November 5, 2020

Exercise 1. We can consider the p-adic integers as a topological abelian group, and hence as an object of Cond(Ab), or we could consider the object $\lim \mathbb{Z}/p^i$ where we form the limit in the category Cond(Ab). Show that these are the same.

Exercise 2. More generally, show that the functor

$$\mathrm{Ab}(\mathrm{CHaus}) \to \mathrm{Cond}(\mathrm{Ab})$$

preserves all limits.

Exercise 3. The category of solid abelian groups has compact, projective generators $\prod_I \mathbb{Z}$ where I is a set. To pin down the structure of this category, all you need to be able to compute, then, is:

$$\mathrm{Hom}(\prod_I \mathbb{Z}, \mathbb{Z}).$$

What is it?

Exercise 4. Recall that, for $S = \lim S_i$ profinite, we defined

$$\mathbb{Z}_{\blacksquare}[S] = \lim \mathbb{Z}[S_i],$$

which is what one would like the free gadget to look like. Let's see why the free gadget in condensed abelian groups differs from this. Let $\mathbb{Z}[S_i]_{\leq n} \subseteq \mathbb{Z}[S_i]$ be the subset consisting of elements $\sum_{s \in S_i} n_s[s]$ with $\sum |n_s| \leq n$. Then the map

$$\mathbb{Z}[S] \to \mathbb{Z}_{\blacksquare}[S]$$

is injective with image given by

$$\bigcup_n \lim_i \mathbb{Z}[S_i]_{\leq n}.$$

Exercise 5. Let $\mathbb{Q}_p\langle t\rangle$ be the convergent power series on the unit disc, i.e.

$$\mathbb{Q}_p\langle t\rangle = \{\sum_{n \geq 0} a_n t^n : a_n \to 0\}$$

Show that the multiplication map

$$\mathbb{Q}_p\langle t\rangle \otimes^{\blacksquare}_{\mathbb{Q}_p[t]} \mathbb{Q}_p\langle t\rangle \to \mathbb{Q}_p\langle t\rangle$$

is an equivalence.

Exercise 6. Compute the tensor product

Exercise 7. The condensed abelian group $\mathbb{Z}_p^\wedge$ is a ring object in Solid, so we could consider $\mathrm{Mod}_{\mathbb{Z}_p^\wedge}(\mathrm{Solid})$. On the other hand, we have an analytic ring $\mathbb{Z}_{p,\blacksquare}$ with free objects given by

$$S \mapsto \lim_i \mathbb{Z}_p[S_i],$$

and so can form $\mathrm{Mod}_{\mathbb{Z}_{p,\blacksquare}}$. What is the relationship between these categories? (They are not the same.)

Exercise 8. Is there any relationship between $\mathrm{Mod}_{\mathbb{Z}_{p,\blacksquare}}$ and the category of derived p-complete abelian groups?

Exercise 9. Suppose $\mathcal{A} = (R, \mathcal{M})$ is an analytic ring. Then $R' := \mathcal{M}(\bullet)$, the free object on a point, is canonically a ring, and there's a natural map

$$(R, \mathcal{M}) \to (R', \mathcal{M}).$$

Show the forgetful functor on (derived or underived) categories of modules is an equivalence.

Exercise 10. How does one compute pushouts in the category of analytic rings?