, \cdots, x_{n-1})$$

will of course never let us slash down to a point. What makes the first a "legitimate way" to slash down the localized affine space turns out to be the condition that

$$\mathfrak{m} = (x_1, \cdots, x_n)^2 \subset (x_1^2, x_2^2, \cdots, x_n^2).$$

Definition 19.1.5. If R is a local noetherian ring, then an **ideal of definition** is an ideal $I \subset R$ such that $\sqrt{I} = \mathfrak{m}$.

Here's another fact from commutative algebra that lets us think about ideals of definitions.

Lemma 19.1.6. *Let* R *be a noetherian local ring and* I *an ideal of* R*. Then* I *is an ideal of definition if and only if* R/I *is an Artin local ring.*

Proof. Note that R/I is easily seen to be a noetherian local ring whose maximal idea is $\mathfrak{m}/I$; since $\mathfrak{m}^r \subset I$ for some $r \gg 0$, we get that $\mathfrak{m}/I$ is a nilpotent ideal in R/I. □

In particular, this does mean that $\mathfrak{m}$ is an ideal of definition, but there could be others; Lemma 19.1.6 says that an ideal of definition is exactly those such that R/I is dimension zero. Since R is noetherian this means that there exists an r such that $\mathfrak{m}^r \subset I$. Ideal of definitions compute dimension via the following algebraic lemma

Lemma 19.1.7. *Let* R *be a local noetherian ring, the following two numbers coincide:*

(1) $\dim(R)$
(2) $\min\{$*number of generators of* I *for any ideal of definition I*$\}$.

From (2), we note that that $\mathfrak{m}$ cannot be generated by less than dim(R)-variables. On the other hand, the minimal number of generators of $\mathfrak{m}$ is exactly equal to the dimension of the vector space $\dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2$. Hence we record the following inequalities

Lemma 19.1.8. *Let* R *be a local noetherian ring, then:*

(1) $\dim(R) \leqslant \dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2$ *and*
(2) if $x \in \mathfrak{m}$ *then*

$$\dim(R) \leqslant \dim(R/xR) + 1;$$

with equality holding if x is not contained in any minimal prime (e.g. x is a nonzero divisor).

Let us summarize the above discussion:

(1) Let X be a scheme. We said that dimension should be an intrinsically local notion so if $x : \operatorname{Spec} K \to X$ is a point, then this corresponds to a prime ideal $\mathfrak{p} \subset A$ where $\operatorname{Spec} A \subset X$ is an open affine.

(2) We said that the dimension is an intrinsically local notion so we look at
$$\operatorname{Spec} A_{\mathfrak{p}}.$$
We said that dimension works best when everything in sight is noetherian so let us assume further that A is noetherian; so we are looking at a noetherian local ring. In this situation we know exactly what $A_{\mathfrak{p}}$ is and, in fact, we know how to "compute" this number in terms of generators of ideals of definitions. We now ask how do we interpret this number.

(3) We know that the number $\dim A_{\mathfrak{p}}$ is an "invariant" or "coordinate independent" way of thinking about the number of parameters needed to slash A down to an Artin local ring. Thinking of $x : \operatorname{Spec} \kappa(\mathfrak{p}) \hookrightarrow \operatorname{Spec} A \to X$, we interpret $\dim A_{\mathfrak{p}}$ as the number of partially-defined functions (defined only on Spec A) on X required to slash down X into the point $\operatorname{Spec} \kappa(\mathfrak{p})$. Hence we can set:
$$\operatorname{codim}(X, x) := \dim A_{\mathfrak{p}};$$
the **local codimension of** X **at** x. This is all we need to know to understand Hartog's lemma for now.

(4) Now how do we measure the dimension of X itself? Let us for a moment assume that X is an integral scheme. Going along with the codimension reasoning, we want to set the dimension of X to be exactly the codimension of the smallest possible subscheme of X, i.e., it should be the codimension of a closed point. This is where we make use that X is furthermore quasicompact so, by the lemma from the end of last class, X has a closed point. So we say that: if X is a noetherian, irreducible scheme then
$$\dim(X) = \sup_{x \in X \text{closed}} \operatorname{codim}(X, x);$$
where x is a closed point of X.

Remark 19.1.9. Beware that there exists an affine surface, or an affine scheme whose ring is dimension two, such that it has closed points of dimensions both one and two. This means that we must really take the supremum as above. But we will see that this is not a problem when we are considering schemes over fields.

19.2. **Proof of Hartog's lemma.**

Theorem 19.2.1 (Algebraic Hartog's Lemma)**.** *Let* A *be a noetherian domain which is also normal. Then*
$$A = \cap_{\mathfrak{p}, \operatorname{cod}_{\mathfrak{p}} A = 1} A_{\mathfrak{p}},$$
within K.

Proof. This proof is taken from Vakil's notes and I find the exposition really beautiful; it illustrates a standard method of argument in algebraic geometry. We have that
$$A \subset \cap_{\mathfrak{p}, \operatorname{cod}_{\mathfrak{p}} A = 1} A_{\mathfrak{p}}.$$
Suppose that that there exists $f \in \operatorname{Frac}(A)$ which is in all of $A_{\mathfrak{p}}$ but not in A; this means that we have a rational function which is regular on all the $\operatorname{Spec} A_{\mathfrak{p}}$ but not on A. Let us consider the closed subscheme
$$\operatorname{Bad}_f \hookrightarrow \operatorname{Spec} A,$$
the closed subscheme of Spec A defined by the ideal
$$I_f := \{r \in A : rf \in A\}.$$
We note that f is regular if and only if $1 \in I_{(f)}$ if and only if $\operatorname{Bad}_f = \emptyset$, hence we should think of this as the locus in Spec A where f fails to be regular. Hence, since f is assumed not be in A we have that I_f is not all of A.

In this case, we can choose a minimal prime ideal containing $I_{(f)}$ and localize A around this prime ideal. I claim that, in doing so, we may assume that we are in the following situation:

(1) A is a noetherian local domain, which is also normal with maximal ideal $\mathfrak{m}$, the dimension of the local ring $A_{\mathfrak{m}}$ is > 1 and $\mathfrak{m}$ is the only prime ideal containing $I_{(f)}$.

Indeed, we first notice that $I_{(f)}$ is stable under localization: for any prime ideal $\mathfrak{p}$, we have that $(I_{(f)})_{\mathfrak{p}}$ is the set of elements $x \in A_{\mathfrak{p}}$ such that $xf \in A_{\mathfrak{p}}$. But now, if $\mathfrak{p}$ is a prime ideal of codimension 1, we have that $1 \in (I_{(f)})_{\mathfrak{p}}$ so the minimal prime must be of codimension $\geqslant 2$. By the minimality assumption, it becomes the only prime ideal containing $I_{(f)}$.

□

20. Lecture 20: Proof of Hartog's finished; divisors

We were in the midst of proving the following result:

Theorem 20.0.1 (Algebraic Hartog's Lemma)**.** *Let* A *be a noetherian domain which is also normal. Then*

$$A = \cap_{\mathfrak{p}, \mathrm{cod}_{\mathfrak{p}} A = 1} A_{\mathfrak{p}},$$

within K.

We went about doing this by contradiction: we suppose that there exists an element $f \in \cap_{\mathfrak{p}, \mathrm{cod}_{\mathfrak{p}} A = 1} A_{\mathfrak{p}}$ which is not in A. We are able to reduce to the following set-up: A is a noetherian local ring which is normal by looking at the ideal I_f which obstructs f from being a regular function.

Proof, continued. Here's the main claim: there exists an element $z \in \mathrm{Frac}(A) \smallsetminus A$ but $z\mathfrak{m} \subset A$. Why is this useful? Well in such a situation we have two cases:

(1) if $z\mathfrak{m} \subset \mathfrak{m}$, we conclude that $\mathfrak{m}$ is a finitely generated A-module (using the noetherian hypothesis) with a faithful action of $A[z]$ (the ring generated in Frac(A) under A and z. But now since A is integrally closed, and $\mathfrak{m}$ is finitely generated, we get $A[z]$ is finitely generated and thus z is an integral element. This means that z must be in A since A was assumed to be integrally closed.

(2) Othewise, $z\mathfrak{m} \not\subset \mathfrak{m}$; but by locality this means that $z\mathfrak{m} = A$. From this I claim that $\dim(A) \leqslant 1$, contradicting the dimension of A. Indeed, since $z\mathfrak{m} = A$, we get that $\mathfrak{m} = A[\frac{1}{z}]$ so that, in particular, $\mathfrak{m}$ is a principal ideal. From this, we use the estimate:

$$\dim(A) \leqslant \dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2$$

to conclude that

$$\dim(A) \leqslant \dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2 \leqslant 1.$$

Let us now construct said element. We use that $\mathfrak{m}$ is the only prime ideal containing I. Recall that the radical $\sqrt{I}$ is computed as

$$\sqrt{I} = \cap_{I \subset \mathfrak{p}, \mathrm{prime}} \mathfrak{p}.$$

Therefore we must have that $\sqrt{I} = \mathfrak{m}$ since $\mathfrak{m}$ is the only prime ideal containing I. Since $\mathfrak{m}$ is finitely generated, this means that there exists $\infty > n \gg 0$ such that $\mathfrak{m}^n \subset I$; let n be the minimal such exponent so that $\mathfrak{m}^{n-1} \smallsetminus I$ is nonempty. Take an element $y \in \mathfrak{m}^{n-1} \smallsetminus I$ and let $z := yf$. Then we note that y is not in I so that $z = yf \notin A$ (which is what we want out of z). But then:

$$y\mathfrak{m} \subset \mathfrak{m}^n \subset I,$$

by the assumption on A; multiplying everything through by f we get

$$z\mathfrak{m} \subset If \subset A,$$

as desired.

□

Here's the global Hartog's theorem the main point is that the ring of global sections of a normal scheme is still normal:

Corollary 20.0.2. *Let* X *be a noetherian normal scheme and let* $\mathrm{U} \subset \mathrm{X}$ *be an open subscheme with complement* z *such that for any for any* $x : \operatorname{Spec} k \to \mathrm{Z}$, *we have that* $\operatorname{codim}(\mathrm{X}, x) \geqslant 2$. *Then, the restriction map*

$$\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \to \Gamma(\mathrm{U}, \mathcal{O}_{\mathrm{U}})$$

is an isomorphism.

20.1. **The Wild World of Divisors.** Now that we have Hartog's theorem, we can now make strides into some serious geometric territories. The theory of divisors was motivated by a very old problem in complex analysis. We explain its roots:

Remark 20.1.1. Going all the way back to Weierstrass, let us recall that we have the following result from complex analysis:

Theorem 20.1.2 (Weierstrass)**.** *Let* $\Omega \subset \mathbf{C}$ *be a complex domain and* $\mathrm{M} \subset \Omega$ *a discrete set, then there exists a holomorphic function* f *on all of* Ω *with* M *contained in the zero's of* f.

There is also a version, due to Mittag-Leffler, which presents a subset of a domain as a zero's and poles of holomorphic functions: basically we partition $\mathrm{M} = \mathrm{M}_{\text{poles}} \cup \mathrm{M}_{\text{zeros}}$ and we want to find two holomorphic functions f and g with zero's and $\mathrm{M}_{\text{poles}}$ and $\mathrm{M}_{\text{zeros}}$ respectively and set $\frac{f}{g}$ as the solution.

How does one think about the results of Weierstrass and Mittag-Leffler? Well we do know that if we focus around the point $x \in \Omega$ then we can find a function with x as a zero or as a pole; so the problem is very much a local one. Here's a natural reformulation that was given by Pierre Cousin in 1895: suppose for a moment that X is a complex manifold and with a cover $\mathcal{U} = \{\mathrm{U}_\alpha\}$ and, on each U_α, a nonzero meromorphic function f_α (we are ultimately interested in the zero's and poles of this function) such that the ratios

$$\frac{f_\alpha}{f_\beta}$$

are holomorphic and nowhere vanishing. Then, does there exists a global meromorphic function on X such that

$$\frac{f}{f_\alpha}$$

is holomorphic and nonvanishing on U_α? As expected, this problem can be solved naturally using some (these days) elementary sheaf theory: we will se that the exact sequence

$$\Gamma(\mathrm{X}, \mathcal{M}^\times) \to \operatorname{Div}(\mathrm{X}) \to \operatorname{Pic}(\mathrm{X});$$

where the left-most group is the group of meromorphic functions on X and the right most group is the group of line bundles on X with $\otimes$ as the multiplication, up to isomorphism. The Cousin problem basically asks: given an element in $\operatorname{Div}(\mathrm{X})$, when is it obtained from an element of $\Gamma(\mathrm{X}, \mathcal{M}^\times)$. Of course if there is no interesting line bundles on X, then we can solve the Cousin problem!

By the way, if you have seen algebraic topology, here's Cartan's solution to the Cousin problem:

Theorem 20.1.3 (Cartan)**.** *The Cousin problem is affirmative whenever the group* $\mathrm{H}^2(\mathrm{X}; \mathbf{Z}) = 0$. *In particular if* X *is Stein, the Cousin problem is equivalent to vanishing of this cohomology group.*

This result requires a translation between line bundles on X and the second cohomology group. This is surprising: the Cousin problem is of a holomorphic nature, while the solution is topological. It is a great motivation for you to study cohomology if you have not done so already!

Following the discussion above, let me elucidate the format of this theory: for a reasonable scheme X we will be able to assign to it the group $\operatorname{Div}(\mathrm{X})$ of divisors on X; there are a couple of perspectives on what this group is: one due to Weil and the other due to Cartier. The one that is due to Weil places the empahsis on the 'physical" vanishing loci of local meromorphic functions, while the one due to Cartier places emphasis on the functions. If X is integral, we can speak of a meromorphic function on X as an element $f \in k(X)$, its generic point. We will be able to assign

$$\mathrm{div}(f) \in \mathrm{Div}(X),$$

which is, roughly speaking, its zero's and pole's appropriately signed. We will take the quotient of Div(X) by mermorphic functions and obtain this group Pic(X) which we will prove to be isomorphic to line bundles. The group Pic(X) is an invariant the scheme X; among other things it lets us tell the difference between schemes. For example

$$\mathrm{Pic}(\mathbf{P}^2_k) = \mathbf{Z},$$

while

$$\mathrm{Pic}(\mathbf{P}^1 \times_k \mathbf{P}^1) = \mathbf{Z} \oplus \mathbf{Z}.$$

20.2. **Weil and Cartier divisors.** Let's get right into it. First, there are at least two reasonable notions of divisors. First we discuss Weil's approach. Let X be a noetherian scheme and $Z \subset X$ be an integral closed subscheme of X; then it has a generic point

$$\mathrm{Spec}\, k(Z) \to X.$$

It has an associated local ring which we denote by $\mathcal{O}_{X,Z}$ and, as discussed in the previous lecture,

$$\mathrm{codim}_Z(X) = \dim \mathcal{O}_{X,Z}.$$

As is standard we denote

Definition 20.2.1. Let X be a noetherian scheme; write:

(1) $X^{(k)}$ to be the set of integral closed subschemes of X of codimension k;
(2) $Z^k(X)$ to be the free abelian group on $X^{(k)}$. That is, they are finite linear combinations

$$\sum_Z n_Z[Z],$$

where $n_Z \in \mathbf{Z}$; we write $Z^k_+(X) \subset Z^k(X)$ to be the submonoid of those elemets where $n_Z \geqslant 0$.
(3) If $k = 1$, an element of $Z^1(X)$ is called a **Weil divisor**; an integral closed subscheme of X of codimension 1 is called a **prime Weil divisor** and an element of $Z^1_+(X)$ is called an **effective Weil divisor**.

Let us contrast this with Cartier divisors; we are willing to be loosen hypotheses for defining Cartier divisors.

Definition 20.2.2 (Cartier)**.** Let X be a scheme. An **effective Cartier divisor** is a closed subscheme $D \subset X$ whose sheaf of ideals $\mathcal{I}_D$ is a line bundle.

Example 20.2.3. A **locally principally closed subscheme** of X is a closed subscheme Z such that its sheaf of ideals $\mathcal{I}$ is locally generated by a single element. In other words for an open cover $\mathcal{U} = \{U_i \hookrightarrow X\}$ of X, we have that $\mathcal{I}_{U_i} = (f)$ where $f \in \Gamma(U_i)$. We note that every effective Cartier divisor is a locally principally closed subscheme but not vice versa: there is no guarantee of the local-freeness condition. In fact:

Lemma 20.2.4. *Let* X *be a scheme and let* $D \hookrightarrow X$ *be a closed subscheme. Then*

(1) the subscheme D *is an effective Cartier divisor;*
(2) for each $x : \mathrm{Spec}\, k \to D$*, there exists an affine open* $\mathrm{Spec}\, A \hookrightarrow X$ *such that* x *factors through it and* $\mathrm{Spec}\, A \times_X D \cong \mathrm{Spec}\, A/f$ *where* f is a nonzero divisor.

Proof. Suppose that D is an effective Cartier divisor; let $\mathcal{I}$ be the corresponding ideal sheaf so that locally an affine (Spec A) we have an exact sequence of A-modules

$$0 \to I \to A \to A/I \to 0;$$

By assumption the ideal corresponding to it is a line bundle, the above exact sequence is thus isomorphic to

$$0 \to \mathrm{A} \xrightarrow{f} \mathrm{A} \to \mathrm{A}/f \to 0,$$

in other words f, the A-generator of I, must be injective.

Now suppose that that second condition is satisfied. Then $\mathcal{I}$ must indeed be a line bundle as local freeness can be checked affine-locallly. □

We will soon compare the two notions, but before we can do that we need to give more examples of Weil divisors.

21. Lecture 20: Divisors II

We have started the theory of divisors:

- a **Weil divisor** is a formal, finite, sum
$$\sum_{\mathrm{Z}\in\mathrm{X}^{(1)}} n_{\mathrm{Z}}[\mathrm{Z}],$$
where each $\mathrm{Z} \in \mathrm{X}^{(1)}$, an integral closed subscheme of codimension 1; this is most reasonable whenever X is a noetherian scheme so that we have the theory of codimension we introduced at our disposal.
- On the other hand an **effective Cartier divisor** is a closed subscheme $\mathrm{Z} \hookrightarrow \mathrm{X}$ whose ideal sheaf $\mathcal{I}_{\mathrm{Z}}$ is invertible.

Before we go further, we construct some Weil divisors of interest.

21.1. **Examples of Weil divisors: principal divisors.** At this point, many textbooks will give further restrictions on X like X being normal. We will not — when the time comes it will be more obvious what the role of Hartog's is — we now only assume that X is *integral* in order for us to speak of $f \in \mathrm{K}(\mathrm{X})$. First we need to define the order of vanishing of an element. Recall that if R is a ring, then

$$\mathrm{length}_{\mathrm{R}}(\mathrm{M}) = \sup\{n : 0 = \mathrm{M}_0 \subset \cdots \mathrm{M}_j \subset \mathrm{M}_{j+1} \cdots \subset \mathrm{M}_n = \mathrm{M}\}.$$

where $\mathrm{M}_j \neq \mathrm{M}_{j+1}$. We note that length is a straightforward generalization of the notion of dimension of a vector space.

Definition 21.1.1. Let K be a field and $\mathrm{R} \subset \mathrm{K}$ is a local, noetherian ring of dimension 1 with fraction field K. Then the **order of vanishing along** R is a map

$$\mathrm{ord}_{\mathrm{R}} : \mathrm{K}^{\times} \to \mathbf{Z}$$

given by

$$f \mapsto \mathrm{length}_{\mathrm{R}}(\mathrm{R}/f) \qquad \text{if } f \in \mathrm{R}$$

and

$$\tfrac{f}{g} \mapsto \mathrm{length}_{\mathrm{R}}(\mathrm{R}/f) - \mathrm{length}_{\mathrm{R}}(\mathrm{R}/g) \qquad \text{if } f, g \in \mathrm{R}.$$

Remark 21.1.2. Suppose that f is invertible in R. Then $\mathrm{R}/f = \mathrm{R}/f[1/f] = 0$ and thus has length 0. This corresponds to the idea that if f is invertible and therefore *has no zero* on Spec R, then its order of vanishing should be zero.

Remark 21.1.3. First a notation: if R is a ring we write $\mathrm{R}^{\times}$ the abelian group of multiplicatively invertible elements in R. Recall that a valuation ring V is a local domain such that for any $x \in \mathrm{Frac}(\mathrm{V}) = \mathrm{K}$ either x in V or x^{-1} in V. If we set $\mathrm{K}^{\times}/\mathrm{V}^{\times} =: \Gamma$ then it is naturally a totally ordered abelian group (written additively): we say that $\gamma \geqslant \gamma'$ if and only if $\gamma - \gamma'$ is in the image of $\mathrm{V} \smallsetminus \{0\}$. We then have a map

$$\nu : \mathrm{K}^{\times} \to \Gamma,$$

which is the (additive) **valuation**; restricted to

$$V \smallsetminus \{0\} \to \Gamma_{\geqslant 0},$$

this map is a nonarchimedean valuation: 1) $\nu(a) = 0$ if and only if $a \in V^{\times}$, 2) $\nu(ab) = \nu(a)+\nu(b)$ and 3) $\nu(a+b) \geqslant \min(\nu(a), \nu(b))$. You are probably most familiar with discrete valuations rings: these are exactly those whose $\Gamma = \mathbf{Z}$. Prominent examples are

$$k[[t]] \smallsetminus 0 \to \mathbf{Z} \qquad t^r \mapsto r.$$

and

$$\mathbf{Z}_p \smallsetminus 0 \to \mathbf{Z} \qquad \nu(p^r a'/b') = r; \text{ where } a', b' \text{ are coprime to } p.$$

We note that in the event that V is discrete valuation ring (which necessarily means it is dimension one) and K is its fraction field, the above order of vanishing coincides with the valuation. This is one reason why textbooks require "regularity in codimension one" which is true if and only if every codimension one point is a discrete valuation ring (at least in the noetherian situation) to set up order of vanishing. We see that this is not really necessary. We will postpone to the exercises the relationship between normality and being regular in codimension one.

Example 21.1.4. Let us consider a rational function $\frac{f}{g}$ on $\mathbf{A}^1_k$. This is just an expression where f and g are polynomials in one variable. For example take

$$\tfrac{x^5}{x-1}.$$

The generic point of $\mathbf{A}^1_k$ was seen to be $k(x)$ and a dimension 1 local noetherian ring of $k(x)$ are just local rings of closed points of $\mathbf{A}^1_k$. Consider the local rings

$$\mathcal{O}_{\mathbf{A}^1,0} = k[x]_{(x)} \qquad \mathcal{O}_{\mathbf{A}^1,1} = k[x]_{(x-1)}$$

which are both noetherian rings of dimension 1 living in $k(t)$. Then we see that

$$\mathrm{ord}_{\mathcal{O}_{\mathbf{A}^1,0}}(\tfrac{x^5}{x-1}) = \mathrm{length}_{\mathcal{O}_{\mathbf{A}^1,0}} k[x]_{(x)}/(\tfrac{x^5}{x-1}) = \mathrm{length}_{\mathcal{O}_{\mathbf{A}^1,0}} k[x]_{(x)}/(x^5) = 5;$$

while

$$\mathrm{ord}_{\mathcal{O}_{\mathbf{A}^1,1}}(\tfrac{x^5}{x-1}) = -1.$$

Construction 21.1.5. Let X be a noetherian, integral scheme and $f \in K(X)^{\times}$. The **Weil divisor assoicated to** f is given by

$$\mathrm{div}(f) = \Sigma_{Z \in X^{(1)}} \mathrm{ord}_{K(Z)}(f)[Z].$$

We record the next, which follows from definitions:

Lemma 21.1.6. *Let* X *be a noetherian, integral scheme and* $f, g \in K(X)^{\times}$ *and let* $Z \hookrightarrow X$ *be a prime divisor. Then*

$$\mathrm{ord}_Z(fg) = \mathrm{ord}_Z(f) + \mathrm{ord}_Z(g).$$

Definition 21.1.7. Let X be a noetherian, integral scheme and $f, g \in K(X)^{\times}$ and let $Z \hookrightarrow X$ be a prime divisor. We say that

(1) f **has a pole** along Z if $-\mathrm{ord}_Z(f) > 0$;
(2) f **has a zero** along Z if $\mathrm{ord}_Z(f) > 0$.

Remark 21.1.8. The interpretation of $f \in K(X)^{\times}$ as having a zero along Z is best done when Z corresponds to a discrete valuation ring. Indeed, we have that $\mathrm{ord}_Z(f) \geqslant 0$ if and only if $\mathcal{O}_{X,Z}$ in this situation.

Lemma 21.1.9. *The sum defining* $\mathrm{div}(f)$ *is finite.*

Proof. We note that *a priori*, f is an element of $K(X)^\times$. Since the generic point lives in every open affine subscheme of X, we may assume that $f \in A^\times$ (an invertible element) where $\operatorname{Spec} A \hookrightarrow X$ is an open affine of X. Therefore, the prime divisors that appear in the definition $\operatorname{ord}(f)$ from Construction 21.1.5 must not intersection Spec A non-trivially. But now $X \setminus U$ is also a noetherian scheme and, in particular, quasicompact. Therefore there can only be finitely many irreducible components of $X \setminus U$. Now, we conclude by noting that the only way $\operatorname{ord}_Z(f)$ can be nonzero is if $Z \hookrightarrow X \setminus U$.

□

Wit this we have the map

$$\operatorname{div} : K(X)^\times \to Z^1(X).$$

Definition 21.1.10. Let X be a noetherian integral scheme. Then, the **Weil divisor class group** or the **Chow group of codimension** 1 **cycles** is the cokernel of map div, i.e., it fits into a short exact sequence

$$K(X)^\times \xrightarrow{\operatorname{div}} Z^1(X) \to CH^1(X) \to 0.$$

21.2. **Some computations of** $CH^1(X)$**.** We have that $CH^1(X)$ is defined via the exact sequence

Example 21.2.1 (Vanishing CH^1)**.** Let A be a ring and consider $\operatorname{Spec} A = X$. Assume that A is a unique factorization domain such as:

- principal ideal domains such as $\mathbf{Z}$;
- polynomial rings over a unique factorization domain;
- formal power series over a field;
- a result of Auslander and Buschbaum states that a regular local ring (one such that $\dim(R) = \dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$) is a unique factorization domain.

Then every codimension 1 integral subscheme of Spec A is principal and thus $CH^1(X) = 0$. In particular, $CH^1(\mathbf{A}^n_k) = 0$. In fact:

Lemma 21.2.2. *Let* A *be a noetherian domain. Then* A *is a UFD if and only if* $X = \operatorname{Spec} A$ *is normal and* $CH^1(\operatorname{Spec} A) = 0$.

Proof sketch. The interesting part is: if $CH^1(\operatorname{Spec} A) = 0$ and A is a normal then A is a UFD. To say that the Chow group vanishes is to say that for any prime divisor Y we can write $Y = \operatorname{div}(f)$ where $f \in \operatorname{Frac}(A)$. Since X is affine Y corresponds to a prime ideal $\mathfrak{p}$ and it will suffice to show that f generates $\mathfrak{p}$. I will say why f is in A, which already requires normality.

To see this, the above expression says that $\operatorname{ord}_Y(f) = 1$ indicating that it is in the maximal idea of $A_\mathfrak{p}$ and for any other prime divisor Y, we have that $\operatorname{ord}_Y(f) = 0$ indicating that it is among the units of $A_\mathfrak{q}$ where $\mathfrak{q}$ corresponds to the prime ideal of A. This means that $f \in A_\mathfrak{p}$ for all prime ideals of height 1

□

To state the next result, we note the easiest possible functoriality of CH^1: if $j : U \hookrightarrow X$ is an open immersionm then we have a pullback map

$$j^* : CH^1(X) \to CH^1(U).$$

Indeed, prime divisors are seen easily to restrict to prime divisors on U (a prime divisor $Z \hookrightarrow X$ has a generic point $\operatorname{Spec} K(Z)$ and either factors through U or it does not) and we see that we can regard $f \in K(X)^\times$ as an element of $K(U)^\times$ as well. Therefore, the map $Z^1(X) \to Z^1(U)$ factors through the relation given by div and descends to the map j^* above.

Lemma 21.2.3 (Excision)**.** *Let* X *be a noetherian scheme and let* $Z \hookrightarrow X$ *be a closed subscheme which does not contain any irreducible component of* X *(so if* X *is integral, then this is equivalent* *to asking that* $X \smallsetminus Z$ *is nonempty). Let* $Z_1, \cdots, Z_r$ *be the irreducible components of* X *which are of codimension* 1*. Then we have an exact sequence*

$$\bigoplus_{1\leqslant i\leqslant r} \mathbf{Z}[Z_i] \to \mathrm{CH}^1(X) \to \mathrm{CH}^1(U) \to 0.$$

Proof. It suffices to prove the result when Z being an integral closed subscheme; we note that although $\mathbf{Z}[Z_i] \hookrightarrow Z^1(X)$ is an injection, we make no assertion about the exactness on the left of the above sequence. Now:

(1) the kernel of the map $\mathrm{CH}^1(X) \to \mathrm{CH}^1(U)$ are exactly those divisors whose support is contained in Z, therefore the kernel is the subgroup of $\mathrm{CH}^1(X)$ generated by Z.
(2) I claim that the map $\mathrm{CH}^1(X) \to \mathrm{CH}^1(U)$ is surjective. Indeed, if $W \hookrightarrow U$ is a prime divisor in U, then taking its closure we obtain a prime divisor in X since its local ring will still be of dimension 1. Therefore we have the desired surjection.

□

Proposition 21.2.4. *There is a canonical degree map*

$$\mathrm{CH}^1(\mathbf{P}^n_k) \xrightarrow{\deg} \mathbf{Z}$$

which is an isomorphism.

Proof. Let $H \subset \mathbf{P}^n_k$ be a hyperplane, a closed subscheme isomorphic to $\mathbf{P}^{n-1}_k$ whose complement is $\mathbf{A}^n$, one of the affine charts. Then we apply the above exact sequence from Lemma 21.2.3 to get

$$\mathbf{Z}[H] \to \mathrm{CH}^1(\mathbf{P}^n_k) \to 0;$$

whence $\mathbf{Z}$ surjects onto $\mathrm{CH}^1(\mathbf{P}^n_k)$. It suffices to prove injectivity which lets us define the degree map as well. Indeed, from the homework in this week every prime divisor of $\mathbf{P}^n$ corresponds to a hypersurface of degree d in $\mathbf{P}^n$ which we have seen to be describable in terms of either (1) global sections of $\mathcal{O}(d)$ or (2) a homogeneous polynomial of degree d in $n+1$-variables. The degree map does

$$\Sigma n_i D_i \mapsto \Sigma n_i \deg(D_i).$$

Now we note that any rational function on $\mathbf{P}^n$ is a quotient of homogeneous polynomials of the same degree and thus is sent to zero under the degree map and thus have $\mathrm{CH}^1(\mathbf{P}^n_k) \xrightarrow{\deg} \mathbf{Z}$. The composite $\mathbf{Z}[H] \to \mathrm{CH}^1(\mathbf{P}^n_k) \to \mathbf{Z}$ is evidently the identity and thus have the desired injection. □

21.3. **Exercises.**

Exercise 21.3.1. *Prove that any prime divisor in* $\mathbf{P}^n_k$ *corresponds to a homogeneous equation in* $n+1$*-variables. In particular such a prime divisor corresponds to a hypersurface cut out by this polynomial equation.*

Exercise 21.3.2. *Let* $A = k[x,y,z]/(xy - z^2)$*. Prove that* $\mathrm{CH}^1(X) \cong \mathbf{Z}/2$ *using excision.*

Exercise 21.3.3. *Let* $Z \hookrightarrow \mathbf{P}^n_k$ *be the closed cut out by a hypersurface of degree d. Compute* $\mathrm{CH}^1(Z) \cong \mathbf{Z}/d\mathbf{Z}$ *again using excision.*

22. Lecture 21: Weil versus Cartier divisors

For starters, let X be an integral, noetherian scheme. Last time, we wanted to give some examples of CH^1. We asserted that the pullback map $\mathrm{CH}^1(X) \to \mathrm{CH}^1(X \times \mathbf{A}^1)$ is an isomorphism in nice cases. First we note that this pullback is indeed well-defined: suppose that $Z \hookrightarrow X$ is a prime divisor then $Z \times \mathbf{A}^1 \hookrightarrow X \times \mathbf{A}^1$ is codimension one: if $\mathcal{O}_{X,Z}$ is the local ring corresponding to Z and $\mathfrak{m}$ is its maximal ideal, then the local ring of $Z \times \mathbf{A}^1$ is given by $(\mathcal{O}_{X,Z}[t])_{\mathfrak{m}\mathcal{O}_{X,Z}[t]}$ which is again dimension one. Furthermore, if $f \in K(X)^\times$, then we can regard f as an element of $K(X)(t)$. Therefore, we have an induced pullback map

$$\mathrm{CH}^1(X) \to \mathrm{CH}^1(X \times \mathbf{A}^1).$$

Remark 22.0.1. In fact CH^1 is functorial for *flat morphisms*, the details of which we will discuss next semester.

Theorem 22.0.2 (Homotopy invariance)**.** *Let* X *be an noetherian, integral, normal scheme, then the pullback map* $\mathrm{CH}^1(\mathrm{X}) \to \mathrm{CH}^1(\mathrm{X} \times \mathbf{A}^1)$ *is an isomorphism.*

Example 22.0.3. Let $\mathcal{E}$ be a vector bundle on a scheme X, then we have the projection map $p : \mathbf{V}(\mathcal{E}) \to \mathrm{X}$. One can show that the induced map $p^* : \mathrm{CH}^1(\mathrm{X}) \to \mathrm{CH}^1(\mathbf{V}(\mathcal{E}))$ is an isomorphism via homotopy invariance and excision.

Having this, we can combine with the excision sequence to get a computation of $\mathrm{CH}^1(\mathbf{P}^1_k \times_k \mathbf{P}^1_k)$.

Example 22.0.4. We claim that we have the exact sequence:

$$0 \to \mathbf{Z}[\mathbf{P}^1_k] \to \mathrm{CH}^1(\mathbf{P}^1 \times \mathbf{P}^1) \to \mathrm{CH}^1(\mathbf{A}^1 \times \mathbf{P}^1) \to 0.$$

We need to prove that the map $\mathbf{Z}[\mathbf{P}^1_k] \to \mathrm{CH}^1(\mathbf{P}^1 \times \mathbf{P}^1)$ is injective. But this map coincides with the pullback map

$$\mathrm{CH}^1(\mathbf{P}^1) \to \mathrm{CH}^1(\mathbf{P}^1 \times \mathbf{P}^1).$$

Further composing with the restriction to $\mathrm{CH}^1(\mathbf{P}^1 \times \mathbf{A}^1)$, we see that this agrees with the pullback map $\mathrm{CH}^1(\mathbf{P}^1) \to \mathrm{CH}^1(\mathbf{P}^1 \times \mathbf{A}^1)$ which we know is an isomorphism. Therefore this map is injective.

Now using homotopy invariance the other way, we see that the last term in the exact sequence $\mathrm{CH}^1(\mathbf{A}^1 \times \mathbf{P}^1)$ is isomorphic to CH^1 of the other $\mathbf{P}^1$. Using this isomorphism, we can split the exact sequence via the pullback map and conclude that the above exact sequence splits and thus:

$$\mathrm{CH}^1(\mathbf{P}^1 \times \mathbf{P}^1) \simeq \mathbf{Z} \oplus \mathbf{Z}.$$

22.1. **Comparison and homotopy invariance.** The homotopy invariance statement is quite powerful, especially when combined with excision. I want to motivate the comparison result between Cartier and Weil divisors using the injectivity part which I will now start discussing. We want to prove that $\pi^* : \mathrm{CH}^1(\mathrm{X}) \to \mathrm{CH}^1(\mathrm{X} \times \mathbf{A}^1)$ is injective. It would be nice if we have a pullback map $z^* : \mathrm{CH}^1(\mathrm{X} \times \mathbf{A}^1) \to \mathrm{CH}^1(\mathrm{X})$ which is pullback along the zero section $z : \mathrm{X} \to \mathrm{X} \times \mathbf{A}^1$. However, it not at all clear that this map is defined: indeed if $\mathrm{Z} \subset \mathrm{X} \times \mathbf{A}^1$ is a prime divisor such that $\mathrm{Z} \to \mathrm{X} \times \mathbf{A}^1 \to \mathrm{X}$ is dominant onto X, then this putative pullback map will not send Z to a Weil divisor. This points at one of the difficulties of Weil divisor theory — its functoriality is not immediate. What saves us eventually is the fact that we *move* Z into a cycle which admits a pullback to X after quotienting out by div. This explicitly says that

$$\mathrm{Z} = \mathrm{D} + \mathrm{div}(f),$$

where $z^{-1}(\mathrm{D})$ is a Cartier divisor on X. Since we will use it again and again, we adopt the following terminology: if D, D' are Weil divisors then we say that D is **linearly equivalent** to D' if they define the same class in $\mathrm{CH}^1(\mathrm{X})$. Equivalently, we can write

$$\mathrm{D} = \mathrm{D}' + \mathrm{div}(f).$$

Hence, being linearly equivalent to zero means I can express D as the zero's and poles of a rational function on X.

First, let us construct a comparison map between Cartier and Weil divisors. We recall the following lemma from two classes back:

Lemma 22.1.1. *Let* X *be a scheme and let* $\mathrm{D} \hookrightarrow \mathrm{X}$ *be a closed subscheme. Then*

(1) the subscheme D *is an effective Cartier divisor;*
(2) for each $x : \mathrm{Spec}\, k \to \mathrm{D}$*, there exists an affine open* $\mathrm{Spec}\,\mathrm{A} \hookrightarrow \mathrm{X}$ *such that* x *factors through it and* $\mathrm{Spec}\,\mathrm{A} \times_{\mathrm{X}} \mathrm{D} \cong \mathrm{Spec}\,\mathrm{A}/f$ *where* f is a nonzero divisor.

While we have also made sense of effective Cartier divisors, we need also to define their "negatives" in line with the the theory of Weil divisors. To do this, we not that $\mathrm{Div}_+(Z)$ is a monoid whose addition is defined as follows: if $D, D' \hookrightarrow X$ are effective Cartier divisors then their ideal sheaves $\mathcal{I}_D, \mathcal{I}_{D'}$ are line bundles. Therefore we can tensor them and get $\mathcal{I}_D \otimes \mathcal{I}_{D'}$ which is again a line bundle and hence defines a new Cartier divisor which we denote by $D + D'$. The neutral element of this monoid is the Cartier divisor is formally given by the the empty subscheme (by usual convention this is a subscheme of any codimension) and so the ideal sheaf is $\mathcal{O}_X$. We can formally add inverses or, more formally, take its **group completion** and obtain the abelian group $\mathrm{Div}(X)$ which has the expected universal property: if A is an abelian group and $\mathrm{Div}_+(X) \to A$ is a morphism of monoids, then it factors through an abelian group map $\mathrm{Div}(X) \to A$.

Construction 22.1.2. We now construct

$$\mathrm{cyc} : \mathrm{Div}_+(X) \to Z^1(X).$$

This construction is called the **Weil divisor associated to an effective Cartier divisor**. Indeed, let $D \in \mathrm{Div}_+(X)$ be an effective Weil divisor. Then the we send it to the Weil divisor given by

$$\mathrm{cyc}(D) = \sum_{Z \in X^{(1)}} \nu_Z(D)[Z]$$

where:

Case 1 If the generic point $\eta_Z : \mathrm{Spec}\, K(Z) \to Z \to X$ factors through D then we set $\nu_Z(D) = 0$. In particular, we send the empty divisor/neutral element to zero.

Case 2 If the generic point does factor through D, then by the above there exists an affine open $\mathrm{Spec}\, A \subset X$ such that η_Z factors through it and $\mathrm{Spec}\, A \times_X D \cong \mathrm{Spec}\, A/f$. But now, via the map $\eta_Z^\sharp : A \to K(Z)$, we get an element $f \in K(Z)^\times$ and we can compute its $\mathrm{ord}_Z(f)$ which we set to be $\nu_Z(D)$. We then extend this map linearly $\mathrm{Div}_+(X) \to Z^1(X)$ which is a map of monoids. But since the target is a group it factors through the group completion of $\mathrm{Div}_+(X)$ and defines a map

$$\mathrm{Div}(X) \to Z^1(X).$$

Example 22.1.3. Let A be a domain. Then a nonzero divisor $f \in A$ defines a Cartier divisor by A/f. The map $\mathrm{cyc}([\mathrm{Spec}\, A/f])$ sends it to $\mathrm{div}(f)$. Locally, the map cyc is just really the map div.

As promised, I wanted to discuss the relationship between normality and regularity in codimension one.

Lemma 22.1.4. *Let* A *be a ring. The following are equivalent:*

(1) A *is a discrete valuation ring;*
(2) A *is a noetherian valuation ring;*
(3) A *is a regular local ring of dimension one;*
(4) A *is a noetherian local, normal domain of dimension one.*

In all these cases $\mathfrak{m}$*, the maximal ideal, is generated by a choice uniformizer* π *which determines a unique expression* $u\pi^n$*, where* $u \in A^\times$ *for any element in* A*. In particular,* $\mathrm{ord}_A(u\pi^n) = n$.

Proof. See [Stacks, Tag 00PD] for a proof. □

The point here is that we can interpret ord as a valuation and hence determine whether or not an element in the fraction field of A is, in fact, in A using ord.

Lemma 22.1.5. *Let* X *be a noetherian, integral scheme. The above construction is independent of choices and descends to a map*

$$\mathrm{cyc} : \mathrm{Div}(X) \to Z^1(X).$$

If X *is normal, then this map is always injective.*

Proof. The check that this construction is independent of choices will be left to the reader. Suppose that $\mathrm{cyc}(\mathrm{D}) = 0$ so that $\sum_{\mathrm{Z}\in \mathrm{X}^{(1)}} \nu_{\mathrm{Z}}(\mathrm{D})[\mathrm{Z}] = 0$ which means that $\nu_{\mathrm{Z}}(\mathrm{D}) = 0$ for all $\mathrm{Z} \in \mathrm{X}^{(1)}$. We claim that D is locally defined by an invertible element $f \in \mathrm{A}^{\times}$: if so then $\mathcal{I}_{\mathrm{D}} \cong \mathcal{O}_{\mathrm{X}}$ and hence we are looking at the neutral element. We note that this is a local claim so we might as well go local. In this case, we are in Example 22.1.3 and the fact that $\nu_{\mathrm{Z}}(\mathrm{D}) = 0$ means that $\mathrm{ord}(f) = 0$. Since X is normal, this means that f is in $\mathrm{A}_{\mathfrak{p}}$ for all where $\mathfrak{p}$ is a a height one prime. But by Hartog's theorem (using normality here!), f is in A. It is furthermore invertible since it has zero valuation. □

We can further ask when this map is an isomorphism.

Definition 22.1.6. We say that a ring R is **locally factorial** if every $\mathrm{R}_{\mathfrak{p}}$ is a unique factorization domain. We say that a scheme X is **locally factorial** if for any affine open $\operatorname{Spec} \mathrm{A} \hookrightarrow \mathrm{X}$, A is a locally factorial ring.

Remark 22.1.7. It is well known that unique factorization domains are normal domains. Hence we are imposing a stronger condition.

Remark 22.1.8. If R is a noetherian ring. Then, R is factorial if and only if every height one prime ideal of R is principal. Geometrically this means that a locally noetherian scheme X is principal if for every point $x : \operatorname{Spec} k \to \mathrm{R}$ and every closed integral subscheme of codimension one $\mathrm{Z} \hookrightarrow \mathrm{X}$, there exists an affine open $\operatorname{Spec} \mathrm{A} \subset \mathrm{X}$ containing x such that $\operatorname{Spec} \mathrm{A} \times_{\mathrm{X}} \mathrm{Z} \cong \operatorname{Spec} \mathrm{A}/f$ where f is a nonzero divisor of A.

Lemma 22.1.9. *Let* X *be a noetherian, integral scheme. Further assume that* X *is locally factorial, then the map*

$$\mathrm{cyc} : \mathrm{Div}(\mathrm{X}) \to \mathrm{Z}^1(\mathrm{X}),$$

is an isomorphism.

Proof. Since X is locally factorial, we see that every prime divisor is, in fact, an effective Cartier divisor. Since $\mathrm{Z}^1(\mathrm{X})$ is generated by prime divisors we are done. □

23. Divisors and line bundles

Let X be a noetherian integral scheme. We have constructed a map $\mathrm{Div}(\mathrm{X}) \to \mathrm{Z}^1(\mathrm{X})$. Such a map is injective whenever X is normal and it is an isomorphism whenever X is locally factorial. An important class of schemes for which all these conditions hold are **regular schemes**: they are noetherian schemes such that for each $x : \operatorname{Spec} k \to \mathrm{X}$ there exists an open affine $\operatorname{Spec} \mathrm{A} \hookrightarrow \mathrm{X}$ containing x such that the localizations $\mathrm{A}_{\mathfrak{p}}$ are regular local rings. This means that we have an equality

$$\dim(\mathrm{R}_{\mathfrak{p}}) = \dim_{\kappa(\mathfrak{p})} \mathfrak{m}/\mathfrak{m}^2.$$

Remark 23.0.1. The cusp and nodes — localized at the cusp and node respectively — are not regular local rings. In fact, noetherian regular local rings of dimension one must are the same as normal ones.

Next, I want to discuss the following reformulation of Cartier divisors.

Construction 23.0.2. Let $\mathrm{Z} \hookrightarrow \mathrm{X}$ be an effective Cartier divisor. Then, its ideal sheaf $\mathcal{I}_{\mathrm{Z}}$ is a line bundle on X. We set

$$\mathcal{O}_{\mathrm{S}}(\mathrm{D}) := \mathcal{I}_{\mathrm{Z}}^{\vee} = \mathcal{I}_{\mathrm{Z}}^{\otimes -1}.$$

It comes equipped with a canonical section $\mathrm{can} : \mathcal{O}_{\mathrm{X}} \to \mathcal{O}_{\mathrm{X}}(\mathrm{D})$ classifying the map $\mathcal{I}_{\mathrm{Z}} \to \mathcal{O}_{\mathrm{X}}$.

Remark 23.0.3. By definition, we have an exact sequence

$$0 \to \mathcal{O}_{\mathrm{S}}(-\mathrm{D}) \to \mathcal{O}_{\mathrm{X}} \to i_*\mathcal{O}_{\mathrm{D}} \to 0,$$

where $i : \mathrm{D} \hookrightarrow \mathrm{X}$ is the closed immersion. We can always associate this exact sequence to an effective Cartier divisor on X and this exact sequence will be very useful in studying cohomology in the next semester.

Definition 23.0.4. Let X be a scheme and $\mathcal{L}$ a line bundle on X then a section $s \in \Gamma(\mathrm{X}, \mathcal{L})$ is a **regular section** if for any affine open $\operatorname{Spec} \mathrm{A} \subset \mathrm{X}$, the map

$$\mathrm{A} \to \mathcal{L}|_{\mathrm{A}} \qquad f \mapsto sf$$

is injective.

Construction 23.0.5. a line bundle with a section $\mathrm{D} = (\mathcal{L}, s)$, we can convert this by Serre's theorem into the data of (U_i, f_i) where each f_i is a nonzero divisor on U_i. We can assign such data to a divisor in the following way: for each prime divisor $\mathrm{Z} \in \mathrm{X}^{(1)}$, we pick an i such that the generic point of Z is in U_i. With this, we can take $\operatorname{ord}_{\mathrm{Z}}(f_i) := \operatorname{ord}_{\mathrm{Z}}(\mathrm{D})$. We set

$$\operatorname{cyc}((\mathcal{L}) = \Sigma_{\mathrm{Z} \in \mathrm{X}^{(1)}} \operatorname{ord}_{\mathrm{Z}}(\mathrm{D})[\mathrm{Z}].$$

Remark 23.0.6. A global section s of $\mathcal{O}_{\mathrm{X}}$ is a regular section if and only if s is a nonzero divisor on every affine open.

Remark 23.0.7. The section in the above construction is regular.

The next lemma states that we can regard effective Cartier divisors are isomorphism classes of line bundles equipped with a regular section. By isomorphism we mean the following: $(\mathcal{L}, s) \to (\mathcal{L}', s')$ is an invertible morphism of quasicoherent sheaves $\mathcal{L} \to \mathcal{L}'$ which intertwines the sections s and s'.

Lemma 23.0.8. *Let* X *be a scheme, then there is a canonical bijection between:*

(1) *effective Cartier divisors on* X*;*
(2) *isomorphism classes of pairs* $(\mathcal{L}, s)$ *and* $s \in \Gamma(\mathrm{X}, \mathcal{L})$ *a regular section.*

Given by

$$\mathrm{Z} \hookrightarrow \mathrm{X} \mapsto (\mathcal{O}_{\mathrm{X}}(\mathrm{D}), \mathrm{can});$$

and

$$(\mathcal{L}, s) \mapsto \mathrm{V}(s).$$

Proof. Since we know that locally every effective Cartier divisor is of the form $\operatorname{Spec} \mathrm{A}/f$ where $f \in \mathrm{A}$ is a nonzero divisor, the result follows from a local check. □

Therefore, we have the forgetful map

$$\operatorname{Div}_+(\mathrm{X}) \to \operatorname{Pic}(\mathrm{X});$$

using again that this is a morphism of monoids and the right hand side is a group we get a map

$$\operatorname{Div}(\mathrm{X}) \to \operatorname{Pic}(\mathrm{X}).$$

Theorem 23.0.9. *Let* X *be an integral noetherian scheme which is locally factorial. Then there is a canonical isomorphism:*

$$\mathrm{CH}^1(\mathrm{X}) \xrightarrow{\cong} \operatorname{Pic}(\mathrm{X})$$

Proof. We already have an isomorphism $\mathrm{Z}^1(\mathrm{X}) \cong \operatorname{Div}(\mathrm{X})$ for these schemes. The result then follows from the following claim:

- two Weil divisors are linearly equivalent if and only if they define the same line bundles.

□

Corollary 23.0.10. *Given any morphism* $f : \mathrm{X} \to \mathrm{Y}$ *of integral noetherian, locally factorial schemes, we get a map* $f^* : \mathrm{CH}^1(\mathrm{X}) \to \mathrm{CH}^1(\mathrm{Y})$.

Let us prove homotopy invariance of CH^1.

Homotopy invariance of CH^1. Since X is an integral, noetherian, locally factorial scheme we note that $\mathrm{X} \times \mathbf{A}^1$ is too. The point here is that if R is a UFD, then so is R[t]. Now, by the previous corollary, we have a map $z^* : \mathrm{CH}^1(\mathrm{X} \times \mathbf{A}^1) \to \mathrm{CH}^1(\mathrm{X})$. This map is induced by the pullback along the zero section $z : \mathrm{X} \to \mathrm{X} \times \mathbf{A}^1$. Since $\pi \circ z = \mathrm{id}$ where $\pi : \mathrm{X} \times \mathbf{A}^1 \to \mathrm{X}$ is the projection, we get that $z^* \circ \pi^* = \mathrm{id}$ and hence π^* is injective. To prove surjectivity.

□

23.1. **Divisors versus line bundles.**

Lemma 23.1.1. *Let* X *be a noetherian integral scheme. If* X *is normal, then the above map descends to an injection:*

$$\mathrm{Pic}(\mathrm{X}) \to \mathrm{CH}^1(\mathrm{X}).$$

It is an isomorphism whenever the local rings of X *are all unique factorization domains.*

We now start the discussion on the relationship between Cartier divisors, Weil divisors and the Picard group of line bundles. Here's a construction that relates the two. We recall:

The section that comes with $\mathcal{O}_{\mathrm{X}}(\mathrm{D})$ has the following pleasant property.

24. Final project ideas

24.1. **27 lines on a cubic.**

One of the early achievements of algebraic geometry is the following result:

Theorem 24.1.1. *Let k be a field and $\mathrm{X} \subset \mathbf{P}^3_k$ be a smooth cubic surface. Then* X *contains exactly 27 different lines of $\mathbf{P}^3_k$.*

Give a full exposition of this proof.

24.2. **The projective space as a quotient.**

If X is a scheme and G is an algebraic group acting on X, we can construct a scheme-theoretic quotient X/G using **geometric invariant theory**. This gives yet another description of $\mathbf{P}^n$; give an exposition of the following result (without delving into all of geometric invariant theory):

Theorem 24.2.1. *There is a canonical isomorphism*

$$\mathbf{A}^n/\mathbf{G}_m \cong \mathbf{P}^{n-1}.$$

24.3. **The moduli of hypersurfaces.**

We have seen in class that we can $\mathrm{H}^0(\mathbf{P}^n; \mathcal{O}(d))$ classify degree d hypersurfaces in $\mathbf{P}^n$. Construct this formally as a scheme; its functor of points, which we denote by $\mathcal{H}_{d,n}$ is given as follows

$$\mathcal{H}_{d,n} : \mathrm{A} \mapsto \{\mathrm{X} \hookrightarrow \mathbf{P}^n_{\mathrm{A}} : \mathrm{X} \text{ is flat, finitely presented over } \mathrm{Spec}\,\mathrm{A} \text{ and fibers are hypersurfaces of degree } d\}.$$

You need to explain the meaning of flatness and why it is a key hypothesis here.

24.4. **The Grassmanian as a scheme.**

We are familiar with $\mathrm{Gr}(n, k)$, the Grassmanian classifying k-dimensional subspaces of n-dimensional vector spaces over a field. This is an extremely interesting algebro-geometric object. Construct the Grassmanian as a scheme, similar to what we did in projective space.

24.5. **Algebraic stacks and Grothendieck topology.**

In class we have introduced the notion of a Zariski stack. More conventionally a stack should have 1) descent with respect to the étale topology and 2) be a functor valued in groupoids. The task of this project is to make sense of this notion and prove the following result

Theorem 24.5.1. *The functor which sends a ring* A *to the groupoid of line bundles over* Spec A *is a stack.*

24.6. **Deformation theory: flat families.** As we have seen, there are many ways that schemes can deform along families. One of the key requirements for this idea to be useful is the idea of "flat families." Explain (no need to prove) the following result:

Lemma 24.6.1. *Let k be a field and B a reduced k-scheme. Let $\mathcal{X} \subset \mathbf{A}^n_{\mathrm{B}}$ be a closed subscheme and $b \in \mathrm{B}$ a closed point. Then the following are equivalent:*

(1) $\mathcal{X}$ is flat over B;
(2) for any nonsingular, one-dimensional k-scheme B' with a closed point $0 \in \mathrm{B}$ and a pointed morphism

$$(\mathrm{B}', 0) \to (\mathrm{B}, b),$$

the fiber X_b is the flat limit of $\mathrm{X}_{\varphi(b')}, b' \in \mathrm{B}$.

In other words, explain what this result means and use it to give many examples of flat limits.

24.7. **Twisted projective spaces.** Let k be a field. Then a **twisted projective space** or a **Severi-Brauer variety** is a smooth proper k-scheme X such that there exists a separable extension k'/k for which $\mathrm{X}_{k'} \cong \mathbf{P}^n_{k'}$. All such objects are constructed out of central simple algebras which are generalizations of Hamilton's quarternion. Give a construction of the Severi-Brauer varieties using the language of this class. This is one of the most fascinating links between number theory, pure algebra and algebraic geometry.

24.8. **Algebraic geometry and differential geometry.** Let M be a suitably nice manifold. The **Serre-Swan correspondence** asserts that there is an equivalence of categories between projective modules over the ring of C^∞-functions on M and vector bundles over M. Give an exposition of this result.

24.9. **Algebraic geometry and complex geometry.** Work within algebraic/complex geometry over the complex numbers to establish the exponential sheaf sequence

$$0 \to 2\pi i\mathbf{Z} \to \mathcal{O} \to \mathcal{O}^\times \to 0;$$

and discuss the solution of the Cousin problem using the language of divisors. This project will probably require you to delve a bit deeper into complex geometry.

24.10. **Bézout's theorem.** Let k be a field and let $\mathrm{C}_1, \mathrm{C}_2 \subset \mathbf{P}^2_k$ be integral closed subschemes which are codimension 1, i.e., they are curves. We can associate to $\mathrm{C}_1, \mathrm{C}_2$ their degrees and to the intersection $\mathrm{C}_1 \cap \mathrm{C}_2$ the intersection product $\mathrm{C}_1 \cdot \mathrm{C}_2$. Explain these apparatus and prove:

Theorem 24.10.1. *We have*

$$\mathrm{C}_1 \cdot \mathrm{C}_2 = \deg(\mathrm{C}_1) \cdot \deg(\mathrm{C}_2).$$

24.11. **The classification of curves up to birational equivalence.**

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