Descent for semiorthogonal decompositions

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Abstract

We prove descent theorems for semiorthogonal decompositions using techniques from derived algebraic geometry. Our methods allow us to capture more general filtrations of derived categories and even marked filtrations, where one descends not only admissible subcategories but also preferred objects.

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1 Introduction

The theory of semiorthogonal decompositions is a crucial tool in understanding derived categories of schemes, especially derived categories of smooth proper schemes over fields. This paper concerns the construction of new semiorthogonal decompositions from known ones in various situations that involve "twists" or "descent". We expect our work will be useful for future explorations of the relationship between derived categories and rationality as studied in [3]. We consider two questions.

Question 1.1 (Absolute descent)**.** Suppose that G is a group scheme acting on a scheme X and suppose that there is a semiorthogonal decomposition

$$\mathcal{P}\mathrm{erf}(X) \simeq \langle \mathcal{E}_0, \ldots, \mathcal{E}_r \rangle.$$

When does $\mathcal{P}\mathrm{erf}([X/G])$ admit a descended semiorthogonal decomposition?

Here $[X/G]$ is the stack quotient of X by G. A sufficient condition on the action, namely that it be upper triangular with respect to the semiorthogonal decomposition, was first obtained by Elagin in [19].

Question 1.2 (Relative descent)**.** Let X and Y be k-schemes for some field k and suppose that X and Y become isomorphic over some separable extension l/k. If X admits a k-linear semiorthogonal decomposition

$$\mathcal{P}\mathrm{erf}(X) \simeq \langle \mathcal{E}_0, \ldots, \mathcal{E}_r \rangle,$$

then $\mathcal{P}\mathrm{erf}(Y_l)$ admits an l-linear semiorthogonal decomposition $\mathcal{P}\mathrm{erf}(Y_l) \simeq \langle (\mathcal{E}_0)_l, \dots, (\mathcal{E}_r)_l \rangle$.[1] When does this semiorthogonal decomposition on Y_l descend to Y?

For example, Blunk–Sierra–Smith [11] constructed a semiorthogonal decomposition on the blowup of $\mathbb{P}^2$ at 3 points which descends to any twisted form. Similarly, Bernardara showed [9] that Beilinson's semiorthogonal decomposition of the derived category of $\mathcal{P}\mathrm{erf}(\mathbb{P}^n_k)$ descends to $\mathcal{P}\mathrm{erf}(Y)$ if Y is the Severi–Brauer variety of a central simple algebra of degree $n+1$ over k.

Note that Question 1.2 can be reduced to Question 1.1 by replacing l with its Galois closure, but we will see that there is some advantage in treating the relative case separately. In both cases, the philosophical answer is that the semiorthogonal decomposition should descend as long as the group action or descent data preserves the semiorthogonal decomposition.

One of the goals of this paper is to make this intuition precise. In particular, we wanted to understand why it is enough to check only 1-categorical information in the results of Elagin [19], Auel–Bernardara [3], and Ballard–Duncan–McFaddin [6] when in principal one has to glue higher homotopical objects, a process which requires higher-degree analogues of the cocycle condition in general.

The starting point of our approach is the idea that a semiorthogonal decomposition constitutes a special kind of filtration on the derived category. We will see that after descending the filtration admissibility comes along for the ride.

Fix a base scheme S and let $\mathbf{Cat}$ be the stack[2] which assigns to each affine $\operatorname{Spec} R \to S$ the ∞-category $\mathbf{Cat}(R) = \mathrm{Cat}_R$ of small idempotent complete R-linear stable ∞-categories.[3] Now, fix a poset P and let $\mathbf{Filt}_P$ be the prestack of linear categories equipped with P-shaped filtrations.

Theorem 1.3 (Filtrations, Theorem 2.17 and Proposition 2.19). *Let P be a poset. The prestack $\mathbf{Filt}_P$ of P-shaped filtrations is a stack. Moreover, the forgetful functor $\mathbf{Filt}_P \to \mathbf{Cat}$ has discrete fibers.*

The proof of Theorem 1.3 is rather formal, although it has important consequences for the questions above. The next result is deeper. Let P be a poset and let $\mathbf{Sod}_P \subseteq \mathbf{Filt}_P$ be the subprestack of P-shaped semiorthogonal decompositions.

Theorem 1.4 (Semiorthogonal decompositions, Corollary 3.17). *For any poset P, the prestack $\mathbf{Sod}_P$ is a stack.*

This theorem says that the only obstruction to descending a semiorthogonal decomposition is descending the associated filtration. It is perhaps one of the main insights of this paper and demonstrates the local nature of admissibility which, on first pass, is rather surprising. The proof of Theorem 1.4 highlights the power of working in a higher categorical setting where problems of descent and base change can be formulated and solved in an elegant manner.

While writing this paper, we were made aware of a different approach to variants of our results by Belmans–Okawa–Ricolfi [7]. These authors also studied a version of the stack which

[1] In this paper, $\mathcal{P}\mathrm{erf}(X)$ will denote the small idempotent complete stable ∞-category of perfect complexes of $\mathcal{O}_X$-modules on X. The homotopy category of $\mathcal{P}\mathrm{erf}(X)$ is thus the usual triangulated category $\mathrm{Perf}(X)$ of perfect complexes on X. When X is regular, noetherian, and quasi-separated, $\mathcal{P}\mathrm{erf}(X) \simeq \mathcal{D}^b(X)$, the bounded derived category of coherent sheaves on X, but in general these are different.

[2] In this paper our stacks are fppf stacks.

[3] We will work with stable ∞-categories instead of dg categories out of personal preference. The theory could also be developed in the language of the equivalent theory of dg categories.

in our notation would be denoted by $\mathbf{Sod}_{[n]}^{\mathcal{P}\mathrm{erf}(X)}$ where X is scheme of characteristic zero. They proved fppf descent in this setting using a completely different approach, namely, by studying Fourier-Mukai kernels instead of appealing to machinery derived algebraic geometry. While their descent results are more restricted, they were able to access some deep geometric information about this stack. Among other things they proved that these stacks are, in fact, algebraic spaces; we conclude only that these stacks are discrete in this paper.

Theorems 1.3 and 1.4 provide complete answers to Questions 1.1 and 1.2 — the stack $\mathbf{Sod}_P$ controls the descent problems posed by both questions. See also Sections 2.5 and 4.3. As we illustrate in Section 5 these problems are actually quite tractable in practice. Moreover, we also give results on descending individual objects in the pieces of semiorthogonal decompositions.

Theorem 1.3 is proved in Section 2 and Theorem 1.4 in Section 3. The twisted Brauer space perspective is developed in Section 4 and many examples are given in Section 5.

Previously, Elagin studied descent for semiorthogonal decompositions in [19], showing that one could descend semiorthogonal decompositions of triangulated categories along certain comonads. The key condition of Elagin's main theorem is that the comonad should be upper triangular with respect to the semiorthogonal decomposition, meaning in other words that the comonad respects the filtration coming from the semiorthogonal decomposition. Elagin's work was recently revisited by Shinder who gave a new proof [35]. A similar approach is given in work of Bergh and Schnürer [8].

In practice, there are many cases where working algebraic geometers have established descent for semiorthogonal decompositions or exceptional collections in interesting settings by hand. We list some here. Historically, the first is Bernardara's work mentioned above on Severi–Brauer schemes [9] followed by work of Blunk–Sierra–Smith [11] on degree six del Pezzo surfaces, unpublished work of Blunk [12] on some twisted Grassmannians, and work of Baek [5] on twisted Grassmannians in general. Perhaps the two most impressive works in this direction are the paper of Auel and Bernardara on derived categories of del Pezzo varieties over general fields [3] and the work of Ballard–Duncan–McFaddin on toric varieties [6]. In these the authors construct explicit vector bundles generating their semiorthogonal decompositions. For more on the connection of our work with this previous work, see Section 5.

Under the hood, our approach bears some similarity to the recent work of Scherotzke–Sibilla–Talpo in [34] who prove that ∞-categories equipped with finite semiorthogonal decompositions indexed by possibly varying index sets admit certain limits. In our work, the indexing sets will be fixed.

The main ideas in this paper go back to 2013 when a first draft of the paper was produced by the first author. However, at that time, Alexander Kuznetsov pointed out Elagin's work and it was decided not to pursue the project further. In the meantime, the problem of descent for semiorthogonal decompositions has returned again and again and it seemed like those early results were worth making public after all. This has been done here with many simplifications and extensions.

Notation. Let $\mathcal{S}$ be the ∞-category of spaces and let $\mathcal{S}\mathrm{p}$ be the ∞-category of spectra. If $\mathcal{C}$ is an ∞-category and $x, y \in \mathcal{C}$, then $\mathrm{Map}_{\mathcal{C}}(x, y)$ denotes the mapping space from x to y and, if $\mathcal{C}$ is stable, then $\mathbf{Map}_{\mathcal{C}}(x, y)$ denotes the mapping spectrum.

Conventions. In this paper, a **prestack** on a small ∞-category $\mathcal{C}$ will mean a functor $\mathcal{C}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ where $\widehat{\mathrm{Cat}}_\infty$ is the ∞-category of possibly large ∞-categories. If τ is a topology on $\mathcal{C}$, by a **τ-stack** we mean a prestack which satisfies τ-descent. Functors of the form $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ will be called **presheaves** and those that satisfy τ-descent are **τ-sheaves**. If X is a presheaf on Aff_R, the category of affine schemes over a commutative ring R, there is a symmetric monoidal stable ∞-category $\mathcal{P}\mathrm{erf}(X)$ of perfect complexes on X defined as $\lim_{\operatorname{Spec} S \to X} \mathcal{P}\mathrm{erf}(\operatorname{Spec} S)$; see Example 2.3.

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2 The stack of filtrations and variants

In this section we prove that the ∞-category of possibly marked filtered idempotent complete stable ∞-categories forms an fppf sheaf.

2.1 Background on stable ∞-categories

We give a brief set of definitions and remarks about stable ∞-categories. For details, see [29, Chapter 1].

Definition 2.1. (a) An ∞-category $\mathcal{C}$ is **stable** if it is pointed, admits finite limits and finite colimits, and if the suspension functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ is an equivalence.

(b) A functor $F\colon \mathcal{C} \to \mathcal{D}$ between two stable ∞-categories is **exact** if it preserves finite limits or equivalently (for stable ∞-categories) finite colimits.

(c) An ∞-category $\mathcal{C}$ is **idempotent complete** if every idempotent in $\mathcal{C}$ admits a splitting.

Remark 2.2. If $\mathcal{C}$ is a stable ∞-category, then the homotopy category $\mathrm{Ho}(\mathcal{C})$, which is just an ordinary category, admits a canonical triangulated category structure [29, Theorem 1.1.2.14]. A functor F as in (b) is exact if and only if the functor $\mathrm{Ho}(F)\colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ is exact. Additionally, $\mathcal{C}$ is idempotent complete if and only if $\mathrm{Ho}(\mathcal{C})$ is idempotent complete.

Example 2.3. (1) If R is a commutative ring, then $\mathcal{D}(R)$ and $\mathcal{P}\mathrm{erf}(R)$ are a idempotent complete stable ∞-category. In this case, $\mathcal{P}\mathrm{erf}(R) \subseteq \mathcal{D}(R)$ is the full subcategory of compact objects, i.e., those objects M such that the functor $\mathrm{Hom}_{\mathrm{D}(R)}(M, -)$ preserves arbitrary coproducts, where $\mathrm{D}(R) = \mathrm{Ho}(\mathcal{D}(R))$.

(2) If S is an algebraic stack, then we can define $\mathcal{P}\mathrm{erf}(S)$ by right Kan extension. Namely,

$\mathcal{P}\mathrm{erf}(S)$ is the value at S of the diagonal functor in the commutative diagram

$$\begin{array}{ccc} \mathrm{Aff}^{\mathrm{op}} & \xrightarrow{\mathcal{P}\mathrm{erf}(-)} & \mathrm{Cat}_\infty^{\mathrm{perf}} \\ \downarrow & \overset{\mathrm{R}\mathcal{P}\mathrm{erf}(-)}{\nearrow} & \\ \mathcal{P}(\mathrm{Aff})^{\mathrm{op}}, & & \end{array}$$

where $\mathcal{P}(\mathrm{Aff})$ is the ∞-category of presheaves of spaces on Aff. Practically speaking, we compute

$$\mathcal{P}\mathrm{erf}(S) = \lim_{\operatorname{Spec} R \to S} \mathcal{P}\mathrm{erf}(R),$$

although since $\mathrm{Aff}^{\mathrm{op}}$ is not small, care needs to be taken to ensure that this limit exists in $\mathrm{Cat}_\infty^{\mathrm{perf}}$. When S is quasi-compact and quasi-separated with quasi-affine diagonal, for example $[X/G]$ where G is an affine algebraic group acting on a quasicompact quasiseparated scheme X, this limit does exist. Indeed, there is a cover $U \to S$ where U is affine and where each $U \times_S \cdots \times_S U$ is quasicompact and quasi-affine. Then, $\mathcal{P}\mathrm{erf}(S) \simeq \lim_\Delta \mathcal{P}\mathrm{erf}(\check{C}(U))$, where $\check{C}(U)$ is the Čech complex of $U \to S$.

(3) The ∞-categories $\mathcal{S}\mathrm{p}$ of spectra and $\mathcal{S}\mathrm{p}^\omega$ of finite spectra are stable ∞-categories. The former plays the role of $\mathcal{D}(\mathbb{S})$ while the latter plays the role of $\mathcal{P}\mathrm{erf}(\mathbb{S})$, where $\mathbb{S}$ is the sphere spectrum, the initial commutative ring in stable homotopy theory.

(4) If C is a pretriangulated dg category, then there is a naturally associated stable ∞-category $\mathrm{N}_{\mathrm{dg}}(C)$.

The theory of idempotent complete stable ∞-categories and exact functors is organized into an ∞-category $\mathrm{Cat}_\infty^{\mathrm{perf}}$.[4] Moreover, this ∞-category admits a natural symmetric monoidal structure where, for example, if A and B are rings, then $\mathcal{P}\mathrm{erf}(A) \otimes_{\mathcal{P}\mathrm{erf}(\mathbb{Z})} \mathcal{P}\mathrm{erf}(B) \simeq \mathcal{P}\mathrm{erf}(A \otimes_{\mathbb{Z}}^{\mathrm{L}} B)$, where $A \otimes_{\mathbb{Z}}^{\mathrm{L}} B$ denotes the derived tensor product viewed for example as a dg algebra.

If R is a commutative ring, then $\mathcal{P}\mathrm{erf}(R)$ admits a natural symmetric monoidal structure (which on the homotopy category gives the derived tensor product of R-modules), whence we may view $\mathcal{P}\mathrm{erf}(R)$ as a "highly-structured" commutative ring, more precisely an E_∞-ring, object in $\mathrm{Cat}_\infty^{\mathrm{perf}}$. We thus form categories of modules over them [29, Chapter 3] and define

$$\mathrm{Cat}_R = \mathrm{Mod}_{\mathcal{P}\mathrm{erf}(R)}(\mathrm{Cat}_\infty^{\mathrm{perf}}).$$

The objects of Cat_R are R-linear idempotent complete stable ∞-categories. We will just call these ***R*-linear categories** for simplicity. Note that Cat_R is equivalent to the theory of idempotent complete stable ∞-categories and exact functors enriched in $\mathcal{D}(R)$.

Remark 2.4. We can also consider R-linear dg categories. There is a model structure on the category dgcat_R of R-linear dg categories where the weak equivalences are the Morita equivalences: a functor $C \to E$ of dg categories is a weak equivalence if the induced functor

[4]For technical purposes later, we will need to use that $\mathrm{Cat}_\infty^{\mathrm{perf}}$ admits a natural $(\infty, 2)$-categorical structure. Indeed, for two idempotent complete stable ∞-categories there is an idempotent complete stable ∞-category $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ of exact functors. The underlying ∞-groupoid (obtained by forgetting all non-invertible natural transformations between exact functors) $\iota\,\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ is naturally equivalent to the mapping space $\mathrm{Map}_{\mathrm{Cat}_\infty^{\mathrm{perf}}}(\mathcal{C}, \mathcal{D})$.

$\mathrm{D}_{\mathrm{dg}}(C) \to \mathrm{D}_{\mathrm{dg}}(E)$ on dg module categories is an equivalence. The ∞-category associated to this model category is equivalent to Cat_R. Unfortunately, no published reference for this fact is known, but see [16] for an unpublished version. An analogous statement, due to Shipley [36], says that the homotopy theory of dg algebras agrees with the homotopy theory of $\mathbb{E}_1$-ring spectra over R if R is a commutative ring.

The assignment $R \mapsto \mathrm{Cat}_R$ can be given the structure of a presheaf

$$\mathbf{Cat}\colon \mathrm{CAlg} \simeq \mathrm{Aff}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$$

of large ∞-categories on the category $\mathrm{Aff}^{\mathrm{op}}$ of affine schemes (over $\mathbb{Z}$) with

$$\mathbf{Cat}(\mathrm{Spec}\, R) = \mathbf{Cat}(R) = \mathrm{Cat}_R.$$

We call this the **prestack of linear categories**. We can also work relative to a base commutative ring R and restrict this to a presheaf on $\mathrm{Aff}_R = \mathrm{CAlg}_R^{\mathrm{op}}$, the category of affine R-schemes.

Now, let S be an algebraic stack. We let $\mathrm{Cat}_S = \mathbf{Cat}(S)$ be the value on S of the right Kan extension of $\mathbf{Cat}\colon \mathrm{Aff}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ along the inclusion $\mathrm{Aff}^{\mathrm{op}} \to \mathcal{P}(\mathrm{Aff})^{\mathrm{op}}$. We will call the objects of Cat_S simply $\mathcal{O}_S$-linear categories.

Warning 2.5. There is a canonical functor $\mathrm{Mod}_{\mathcal{P}\mathrm{erf}(S)}(\mathrm{Cat}_\infty^{\mathrm{perf}}) \to \mathrm{Cat}_S$. We warn the reader that this functor is *not* always an equivalence. Algebraic stacks for which this is the case are called **1-affine** [21, Definition 1.3.7]. Examples of 1-affine algebraic stacks include

(a) quasi-compact quasi-separated schemes, or more generally

(b) quasi-compact quasi-separated algebraic spaces [21, Theorem 2.1.1], and even

(c) stack quotients of quasi-compact quasi-separated algebraic spaces by affine algebraic groups of finite type [21, Theorem 2.2.2].

For many results in this paper we require that S is 1-affine so that we can view an $\mathcal{O}_S$-linear category more concretely as a small idempotent complete stable ∞-category with extra structure, namely with the structure of a $\mathcal{P}\mathrm{erf}(S)$-module structure. On the other hand, defining Cat_S via right Kan extension has the advantage that it inherits descent properties from its value on affine schemes; see [21, Theorem 1.5.7].

2.2 Filtrations on stable ∞-categories

We will work everywhere relative to a fixed poset.

Definition 2.6. A **poset** is a partially ordered set, i.e., a set P together with a binary relation $\leqslant$ satisfying the reflexivity, antisymmetry, and transitivity conditions. When we say that an ∞-category is a poset, we mean that it is equivalent to the nerve of a poset. Equivalently, a poset is an ∞-category $\mathcal{C}$ such that for each pair of objects $x, y \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(x, y)$ is either empty or contractible. We will make no notational distinction between considering a poset P as an ordinary category or as an ∞-category. The posets that we care about in this paper are P is **filtered**: every finite set of elements of P has an upper bound.

Let P be a poset.

Example 2.7. For example, P could be

(i) the totally ordered set with $n+1$ elements $[n] = \{0 < 1 < \cdots < n\}$ for an integer $n \geqslant 0$;

(ii) products such as $[m] \times [n]$, for integers $m, n \geqslant 0$, with the product partial order: $(i, j) \leqslant (l, k)$ if and only if $i \leqslant l$ and $j \leqslant k$;

(iii) $\mathbb{N} = \{0, 1, 2, \ldots\}$ or $\mathbb{Z}$ with the usual total orders;

(iv) the set of finite subsets of a given set with the partial order given by set containment.

Let S be a 1-affine algebraic stack. We will study P-shaped filtrations on $\mathcal{O}_S$-linear categories.

Definition 2.8. The ∞-category $\mathrm{Filt}_P\mathrm{Cat}_S$ of **P-shaped filtrations of $\mathcal{O}_S$-linear categories** is the full subcategory of the functor category $\mathrm{Fun}(P, \mathrm{Cat}_S)$ on those functors $\mathrm{F}_\star\mathcal{C} \colon P \to \mathrm{Cat}_S$ such that for $p \leqslant q$ in P the induced map $\mathrm{F}_p\mathcal{C} \to \mathrm{F}_q\mathcal{C}$ is fully faithful.[5]

Example 2.9. Evaluation at 0 gives an equivalence $\mathrm{Filt}_{[0]}\mathrm{Cat}_S \to \mathrm{Cat}_S$.

In the previous definition, there is no ambient $\mathcal{O}_S$-linear category that is being filtered.

Definition 2.10. A **P-shaped filtration on an $\mathcal{O}_S$-linear category** $\mathcal{C}$ is a P-shaped filtration $\mathrm{F}_\star\mathcal{C}$ equipped with a functor

$$\mathrm{F}_\infty\mathcal{C} = \operatorname*{colim}_P \mathrm{F}_\star\mathcal{C} \to \mathcal{C}$$

such that for each $p \in P$ the induced functor $\mathrm{F}_p\mathcal{C} \to \mathcal{C}$ is fully faithful. If the functor $\mathrm{F}_\infty\mathcal{C} \to \mathcal{C}$ is moreover an equivalence, we say that the filtration is **exhaustive**. To give a precise definition of the ∞-category of P-shaped filtrations on $\mathcal{C}$, we first define the lax pullback

$$\mathrm{Filt}_P\mathrm{Cat}_S \overrightarrow{\times}_{\mathrm{Cat}_S} \Delta^0$$

as the pullback

$$\begin{array}{ccc} \mathrm{Filt}_P\mathrm{Cat}_S \overrightarrow{\times}_{\mathrm{Cat}_S} \Delta^0 & \longrightarrow & (\mathrm{Cat}_S)^{\Delta^1} \\ \downarrow & & \downarrow {\scriptstyle (\partial_1, \partial_0)} \\ \mathrm{Filt}_P\mathrm{Cat}_S & \xrightarrow{(\mathrm{colim}_P, \mathcal{C})} & \mathrm{Cat}_S \times \mathrm{Cat}_S. \end{array} \tag{1}$$

Thus, $\mathrm{Filt}_P\mathrm{Cat}_S \overrightarrow{\times}_{\mathrm{Cat}_S} \Delta^0$ is the ∞-category consisting of pairs $(\mathrm{F}_\star\mathcal{C}, \mathcal{C})$ of a P-shaped filtration $\mathrm{F}_\star\mathcal{C}$, an $\mathcal{O}_S$-linear category $\mathcal{C}$, and a functor $\mathrm{colim}_P \mathrm{F}_\star\mathcal{C} \to \mathcal{C}$. We let

$$\mathrm{Filt}_P^{\mathcal{C}} \subseteq \mathrm{Filt}_P\mathrm{Cat}_S \overrightarrow{\times}_{\mathrm{Cat}_S} \Delta^0$$

be the full subcategory where each induced functor $\mathrm{F}_p\mathcal{C} \to \mathcal{C}$ is fully faithful. We let

$$\mathrm{ExFilt}_P^{\mathcal{C}} \subseteq \mathrm{Filt}_P^{\mathcal{C}}$$

be the full subcategory of exhaustive P-shaped filtrations on $\mathcal{C}$.

[5]Note that, by 1-affineness, a functor $\mathcal{E} \to \mathcal{C}$ of $\mathcal{O}_S$-linear categories is fully faithful if and only if corresponding functor $\mathrm{Ho}(\mathcal{E}) \to \mathrm{Ho}(\mathcal{C})$ of triangulated homotopy categories is fully faithful.

Warning 2.11. Definition 2.10 is usually only interesting if P is filtered so that the mapping spaces in $\mathcal{C}$ can be computed as a filtered colimit of mapping spaces in each $\mathrm{F}_p\mathcal{C}$. In particular, a filtered colimit of fully faithful functors is fully faithful. But, the present notion also lets us use discrete sets such as $\{0,1\}$ as our indexing sets. A $\{0,1\}$-shaped filtration on an $\mathcal{O}_S$-linear category $\mathcal{C}$ is just the data of two full subcategories $\mathrm{F}_0\mathcal{C} \subseteq \mathcal{C}$ and $\mathrm{F}_1\mathcal{C} \subseteq \mathcal{C}$ with no imposed relation. This might be useful in some situations, so we will work in this generality.

Example 2.12. (a) A sequence $\mathrm{F}_0\mathcal{C} \subseteq \mathrm{F}_1\mathcal{C} \subseteq \cdots \subseteq \mathrm{F}_n\mathcal{C}$ of $n+1$ full subcategories of $\mathcal{C}$ defines an $[n]$-shaped filtration of $\mathcal{C}$. It is exhaustive if and only if the last inclusion is an equivalence $\mathrm{F}_n\mathcal{C} \simeq \mathcal{C}$.

(b) The Beilinson filtration on $\mathcal{P}\mathrm{erf}(\mathbb{P}^n)$ with $\mathrm{F}_p\mathcal{P}\mathrm{erf}(\mathbb{P}^n) = \langle \mathcal{O}(0), \dots, \mathcal{O}(p)\rangle$ gives an exhaustive $[n]$-shaped filtration of $\mathcal{P}\mathrm{erf}(\mathbb{P}^n)$.

(c) Let X be a qcqs scheme. Let $Z_0 \subseteq Z_1 \subseteq \cdots$ be an $\mathbb{N}$-indexed sequence of closed subsets of X, each with quasi-compact complement. Then, $\mathcal{P}\mathrm{erf}(X \text{ on } Z_\star)$ defines an $\mathbb{N}$-shaped filtration on $\mathcal{P}\mathrm{erf}(X)$. It is exhaustive if and only if each generic point of X is contained in Z_p for some finite p.

As we will see, the following simple lemma turns out to be the secret sauce.

Lemma 2.13. *Let S be a 1-affine algebraic stack and let $\mathcal{C}$ be an $\mathcal{O}_S$-linear category. For any poset P, the ∞-category $\mathrm{Filt}_P^{\mathcal{C}}$ is a poset. In particular, $\mathrm{ExFilt}_P^{\mathcal{C}}$ is a poset as well.*

Proof. Let $\mathrm{Sub}^{\mathcal{C}} \subseteq (\mathrm{Cat}_R)_{/\mathcal{C}}$ be the full subcategory on the fully faithful inclusions. Then,

$$\mathrm{Filt}_P^{\mathcal{C}} \simeq \mathrm{Fun}(P, \mathrm{Sub}^{\mathcal{C}}).$$

Since the ∞-category of functors from one poset to another forms a poset, it now suffices to see that $\mathrm{Sub}^{\mathcal{C}}$ is a poset. Let $\mathcal{D}_0$ and $\mathcal{D}_1$ be two full subcategories of $\mathcal{C}$. There is a fiber sequence

$$\mathrm{Map}_{\mathrm{Sub}^{\mathcal{C}}}(\mathcal{D}_0, \mathcal{D}_1) \to \mathrm{Map}_{\mathrm{Cat}_R}(\mathcal{D}_0, \mathcal{D}_1) \to \mathrm{Map}_{\mathrm{Cat}_R}(\mathcal{D}_0, \mathcal{C})$$

of spaces, where the left hand term is the fiber over the fixed inclusion $\mathcal{D}_0 \hookrightarrow \mathcal{C}$. Since $\mathcal{D}_1 \to \mathcal{C}$ is fully faithful, the right map is an inclusion of connected components. Thus, the fibers are either empty or contractible. Therefore, $\mathrm{Sub}^{\mathcal{C}}$ is a poset. Hence, $\mathrm{Fun}(P, \mathrm{Sub}^{\mathcal{C}})$ is a poset, which is what we wanted to prove. □

2.3 Marked filtrations

Marked filtrations are filtrations in which we also specify an object from each layer in the filtration.

Definition 2.14. Let S be a 1-affine algebraic stack. Suppose that $\mathrm{F}_\star\mathcal{C} \to \mathcal{C}$ is a filtration on an $\mathcal{O}_S$-linear category $\mathcal{C}$. A **marking** of $\mathrm{F}_\star\mathcal{C} \to \mathcal{C}$ is the choice of an object $M_p \in \mathrm{F}_p\mathcal{C}$ for each $p \in P$. The pair $(\mathrm{F}_\star\mathcal{C} \to \mathcal{C}, M_*)$ will be called a **marked P-shaped filtration on** $\mathcal{C}$. A marked P-shaped filtration on $\mathcal{C}$ is called **exhaustive** if the underlying filtration is exhaustive.

We will denote by $\mathrm{MFilt}_P^{\mathcal{C}}$ (resp. $\mathrm{MExFilt}_P^{\mathcal{C}}$) the ∞-category of marked (resp. marked exhaustive) filtrations on $\mathcal{C}$.

In contrast to Lemma 2.13, neither the ∞-category $\mathrm{MExFilt}_P^{\mathcal{C}}$ nor $\mathrm{MFilt}_P^{\mathcal{C}}$ is a poset in general. For example, there is a natural equivalence $\mathrm{MExFilt}_{[0]}^{\mathcal{C}} \to \iota\mathcal{C}$, where $\iota\mathcal{C}$ is the space of objects in $\mathcal{C}$, given by taking the marked object.

Remark 2.15. There are many variants one can consider, for example by marking only certain parts of the filtration, or by marking the quotients $\frac{\mathrm{F}_q\mathcal{C}}{\mathrm{F}_p\mathcal{C}}$ for $p \leqslant q$. We leave it to the reader to spell out the theory in these cases.

2.4 Descent for the stack of filtrations

Central to our results is following theorem which is essentially due to Jacob Lurie. We include a proof, which relies heavily on [31], because Lurie works in a slightly different setting.

Theorem 2.16. *The functor*

$$\mathbf{Cat}\colon \mathrm{Aff}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$$

satisfies fppf descent.

Proof. To begin with, for any commutative ring R, we have inclusions of subcategories

$$\mathrm{Cat}_R \subseteq \mathrm{LinCat}_R^{\mathrm{cg}} \subseteq \mathrm{LinCat}_R^{\mathrm{st}}$$

where $\mathrm{LinCat}_R^{\mathrm{st}} = \mathrm{Mod}_{\mathcal{M}\mathrm{od}(R)}(\mathrm{Pr}^{\mathrm{L}})$ [31, Variant D.1.5.1] and $\mathrm{LinCat}_R^{\mathrm{cg}}$ is the full subcategory of those objects which are furthermore compactly generated [28, 5.5.7]. To explain the first inclusion, note that the functor of taking ind-objects

$$\mathrm{Ind} : \mathrm{Cat}_R \to \mathrm{LinCat}_R^{\mathrm{st}}$$

factors through $\mathrm{LinCat}_R^{\mathrm{cg}}$ and identifies as the subcategory of $\mathrm{LinCat}_R^{\mathrm{st}}$ where the objects are compactly generated but the functors are those which additionally preserve compact objects (this holds by the R-linear version of [28, Proposition 5.5.7.10]).

Now, [31, Theorem D.3.6.2] implies that the functor $R \mapsto \mathrm{LinCat}_R^{\mathrm{st}}$ is an fppf sheaf. Indeed, *loc. cit.* proves that it is a sheaf with respect to the universal descent topology which is finer than the fppf topology by [31, Proposition D.3.3.1] (the cardinality assumption is trivially satisfied by morphisms of finite presentation). It then suffices to prove that

(a) $R \mapsto \mathrm{LinCat}_R^{\mathrm{cg}}$ is an fppf sheaf,

(b) for any ring R and a colimit-preserving functor $F : \mathcal{C} \to \mathcal{D}$ in $\mathrm{LinCat}_R^{\mathrm{cg}}$ if $R \to S$ is a faithfully flat morphism of finite presentation, then if $F \otimes_R S$ preserves compact objects, F does as well.

Indeed, we claim that the proof of [31, Theorem D.5.3.1.b] proves (a). Since $R \mapsto \mathrm{LinCat}_R^{\mathrm{st}}$ is an fppf sheaf, it suffices to prove that the property of being compactly generated is local for the fppf topology. To this end, for each commutative ring R and $\mathcal{C} \in \mathrm{LinCat}_R^{\mathrm{cg}}$, define the presheaf (on Aff_R)

$$\chi_M(R') = \begin{cases} * & \text{if } \mathcal{C} \otimes_S R' \text{ is compactly generated} \\ \emptyset & \text{otherwise.} \end{cases}$$

The first half of the proof of [31, Theorem D.5.3.1(b)] shows that χ_M is an Nisnevich sheaf on the small Nisnevich site[6] for each R', hence it is a Nisnevich sheaf on Aff_R[7].

Now (beginning at the end of page 2153), Lurie claims that it is a sheaf for the finite étale topology. However the argument proves that it is in fact a sheaf for the finite flat topology since, in the notation of the proof, one only needs that B is finitely generated and projective as an A-module. But now, finite flat descent and Nisnevich descent implies fppf descent by [38, Tag 05WM] (see also [22, Corollaire 17.16.2]).

Now, we prove (b). Suppose that $M \in \mathcal{C}$. Define the presheaf (on Aff_S)

$$\chi_M(R') = \begin{cases} * & \text{if } F(M) \otimes_S R' \text{ is compact and} \\ \emptyset & \text{otherwise.} \end{cases}$$

This makes sense as $\mathcal{D} \to \mathcal{D} \otimes_{R'} R''$ preserves compact objects for $R' \to R''$ a map of commutative R-algebras. Arguing as above, one proves that χ_M is an fppf sheaf. Now, the functor $\mathcal{C} \to \mathcal{C} \otimes_R S$ preserves compact objects, hence the composite $\mathcal{C} \to \mathcal{D} \otimes_R S$ preserves compact objects, whence $\chi_M(S) \simeq *$ and thus $\chi_M(R) = *$ by fppf descent. This proves (b) and hence the theorem. $\square$

From this, we will prove that various prestacks classifying filtrations are actually fppf stacks. Let us fix a quasicompact quasiseparated 1-affine algebraic stack S and an $\mathcal{O}_S$-linear category $\mathcal{C}$. We consider the prestacks

$$\mathrm{Aff}_S^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$$

given by

(a) $\mathbf{Filt}_P \colon \operatorname{Spec} R \mapsto \mathrm{Filt}_P(\mathrm{Cat}_R)$,

(b) $\mathbf{Filt}_P^{\mathcal{C}} \colon \operatorname{Spec} R \mapsto \mathrm{Filt}_P^{\mathcal{C} \otimes_{\mathcal{P}\mathrm{erf}(S)} \mathcal{P}\mathrm{erf}(R)}(\mathrm{Cat}_R)$,

(c) $\mathbf{ExFilt}_P^{\mathcal{C}} \colon R \mapsto \mathrm{ExFilt}_P^{\mathcal{C} \otimes_{\mathcal{P}\mathrm{erf}(S)} \mathcal{P}\mathrm{erf}(R)}(\mathrm{Cat}_R)$,

as well as their marked variants $\mathbf{MFilt}_P, \mathbf{MFilt}_P^{\mathcal{C}}$ and $\mathbf{MExFilt}_P^{\mathcal{C}}$.

Theorem 2.17. *Let P be a poset.*

(i) *The prestack $\mathbf{Filt}_P$ satisfies fppf descent. Consequently, $\mathbf{Filt}_P^{\mathcal{C}}$ and $\mathbf{ExFilt}_P^{\mathcal{C}}$ also satisfy fppf descent.*

(ii) *The prestack $\mathbf{MFilt}_P$ satisfies fppf descent. Consequently, $\mathbf{MFilt}_P^{\mathcal{C}}$ and $\mathbf{MExFilt}_P^{\mathcal{C}}$ also satisfy fppf descent.*

Proof. To see part (i), note that the prestack $\operatorname{Spec} R \mapsto \mathrm{Fun}(P, \mathbf{Cat}(R))$ is an fppf stack since it is the mapping prestack $\mathrm{Fun}(P, \mathbf{Cat})$, and because $\mathbf{Cat}$ is an fppf stack by Theorem 2.16. To prove that $\mathbf{Filt}_P \subseteq \mathrm{Fun}(P, \mathbf{Cat})$ is an fppf stack, it is enough to show that if $\mathrm{F}_\star\mathcal{C} \colon P \to \mathrm{Cat}_R$ is a diagram which becomes a diagram of fully faithful maps after base changing along a faithfully flat map $S \to T$, then each $\mathrm{F}_p\mathcal{C} \to \mathrm{F}_q\mathcal{C}$ is already fully faithful. But, this

[6] By the small Nisnevich site we mean the category of étale R-schemes $\mathrm{\acute{E}t}_R$ equipped with the Nisnevich topology.

[7] One way to see this is that to be a Nisnevich sheaf, it suffices to check an excision condition involving objects on the small Nisnevich site; see [4, Appendix A] for the most general statement.

follows from the fact that mapping spaces themselves may be calculated flat locally. (See for example [30, Corollary 6.11].) Similarly, the prestack $\mathbf{Cat}^{\Delta^1}\colon \operatorname{Spec} R \mapsto \mathrm{Cat}_R^{\Delta^1}$ is $\mathrm{Fun}(\Delta^1, \mathbf{Cat})$ and so defines an fppf stack. Since stacks are stable under pullbacks in prestacks, we get that $\mathbf{Filt}_P \times_{\mathbf{Cat}\times\mathbf{Cat}} \mathbf{Cat}^{\Delta^1} \simeq \mathbf{Filt}_p \overrightarrow{\times}_{\mathbf{Cat}} \Delta^0$ is an fppf stack. Now, $\mathbf{Filt}_P^{\mathcal{C}} \subseteq \mathbf{Filt}_P \times_{\mathbf{Cat}\times\mathbf{Cat}} \mathbf{Cat}^{\Delta^1}$ is defined by the condition that each $\mathrm{F}_p\mathcal{C} \to \mathcal{C}$ is fully faithful. This is fppf-local, so the fact that $\mathbf{Filt}_P^{\mathcal{C}}$ is an fppf stack follows. The fact that $\mathbf{ExFilt}_P^{\mathcal{C}}$ is an fppf stack again follows from the fact that objects in Cat_S satisfy fppf descent by[30, Corollary 6.11]. Specifically, if $\operatorname{colim}_P \mathrm{F}_p\mathcal{C} \to \mathcal{C}$ is fppf locally an equivalence, then it is an equivalence. Indeed, set $\mathcal{D} = \operatorname{colim}_P \mathrm{F}_p\mathcal{C}$ and fix a faithfully flat map $R \to S$. Then, $\mathcal{D} \simeq \lim_\Delta \mathcal{D} \otimes_{\mathcal{P}\mathrm{erf}(R)} \mathcal{P}\mathrm{erf}(S^{\otimes_R \bullet+1})$ and $\mathcal{C} \simeq \lim_\Delta \mathcal{C} \otimes_{\mathcal{P}\mathrm{erf}(R)} \mathcal{P}\mathrm{erf}(S^{\otimes_R \bullet+1})$. The natural transformation $\mathcal{D} \otimes_{\mathcal{P}\mathrm{erf}(R)} \mathcal{P}\mathrm{erf}(S^{\otimes_R \bullet+1}) \to \mathcal{C} \otimes_{\mathcal{P}\mathrm{erf}(R)} \mathcal{P}\mathrm{erf}(S^{\otimes_R \bullet+1})$ is a degree-wise equivalence by hypothesis, thus it is an equivalence in the limit.

The prestack $\mathbf{MFilt}_P$ is computed as pullback of prestacks

$$\begin{array}{ccc} \mathbf{MFilt}_P & \longrightarrow & \mathbf{Filt}_P \\ \downarrow & & \downarrow \\ \prod_P \mathbf{Cat}_* & \xrightarrow{\prod u} & \prod_P \mathbf{Cat}. \end{array}$$

Here, the right vertical arrow is induced by the inclusion of the vertices of P into P and the bottom horizontal arrow forgets the base point. Since stacks are closed under pullbacks and products in prestacks, we see that $\mathbf{MFilt}_P$ is an fppf stack. The fact that $\mathbf{MFilt}_P^{\mathcal{C}}$ and $\mathbf{MExFilt}_P^{\mathcal{C}}$ are fppf stacks follow by the same argument as for their unmarked counterparts. □

Remark 2.18. For any of the prestacks $\mathbf{F}$ appearing in Theorem 2.17, we obtain presheaves by taking maximal subgroupoids. We will decorate the resulting presheaves by $\iota\mathbf{F}$. Then Theorem 2.17 tells us that $\iota\mathbf{F}$ are fppf sheaves since the formation of maximal subgroupoids preserves limits.

By construction, there is canonical morphism of prestacks $u\colon \mathbf{Filt}_P \to \mathbf{Cat}$ given by taking the colimit.

Proposition 2.19. *The fibers of $u\colon \mathbf{Filt}_P \to \mathbf{Cat}$ are posets.*

Proof. This is an immediate consequence of Lemma 2.13 since the fiber over $\mathcal{C}$ is precisely $\mathbf{ExFilt}_P^{\mathcal{C}}$. □

Corollary 2.20. *The sheaves of spaces $\iota\mathbf{Filt}_P^{\mathcal{C}}$ and $\iota\mathbf{ExFilt}_P^{\mathcal{C}}$ are 0-truncated.*

In other words, these are sheaves of sets.

2.5 Filtrations on quotient stacks

As an application of our methods, we will prove the following result which is a version of a theorem of Elagin in [19].

Theorem 2.21. *Let P be a poset, S a qcqs scheme. Let G be a flat affine algebraic S-group scheme of finite presentation and let X be a qcqs S-scheme with an action of G. Let $\mathrm{F}_\star\mathcal{P}\mathrm{erf}(X) \to \mathcal{P}\mathrm{erf}(X)$ be a P-filtration. If G preserves the filtration on $\mathcal{P}\mathrm{erf}(X)$, then there is an induced filtration $\mathrm{F}_\star\mathcal{P}\mathrm{erf}([X/G]) \to \mathcal{P}\mathrm{erf}([X/G])$.*

Example 2.22. Let S be a qcqs scheme and let $\mathbb{G}_{m,S}$ act on $\mathbb{P}^1_S$ with some weight; we will suppress the base S for this discussion. Consider the [1]-shaped Beilinson filtration defined on $\mathcal{P}\mathrm{erf}(\mathbb{P}^1)$ via

$$0 \mapsto \mathrm{F}_0\mathcal{P}\mathrm{erf}(\mathbb{P}^1) = \langle \mathcal{O} \rangle \qquad 1 \mapsto \mathrm{F}_1\mathcal{P}\mathrm{erf}(\mathbb{P}^1) = \langle \mathcal{O}, \mathcal{O}(1) \rangle;$$

this filtration will be discussed more extensively in Section 5.1. Since any automorphism of $\mathbb{P}^1$ preserves $\mathcal{O}(1)$, Theorem 2.21 implies that there is a [1]-shaped filtration on perfect complexes over the stack $[\mathbb{P}^1/\mathbb{G}_m]$ (concretely, $\mathbb{G}_m$-equivariant complexes on $\mathbb{P}^1$) which is compatible with the quotient map $p\colon \mathbb{P}^1 \to [\mathbb{P}^1/\mathbb{G}_m]$ in the sense that the filtration $\mathrm{F}_i\mathcal{P}\mathrm{erf}([\mathbb{P}^1/\mathbb{G}_m])$ pulls back under p^* to $\mathrm{F}_i\mathcal{P}\mathrm{erf}(\mathbb{P}^1)$. Combined with Corollary 3.18 we see that the semiorthogonal decomposition on $\mathcal{P}\mathrm{erf}(\mathbb{P}^1)$ constructed by Beilinson induces a semiorthogonal decomposition of perfect complexes on the quotient stack. Specifically, the graded pieces of the filtration above are each given by

$$\mathrm{gr}_i\mathcal{P}\mathrm{erf}([\mathbb{P}^1/\mathbb{G}_m]) \simeq \mathcal{P}\mathrm{erf}(\mathbf{B}\mathbb{G}_m),$$

which is the derived category of perfect graded complexes, for $i = 0, 1$.

Remark 2.23. Recall that a pure complex Hodge structure of weight i is a graded finite dimensional $\mathbb{C}$-vector space.[8] If we specialize Example 2.22 to the case where $S = \operatorname{Spec}\mathbb{C}$, then $\mathrm{gr}_i\mathcal{P}\mathrm{erf}([\mathbb{P}^1/\mathbb{G}_m])$ is naturally the bounded derived ∞-category of pure complex Hodge structures of weight i; in particular, there is a t-structure whose heart is the abelian category of pure complex Hodge structures of weight i. In order to get pure complex Hodge structures of weight $i \neq 0, 1$, one can simply twist Example 2.22 and consider the [1]-shaped filtration given by

$$0 \mapsto \mathrm{F}_0\mathcal{P}\mathrm{erf}(\mathbb{P}^1) = \langle \mathcal{O}(i) \rangle \qquad 1 \mapsto \mathrm{F}_1\mathcal{P}\mathrm{erf}(\mathbb{P}^1) = \langle \mathcal{O}(i), \mathcal{O}(i+1) \rangle.$$

One can make a similar construction in the more interesting case where $S = \operatorname{Spec}\mathbb{R}$, $X \subseteq \mathbb{P}^2_{\mathbb{R}}$ is the conic curve

$$x^2 + y^2 + z^2 = 0$$

(which has no $\mathbb{R}$-points). In this case, X is a twisted form of $\mathbb{P}^1_{\mathbb{R}}$ and there is an action of an algebraic group $\mathrm{U}(1)$ acting on $\mathbb{P}^1_{\mathbb{R}}$, where $\mathrm{U}(1)$ is a non-split form of $\mathbb{G}_m$ with $\mathbb{R}$-points $\mathrm{U}(1)(\mathbb{R}) \subseteq \mathrm{U}(1)(\mathbb{C}) \cong \mathbb{C}^\times$ given by $S^1 \subseteq \mathbb{C}^\times$. Thus, $[X/\mathrm{U}(1)]$ is an Artin stack over $\operatorname{Spec}\mathbb{R}$ and it is a twisted form of $[\mathbb{P}^1/\mathbb{G}_m]$. There is a [1]-shaped Beilinson filtration on $\mathcal{P}\mathrm{erf}(X)$ which is a twisted form of the Beilinson filtration on $\mathcal{P}\mathrm{erf}(\mathbb{P}^1)$. The graded pieces are given by $\mathrm{gr}_0\mathcal{P}\mathrm{erf}(X) \simeq \mathcal{P}\mathrm{erf}(\mathbb{R})$ and $\mathrm{gr}_1\mathcal{P}\mathrm{erf}(X) \simeq \mathcal{P}\mathrm{erf}(\mathbb{H})$, where $\mathbb{H}$ is the quaternion algebra over $\mathbb{R}$. Again, $\mathrm{U}(1)$ preserves the filtration and Theorem 2.21 on the quotient stack we get a [1]-shaped filtration, which will give rise to a semiorthogonal decomposition by Corollary 3.18, with graded piece

$$\mathrm{gr}_i\mathcal{P}\mathrm{erf}([X/\mathrm{U}(1)])$$

given by the bounded derived ∞-category of pure real Hodge structures of weight i. In particular, there is a bounded t-structure on each graded piece with heart the abelian category of pure real Hodge structures of weight i. To obtain the picture for $i \neq 0, 1$ one takes Serre twists as above. For details on this perspective on Hodge theory, see the work of Simpson, for example [37, Lemma 19].

[8] Indeed a pure complex Hodge structure is a bigraded vector space $V^{*,*}$ and its weight i part is the subspace given by those $V^{p,q}$ such that $p+q = i$. Up to reindexing, a pure weight i complex Hodge structure is thus a graded finite dimensional $\mathbb{C}$-vector space. See [17, Section 2.1].

Thanks to Corollary 2.20, we will see that the Theorem 2.21 is a consequence of basic covering space theory. Let $\mathbf{B}G$ be the classifying stack $[\operatorname{Spec} S/G]$. It is 1-affine by Warning 2.5. Since $\mathbf{Cat}$ is an fppf stack by Theorem 2.16, there is a morphism of stacks[9]

$$\mathbf{B}G \to \mathbf{Cat}$$

classifying the G-action on $\mathcal{P}\mathrm{erf}(X)$. Now, suppose that we have a P-filtration $\mathrm{F}_\star\mathcal{P}\mathrm{erf}(X) \to \mathcal{P}\mathrm{erf}(X)$. This is equivalent to giving a commutative diagram of stacks

$$\begin{array}{ccc} * & \longrightarrow & \mathbf{Filt}_P \\ \downarrow & & \downarrow \\ \mathbf{B}G & \longrightarrow & \mathbf{Cat}. \end{array} \tag{2}$$

Lemma 2.24. *A lift $\mathbf{B}G \to \mathbf{Filt}_P$ filling in the diagram (2) exists if and only if a lift in*

$$\begin{array}{ccc} & & \pi_0\mathbf{Aut}_{\mathbf{Filt}_P}(\mathrm{F}_\star\mathcal{P}\mathrm{erf}(X) \to \mathcal{P}\mathrm{erf}(X)) \\ & \nearrow & \downarrow \\ G & \longrightarrow & \pi_0\mathbf{Aut}_{\mathbf{Cat}}(\mathcal{P}\mathrm{erf}(X)) \end{array} \tag{3}$$

exists in the category of fppf sheaves of groups on Aff_S*.*

Proof. The "only if" direction is clear. Suppose that the filler in (3) exists. We may assume that $S = X$. The horizontal arrows in (2) factor through the underlying sheaves of spaces so we obtain a commutative diagram

$$\begin{array}{ccc} * & \longrightarrow & \iota\mathbf{Filt}_{P,\mathrm{F}_\star\mathcal{P}\mathrm{erf}(X)\to\mathcal{P}\mathrm{erf}(X)} \\ \downarrow & & \downarrow p \\ \mathbf{B}G & \longrightarrow & \iota\mathbf{Cat}_{\mathcal{P}\mathrm{erf}(X)}, \end{array} \tag{4}$$

where $\iota\mathbf{Cat}_{\mathcal{P}\mathrm{erf}(X)}$ (resp. $\iota\mathbf{Filt}_{P,\mathrm{F}_\star\mathcal{P}\mathrm{erf}(X)\to\mathcal{P}\mathrm{erf}(X)}$) denotes the connected component of $\iota\mathbf{Cat}$ (resp. $\iota\mathbf{Filt}_P$) corresponding to the base point $\mathcal{P}\mathrm{erf}(X)$ (resp. $\mathrm{F}_\star\mathcal{P}\mathrm{erf}(X) \to \mathcal{P}\mathrm{erf}(X)$) so it suffices to solve this lifting problem. By Corollary 2.20, the right hand vertical map is 0-truncated and so covering space theory in the ∞-topos of fppf sheaves on Aff_S tells us that the existence of a filler is assured if the map $G \to \pi_1(\iota\mathbf{Cat}_{\mathcal{P}\mathrm{erf}(X)}, \mathcal{P}\mathrm{erf}(X))$ factors through

$$p_*(\pi_1(\iota\mathbf{Filt}_{P,\mathrm{F}_\star\mathcal{P}\mathrm{erf}(X)\to\mathcal{P}\mathrm{erf}(X)}, \mathrm{F}_\star\mathcal{P}\mathrm{erf}(X) \to \mathcal{P}\mathrm{erf}(X))) \subset \pi_1(\iota\mathbf{Cat}_{\mathcal{P}\mathrm{erf}(X)}, \mathcal{P}\mathrm{erf}(X)).$$

This is exactly the existence of a lift as in (3). □

Proof of Theorem 2.21. The statement of Theorem 2.21 asserts the existence of a filtration on $\mathcal{P}\mathrm{erf}([X/G])$. The assumption guarantees by Lemma 2.24 that there is a point in $\mathbf{Filt}_P(\mathbf{B}G)$ lying over $\mathcal{P}\mathrm{erf}(X) \in \mathbf{Cat}(\mathbf{B}G)$. In other words, $\mathrm{F}_\star\mathcal{P}\mathrm{erf}(X) \to \mathcal{P}\mathrm{erf}(X)$ admits the structure of a G-equivariant filtration. Now, applying homotopy G-fixed points, we obtain a filtration $(\mathrm{F}_\star\mathcal{P}\mathrm{erf}(X))^{hG} \to \mathcal{P}\mathrm{erf}(X)^{hG} \simeq \mathcal{P}\mathrm{erf}([X/G])$, as desired. □

[9] Here, we use that $\mathbf{B}G$ is the quotient stack in the fppf topology, following the conventions of [38, Tag 044O].

Remark 2.25. Theorem 2.21 is somewhat surprising since *a priori* it involves manipulating higher-categorical objects which usually involves an infinite list of coherent descent data. The explanation that this is not necessary for filtrations is given by Lemma 2.13. This also explains why Elagin in [19] could stay within the realm of triangulated categories which usually does not interact well with descent problems.

3 Semiorthogonal decompositions

The previous sections dealt with general filtrations. Now, we deal with semiorthogonal decompositions in the sense of [14]. In particular, we prove Theorem 1.4 via Theorem 3.12, which says that a subcategory $\mathcal{A} \subseteq \mathcal{C}$ is admissible if and only if it is fppf-locally admissible.

3.1 Admissibility

Let S be a 1-affine algebraic stack. We review in this section the definitions and standard facts about admissible subcategories.

Definition 3.1 ([14])**.** Let $\mathcal{A} \subseteq \mathcal{C}$ be a fully faithful inclusion of $\mathcal{O}_S$-linear categories. We say that $\mathcal{A}$ is **right-admissible** in $\mathcal{C}$ if the inclusion admits an $\mathcal{O}_S$-linear right adjoint. Similarly, $\mathcal{A}$ is **left-admissible** in $\mathcal{C}$ if the inclusion admits an $\mathcal{O}_S$-linear left adjoint. If the inclusion admits both adjoints, we say that $\mathcal{A} \subseteq \mathcal{C}$ is **admissible**.

Remark 3.2. In the case of greatest interest, $\mathcal{C}$ will be dualizable (i.e., smooth and proper) as an $\mathcal{O}_S$-linear category in which case the three notions of admissibility for a full subcategory $\mathcal{A} \subseteq \mathcal{C}$ agree, and are all furthermore equivalent to the smoothness of $\mathcal{A}$.

Definition 3.3. If $\mathcal{A} \subseteq \mathcal{C}$ is an inclusion of stable ∞-categories, then the left (resp. right) orthogonal of $\mathcal{A}$, denoted by ${}^{\perp}\mathcal{A}$ (resp. $\mathcal{A}^{\perp}$) is the full subcategory of $\mathcal{C}$ spanned by objects $y \in \mathcal{C}$ such that $\mathbf{Map}_{\mathcal{C}}(y, x)$ (resp. $\mathbf{Map}_{\mathcal{C}}(x, y)$) is contractible for all $x \in \mathcal{A}$. If the ambient stable ∞-category is ambiguous, we will write $({}^{\perp}\mathcal{A})_{\mathcal{C}}$ (resp. $(\mathcal{A}^{\perp})_{\mathcal{C}}$) to avoid confusion.

The next well-known proposition furnishes a list of checkable criteria for right-admissibility.

Proposition 3.4 ([14])**.** *Let S be a 1-affine algebraic stack. Suppose that $i\colon \mathcal{A} \subseteq \mathcal{C}$ is a fully faithful $\mathcal{O}_S$-linear functor of $\mathcal{O}_S$-linear categories. Then, the following conditions on i are equivalent.*

(1) *For every x in $\mathcal{C}$ there is a cofiber sequence $y \to x \to z$ where $y \in \mathcal{A}$ and $z \in \mathcal{A}^{\perp}$.*

(2) *There is a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ on $\mathcal{C}$ for which $\mathcal{C}_{\geqslant 0} \simeq \mathcal{A}$.*

(3) *The functor i admits a right adjoint.*

(4) *The inclusion $i'\colon \mathcal{A}^{\perp} \subseteq \mathcal{C}$ admits a left adjoint.*

(5) *The composition $\mathcal{A}^{\perp} \xrightarrow{i'} \mathcal{C} \to \mathcal{C}/\mathcal{A}$ is an equivalence.*

Furthermore, the adjoints appearing in (3) and (4) are automatically $\mathcal{O}_S$-linear.

Proof. One reference for most of the implications is [14, Section 1], but we will sketch the arguments for completeness. The implication (1) $\Rightarrow$ (2) is immediate from the definition of a t-structure [10, Définition 1.3.1].[10] The implication (2) $\Rightarrow$ (3) is given by [10, Proposition 1.3.3]. Let us prove that (3) and (4) are equivalent. Since $i\colon \mathcal{A} \hookrightarrow \mathcal{C}$ has a right adjoint R, we can cook up an endofunctor of $\mathcal{C}$ by the formula

$$L\colon X \mapsto \operatorname{cofib}(iRX \to X)$$

given by taking the counit of the adjunction. The functor L takes X to $\mathcal{A}^\perp$ since, for any $Y \in \mathcal{A}$ we have a cofiber sequence

$$\mathbf{Map}_{\mathcal{C}}(iY, iRX) \to \mathbf{Map}_{\mathcal{C}}(iY, X) \to \mathbf{Map}_{\mathcal{C}}(iY, LX)$$

where the first arrow is an equivalence, hence the last term is contractible. The check that L is indeed the left adjoint is standard. Conversely, if L is a left adjoint to the inclusion $i'\colon \mathcal{A}^\perp \to \mathcal{C}$, then we define the right adjoint as

$$R\colon X \mapsto \operatorname{fib}(X \to i'LX).$$

A similar argument shows that R is right adjoint to i.

Now we prove that (4) implies (5). Let i_0 denote the composition $i_0\colon \mathcal{A}^\perp \to \mathcal{C} \to \mathcal{C}/\mathcal{A}$ and observe that the adjoint $L\colon \mathcal{C} \to \mathcal{A}^\perp$ of (4) vanishes on $\mathcal{A}$ and hence factors through $\mathcal{C}/\mathcal{A}$ to define a functor $L_0\colon \mathcal{C}/\mathcal{A} \to \mathcal{A}^\perp$ [32, Theorem 1.3.3(i)] which can be checked to be the left adjoint to i_0, and fits into the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{L} & \mathcal{A}^\perp \\ \downarrow & \nearrow_{L_0} & \\ \mathcal{C}/\mathcal{A} & & . \end{array} \tag{5}$$

Since L and the projection $\mathcal{C} \to \mathcal{C}/\mathcal{A}$ are essentially surjective, it is easy to see that L_0 is as well. It remains to show that L_0 is fully faithful. Suppose that $X, Y \in \mathcal{C}$ with images $\overline{X}, \overline{Y} \in \mathcal{C}/\mathcal{A}$. Since (4) implies (3), the existence of a right adjoint to i tells us that the filtered category $\mathcal{A}_{/Y}$ admits a final object, namely, RY. From this we compute, using the formula for mapping space in Verdier quotients [32, Theorem 1.3.3(ii)]

$$\begin{aligned} \mathbf{Map}_{\mathcal{C}/\mathcal{A}}(\overline{X}, \overline{Y}) &\simeq \operatorname*{colim}_{Z \in \mathcal{A}_{/LY}} \mathbf{Map}_{\mathcal{C}}(X, \operatorname{cofib}(Z \to Y)) \\ &\simeq \mathbf{Map}_{\mathcal{C}}(X, \operatorname{cofib}(iRY \to Y)) \\ &\simeq \mathbf{Map}_{\mathcal{C}}(X, LY) \\ &\simeq \mathbf{Map}_{\mathcal{C}/\mathcal{A}}(\overline{X}, \overline{LY}). \end{aligned}$$

This proves that (4) implies (5). Now (1) follows from (5) by the standard machinery of Bousfield localization.

The fact that $\mathcal{P}\mathrm{erf}(R)$ is rigid symmetric monoidal, meaning that every object is dualizable, implies that the adjoints above, if they exist, are automatically R-linear; see for example [24, Proposition 4.9(3)]. □

[10] Note that we are working with homological instead of cohomological indexing.

Example 3.5. The requirement that an inclusion be right or left-admissible is very strong. Let k be a field and let $\mathbb{P}^1 = \mathbb{P}^1_k$. If H is a hyperplane in $\mathbb{P}^1$ with complement U, then $\mathcal{O}_H$ is naturally an object of $\mathcal{P}\mathrm{erf}(\mathbb{P}^1)$. Let $\langle H\rangle$ denote the thick k-linear subcategory of $\mathcal{P}\mathrm{erf}(\mathbb{P}^1)$ generated by $\mathcal{O}_H$. The quotient of $\mathcal{P}\mathrm{erf}(\mathbb{P}^1)$ by $\langle H\rangle$ is $\mathcal{P}\mathrm{erf}(U)$. However, it is clear that there can be no fully faithful functor $\mathcal{P}\mathrm{erf}(U) \to \mathcal{P}\mathrm{erf}(\mathbb{P}^1)$ because the mapping spectra in $\mathcal{P}\mathrm{erf}(\mathbb{P}^1)$ are perfect complexes.

3.2 Semiorthogonal decompositions

These were first introduced by Bondal [13] and Bondal–Kapranov [14]. We would like to define semiorthogonal decompositions which are indexed not just by Δ^n or $\mathbb{Z}$ but also by a poset P. The next definition is a naive generalization of the definition that appears in [14, Definition 4.1]:

Definition 3.6. Let $\mathcal{C}$ be an $\mathcal{O}_S$-linear category, let P be a poset, and consider a P-shaped filtration $\mathrm{F}_\star\mathcal{C} \to \mathcal{C}$. We say that the filtration is **admissible** (resp. **right-admissible, left-admissible**) if for every arrow $p \to q$ in P the fully faithful embedding $\mathrm{F}_p\mathcal{C} \hookrightarrow \mathrm{F}_q\mathcal{C}$ is admissible (resp. right-admissible, left-admissible).

We say that an (right, left) admissible filtration $\mathrm{F}_\star\mathcal{C} \to \mathcal{C}$ is a **P-shaped (right, left) semiorthogonal decomposition of** $\mathcal{C}$ if

(1) the filtration is exhaustive and

(2) for each arrow $p \in P$, the subcategory $\mathrm{F}_p\mathcal{C} \subseteq \mathcal{C}$ is (right, left) admissible.

Remark 3.7. In practice, while admissible P-shaped filtrations come up in many situations, P-shaped semiorthogonal decompositions occur usually when P is filtered. Moreover, in this case, each $\mathrm{F}_p\mathcal{C} \subseteq \mathcal{C}$ is automatically admissible (resp. right-admissible, left-admissible) as well by [14, Proposition 4.4], i.e., condition (2) above is superfluous.

We make a comparison between Definition 3.6 with a notion that appears in textbook references (e.g. [25, Definition 1.59]), at least in the case of the finite ordered set $[n]$. See also [14, Proposition 4.4].

Proposition 3.8. *Let $\mathcal{C}$ be an $\mathcal{O}_S$-linear category. Then the following data are equivalent:*

(1) *an $[n]$-shaped semiorthogonal decomposition $\mathrm{F}_\star\mathcal{C}$ of $\mathcal{C}$.*

(2) *A collection of admissible small $\mathcal{O}_S$-linear full subcategories $\{\mathcal{C}_i\}_{0\leqslant i\leqslant n}$ such that*

 (a) *for any $i \leqslant j$, $\mathbf{Map}_{\mathcal{C}}(c_j, c_i) \simeq 0$ or all $c_i \in \mathcal{C}_i$, $c_j \in \mathcal{C}_j$, i.e., $\mathcal{C}_i \subset (\mathcal{C}_j)^\perp$ and*

 (b) *the smallest stable subcategory containing the $\mathcal{C}_i$'s is all of $\mathcal{C}$.*

Proof. It suffices to consider the case of $n = 1$. In this case, given a [1]-shaped semiorthogonal decomposition $\mathrm{F}_0\mathcal{C} \hookrightarrow \mathrm{F}_1\mathcal{C} \simeq \mathcal{C}$ we can take the Verdier quotient $\frac{\mathrm{F}_1\mathcal{C}}{\mathrm{F}_0\mathcal{C}}$ which is canonically equivalent to the right orthogonal of $\mathrm{F}_0\mathcal{C}$ in $\mathrm{F}_1\mathcal{C}$ by Proposition 3.4 ((3)⇒(5) direction) and is an admissible subcategory of $\mathcal{C}$ by assumption. So the collection of admissible subcategories $\{\mathrm{F}_0\mathcal{C}, \frac{\mathrm{F}_1\mathcal{C}}{\mathrm{F}_0\mathcal{C}}\}$ satisfy 2(a). Since the filtration is assumed to be exhaustive, we get 2(b).

Now assume that we have a collection $\{\mathcal{C}_0, \mathcal{C}_1\}$ as in (2). We define the filtration $\mathcal{C}(\star) := \mathcal{C}_0 \to \langle\mathcal{C}_0, \mathcal{C}_1\rangle$ where the $\langle\mathcal{C}_0, \mathcal{C}_1\rangle$ indicate the smallest stable ∞-category containing both $\mathcal{C}_0$ and $\mathcal{C}_1$. Then the inclusion $\mathcal{C}_0 \to \langle\mathcal{C}_0, \mathcal{C}_1\rangle$ has $\mathcal{C}_1$ as the Verdier quotient and Proposition 3.4 ((5)⇒(3) direction) tells us the inclusion does have a right adjoint. The fact that this filtration is exhaustive follows from 2(b), i.e., the equivalence $\langle\mathcal{C}_0, \mathcal{C}_1\rangle \simeq \mathcal{C}$. □

Remark 3.9. If $\mathcal{C}$ is a stable ∞-category, we denote by $\mathrm{K}(\mathcal{C})$ the algebraic K-theory spectrum of $\mathcal{C}$. An $[n]$-shaped semiorthogonal decomposition $\mathcal{C}(\star) \to \mathcal{C}$ induces a decomposition $\mathrm{K}(\mathcal{C}) \simeq \prod_{p=0}^{n} \mathrm{K}(\frac{\mathrm{F}_p\mathcal{C}}{\mathrm{F}_{p-1}\mathcal{C}})$. However, for a general poset P such a decomposition on K-theory is *not* guaranteed.

3.3 Marked variants

We also want to discuss marked variants of semiorthogonal decompositions. Of greatest interest are markings by exceptional objects.

Definition 3.10. Suppose that $\mathcal{C}$ is a small $\mathcal{O}_S$-linear stable ∞-category. An **exceptional object** of $\mathcal{C}$ is an object $e \in \mathcal{C}$ such that $\mathbf{Map}_{\mathcal{C}}(e, e) \simeq \mathcal{O}_S$ as an $\mathcal{O}_S$-algebra.[11] In this case, the thick $\mathcal{O}_S$-linear subcategory of $\mathcal{C}$ generated by e is equivalent to $\mathcal{P}\mathrm{erf}(S)$.

Let P be a poset. A collection of objects $\{e_p\}_{p\in P}$ of $\mathcal{C}$ is an **exceptional sequence** if each e_p is an exceptional object and if $\mathbf{Map}_{\mathcal{C}}(e_q, e_p) \simeq 0$ for $p < q$. The exceptional sequence is called **full** if the objects generate $\mathcal{C}$ as an $\mathcal{O}_S$-linear category. If R is an ordinary commutative ring, then an exceptional sequence in an R-linear category is called **strong** if $\mathrm{Hom}_{\mathcal{C}}(e_p, e_q[n]) = 0$ for $n \neq 0$ and all $p, q \in P$.

Definition 3.11. Let $\mathcal{C}$ be an $\mathcal{O}_S$-linear stable ∞-category. Then a marked P-shaped (right, left) admissible filtration $(\mathrm{F}_\star\mathcal{C} \to \mathcal{C}, M_*)$ is **exceptional** if for each $p \in P$ the object M_p is exceptional. If $P = [n]$, the admissible filtration is an $[n]$-shaped semiorthogonal decomposition, and each M_p is contained in $(\mathrm{F}_{p-1}\mathcal{C})^{\perp}_{\mathrm{F}_p\mathcal{C}}$, we say that $(\mathrm{F}_\star\mathcal{C} \to \mathcal{C}, M_*)$ is an **exceptionally marked** $[n]$**-shaped semiorthogonal decomposition**.

Following [3], we could also study $\mathcal{A}$-exceptional objects for any Azumaya $\mathcal{O}_S$-algebra $\mathcal{A}$. For these, we require $\mathbf{Map}_{\mathcal{C}}(e, e) \simeq \mathcal{A}$ so that e generates a subcategory of $\mathcal{C}$ equivalent to $\mathcal{P}\mathrm{erf}(\mathcal{A})$. We could then define a version of $\{\mathcal{A}_p\}_{p\in P}$-exceptional sequences. Call these **twisted exceptional objects**. We will use the obvious notion of a **full twisted exceptional collection**, which gives rise to the notion of a twisted exceptionally marked semiorthogonal decomposition, whose formulation we leave to the reader.

3.4 The local nature of admissibility

We now prove the following result, which is the main technical theorem of the paper. If S is an algebraic stack, $\mathcal{C}$ is an $\mathcal{O}_S$-linear category, and $T \to S$ is a morphism of algebraic stacks, we let $\mathcal{C}_T = \mathcal{P}\mathrm{erf}(T) \otimes_{\mathcal{P}\mathrm{erf}(S)} \mathcal{C}$.

Theorem 3.12. *Let S be a 1-affine algebraic stack. Suppose that $\mathcal{C}$ is an $\mathcal{O}_S$-linear ∞-category and let $\mathcal{A} \subseteq \mathcal{C}$ be an $\mathcal{O}_S$-linear inclusion of a full subcategory. If $U \to S$ is an fppf cover, then $\mathcal{A} \to \mathcal{C}$ is right-admissible if and only if $\mathcal{A}_U \to \mathcal{C}_U$ is right-admissible. The same holds for left-admissibility and admissibility.*

The next proposition highlights the importance of working in an enhanced setting, as opposed to just with triangulated categories. It states that the notion of right-admissibility is stable under (derived) base change. This generalizes [27, Theorem 5.6].

[11] Here we abuse notation a little and view $\mathbf{Map}_{\mathcal{C}}(x, x)$ as a perfect complex on S.

Proposition 3.13. *Suppose that $i\colon \mathcal{A} \subseteq \mathcal{C}$ are $\mathcal{O}_S$-linear categories, and assume that i is right-admissible. If $\mathcal{F}$ is another $\mathcal{O}_S$-linear stable ∞-category, then*

$$\mathcal{F} \otimes i\colon \mathcal{F} \otimes_{\mathcal{P}\mathrm{erf}(S)} \mathcal{A} \to \mathcal{F} \otimes_{\mathcal{P}\mathrm{erf}(S)} \mathcal{C}$$

is fully faithful and right-admissible. A similar statement holds for left-admissibility.

Proof. Let $r\colon \mathcal{C} \to \mathcal{A}$ be the right adjoint to i. To say that r and i are adjoint is the same as giving a natural transformation $\mathrm{id}_{\mathcal{A}} \to r \circ i$ such that for each $a \in \mathcal{A}$ and $b \in \mathcal{C}$ the induced composition

$$\eta_{a,b}\colon \mathbf{Map}_{\mathcal{C}}(i(a), b) \to \mathbf{Map}_{\mathcal{A}}(ri(a), r(b)) \to \mathbf{Map}_{\mathcal{A}}(a, r(b))$$

is an equivalence. Moreover, if i and r are adjoint, i is fully faithful if and only if the unit natural transformation is an equivalence.

Let $i_{\mathcal{F}}\colon \mathcal{F} \otimes_{\mathcal{P}\mathrm{erf}(S)} \mathcal{A} \to \mathcal{F} \otimes_{\mathcal{P}\mathrm{erf}(S)} \mathcal{C}$ and $r_{\mathcal{F}}\colon \mathcal{A} \otimes_{\mathcal{P}\mathrm{erf}(S)} \mathcal{C} \to \mathcal{F} \otimes_{\mathcal{P}\mathrm{erf}(S)} \mathcal{A}$ be the maps induced by i and r by functoriality of the tensor product. We have an induced natural isomorphism $\mathrm{id}_{\mathcal{F} \otimes_{\mathcal{P}\mathrm{erf}(S)} \mathcal{A}} \to r_{\mathcal{F}} \circ i_{\mathcal{F}}$ (since $\mathrm{id}_{\mathcal{A}} \simeq r \circ i$). Thus, to prove the lemma, it is enough to prove that $i_{\mathcal{F}}$ and $r_{\mathcal{F}}$ are adjoint, since then fully faithfulness then follows from the discussion above.

Consider for $a \in \mathcal{F} \otimes_{\mathcal{P}\mathrm{erf}(S)} \mathcal{A}$ and $b \in \mathcal{F} \otimes_{\mathcal{P}\mathrm{erf}(S)} \mathcal{C}$ the induced map

$$\mathbf{Map}_{\mathcal{F}\otimes\mathcal{C}}(i_{\mathcal{F}}(a), b) \to \mathbf{Map}_{\mathcal{F}\otimes\mathcal{A}}(r_{\mathcal{F}} i_{\mathcal{F}}(a), r_{\mathcal{F}}(b)) \to \mathbf{Map}_{\mathcal{F}\otimes\mathcal{A}}(a, r_{\mathcal{F}}(b)). \tag{6}$$

We would like to show that this is an equivalence. Suppose that $a = x \otimes y$ and $b = w \otimes z$ are pure tensors, i.e., $x, w \in \mathcal{F}$, $y \in \mathcal{A}$, and $z \in \mathcal{C}$. Then, we find the map is equivalent to $\mathbf{Map}_{\mathcal{F}}(x, w) \otimes \eta_{y,z}$ since, for example,

$$\mathbf{Map}_{\mathcal{F}\otimes\mathcal{C}}(i_{\mathcal{F}}(x \otimes y), w \otimes z) \simeq \mathbf{Map}_{\mathcal{F}\otimes\mathcal{C}}(x \otimes i(y), w \otimes z) \simeq \mathbf{Map}_{\mathcal{F}}(x, w) \otimes_{\mathcal{O}_Y} \mathbf{Map}_{\mathcal{C}}(i(y), z).$$

Since $\eta_{y,z}$ is an equivalence, we see that (6) is an equivalence for pure tensors. Since the pure tensors generate $\mathcal{F} \otimes_{\mathcal{P}\mathrm{erf}(S)} \mathcal{A}$ and $\mathcal{F} \otimes_{\mathcal{P}\mathrm{erf}(S)} \mathcal{C}$, a standard thick subcategory argument proves that (6) is an equivalence for all $a \in \mathcal{F} \otimes \mathcal{A}$ and $b \in \mathcal{F} \otimes \mathcal{C}$. ☐

Proof of Theorem 3.12. The necessity of fppf local admissibility follows immediately from the stability of admissibility under base change by Proposition 3.13 with $\mathcal{F} = \mathcal{P}\mathrm{erf}(U)$. It suffices to prove the converse for right-admissibility, since $\mathcal{A} \subseteq \mathcal{C}$ is left-admissible if and only if $\mathcal{A}^{\mathrm{op}} \subseteq \mathcal{C}^{\mathrm{op}}$ is right-admissible.

Let $p\colon U \to S$ be an fppf cover and assume that $\mathcal{A}_U \to \mathcal{C}_U$ admits a right adjoint R. Let $\check{C}_\bullet(p)$ be the simplicial algebraic stack obtained by taking the Čech complex of p; i.e., $\check{C}_n(p) \cong U^{\times_S (n+1)}$. Taking $\mathcal{P}\mathrm{erf}(\check{C}_\bullet(p))$ we obtain a cosimplicial $\mathcal{O}_S$-linear category with $\mathcal{P}\mathrm{erf}(\check{C}_n(p)) \simeq \mathcal{P}\mathrm{erf}(U^{\times_S (n+1)})$. Tensoring with the inclusion $i\colon \mathcal{A} \to \mathcal{C}$, we obtain a natural transformation

$$\mathcal{A}_{\check{C}_\bullet(p)} \to \mathcal{C}_{\check{C}_\bullet(p)}$$

of cosimplicial $\mathcal{O}_S$-linear categories.

For simplicity, write $U_n = U^{\times_S (n+1)}$, the nth term in $\check{C}_\bullet(f)$. By Proposition 3.13, each $i^n\colon \mathcal{A}_{U_n} \to \mathcal{C}_{U_n}$ admits a right adjoint, say r^n.

Claim. For each $q\colon [m] \to [n]$ in Δ, the commutative square

$$\begin{array}{ccc} \mathcal{A}_{U_n} & \xrightarrow{i^n} & \mathcal{C}_{U_n} \\ \downarrow{\scriptstyle q^*_{\mathcal{A}}} & & \downarrow{\scriptstyle q^*_{\mathcal{C}}} \\ \mathcal{A}_{U_m} & \xrightarrow{i^m} & \mathcal{C}_{U_m} \end{array}$$

is right adjointable, i.e., the induced natural transformation

$$q^*_{\mathcal{A}} \circ r^n \to r^m \circ q^*_{\mathcal{C}}$$

is an equivalence.[12] Indeed, this follows from Lemma 3.14 below.

To prove that i admits a right adjoint, it suffices to check the object-wise criterion to be an adjoint by [28, 5.2.7.8]. Namely, it is enough to show that for each $x \in \mathcal{C}$ there exists an element $y \in \mathcal{A}$ and a map $y \to x$ such that for each $w \in \mathcal{A}$ the natural map $\mathbf{Map}_{\mathcal{A}}(w, y) \to \mathbf{Map}_{\mathcal{C}}(i(w), x)$ is an equivalence.

By unstraightening, we can view the functor $\mathcal{A}_{\check{C}_\bullet(p)}\colon \Delta \to \mathbf{Cat}(S)$ as classifying a Cartesian fibration

$$\widetilde{\mathcal{A}} = \int_\Delta \mathcal{A}_{\check{C}_\bullet(p)} \to \Delta^{\mathrm{op}}.$$

Given $x \in \mathcal{C}$ as above, the Beck–Chevalley transformations give $r^\bullet x$ the structure of a section $r(x)$ of $\widetilde{\mathcal{A}}$.[13] In other words, $r(x)$ is a kind of lax cosimplicial object. But, the claim above, that the squares are right adjointable, implies that this section is in fact Cartesian. Thus, $r(x)$ defines an object of $\mathcal{A}$, which is equivalent to the ∞-category of Cartesian sections of $\widetilde{\mathcal{A}} \to \Delta^{\mathrm{op}}$ by [28, Corollary 3.3.3.2].

We also have a map $ir(x) \to x$. Fix $w \in \mathcal{A}$. We have

$$\begin{aligned} \mathbf{Map}_{\mathcal{A}}(w, r(x)) &\simeq \lim_{\Delta} \mathbf{Map}_{\mathcal{A}_{U_\bullet}}(w_{U_\bullet}, r^\bullet(x_{U_\bullet})) \\ &\simeq \lim_{\Delta} \mathbf{Map}_{\mathcal{C}_{U_\bullet}}(\iota^\bullet(w_{U_\bullet}), x_{U_\bullet}) \\ &\simeq \mathbf{Map}_{\mathcal{C}}(i(w), x), \end{aligned}$$

as desired. □

[12] This natural transformation is called the Beck–Chevalley transformation and is constructed for example in [29, Definition 4.7.4.13].

[13] One way to make this precise is to use the classifying 2-category of adjunctions, Adj. Specifically, viewing $\mathbf{Cat}(S)$ as a 2-category where the mapping categories are the ∞-categories $\mathrm{Fun}^{\mathrm{ex}}(-,-)$ of exact $\mathcal{O}_S$-linear functors, we have a forgetful functor $\mathrm{Fun}(\mathrm{Adj}, \mathbf{Cat}(S)) \to \mathrm{Fun}(\Delta^1, \mathbf{Cat}(S))$, where on the left $\mathrm{Fun}(\mathrm{Adj}, \mathbf{Cat}(S))$ is the ∞-category of 2-categorical functors from Adj to $\mathbf{Cat}(S)$. We note that the morphisms in $\mathrm{Fun}(\mathrm{Adj}, \mathbf{Cat}(S))$ are exactly given by squares which are pointwise adjointable [20, Appendix A]. Now, the theorem of Riehl–Verity [33] implies that this functor is fully faithful with essential image exactly those objects of $\mathrm{Fun}(\Delta^1, \mathbf{Cat}(S))$ possessing an adjoint. Now, we can view i as defining a functor $\Delta \to \mathrm{Fun}(\Delta^1, \mathbf{Cat}(S))$. The pointwise adjointability, implies that this functor factors through the subcategory to give a functor $\Delta \to \mathrm{Fun}(\mathrm{Adj}, \mathbf{Cat}(S))$. If $\widetilde{\mathcal{C}} \to \Delta^{\mathrm{op}}$ denotes the unstraightening of $\mathcal{C}_{\check{C}_\bullet(p)}$, then this functor gives a functor $\widetilde{\mathcal{C}} \to \widetilde{\mathcal{A}}$ of ∞-categories over Δ^{op}. The object x defines a (Cartesian) section of $\widetilde{\mathcal{C}} \to \Delta^{\mathrm{op}}$ and we apply the functor to get a section of $\widetilde{\mathcal{A}} \to \Delta$.

Lemma 3.14. *Suppose that $f : U \to S$ be a morphism of schemes. Let $F : \mathcal{D} \to \mathcal{C}$ be a functor of $\mathcal{O}_S$-linear categories and consider the diagram*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle f^*_{\mathcal{D}}}\downarrow & & \downarrow{\scriptstyle f^*_{\mathcal{C}}} \\ \mathcal{D}_U & \xrightarrow{F_U} & \mathcal{C}_U. \end{array}$$

If F admits a right adjoint G, then the square above is right adjointable.

Proof. Suppose that $\{y_i\}$ is a collection of objects in $\mathcal{D}$ which generates $\mathcal{D}$. Then the collection $\{y_i \otimes \mathcal{O}_U\}$ generates $\mathcal{D}_U$. Hence to prove the claim, it suffices to prove that the canonical map

$$\mathbf{Map}_{\mathcal{D}_U}(y_i \otimes \mathcal{O}_U, f^*_{\mathcal{D}} Gx) \to \mathbf{Map}_{\mathcal{D}_U}(y_i \otimes \mathcal{O}_U, G_U f^*_{\mathcal{C}} x),$$

is an equivalence. This follows from the following computation

$$\begin{aligned} \mathbf{Map}_{\mathcal{D}_U}(y_i \otimes \mathcal{O}_U, f^*_{\mathcal{D}} Gx) &= \mathbf{Map}_{\mathcal{D}_U}(y_i \otimes \mathcal{O}_U, Gx \otimes \mathcal{O}_U) \\ &\simeq \mathbf{Map}_{\mathcal{D}}(y_i, Gx) \otimes \mathcal{O}_U \\ &\simeq \mathbf{Map}_{\mathcal{C}}(Fy_i, x) \otimes \mathcal{O}_U \\ &\simeq \mathbf{Map}_{\mathcal{C}_U}(F_U y_i, x \otimes \mathcal{O}_U) \\ &\simeq \mathbf{Map}_{\mathcal{D}_U}(y_i \otimes \mathcal{O}_U, G_U f^*_{\mathcal{C}} x). \end{aligned}$$

Here, the only nontrivial step is the equivalence in the second line which follows from the same argument as in [2, Lemma 2.7]. □

Definition 3.15. Let P be a poset. We denote by $\mathbf{Sod}_P$ the subprestack of $\mathbf{Filt}_P$ spanned by the P-shaped semiorthogonal decompositions. We call the prestack $\mathbf{Sod}_P$ the **stack of P-shaped semiorthogonal decompositions**.

Remark 3.16. Proposition 3.13 implies that $\mathbf{Sod}_P$ is indeed a prestack.

Now, we see that $\mathbf{Sod}_P$ is a stack, which proves Theorem 1.4.

Corollary 3.17. *The prestack $\mathbf{Sod}_P$ is an fppf stack.*

Proof. For each $\operatorname{Spec} R \to S$, we have an inclusion of connected components $\mathbf{Sod}_P(R) \subseteq \mathrm{Filt}_P(R)$. Since $\mathbf{Filt}_P$ is an fppf stack by Theorem 2.17, it suffices to check the effectiveness of descent, which is precisely Theorem 3.12. □

Corollary 3.18. *Let P be a poset, S a qcqs scheme. Let G be a flat affine algebraic S-group scheme of finite presentation and let X be a qcqs S-scheme with an action of G. Suppose that $\mathcal{P}\mathrm{erf}(X)$ admits a semiorthogonal decomposition $\mathrm{F}_\star\mathcal{C} \to \mathcal{C}$. If G preserves the filtration $\mathrm{F}_\star\mathcal{C} \to \mathcal{C}$, then the induced filtration $\mathrm{F}_\star\mathcal{P}\mathrm{erf}([X/G]) \to \mathcal{P}\mathrm{erf}([X/G])$ is a P-shaped semiorthogonal decomposition.*

Proof. This follows immediately from Theorem 3.12 since $X \to [X/G]$ is an fppf cover. □

Remark 3.19. Under the equivalence given by Proposition 3.8 between our form of semiorthogonal decompositions and the usual form, Corollary 3.18 gives a very general form of Elagin's theorem.

4 The twisted Brauer space for filtrations

We introduce in this section tools for constructing obstruction classes in cohomology attached to filtrations.

4.1 Twisted Brauer spaces: recollections

Throughout, S will be a 1-affine algebraic stack and $\mathcal{C}$ will be an $\mathcal{O}_S$-linear category. The twisted Brauer space of $\mathcal{C}$ is a topological space (or, really, simplicial set) $\mathbf{Br}^{\mathcal{C}}(S)$ whose points classify idempotent complete S-linear stable ∞-categories $\mathcal{D}$ which are fppf locally on S equivalent to $\mathcal{C}$, i.e., a **twisted form** of $\mathcal{C}$. Here are some salient features of twisted Brauer spaces; we refer to [1] for proofs. Note that [1] treats the étale case; thanks to Theorem 2.16, the same results work for the fppf analogue.

(i) If $\mathcal{C} = \mathcal{P}\mathrm{erf}(S)$, then $\mathbf{Br}^{\mathcal{C}} \simeq \mathbf{Br}$ is the fppf version of the Brauer space considered in [2, 40]; in particular

$$\pi_i \mathbf{Br}(S) \cong \begin{cases} \mathrm{H}^1_{\mathrm{fppf}}(S;\mathbb{Z}) \times \mathrm{H}^2_{\mathrm{fppf}}(S;\mathbb{G}_m) & \text{if } i=0, \\ \mathrm{H}^0_{\mathrm{fppf}}(S;\mathbb{Z}) \times \mathrm{H}^1_{\mathrm{fppf}}(S;\mathbb{G}_m) & \text{if } i=1, \\ \mathrm{H}^0_{\mathrm{fppf}}(S;\mathbb{G}_m) & \text{if } i=2, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

where the higher homotopy groups are computed at the basepoint $\mathcal{P}\mathrm{erf}(S) \in \mathbf{Br}(S)$.[14]

(ii) There is a one-to-one correspondence between $\mathcal{O}_S$-linear equivalence classes of $\mathcal{O}_S$-linear categories which are fppf locally equivalent to $\mathcal{C}$ and the elements of the set $\pi_0 \mathbf{Br}^{\mathcal{C}}(S)$. Given a point $\mathcal{D}$ of $\mathbf{Br}^{\mathcal{C}}(S)$, the higher homotopy groups of the space $\mathbf{Br}^{\mathcal{C}}(S)$ are given by

$$\pi_i(\mathbf{Br}^{\mathcal{C}}(S), \mathcal{D}) \cong \pi_{i-1}\mathbf{Aut}_{\mathcal{D}}(S),$$

where $\mathbf{Aut}_{\mathcal{D}}(S)$ is the space of derived $\mathcal{O}_S$-linear autoequivalences of $\mathcal{D}$.

(iii) The space $\mathbf{Aut}_{\mathcal{D}}(S)$ is the space of global sections of a sheaf of spaces $\mathbf{Aut}_{\mathcal{D}}$. If $S = \operatorname{Spec} R$ is affine, then the higher homotopy groups of this space are well-understood:

$$\pi_{i-1}\mathbf{Aut}_{\mathcal{D}}(S) \cong \begin{cases} \mathrm{HH}^0(\mathcal{D}/S)^\times & i=2, \\ \mathrm{HH}^{2-i}(\mathcal{D}/S) & i \geqslant 3, \end{cases}$$

where $\mathrm{HH}(\mathcal{D}/S)$ denotes the Hochschild cohomology of $\mathcal{D}$ relative to $S = \operatorname{Spec} R$. This is a result of Toën's [39, Corollary 1.6]. For non-affine S, one computes the space $\mathbf{Aut}_{\mathcal{D}}(S)$ via a local-global spectral sequence. Note that if $\mathcal{D} \simeq \mathcal{P}\mathrm{erf}(X)$ for some scheme X, then $\mathrm{HH}^{2-i}(\mathcal{D}/S) = 0$ for $i \geqslant 3$: schemes do not have non-zero negative Hochschild cohomology groups.

(iv) the twisted Brauer space is the space of S-sections of an fppf sheaf

$$\mathbf{Br}^{\mathcal{C}} \colon S_{\mathrm{fppf}} \to \mathcal{S}.$$

[14] Recall Grothendieck's theorem [23, Section 5, Théorème 11.7] that for a smooth affine group scheme G the natural map $\mathrm{H}^s_{\mathrm{\acute{e}t}}(S,G) \to \mathrm{H}^s_{\mathrm{fppf}}(S,G)$ is an isomorphism for all s.

Since every point of $\mathbf{Br}^{\mathcal{C}}$ is fppf-locally equivalent to $\mathcal{C}$, the sheaf $\mathbf{Br}^{\mathcal{C}}$ is connected as a sheaf of spaces. Thus, the twisted Brauer space is the classifying stack of the sheaf of groups $\mathbf{Aut}_{\mathcal{C}}$, formalizing the way in which it classifies twisted forms of $\mathcal{C}$. Here, the sheaf of groups should be understood in the homotopical context: $\mathbf{Aut}_{\mathcal{C}}$ is like a sheaf of H-spaces and is precisely a sheaf of what are called grouplike A_∞-spaces. Using this perspective, one can often enumerate twisted forms using descent spectral sequences; see [1] for some examples.

4.2 Twisted Brauer spaces for filtrations

We now consider a variant of twisted Brauer spaces where $\mathcal{C}$ is equipped with a filtration.

Definition 4.1. Let P be a poset and let $\mathrm{F}_\star\mathcal{C} \to \mathcal{C}$ be a P-shaped filtration on $\mathcal{C}$. We let $\mathbf{Br}^{\mathrm{F}_\star\mathcal{C}}$ be the fppf sheafification of the corresponding point of $\iota\mathbf{Filt}_P$. This is the **twisted Brauer space of the filtration** $\mathrm{F}_\star\mathcal{C} \to \mathcal{C}$.

Similarly, if $(\mathrm{F}_\star\mathcal{C} \to \mathcal{C}, M_*)$ is a marked P-shaped filtration, we let the **twisted Brauer space of the marked filtration** $\mathbf{Br}^{(\mathrm{F}_\star\mathcal{C}, M_*)}$ be the fppf sheafification of the corresponding point in $\iota\mathbf{MFilt}_P$.

Remark 4.2. We remark that setting $M_i = 0$ for all $i \in P$ is equivalent to having no markings on the filtration so that $\mathbf{Br}^{\mathrm{F}_\star\mathcal{C}} \simeq \mathbf{Br}^{(\mathrm{F}_\star\mathcal{C}, M_*)}$ in this case.

By construction both $\mathbf{Br}^{\mathrm{F}_\star\mathcal{C}}$ and $\mathbf{Br}^{(\mathrm{F}_\star\mathcal{C}, M_*)}$ are fppf sheaves on the big site which, *a priori*, take values in the ∞-category of large spaces. We will see in fact that they are sheaves of small spaces.

We record some basic properties of these gadgets bearing in mind Remark 4.2.

Proposition 4.3. *Let S be a 1-affine algebraic stack, $\mathrm{F}_\star\mathcal{C} \to \mathcal{C}$ a P-shaped filtration, and M_* a marking for the filtration. The fppf sheaves $\mathbf{Br}^{(\mathrm{F}_\star\mathcal{C}, M_*)}$ satisfy the following properties.*

(a) *For any qcqs S-scheme T, the space $\mathbf{Br}^{(\mathrm{F}_\star\mathcal{C}, M_*)}(T)$ is the space of marked filtered idempotent complete T-linear stable ∞-categories $(\mathrm{F}_\star\mathcal{C}, N_*)$ which are fppf-locally equivalent to $(\mathrm{F}_\star\mathcal{C} \otimes_{\mathcal{P}\mathrm{erf}(S)} \mathcal{P}\mathrm{erf}(T), M_* \otimes_{\mathcal{O}_S} \mathcal{O}_T)$.*

(b) *Let $\mathbf{Aut}_{(\mathrm{F}_\star\mathcal{C}, M_*)}$ denote the sheaf of automorphisms of the marked P-filtered stable ∞-category defined by $\mathrm{F}_\star\mathcal{C} \to \mathcal{C}$ and M_*. There is a natural equivalence*

$$\mathbf{BAut}_{(\mathrm{F}_\star\mathcal{C}, M_*)} \simeq \mathbf{Br}^{(\mathrm{F}_\star, M_*)}$$

of fppf sheaves.

(c) *There are forgetful maps*

$$\mathbf{Br}^{(\mathrm{F}_\star\mathcal{C}, M_*)} \to \mathbf{Br}^{\mathrm{F}_\star\mathcal{C}} \to \mathbf{Br}^{\mathcal{C}}$$

of fppf sheaves. A twisted form $\mathcal{D}$ of $\mathcal{C}$ giving a point of $\mathbf{Br}^{\mathcal{C}}(T)$ lifts to $\mathbf{Br}^{\mathrm{F}_\star\mathcal{C}}(T)$ (resp. $\mathbf{Br}^{(F_\star\mathcal{C}, M_*)}(T)$) if and only if the (marked) filtration on $\mathcal{C}$ descends to $\mathcal{D}$.

(d) *The morphism of sheaves $\mathbf{Br}^{\mathrm{F}_\star\mathcal{C}} \to \mathbf{Br}^{\mathcal{C}}$ is 0-truncated, i.e, the fibers are fppf sheaves of sets.*

(e) *On sheaves of fundamental groups, the inclusion*

$$\pi_1(\mathbf{Br}^{\mathrm{F}_\star\mathcal{C}}, \mathrm{F}_\star\mathcal{C}) \hookrightarrow \pi_1(\mathbf{Br}^{\mathcal{C}}, \mathcal{C}) \simeq \pi_0(\mathbf{Aut}_{\mathcal{C}}),$$

corresponds to the inclusion of those automorphisms of $\mathcal{C}$ *which preserve the filtration* $\mathrm{F}_\star\mathcal{C}$.

Proof. (a) Let $\mathcal{G} \subset \mathbf{MFilt}_P$ be the subpresheaf classifying objects $(\mathrm{F}_\star\mathcal{D} \to \mathcal{D}, M_*)$ which are fppf locally equivalent to $(\mathrm{F}_\star\mathcal{C} \to \mathcal{C}, M_*)$. By definition of $\mathbf{Br}^{(\mathrm{F}_\star\mathcal{C}, M_*)}$ the map $\mathbf{Br}^{(\mathrm{F}_\star\mathcal{C}, M_*)} \to \mathbf{MFilt}_P$ factors through the inclusion of $\mathcal{G}$. Furthermore, the subpresheaf $\mathcal{G}$ is actually an fppf sheaf since the condition of a section of $\mathbf{MFilt}_P$ being in $\mathcal{G}$ is fppf local. Since, fppf locally, the map $\mathbf{Br}^{(\mathrm{F}_\star\mathcal{C}, M_*)} \to \mathcal{G}$ is an equivalence on sections we get an equivalence of fppf sheaves.

(b) We note that the fppf sheaf of connected components $\pi_0(\mathbf{BAut}_{(\mathrm{F}_\star\mathcal{C}, M_*)})$ is equivalent to the terminal sheaf since, fppf locally, $(\mathrm{F}_\star\mathcal{C}, M_*)$ is the only equivalence class of objects. We thus have an inclusion of sheaves $\mathbf{BAut}_{(\mathrm{F}_\star\mathcal{C}, M_*)} \hookrightarrow \mathbf{Br}^{(\mathrm{F}_\star\mathcal{C}, M_*)}$ where the former is the maximal subgroupoid of the full subcategory spanned by the global base point, $(\mathrm{F}_\star\mathcal{C}, M_*)$. Hence to prove the claimed equivalence it suffices to apply Ω to both sides (which preserves sheaves). The resulting sheaf in both cases is the sheaf of automorphisms $\mathbf{Aut}_{(\mathrm{F}_\star\mathcal{C}, M_*)}$.

(c) This follows by definition.

(d) This follows from Lemma 2.13.

(e) Let the fiber $\mathbf{Br}^{\mathrm{F}_\star\mathcal{C}} \to \mathbf{Br}^{\mathcal{C}}$ be denoted by $\mathcal{F}$. From part (*d*), we have an exact sequence of sheaves:

$$0 \simeq \pi_2(\mathcal{F}) \to \pi_1(\mathbf{Br}^{\mathrm{F}_\star\mathcal{C}}, \mathrm{F}_\star\mathcal{C}) \hookrightarrow \pi_1(\mathbf{Br}^{\mathcal{C}}, \mathcal{C}) \simeq \pi_0(\mathbf{Aut}_{\mathcal{C}}),$$

where the last isomorphism is [1, Lemma 2.5]. The claim then follows from (b) and the definition of the forgetful maps.

□

Remark 4.4. Out of the above properties, property (d) is the most striking: it tells us that in order to descend a filtration on $\mathcal{C}$ to a filtration on its twisted form the obstruction is purely "discrete." This explains how previous results on descent for semiorthogonal decompositions, such as [19] or [8], could avoid the subtleties of gluing higher categorical objects (such as dg categories); see Theorem 4.6 for a precise statement.

We also note the following simple corollary, which says that there are not too many twisted forms.

Corollary 4.5. $\mathbf{Br}^{(\mathrm{F}_\star\mathcal{C}, M_*)}$ *is a sheaf of small spaces.*

Proof. After Proposition 4.3(b), the argument is the same as in [1, Proposition 2.6]. □

4.3 Descending filtrations on twisted forms

We will use the language of twisted Brauer spaces to explain the following phenomena: to check if filtrations on a scheme induce a compatible filtration on its twisted form one only needs to check 1-categorical compatibilities. This leads to computability of the obstructions in trying to descend semiorthogonal decompositions as illustrated in our examples in Section 5.

Theorem 4.6. *Let S be a 1-affine algebraic stack. Suppose that X and Y are two qcqs S-schemes. Let*

$$\mathrm{F}_\star \mathcal{P}\mathrm{erf}(X) \to \mathcal{P}\mathrm{erf}(X)$$

be a P-shaped filtration for some poset P. Assume further that

(i) *there is a surjective fppf morphism $T \to S$ and an isomorphism $\alpha\colon Y_T \to X_T$ and*

(ii) *the induced filtration $\mathrm{F}_\star \mathcal{P}\mathrm{erf}(Y_T) = \alpha^* q^* \mathrm{F}_\star \mathcal{P}\mathrm{erf}(X)$ on $\mathcal{P}\mathrm{erf}(Y_T)$ satisfies the cocycle condition: if $p_i\colon Y_{T\times_S T} \to Y_T$ are the two projections, then for each $p \in P$ the subcategories*

$$p_1^* \mathrm{F}_p \mathcal{P}\mathrm{erf}(Y_T) \quad \textit{and} \quad p_2^* \mathrm{F}_p \mathcal{P}\mathrm{erf}(Y_T)$$

of $\mathcal{P}\mathrm{erf}(Y_{T\times_S T})$ coincide.

Then, there is a filtration $\mathrm{F}_\star \mathcal{P}\mathrm{erf}(Y) \to \mathcal{P}\mathrm{erf}(Y)$ and an equivalence of filtrations

$$(\mathrm{F}_\star \mathcal{P}\mathrm{erf}(Y_T) \to \mathcal{P}\mathrm{erf}(Y_T)) \simeq (\mathrm{F}_\star \mathcal{P}\mathrm{erf}(X_T) \to \mathcal{P}\mathrm{erf}(X_T)),$$

induced by α. Moreover, if $\mathrm{F}_\star \mathcal{P}\mathrm{erf}(X)$ is a P-shaped semiorthogonal decomposition of $\mathcal{P}\mathrm{erf}(X)$, then the induced filtration $\mathrm{F}_\star \mathcal{P}\mathrm{erf}(Y)$ is a semiorthogonal decomposition of $\mathcal{P}\mathrm{erf}(Y)$.

Proof. By assumption, the S-scheme Y gives a global section $\beta_Y\colon S \to \mathbf{Br}^{\mathcal{P}\mathrm{erf}(X)}$. By Proposition 4.3(c), we need to lift β_Y along the map $\mathbf{Br}^{F_\star \mathcal{P}\mathrm{erf}(X)} \to \mathbf{Br}^{\mathcal{P}\mathrm{erf}(X)}$. We denote by $\mathcal{F}_{\beta_Y}$ the fiber of $\mathbf{Br}^{F_\star \mathcal{P}\mathrm{erf}(X)} \to \mathbf{Br}^{\mathcal{P}\mathrm{erf}(X)}$ over β_Y, which is an fppf sheaf on Sch_Y which is naturally equivalent to $\mathbf{Filt}_P^{\mathcal{C}\otimes_{\mathcal{P}\mathrm{erf}(X)}\mathcal{P}\mathrm{erf}(Y)}$. We proceed to construct a section of the canonical map $\mathcal{F}_{\beta_Y} \to Y$. But, by Proposition 4.3(d) (or Lemma 2.13) the sheaf $\mathcal{F}_{\beta_Y}$ is an fppf sheaf of sets, hence the hypothesis in (ii) suffices to construct the desired section. The final claim follows from Theorem 3.12. ☐

5 Examples

In many good situations, we have a fiber sequence

$$\mathbf{Br}^{\mathrm{F}_\star \mathcal{C}} \to \mathbf{Br}^{\mathcal{C}} \to \mathbf{B}G,$$

of sheaves where, by Proposition 4.3(d), G is a discrete group. In fact, this will hold if and only if the sheaf of subgroups of the sheaf $\pi_0 \mathbf{Aut}_{\mathcal{C}}$ corresponding to automorphisms which preserve the filtration $\mathrm{F}_\star \mathcal{C}$ is normal. (For a case where this does not happen, see Example 5.1.) Furthermore, Proposition 4.3(e) in conjunction with known computations of the homotopy automorphisms of $\mathcal{P}\mathrm{erf}(S)$ will let us compute G. In these cases, we get a theory of *characteristic classes* for filtrations — to a twisted form $\mathcal{D}$ of $\mathcal{C}$ on S we get a class $o(\mathcal{C}(\star)) \in \mathrm{H}^1_{\mathrm{fppf}}(S, \mathrm{G})$ whose vanishing controls whether or not we obtain a filtration on $\mathcal{D}$ which is fppf-locally equivalent to $\mathrm{F}_\star \mathcal{C}$. We will illustrate how this works in this section.

Example 5.1. For a case where the sheaf of subgroups $\pi_0\mathbf{Aut}_{\mathrm{F}_\star\mathcal{C}} \subseteq \pi_0\mathbf{Aut}_{\mathcal{C}}$ is not normal, let $\mathcal{C} \simeq \bigoplus_{i=1}^3 \mathcal{P}\mathrm{erf}(S)$ and consider the [1]-shaped filtration given by $\mathrm{F}_0\mathcal{C} = \oplus_{i=1}^2 \mathcal{P}\mathrm{erf}(S)$ and $\mathrm{F}_1\mathcal{C} = \mathcal{C}$. Then, there is an exact sequence $0 \to \mathbb{Z}^{\oplus 3} \to \pi_0\mathbf{Aut}_{\mathcal{C}} \to S_3 \to 0$, where S_3 is the symmetric group on 3 letters. Here, the $\mathbb{Z}^{\oplus 3}$ corresponds to separately suspending in each copy of $\mathcal{P}\mathrm{erf}(S)$. There is also an exact sequence

$$0 \to \mathbb{Z}^{\oplus 3} \to \pi_0\mathbf{Aut}_{\mathrm{F}_\star\mathcal{C}} \to \mathbb{Z}/2 \to 0.$$

Since $\mathbb{Z}/2$ is not normal in S_3, we see that $\pi_0\mathbf{Aut}_{\mathrm{F}_\star\mathcal{C}}$ is not normal in $\pi_0\mathbf{Aut}_{\mathcal{C}}$.

5.1 From Beilinson to Bernardara

Let $[n]$ be the poset $\{0 \to 1 \to \cdots \to i \to \cdots n\}$. Beilinson's description of the derived category of $\mathbb{P}^n_S$ gives an $[n]$-shaped filtration of $\mathcal{P}\mathrm{erf}(\mathbb{P}^n_S)$ with

$$i \mapsto \mathrm{F}_i\mathcal{P}\mathrm{erf}(\mathbb{P}^n_S) = \langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(i)\rangle.$$

We call this the **Beilinson filtration**.

Lemma 5.2. *There is a fiber sequence*

$$\mathbf{Br}^{\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n_S)} \to \mathbf{Br}^{\mathcal{P}\mathrm{erf}(\mathbb{P}^n_S)} \to \mathbf{B}\mathbb{Z}$$

of fppf sheaves of spaces on S.

Proof. Since, by a result of Bondal and Orlov [15, Theorem 3.1], we know the homotopy automorphisms of $\mathcal{P}\mathrm{erf}(\mathbb{P}^n_S)$, the sheaf of automorphisms of $\mathcal{P}\mathrm{erf}(\mathbb{P}^n_S)$ has homotopy sheaves

$$\pi_i\mathbf{Aut}_{\mathcal{P}\mathrm{erf}(\mathbb{P}^n_S)} \cong \begin{cases} \mathrm{PGL}_{n+1} \times \mathbb{Z} \times \mathrm{Pic}_{\mathbb{P}^n/S} & \text{if } i = 0, \\ \mathbb{G}_m & \text{if } i = 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Here, PGL_{n+1} acts by automorphisms of the S-scheme $\mathbb{P}^n_S$, the group $\mathbb{Z}$ acts by suspension $\mathcal{F} \mapsto \mathcal{F}[1]$ in $\mathcal{P}\mathrm{erf}(\mathbb{P}^n_S)$, and the relative Picard scheme $\mathrm{Pic}_{\mathbb{P}^n/S} \cong \mathbb{Z}$ acts by tensoring with line bundles. Similarly, for the Beilinson filtration $\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n_S)$, the sheaf of automorphisms is the subsheaf of groups on the connected components that preserve the filtration. By Proposition 4.3.e, this eliminates only the non-zero elements of $\mathrm{Pic}_{\mathbb{P}^n_S/S}$ since they do not preserve the Beilinson filtration. Thus, $\mathbf{Aut}_{\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n_S)}$ has homotopy sheaves

$$\pi_i\mathbf{Aut}_{\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n_S)} \cong \begin{cases} \mathrm{PGL}_{n+1} \times \mathbb{Z} & \text{if } i = 0, \\ \mathbb{G}_m & \text{if } i = 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

It follows that there is a fiber sequence

$$\mathbf{Aut}_{\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n_S)} \to \mathbf{Aut}_{\mathcal{P}\mathrm{erf}\mathbb{P}^n_S} \to \mathbb{Z},$$

and this sequence deloops to give a fiber sequence as claimed. ☐

The next theorem generalizes the main result of [9].

Theorem 5.3. *Let S be a qcqs 1-affine algebraic stack and let $P \to S$ be a Severi–Brauer scheme associated to an Azumaya algebra $\mathcal{A}$ of degree $(n+1)$ with Brauer class α. There exists a natural semiorthogonal decomposition*

$$\mathcal{P}\mathrm{erf}(P) \simeq \langle \mathcal{P}\mathrm{erf}(S), \mathcal{P}\mathrm{erf}(S,\alpha), \cdots, \mathcal{P}\mathrm{erf}(S,\alpha^{\otimes n})\rangle.$$

Proof. Let $f\colon S \to \mathbf{Br}^{\mathcal{P}\mathrm{erf}(\mathbb{P}^n_S)}$ classify $\mathcal{P}\mathrm{erf}(P)$. By Lemma 5.2, to descend the Beilinson filtration to $\mathcal{P}\mathrm{erf}(P)$, it suffices to prove that the composite $S \to \mathbf{B}\mathbb{Z}$ is null. But f factors through the map $\mathbf{BPGL}_{n+1} \to \mathbf{Br}^{\mathcal{P}\mathrm{erf}(\mathbb{P}^n_X)}$ which classifies $\mathcal{P}\mathrm{erf}$ of the universal PGL_{n+1}-torsor. It suffices to prove that

$$\mathrm{H}^1_{\mathrm{fppf}}(\mathbf{BPGL}_{n+1};\mathbb{Z}) = 0$$

in the case where $S = \operatorname{Spec}\mathbb{Z}$. We may use the spectral sequence

$$\mathrm{E}_1^{st} = \mathrm{H}^s_{\mathrm{fppf}}(\mathrm{PGL}_{n+1}^{\times t},\mathbb{Z}) \Rightarrow \mathrm{H}^{s+t}_{\mathrm{fppf}}(\mathbf{BPGL}_{n+1},\mathbb{Z})$$

associated to the simplicial scheme

$$* \Longleftarrow \mathrm{PGL}_{n+1} \Lleftarrow \mathrm{PGL}_{n+1}\times\mathrm{PGL}_{n+1} \;\substack{\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow}\; \cdots,$$

to compute fppf cohomology. The only groups that might contribute to $\mathrm{H}^1_{\mathrm{fppf}}(\mathbf{BPGL}_{n+1},\mathbb{Z})$ from the E_1-page are $\mathrm{E}_1^{0,1} = \mathrm{H}^0_{\mathrm{fppf}}(\mathrm{PGL}_{n+1};\mathbb{Z})$ and $\mathrm{E}_1^{1,0} = \mathrm{H}^1_{\mathrm{fppf}}(*;\mathbb{Z}) = 0$. The latter is zero since $* = \operatorname{Spec}\mathbb{Z}$ is normal (see [18, 2.1]). On the other hand, $\mathrm{H}^0_{\mathrm{fppf}}(\mathrm{PGL}_{n+1},\mathbb{Z}) \cong \mathbb{Z}$. Now, in low degrees, the d_1-differentials give a complex

$$\mathrm{E}_1^{0,0} \to \mathrm{E}_1^{0,1} \to \mathrm{E}_1^{0,2}$$

and the cohomology in the middle term is $\mathrm{E}_2^{0,1}$. One can check easily that this complex is isomorphic to $\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z}$. Hence, $\mathrm{E}_2^{0,1} = 0$ so that $\mathrm{H}^1_{\mathrm{fppf}}(\mathbf{BPGL}_{n+1},\mathbb{Z}) = 0$. This proves that we can descend the Beilinson filtration to obtain a Δ^n-shaped semiorthogonal decomposition $\mathrm{F}_\star\mathcal{P}\mathrm{erf}(P)$ of $\mathcal{P}\mathrm{erf}(P)$.

Each graded piece

$$\frac{\mathrm{F}_p\mathcal{P}\mathrm{erf}(P)}{\mathrm{F}_{p-1}\mathcal{P}\mathrm{erf}(P)}$$

is a twisted form of $\mathcal{P}\mathrm{erf}(S)$ and thus of the form $\mathcal{P}\mathrm{erf}(S,\beta_p)$ for some β_p. To complete the theorem, it suffices to see that $\beta_p = \alpha^{\otimes p}$. But, we see by reducing to the universal case $S = \operatorname{Spec}\mathbb{Z}$ that $\beta_p = \alpha^{\otimes a_p}$ for some a_p and then we can find $a_p = p$, for example by referring to Bernardara [9]. □

5.2 Marking the Beilinson filtration

Now consider the marked version of the above picture, where we mark $\mathrm{F}_p\mathcal{P}\mathrm{erf}(\mathbb{P}^n)$ by $\mathcal{O}(p)$. We can in fact describe the sheaf $\mathbf{Br}^{(\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n),\mathcal{O}(*))}$. To do so, consider the maximal torus $T_{\mathrm{PGL}_{n+1}}$ of PGL_{n+1}.

Proposition 5.4. (i) *We have an equivalence of sheaves*

$$\mathbf{Br}^{(\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n),\mathcal{O}(*))} \simeq \mathbf{B}G,$$

where G is a central extension of PGL_{n+1} *by its maximal torus, i.e., we have an exact sequence of groups*

$$1 \to T_{\mathrm{PGL}_{n+1}} \to G \to \mathrm{PGL}_{n+1} \to 1$$

and $T_{\mathrm{PGL}_{n+1}}$ *is in the center of* G*.*

(ii) *If* $(\mathrm{F}_\star\mathcal{D}, M_*)$ *is a twisted form of* $(\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n), \mathcal{O}(*))$*, then* $\mathrm{F}_\star\mathcal{D}$ *is equivalent to the Beilinson filtration on* $\mathcal{P}\mathrm{erf}(P)$ *for some Severi–Brauer scheme* $P \to S$ *associated to an Azumaya algebra of degree* $n+1$*.*

(ii) *If* $P \to S$ *is a Severi–Brauer scheme associated to an Azumaya algebra* $\mathcal{A}$ *of degree* $n+1$ *on* S*, then* $\mathrm{F}_\star\mathcal{P}\mathrm{erf}(P) \in \pi_0\mathbf{Br}^{\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n)}(S)$ *lifts to* $\pi_0\mathbf{Br}^{(\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n),\mathcal{O}(*))}$ *if and only if* $\mathcal{A} \cong \mathcal{E}\mathrm{nd}(\mathcal{V})$ *for some vector bundle* $\mathcal{V}$ *on* S*.*

Proof. By construction, there is a fiber sequence

$$\mathbf{Aut}_{(\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n),\mathcal{O}(*))} \to \mathbf{Aut}_{\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n)} \to \prod_{0\leqslant p\leqslant n} \iota\mathrm{F}_p\mathcal{P}\mathrm{erf}(\mathbb{P}^n)$$

of sheaves of spaces where the right map sends a filtered automorphism φ to the $(n+1)$-tuple $(\varphi(\mathcal{O}(0)), \ldots, \varphi(\mathcal{O}(n)))$. The left-hand term is the fiber over $(\mathcal{O}(0), \ldots, \mathcal{O}(n))$. It follows that there is an exact sequence (of sheaves of abelian groups)

$$0 \to \pi_1\mathbf{Aut}_{(\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n),\mathcal{O}(*))} \to \pi_1\mathbf{Aut}_{\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n)} \xrightarrow{a} \prod_{0\leqslant p\leqslant n} \mathbb{G}_m$$
$$\to \pi_0\mathbf{Aut}_{(\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n),\mathcal{O}(*))} \xrightarrow{b} \pi_0\mathbf{Aut}_{\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n)}.$$

We already know that $\pi_1\mathbf{Aut}_{\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n)} \cong \mathbb{G}_m$, which appears as the natural automorphisms of the identity on $\mathcal{P}\mathrm{erf}(\mathbb{P}^n)$. These natural isomorphisms of $\mathcal{P}\mathrm{erf}(\mathbb{P}^n)$ act as $\mathbb{G}_m$ on each $\mathcal{O}(p)$. Thus, the map a in the diagram is the diagonal embedding of $\mathbb{G}_m$ in $\prod_{0\leqslant p\leqslant n}\mathbb{G}_m$. It follows that $\pi_1\mathbf{Aut}_{(\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n),\mathcal{O}(*))} = 0$. We also can see directly that the image of the map b in $\pi_0\mathbf{Aut}_{\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n)} \cong \mathrm{PGL}_{n+1} \times \mathbb{Z}$ is PGL_{n+1} since the copy of $\mathbb{Z}$ appears as the suspension operation on $\mathcal{P}\mathrm{erf}(\mathbb{P}^n)$ which does not preserve the marking. This proves part (i).

For part (ii), we see from the map $G \to \mathrm{PGL}_{n+1}$ that any twisted form $(\mathrm{F}_\star\mathcal{D}, N_*)$ of $(\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^n), \mathcal{O}(*))$ has the property that $\mathrm{F}_\star\mathcal{D}$ is equivalent to the Beilinson filtration on a Severi–Brauer scheme $P \to S$ for a degree $(n+1)$ Azumaya algebra over S.

Using the exact sequence $1 \to T_{\mathrm{PGL}_{n+1}} \to G \to \mathrm{PGL}_{n+1} \to 1$, we see that a lift of a class $P \in \mathrm{H}^1_{\mathrm{fppf}}(S, \mathrm{PGL}_{n+1})$ to $\mathrm{H}^1_{\mathrm{fppf}}(S, G)$ exists if and only if the obstruction class $\mathrm{ob}(P) \in \mathrm{H}^2_{\mathrm{fppf}}(S, T_{\mathrm{PGL}_{n+1}})$ vanishes. We leave it to the reader to check that under the natural isomorphism $\prod_{0<p\leqslant n}\mathbb{G}_m \cong T_{\mathrm{PGL}_{n+1}}$ the obstruction class of P is $(\alpha, \alpha^{\otimes 2}, \ldots, \alpha^{\otimes n})$. Thus, a lift exists if and only if the Azumaya algebra $\mathcal{A}$ has trivial Brauer class, which happens if and only if $\mathcal{A} \cong \mathcal{E}\mathrm{nd}(\mathcal{V})$ for some vector bundle $\mathcal{V}$, which is what we wanted to prove. □

In particular, we recover the well-known fact that if the marked filtration descends to the Severi–Brauer variety of an Azumaya algebra $\mathcal{A}$, then $\mathcal{A}$ is the sheaf of endomorphisms of a vector bundle.

5.3 Involution surfaces

Now we study twisted forms of $\mathbb{P}^1 \times \mathbb{P}^1$.

Definition 5.5. An **involution surface** is an S-scheme X which is fppf-locally isomorphic to $\mathbb{P}^1_S \times_S \mathbb{P}^1_S$.

Involution surfaces are classified by pairs $(T, \mathcal{A})$ consisting of a $\mathbb{Z}/2$-Galois extension T of S and a quaternion Azumaya algebra $\mathcal{A}$ over T. The associated surface $X(T, \mathcal{A})$ is $\mathrm{Re}_{T/S}\mathrm{SB}(\mathcal{A})$, the Weil restriction from T to S of the Severi–Brauer variety of $\mathcal{A}$. See [3, Example 3.3] or [26, Section 15.B].

Example 5.6. If the quadratic extension T is split, then $T = S \coprod S$ and $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are quaternion Azumaya algebras over S. In this case, $X(T, \mathcal{A}) \cong \mathrm{SB}(\mathcal{A}_1) \times \mathrm{SB}(\mathcal{A}_2)$, the product of the Severi–Brauer schemes of $\mathcal{A}_1$ and $\mathcal{A}_2$.

Since $\mathbb{P}^1 \times \mathbb{P}^1$ is the zero-locus of a quadric form in four variables, its automorphism group is the smooth algebraic group PO_4. Now, note that there is a natural inclusion

$$\mathrm{PGL}_2 \times \mathrm{PGL}_2 \hookrightarrow \mathrm{PO}_4$$

which extends to an exact sequence

$$0 \to \mathrm{PGL}_2 \times \mathrm{PGL}_2 \to \mathrm{PO}_4 \to \mathbb{Z}/2 \to 0.$$

In nonabelian cohomology, the map $\mathrm{H}^1(\mathrm{Spec}\, k, \mathrm{PO}_4) \to \mathrm{H}^1(\mathrm{Spec}\, k, \mathbb{Z}/2)$ classifies the quadratic extension ℓ and if ℓ is trivial, then the fibers give the pairs $\mathcal{A}_1$ and $\mathcal{A}_2$ as in Example 5.6.

The Picard scheme $\mathrm{Pic}_{\mathbb{P}^1 \times \mathbb{P}^1/S}$ is discrete and isomorphic to the constant sheaf $\mathbb{Z}^2$. We let $\mathcal{O}(i,j) = p_1^*\mathcal{O}(i) \otimes p_2^*\mathcal{O}(j)$, where p_1 is projection onto the first factor and p_2 projection onto the second factor. These give all isomorphism classes of line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ if S is the spectrum of a field. By the theorem of Bondal and Orlov [15, Theorem 3.1], which applies since the canonical class of $\mathbb{P}^1 \times \mathbb{P}^1$ is antiample, it follows that there is an exact sequence

$$0 \to \mathbb{Z} \times \mathbb{Z}^{\oplus 2} \to \pi_0 \mathbf{Aut}_{\mathcal{P}\mathrm{erf}(\mathbb{P}^1 \times \mathbb{P}^1)} \to \mathrm{PO}_4 \to 0.$$

The rank three kernel corresponds to tensoring with line bundles and with translation in the derived category.

Now, consider the following two filtrations on $\mathcal{P}\mathrm{erf}(\mathbb{P}^1 \times \mathbb{P}^1)$. First is the $[1] \times [1]$-shaped filtration $\mathrm{F}_\star \mathcal{P}\mathrm{erf}(\mathbb{P}^1 \times \mathbb{P}^1)$ corresponding to the admissible subcategories

$$\begin{array}{ccccc} & & \langle \mathcal{O}(0,0), \mathcal{O}(0,1)\rangle & & \\ & \nearrow & & \searrow & \\ \langle \mathcal{O}(0,0)\rangle & & & & \langle(\mathcal{O}(0,0), \mathcal{O}(0,1), \mathcal{O}(1,0), \mathcal{O}(1,1)\rangle \\ & \searrow & & \nearrow & \\ & & \langle \mathcal{O}(0,0), \mathcal{O}(1,0)\rangle & & \end{array}$$

The second is the $[2]$-shaped filtration $\mathrm{G}_\star \mathcal{P}\mathrm{erf}(\mathbb{P}^1 \times \mathbb{P}^1)$ corresponding to the semiorthogonal decomposition

$$\langle \mathcal{O}(0,0), \mathcal{O}(0,1) \oplus \mathcal{O}(1,0), \mathcal{O}(1,1)\rangle.$$

There is a natural map $\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^1\times\mathbb{P}^1) \to \mathrm{G}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^1\times\mathbb{P}^1)$ over the collapse map $[1]\times[1]\to[2]$ which sends the vertices $(0,1)$ and $(1,0)$ to 1.

Now, we consider the maps

$$\mathbf{Br}^{\mathrm{F}_\star} \to \mathbf{Br}^{\mathrm{G}_\star} \to \mathbf{Br}^{\mathbb{P}^1\times\mathbb{P}^1},$$

where we have made the evident abbreviations to cut down on notation.

Proposition 5.7. (a) *The sequence $\mathbf{Br}^{\mathrm{G}_\star} \to \mathbf{Br}^{\mathbb{P}^1\times\mathbb{P}^1} \to \mathbb{Z}^{\oplus 2}$ is a fiber equivalence. If X is any involution surface, the filtration $\mathrm{G}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^1\times\mathbb{P}^1)$ descends to $\mathcal{P}\mathrm{erf}(X)$.*

(b) *The map $\mathbf{Br}^{\mathrm{F}_\star} \to \mathbf{Br}^{\mathrm{G}_\star}$ is a $\mathbb{Z}/2$-torsor. Thus, if $\mathrm{G}_\star\mathcal{D}$ is a twisted form of $\mathrm{G}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^1\times\mathbb{P}^1)$, there is a canonical obstruction class $o\in \mathrm{H}^1_{\mathrm{fppf}}(S,\mathbb{Z}/2)$ which vanishes if and only if the filtration $\mathrm{G}_\star\mathcal{D}$ can be refined to a filtration $\mathrm{F}_\star\mathcal{D}$ which is a twist of $\mathrm{F}_\star\mathcal{P}\mathrm{erf}(\mathbb{P}^1\times\mathbb{P}^1)$.*

Proof. We leave the proof to the reader who should follow the lines of the proof of Theorem 5.3. □

In general, if X is an involution surface over S, the obstruction class $\mathrm{H}^1_{\mathrm{fppf}}(S,\mathbb{Z}/2)$ associated to the problem of lifting the canonical filtration $\mathrm{G}_\star\mathcal{P}\mathrm{erf}(X)$ to $\mathrm{F}_\star\mathcal{P}\mathrm{erf}(X)$ is precisely the class of $T\to S$. See [3].

5.4 Descending exceptional blocks

We want to explain how to prove the main descent theorem of Ballard–Duncan–McFaddin in the language of this paper (see [6, Theorem 2.15]).

Let $\mathcal{E}$ be an $\mathcal{O}_S$-linear category with a P-shaped full exceptional collection $\{e_p\}_{p\in P}$. Suppose that a group G acts on $\mathcal{E}$. We say that the full exceptional collection is G-stable if for each $g\in G$ and e_p the object $g\cdot e_p$ is in $\{e_p\}_{p\in P}$.

Theorem 5.8 ([6]). *Let X be a smooth proper S-scheme, $T\to S$ a G-Galois fppf cover, $\mathcal{E}_T\subseteq\mathcal{P}\mathrm{erf}(X_T)$ an admissible $\mathcal{O}_T$-linear subcategory with a G-stable P-shaped full exceptional collection $\{e_p\}_{p\in P}$. Then, $\mathcal{E}_\ell$ descends to $\mathcal{E}\subseteq\mathcal{P}\mathrm{erf}(X)$ and $\mathcal{E}$ admits a full twisted exceptional collection.*

Proof. By hypothesis, G preserves $\mathcal{E}_T$ so it descends to $\mathcal{E}\subseteq\mathcal{P}\mathrm{erf}(X)$ by Theorem 4.6. Arguing as in [6, Lemma 2.12], the G-orbits of the objects of $\{e_p\}_{p\in P}$ are in fact orthogonal exceptional objects. We can assume that $\{e_p\}_{p\in P}$ is in fact a single G-orbit, which is orthogonal. But, then, $\mathcal{E}\simeq\mathcal{P}\mathrm{erf}(T)^n$. The descended version is then a twisted form of $\mathcal{P}\mathrm{erf}(S)^n$. Any such admits a full twisted exceptional collection for example by [3, Theorem 2.16] or by using the twisted Brauer space for $\mathcal{P}\mathrm{erf}(S)^n$. □

With more work one can descend individual vector bundles by using markings. We leave this to the reader.

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