1. Final project ideas

1.1. **27 lines on a cubic.** One of the early achievements of algebraic geometry is the following result:

Theorem 1.1.1. *Let k be a field and $\mathrm{X} \subset \mathbf{P}^3_k$ be a smooth cubic surface. Then X contains exactly 27 different lines of $\mathbf{P}^3_k$.*

Give a full exposition of this proof.

1.2. **The projective space as a quotient.** If X is a scheme and G is an algebraic group acting on X, we can construct a scheme-theoretic quotient X/G using **geometric invariant theory**. This gives yet another description of $\mathbf{P}^n$; give an exposition of the following result (without delving into all of geometric invariant theory):

Theorem 1.2.1. *There is a canonical isomorphism*

$$\mathbf{A}^n/\mathbf{G}_m \cong \mathbf{P}^{n-1}.$$

1.3. **The moduli of hypersurfaces.** We have seen in class that we can $\mathrm{H}^0(\mathbf{P}^n; \mathcal{O}(d))$ classify degree d hypersurfaces in $\mathbf{P}^n$. Construct this formally as a scheme; its functor of points, which we denote by $\mathcal{H}_{d,n}$ is given as follows

$$\mathcal{H}_{d,n} : \mathrm{A} \mapsto \{\mathrm{X} \hookrightarrow \mathbf{P}^n_{\mathrm{A}} : \mathrm{X} \text{ is flat, finitely presented over } \operatorname{Spec} \mathrm{A} \text{ and fibers are hypersurfaces of degree } d\}.$$

You need to explain the meaning of flatness and why it is a key hypothesis here.

1.4. **The Grassmanian as a scheme.** We are familiar with $\mathrm{Gr}(n, k)$, the Grassmanian classifying k-dimensional subspaces of n-dimensional vector spaces over a field. This is an extremely interesting algebro-geometric object. Construct the Grassmanian as a scheme, similar to what we did in projective space.

1.5. **Algebraic stacks and Grothendieck topology.** In class we have introduced the notion of a Zariski stack. More conventionally a stack should have 1) descent with respect to the étale topology and 2) be a functor valued in groupoids. The task of this project is to make sense of this notion and prove the following result

Theorem 1.5.1. *The functor which sends a ring* A *to the groupoid of line bundles over* Spec A *is a stack.*

1.6. **Deformation theory: flat families.** As we have seen, there are many ways that schemes can deform along families. One of the key requirements for this idea to be useful is the idea of "flat families." Explain (no need to prove) the following result:

Lemma 1.6.1. *Let k be a field and* B *a reduced k-scheme. Let $\mathcal{X} \subset \mathbf{A}^n_{\mathrm{B}}$ be a closed subscheme and $b \in \mathrm{B}$ a closed point. Then the following are equivalent:*

(1) $\mathcal{X}$ is flat over B;

(2) for any nonsingular, one-dimensional k-scheme B' with a closed point $0 \in \mathrm{B}$ and a pointed morphism

$$(\mathrm{B}', 0) \to (\mathrm{B}, b),$$

the fiber X_b is the flat limit of $\mathrm{X}_{\varphi(b')}, b' \in \mathrm{B}$.

In other words, explain what this result means and use it to give many examples of flat limits.

1.7. **Twisted projective spaces.** Let k be a field. Then a **twisted projective space** or a **Severi-Brauer variety** is a smooth proper k-scheme X such that there exists a separable extension k'/k for which $\mathrm{X}_{k'} \cong \mathbf{P}^n_{k'}$. All such objects are constructed out of central simple algebras which are generalizations of Hamilton's quarternion. Give a construction of the Severi-Brauer varieties using the language of this class. This is one of the most fascinating links between number theory, pure algebra and algebraic geometry.

1.8. **Algebraic geometry and differential geometry.** Let M be a suitably nice manifold. The **Serre-Swan correspondence** asserts that there is an equivalence of categories between projective modules over the ring of C^{∞}-functions on M and vector bundles over M. Give an exposition of this result.

1.9. **Algebraic geometry and complex geometry.** Work within algebraic/complex geometry over the complex numbers to establish the exponential sheaf sequence

$$0 \to 2\pi i\mathbf{Z} \to \mathcal{O} \to \mathcal{O}^{\times} \to 0;$$

and discuss the solution of the Cousin problem using the language of divisors. This project will probably require you to delve a bit deeper into complex geometry.

1.10. **Bézout's theorem.** Let k be a field and let $C_1, C_2 \subset \mathbf{P}^2_k$ be integral closed subschemes which are codimension 1, i.e., they are curves. We can associate to C_1, C_2 their degrees and to the intersection $C_1 \cap C_2$ the intersection product $C_1 \cdot C_2$. Explain these apparatus and prove:

Theorem 1.10.1. *We have*

$$C_1 \cdot C_2 = \deg(C_1) \cdot \deg(C_2).$$

1.11. **The classification of curves up to birational equivalence.** Using what we have now, we can prove the following equivalence of categories which classifies curves up to birational equivalences:

Theorem 1.11.1. *Let k be a field, then there is an equivalence of categories between*

(1) finitely generated field extensions K/k of transcendence degree 1 and,
(2) the category of curves and dominant rational maps.

Give an exposition of this result.

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