Scheiderer motives and equivariant higher topos theory

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Abstract

We give an algebro-geometric interpretation of C_2-equivariant stable homotopy theory by means of the b-topology introduced by Claus Scheiderer in his study of 2-torsion phenomena in étale cohomology. To accomplish this, we first revisit and extend work of Scheiderer on equivariant topos theory by functorially associating to a ∞-topos $\mathcal{X}$ with G-action a presentable stable ∞-category $\mathrm{Sp}^G(\mathcal{X})$, which recovers the ∞-category Sp^G of genuine G-spectra when $\mathcal{X}$ is the terminal G-∞-topos. Given a scheme X with $\frac{1}{2} \in \mathcal{O}_X$, our construction then specializes to produce an ∞-category $\mathrm{Sp}_b^{C_2}(X)$ of "b-sheaves with transfers" as b-sheaves of spectra on the small étale site of X equipped with certain transfers along the extension $X[i] \to X$; if X is the spectrum of a real closed field, then $\mathrm{Sp}_b^{C_2}(X)$ recovers Sp^{C_2}. On a large class of schemes, we prove that, after p-completion, our construction assembles into a premotivic functor satisfying the full six functors formalism. We then introduce the b-variant $\mathrm{SH}_b(X)$ of the ∞-category $\mathrm{SH}(X)$ of motivic spectra over X (in the sense of Morel-Voevodsky), and produce a natural equivalence of ∞-categories $\mathrm{SH}_b(X)^\wedge_p \simeq \mathrm{Sp}_b^{C_2}(X)^\wedge_p$ through amalgamating the étale and real étale motivic rigidity theorems of Tom Bachmann. This involves a purely algebro-geometric construction of the C_2-Tate construction, which may be of independent interest. Finally, as applications, we deduce a "b-rigidity" theorem, use the Segal conjecture to show étale descent of the 2-complete b-motivic sphere spectrum, and construct a parametrized version of the C_2-Betti realization functor of Heller-Ormsby.

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Contents

1 Introduction

1.1 Motivation from algebraic geometry: compactifying the étale topos of a real variety

Grothendieck introduced the étale topology into algebraic geometry in order to construct a cohomology theory $H^*_{\text{ét}}$ for algebraic varieties that resembles the theory of singular cohomology H^* for topological spaces. The analogy between the two theories is closest when working over a separably closed field k. For instance, when the base scheme is the field $\mathbb{C}$ of complex numbers, the **Artin comparison theorem** [73, Théoremè XI.4.4] states that given a complex variety X and a finite abelian group M, there is a canonical isomorphism

$$H^*_{\text{ét}}(X; M) \cong H^*(X(\mathbb{C}); M),$$

where we endow $X(\mathbb{C})$ with the complex analytic topology. More generally, if we let $\operatorname{cd}_\ell(X_{\text{ét}})$ denote the étale ℓ-cohomological dimension of a k-variety X, then we have the inequality $\operatorname{cd}_\ell(X_{\text{ét}}) \leqslant 2\dim(X)$ for all primes l [73, Corollaire X.4.3], as predicted by Artin's theorem when $k = \mathbb{C}$.

If we no longer suppose that k is separably closed, we now expect properties of the absolute Galois group $G = \operatorname{Gal}(\overline{k}/k)$ to manifest in the étale cohomology of k-varieties; correspondingly, the role of singular cohomology on the topological side should be replaced by G-equivariant Borel cohomology H^*_G. For instance, suppose that $k = \mathbb{R}$ is the field of real numbers, so that $G = C_2$. Then parallel to Artin's theorem, Cox proved that for a real variety X and a finite abelian group M, there is a canonical isomorphism between étale cohomology and (Borel-style) C_2-equivariant cohomology:

$$H^*_{\text{ét}}(X; M) \cong H^*_{C_2}(X(\mathbb{C}); M),$$

using the C_2-action on $X(\mathbb{C})$ given by complex conjugation [Cox79]. In particular, we may now have $\operatorname{cd}_2(X_{\text{ét}}) = \infty$ as a consequence of the infinite 2-cohomological dimension of C_2 itself (in topological terms, the infinite mod 2 cohomology of $\mathbb{R}P^\infty$). More precisely, if we let $X(\mathbb{C})/C_2$ to be the quotient of the topological space $X(\mathbb{C})$ with the analytic topology by its natural C_2-action, then Cox deduces a long-exact sequence of cohomology groups [Cox79, Proposition 1.2]

$$\cdots \to H^k(X(\mathbb{C})/C_2, X(\mathbb{R}); M) \to H^k_{\text{ét}}(X; M) \to H^k_{C_2}(X(\mathbb{R}); M) = H^k(X(\mathbb{R}) \times BC_2; M) \to \cdots$$

from which it follows that we have an isomorphism

$$H^n_{\text{ét}}(X;\mathbb{Z}/2) \xrightarrow{\cong} \bigoplus_{i=0}^{\dim(X)} H^i(X(\mathbb{R});\mathbb{Z}/2) = H^*(X(\mathbb{R});\mathbb{Z}/2)$$

for all $n > 2\dim(X)$. Furthermore, these results extend in a straightforward way from $\mathbb{R}$ to an arbitrary real closed field if one instead uses (C_2-equivariant) semialgebraic cohomology [Sch94, Chapter 15].

In his book [Sch94], Scheiderer generalized the Cox exact sequence to one involving abelian étale sheaves A over a base scheme S on which 2 is invertible, in which the role of $H^*(X(\mathbb{R});M)$ is replaced by **real étale cohomology** . His theorems to this effect operate at the level of topoi, from which the exact sequence is a direct corollary. In picturesque terms, the main idea is to "compactify the (small) étale topos $\widetilde{S_{\text{ét}}}$ by gluing in the (small) real étale topos $\widetilde{S_{\text{rét}}}$ at ∞", where the notion of gluing in question is that of a **recollement** of topoi [72, Définition 9.1.1]; see also [QS19, §1-2], [Lur17a, §A.8], and [BG16] for treatments of this subject in the higher categorical setting.

In more detail, Scheiderer first describes a generic gluing procedure that receives as input any topos $\mathcal{X}$ with C_p-action and outputs the "genuine toposic C_p-orbits" $\widetilde{\mathcal{X}_{C_p}}$; in this paper, we recapitulate his construction in the setting of ∞-topoi (§2). His key insight was then that for the C_2-topos $\widetilde{S[i]_{\text{ét}}}$, the formation of toposic C_2-orbits yields $\widetilde{S_{\text{ét}}}$ by Galois descent, the formation of toposic C_2-fixed points yields $\widetilde{S_{\text{rét}}}$, and with respect to these identifications, the gluing functor $\rho : \widetilde{S_{\text{ét}}} \to \widetilde{S_{\text{rét}}}$ identifies with real étale sheafification. Therefore, the glued topos $(\widetilde{S[i]_{\text{ét}}})_{C_2}$ is equivalent to $\widetilde{S_b}$, where the b**-topology** on the small étale site is defined to be the intersection of the étale and real étale topologies. By recollement theory, $\widetilde{S_b}$ thus contains $\widetilde{S_{\text{ét}}}$ as an open subtopos with closed complement $\widetilde{S_{\text{rét}}}$.

Apart from its role in obtaining the Cox exact sequence, the b-topos turns out to have many excellent formal properties; in particular, Scheiderer proves that $\mathrm{cd}_2(S_b)$ is either $\mathrm{cd}_2(S[i]_{\text{ét}})$ or $\mathrm{cd}_2(S[i]_{\text{ét}})+1$ [Sch94, Corollary 7.18]. This result leads us to think of $\widetilde{S_b}$ as a suitable compactification of $\widetilde{S_{\text{ét}}}$ when the *virtual* étale 2-cohomological dimension of S is finite.

1.2 Motivation from algebraic topology: compactifying the topos of equivariant sheaves

Let G be a finite group and X a topological space with properly discontinuous G-action. The ∞-topos $\mathrm{Shv}_G(X)$ of G-equivariant sheaves of spaces[1] on X generally fails to possess good finiteness properties if G does not act freely. To explain, consider the simple example of $X = *$ with trivial G-action, so that

$$\mathrm{Shv}_G(X) \simeq \mathrm{Spc}^{BG} = \mathrm{Fun}(BG, \mathrm{Spc}),$$

where BG is the one-object groupoid with G as its endomorphisms and Spc is the ∞-category of spaces. Since the formation of homotopy fixed points fails to commute with homotopy colimits in general, we see that the unit is not a compact object in Spc^{BG}.

On the other hand, we have the homotopy theory for G-spaces given by taking a category of (nice) topological spaces with G-action and inverting those morphisms that induce weak homotopy equivalences on all fixed points. Let Spc_G denote the resulting ∞-category, whose objects we henceforth term G-spaces (as opposed to spaces with G-action). By Elmendorf's theorem [Elm83], we have an equivalence

$$\mathrm{Spc}_G \simeq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathrm{Spc})$$

of the ∞-category of G-spaces with presheaves of spaces on the orbit category $\mathcal{O}_G$. In Spc_G, taking G-fixed points amounts to evaluation on the orbit G/G, which *does* preserve all colimits. Therefore, the unit in Spc_G is a compact object, in contrast to Spc^{BG}. Moreover, right Kan extension along the inclusion $BG \subset \mathcal{O}_G^{\mathrm{op}}$ (as the full subcategory on the orbit $G/1$) yields a fully faithful embedding $\mathrm{Spc}^{BG} \to \mathrm{Spc}_G$, which presents Spc^{BG} as an open subtopos of Spc_G. We are thus entitled to view Spc_G as a compactification of Spc^{BG}. Indeed, if $G = C_p$, then Scheiderer's construction applied to the ∞-topos Spc with trivial C_p-action yields

[1] In this paper, we will always use 'space' as a synonym for Kan complex, i.e., ∞-groupoid, as opposed to a topological space.

Spc_{C_p} as its genuine toposic C_p-orbits. In particular, the b-∞-topos of a real closed field identifies with Spc_{C_2}.

Let us now pass to stabilizations. We then have the ∞-category of **Borel G-spectra** $\mathrm{Sp}^{BG} = \mathrm{Fun}(BG, \mathrm{Sp})$, and the ∞-category of **naive G-spectra** $\mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathrm{Sp})$. As is well-known in equivariant stable homotopy theory, the latter ∞-category fails to be an adequate homotopy theory of G-spectra as it does not possess enough dualizable objects (for instance, one does not have equivariant Atiyah duality in this ∞-category). To correct this deficiency, one passes to the ∞-category Sp^G of **genuine G-spectra**, which is obtained from Spc_G by inverting the real representation spheres. In Sp^G, the unit remains compact; indeed, the compact objects now coincide with the dualizable objects. Finally, we still have a fully faithful embedding $\mathrm{Sp}^{BG} \to \mathrm{Sp}^G$ as a right adjoint, whose essential image consists of the **cofree** genuine G-spectra.

1.3 What is this paper about?

Our first goal in this paper is to unify the above two perspectives on compactifications of ∞-topoi in the stable regime. In other words, having seen that the b-topos is analogous to the ∞-topos Spc_{C_2} of C_2-spaces, we aim to answer the following:

Question 1.1. What is the analogue of the passage from G-spaces to genuine G-spectra for the b-topos?

To this end, we will construct a stable ∞-category of **b-sheaves of spectra with transfers** (Definition 3.18) on a scheme S, which we denote by $\mathrm{Sp}_b^{C_2}(S)$. This ∞-category agrees with Sp^{C_2} when $S = \mathrm{Spec}\, k$ for k a real closed field.

Our second goal is to then relate our construction to a variant of Morel-Voevodsky's stable $\mathbb{A}^1$-homotopy theory of schemes $\mathrm{SH}(S)$, in which the role of the Nisnevich topology is replaced by that of the finer b-topology (on the big site Sm_S of smooth schemes over S). In fact, given any topology τ finer than the Nisnevich topology, define the full subcategory $\mathrm{SH}_\tau(S) \subset \mathrm{SH}(S)$ of τ **motivic spectra**[2] to be the local objects with respect to the class of morphisms

$$\{\Sigma^{p,q}\Sigma^\infty_+ U_\bullet \to \Sigma^{p,q}\Sigma^\infty_+ T : p, q \in \mathbb{Z},\ U_\bullet \to T \text{ is the Čech nerve of a } \tau\text{-hypercover } U \to T \text{ in } \mathrm{Sm}_S\}.$$

We wish to relate $\mathrm{SH}_b(S)$. To this end, two recent *rigidity theorems* of Bachmann allow us to isolate the ∞-categories of real étale and (p-complete) étale motivic spectra in terms of the corresponding ∞-categories of sheaves of spectra on the *small* site.

Theorem 1.2 ([Bac18a])**.** *Suppose S is a scheme of finite dimension. Then the morphism of sites*

$$(\mathrm{\acute{E}t}_S, \mathrm{r\acute{e}t}) \to (\mathrm{Sm}_S, \mathrm{r\acute{e}t})$$

induces an equivalence[3]

$$\mathrm{Sp}(\widetilde{S_{\mathrm{r\acute{e}t}}}) \simeq \mathrm{SH}_{\mathrm{r\acute{e}t}}(S). \tag{1}$$

Under this equivalence, the localization functor $L_{\mathrm{r\acute{e}t}}$ is given by inverting the element $\rho \in \pi_{-1,-1}(S^0)$ defined by the unit $-1 \in \mathcal{O}^\times(S)$.

Remark 1.3. In [Bac18a], the schemes involved are assumed to be noetherian. Using continuity for the functor $S \mapsto \mathrm{Sp}(\widetilde{S}_{\mathrm{r\acute{e}t}})$ (see Lemma B.9) and continuity for SH, which persists for $\mathrm{SH}_{\mathrm{r\acute{e}t}}$ (by the same argument with real étale covers as in Lemma B.9), we have removed this hypothesis in the statement of the above theorem. This is implicit in Bachmann's formulation for the étale case below, which we cite verbatim.

For the next result, we denote by $\mathrm{Sch}_B^{p\text{-fin}}$ the category of schemes that are "locally p-étale finite" (see [Bac18b, Definition 5.8]) over a base scheme B.

Theorem 1.4 ([Bac18b])**.** *Suppose $S \in \mathrm{Sch}^{p\text{-fin}}_{\mathbb{Z}[\frac{1}{p}]}$. Then the morphism of sites*

$$(\mathrm{\acute{E}t}_S, \mathrm{\acute{e}t}) \to (\mathrm{Sm}_S, \mathrm{\acute{e}t})$$

[2] Note that we consider the hypercomplete version of the theory.

[3] For our formulation of Bachmann's theorem, we use in addition the hypercompleteness of the real étale site (see Appendix B.1).

induces an equivalence after p-completion

$$\mathrm{Sp}(\widehat{S_{\text{ét}}})^{\wedge}_{p} \simeq \mathrm{SH}_{\text{ét}}(S)^{\wedge}_{p}, \tag{2}$$

where we consider hypercomplete *étale sheaves.*

Remark 1.5. We do not expect Theorem 1.4 to hold integrally, in view of the equivalence of Nisnevich and étale motives after rationalization.

Roughly speaking, we will amalgamate Bachmann's two theorems to construct an equivalence

$$\mathrm{Sp}^{C_2}_{b}(S)^{\wedge}_{p} \simeq \mathrm{SH}_{b}(S)^{\wedge}_{p}$$

under the same hypotheses as Theorem 1.4.

Remark 1.6. In particular, we obtain a fully faithful embedding of $\mathrm{Sp}^{C_2}_{b}(S)^{\wedge}_{p}$ into $\mathrm{SH}(S)^{\wedge}_{p}$. In [BS20], Behrens and the second author showed that the right adjoint to C_2-Betti realization gives a fully faithful embedding $(\mathrm{Sp}^{C_2})^{\wedge}_{p} \hookrightarrow \mathrm{SH}^{\mathrm{cell}}(\mathbb{R})^{\wedge}_{p}$ of p-complete genuine C_2-spectra into p-complete *cellular* real motivic spectra. Our theorems here thus specialize to a non-cellular version of this result when $S = \operatorname{Spec}\mathbb{R}$ (upon identifying b-localization with C_2-Betti realization).

1.4 Main results

Let us now state our main results in greater detail. Roughly, this paper divides into two parts: §2-3 concerns equivariant higher topos theory and its application to establishing the formalism of six operations, while §4-6 builds a connection with motivic homotopy theory in the b-topology. The starting point for equivariant higher topos theory is the following definition.

Definition 1.7. Let G be a finite group. A **G-∞-topos** is a functor $BG \to \mathscr{T}\mathrm{op}^R$, where $\mathscr{T}\mathrm{op}^R$ is the ∞-category of ∞-topoi and geometric morphisms thereof.

Given a G-∞-topos $\mathcal{X}$, we will construct its genuine stabilization $\mathrm{Sp}^G(\mathcal{X})$ in two stages (see §2 for details):

1. We define the **toposic genuine G-orbits** $\mathcal{X}_G$ as a certain lax colimit in $\mathscr{T}\mathrm{op}^R$ (Construction 2.18); this recovers Scheiderer's construction when $G = C_p$ (Example 2.44). Ranging over all subgroups $H \leqslant G$, these ∞-topoi assemble into a presheaf

$$\mathcal{X}_{(-)} : (\mathcal{O}_G)^{\mathrm{op}} \to \mathscr{T}\mathrm{op}^L \simeq (\mathscr{T}\mathrm{op}^R)^{\mathrm{op}},$$

   which constitutes an example of a **G-∞-category**.

2. Given any G-∞-category C that admits finite G-limits, Nardin has constructed a suitable candidate for its genuine stabilization $\mathrm{Sp}^G(\mathrm{C})$ as the ∞-category of G-commutative monoids (or parametrized Mackey functors) in the fiberwise stabilization of C. We apply his construction to define $\mathrm{Sp}^G(\mathcal{X}) = \mathrm{Sp}^G(\mathcal{X}_{(-)})$.

If $G = C_p$, then we also have an alternative description of $\mathrm{Sp}^{C_p}(\mathcal{X})$ that generalizes Glasman's description of genuine C_p-spectra as a recollement obtained by gluing along the C_p-Tate construction $(-)^{tC_p} : \mathrm{Sp}^{BC_p} \to \mathrm{Sp}$ [Gla15].

Theorem 1.8 (Theorem 2.47)**.** *Let $\mathcal{X}$ be a C_p-∞-topos and let $\mathcal{X}_{h_\diamond C_p}$ and $\mathcal{X}^{h_\diamond C_p}$ denote the toposic colimit and limit of the C_p-action on $\mathcal{X}$ (i.e., as taken in $\mathscr{T}\mathrm{op}^R$ instead of Cat_∞). Then there exists a functor*

$$\Theta^{\mathrm{Tate}} : \mathrm{Sp}(\mathcal{X}_{h_\diamond C_p}) \to \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})$$

whose right-lax limit is canonically equivalent to $\mathrm{Sp}^{C_p}(\mathcal{X})$.

One payoff of the recollement perspective is the construction of a symmetric monoidal structure on $\mathrm{Sp}^{C_p}(\mathcal{X})$, which we explain in §2.4. This symmetric monoidal structure will be essential for our subsequent construction of a new six functors formalism when we specialize to our main example of a G-∞-topos: the C_2-∞-topos $\widehat{S[i]_{\text{ét}}}$ of hypercomplete étale sheaves of spaces on $S[i]$ for a scheme S on which 2 is invertible.

Definition 1.9 (Definition 3.18)**.** Let $\mathrm{Sp}_b^{C_2}(S) = \mathrm{Sp}^{C_2}(\widehat{S[i]}_{\text{ét}})$ be the ∞-category of **hypercomplete b-sheaves of spectra with transfers**.

Building upon Bachmann's results, we endow $\mathrm{Sp}_b^{C_2}(-)_p^\wedge$ with the structure of a premotivic functor satisfying the full six functors formalism.

Theorem 1.10 (Theorem 3.27)**.** *For any prime p, there is a six functors formalism*

$$(\mathrm{Sp}_b^{C_2})_p^\wedge : (\mathrm{Sch}^{\mathrm{ft}}_{\mathbb{Z}[\frac{1}{p},\frac{1}{2}]})^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}}),$$

which assigns to a real closed field the p-completion of the ∞-category of genuine C_2-spectra Sp^{C_2}.

That $(\mathrm{Sp}_b^{C_2})_p^\wedge$ assembles into a full six functors formalism crucially relies on the maneuver of genuine stabilization, which guarantees a sufficient supply of dualizable objects; see Definition 3.15 for the precise dualizability assumption that we require. In particular, the Tate motive $\mathbb{1}(1)$ is invertible in $(\mathrm{Sp}_b^{C_2})_p^\wedge$, even though the construction of $(\mathrm{Sp}_b^{C_2})_p^\wedge$ as a premotivic functor does not *a priori* enforce this (cf. Lemma 3.25).

Next, we relate our construction to b-motivic spectra. In order to facilitate the comparison, we first express the C_2-Tate construction in purely algebro-geometric terms.

Theorem 1.11 (Theorem 4.17)**.** *Let k be a real closed field. Under the equivalences $(\mathrm{Sp}^{BC_2})_p^\wedge \xrightarrow{\simeq} \mathrm{SH}_{\text{ét}}(k)_p^\wedge$ and $(\mathrm{Sp})_p^\wedge \xrightarrow{\simeq} \mathrm{SH}_{\text{rét}}(k)_p^\wedge$, we have a canonical equivalence of functors*

$$(-)^{tC_2} \simeq L_{\text{rét}} i_{\text{ét}} : \mathrm{SH}_{\text{ét}}(k)_p^\wedge \to \mathrm{SH}_{\text{rét}}(k)_p^\wedge.$$

To prove Theorem 4.17, we extend the construction of the C_2-Betti realization functor to a real closed field in §4.1 by means of semialgebraic topology, and then prove that its right adjoint is fully faithful after p-completion.

Finally, we obtain the following identification:

Theorem 1.12 (Theorem 5.8)**.** *Let S be a scheme such that $\frac{1}{2} \in \mathcal{O}_S$. Then there is a canonical strong symmetric monoidal functor*

$$C_S : \mathrm{Sp}_b^{C_2}(S) \to \mathrm{SH}_b(S),$$

such that if S is locally p-étale finite and $\frac{1}{p} \in S$, then C_S is an equivalence after p-completion.

The proof of Theorem 5.8 follows at once from the identification of Theorem 4.17 after we construct a transformation $\Theta^{\mathrm{Tate}} \Rightarrow \Theta^{\mathrm{mot}} = L_{\text{rét}} i_{\text{ét}}$ that is suitably stable under base change.

We conclude with three applications of our results. First, in §6.1, we discuss a version of rigidity for the b-topology which, unlike the étale topology, is not quite an identification of the stable motivic category with sheaves of spectra on the small site. Indeed, as Theorem 5.8 indicates, one needs to adjoin more transfers. In §6.2, we describe yet another realization functor out of SH, which in this case is a parametrized version of the C_2-Betti realization functor of Heller-Ormsby. Lastly, in §6.3, we apply Segal's conjecture for the C_2-equivariant sphere to deduce a surprising result: the b-sphere satisfies étale descent after 2-completion. We do not know a way to see this result without appealing to equivariant stable homotopy theory.

Remark 1.13. In [Hoy17], a full six functors formalism has been constructed for equivariant motivic homotopy theory where the group of equivariance is a rather general linear algebraic group. One should view *loc. cit.* as extending motivic homotopy theory from schemes to quotient stacks. In contrast, this paper addresses a different sort of equivariance, namely, *Galois equivariance*. In the context of motivic homotopy theory, this was first studied by Heller and Ormsby in [HO16], where the authors established a connection between C_2-equivariant homotopy theory and motivic homotopy theory over the real numbers. In this light, one contribution of the present paper is to vastly extend the line of inquiry in *loc. cit.* to the parametrized setting. In particular, we believe that the current paper provides an abstract framework via the six functors formalism for studing Real algebraic cycles (in the style of [CD16, §7.1] and [Cis19]). In a future work, we will elaborate on how our formalism is a natural home for various Real cycle class maps recently introduced by various authors in [BW20] and [HWXZ19].

1.5 Conventions and notation

1.5.1 Algebraic geometry

1. By convention, all schemes that appear in this paper are quasicompact and quasiseparated (qcqs). If S is a base scheme, we write Sch'_S to indicate some full subcategory of schemes over S (e.g., finite type S-schemes). We add the superscript fin.dim (resp. noeth) to various categories of schemes to indicate the intersection with finite dimensional (resp. noetherian) schemes.

2. For a ring A, resp. scheme X, let $\mathrm{Sper}\, A$, resp. X_r denote the real spectrum, regarded as a topological space in the usual way (cf. [Sch94, 0.4]).

3. If X is a scheme and τ is topology on $\acute{\mathrm{E}}\mathrm{t}_X$, then we write $\widetilde{X_\tau}$ for the ∞-topos of τ-sheaves of spaces on the small étale site of X (e.g., $\widetilde{X_{\acute{e}t}}$). Abusively, we write $\widetilde{X}_{\mathrm{pre}}$ for the ∞-topos of presheaves on $\acute{\mathrm{E}}\mathrm{t}_X$. We also write $\widehat{X_\tau}$ for its hypercompletion, and we have a localization $(-)^h : \widetilde{X_\tau} \to \widehat{X_\tau}$.

4. Given a presentable ∞-category $\mathcal{E}$, we write $\mathrm{Shv}_\tau(\mathrm{C}, \mathcal{E})$ for the ∞-category of $\mathcal{E}$-valued sheaves on (C, τ). We also suppress the decoration $\mathcal{E}$ if $\mathcal{E} = \mathrm{Spc}$ is the ∞-category of spaces.

5. If C is a small ∞-category and $X \in \mathrm{C}$, then we denote by $h_X \in \mathrm{PShv}(\mathrm{C})$ the presheaf corresponding to X under the Yoneda embedding.

6. Let C be an ∞-category and $R \subset \mathrm{C}_{/X}$ be a sieve. By the correspondence in [Lur09, Proposition 6.2.2.5], this determines and is determined by a monomorphism $R \hookrightarrow h_X$.

1.5.2 (Higher) category theory

7. Let Cat_∞ and $\widehat{\mathrm{Cat}}_\infty$ denote the ∞-category of small, resp. large ∞-categories, and let $\mathrm{Pr}^{\mathrm{L}}_\infty \subset \widehat{\mathrm{Cat}}_\infty$ denote the subcategory of presentable ∞-categories and colimit-preserving functors thereof.

8. Given a geometric morphism $f^* : \mathcal{X} \rightleftarrows \mathcal{Y} : f_*$ of ∞-topoi, we will abuse notation and also write
$$f^* : \mathrm{Sp}(\mathcal{X}) \rightleftarrows \mathrm{Sp}(\mathcal{Y}) : f_*$$
for the induced adjunction on stabilizations. Note that since both f^* and f_* are left-exact, they are computed via postcomposition by the unstable f^* and f_* at the level of spectrum objects.

9. If C is a stable presentable symmetric monoidal ∞-category, we write $\mathrm{C}^\wedge_p$ for the p-completion of C (i.e., the Bousfield localization at $\mathbb{1}/p$). We have a localization functor $L_p : \mathrm{C} \to \mathrm{C}^\wedge_p$ and a fully faithful right adjoint $i_p : \mathrm{C}^\wedge_p \to \mathrm{C}$. We use [MNN17] as our basic reference for this material.

10. Let Top be the ordinary category of topological spaces. Then there is a functor $\mathrm{Sing} : \mathrm{Top} \to \mathrm{Spc}$ which sends a topological space X to $|\mathrm{Hom}(\Delta^\bullet, X)|$, where Δ^n is the standard n-simplex.

11. Given a functor $f : S \to \widehat{\mathrm{Cat}}_\infty$, we write $\int f \to S^{op}$ for the associated cartesian fibration [Lur09, §3.2].

12. Given a cartesian fibration $X \to S^{\mathrm{op}}$, we write $X^\vee \to S$ for the dual cocartesian fibration [BGN18].

13. Following standard terminology in parametrized higher category theory, a **G-∞-category** is a cocartesian fibration $\mathrm{C} \to \mathcal{O}_G^{\mathrm{op}}$, and a **$G$-functor** $\mathrm{C} \to \mathrm{D}$ is a functor that preserves cocartesian edges.[4] We refer to [QS19, Appendix A] for a quick primer on the theory of G-∞-categories. Let $\widehat{\mathrm{Cat}}_{\infty, \mathcal{O}_G}$ denote the ∞-category of G-∞-categories (so $\widehat{\mathrm{Cat}}_{\infty, \mathcal{O}_G} \simeq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \widehat{\mathrm{Cat}}_\infty))$).

14. Since $\widehat{\mathrm{Cat}}_{\infty, \mathcal{O}_G}$ is cartesian-closed, it has an internal hom: given G-∞-categories C and D, we let $\underline{\mathrm{Fun}}_G(\mathrm{C}, \mathrm{D})$ denote the G-∞-category of G-functors $\mathrm{C} \to \mathrm{D}$. The second author proffered an explicit construction of this internal hom at the level of marked simplicial sets in [Sha18, §3].

[4] To avoid confusion with the notion of a G-∞-topos, it would perhaps be more appropriate to call such objects $\mathcal{O}_G$-∞-categories and $\mathcal{O}_G$-functors, but we will stick to this terminology.

15. $\underline{\mathrm{Fun}}_G(\mathrm{C},\mathrm{D})$ serves as a parametrized enhancement of the ∞-category $\mathrm{Fun}_G(\mathrm{C},\mathrm{D})$ of G-functors $\mathrm{C}\to\mathrm{D}$. In general, if a given construction admits some sort of parametrized enhancement, then we will distinguish between the two possibilities by means of this 'underline' notation.

16. We adopt the conventions of [QS19, §1] for recollements. In particular, if $\mathcal{X}$ is an ∞-category, then a recollement on $\mathcal{X}$ is specified by a pair $(\mathcal{U},\mathcal{Z})$ where $\mathcal{U}$ is the open part and $\mathcal{Z}$ is the closed part, and we have the recollement adjunctions

$$\mathcal{U} \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathcal{X} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{Z}$$

in which the composite i^*j_* is said to be the **gluing functor** of the recollement. Conversely, given a left-exact functor $\phi:\mathcal{U}\to\mathcal{Z}$, the **right-lax limit**

$$\mathcal{X}=\mathcal{U}\times_{\phi,\mathcal{Z},\mathrm{ev}_1}\mathrm{Fun}(\Delta^1,\mathcal{Z})=\mathcal{U}\overrightarrow{\times}\mathcal{Z}$$

admits a recollement given by $(\mathcal{U},\mathcal{Z})$ with gluing functor equivalent to ϕ. Note also that in the stable setting a recollement is completely determined by a localizing and colocalizing stable subcategory [BG16].

1.5.3 Motivic homotopy theory

17. We denote by $\mathbb{T}$ the pointed motivic space $\mathbb{T}:=\frac{\mathbb{A}^1}{\mathbb{A}^1\setminus 0}$. We have a canonical equivalence $\mathbb{T}\xrightarrow{\simeq}\mathbb{P}^1$ of pointed motivic spaces.

18. Let τ be a Grothendieck topology and X a scheme. We will denote by

$$\mathrm{H}_\tau(X),\ \mathrm{H}_\tau(X)_\bullet,\ \mathrm{SH}^{S^1}_\tau(X),\ \mathrm{SH}_\tau(X)$$

the unstable, pointed, S^1-stable and $\mathbb{T}$-stable motivic ∞-categories defined with respect to the hypercomplete τ-topology. By definition, these are the localizations of the usual motivic categories

$$\mathrm{H}(X),\ \mathrm{H}(X)_\bullet,\ \mathrm{SH}^{S^1}(X),\ \mathrm{SH}(X)$$

at τ-hypercovers (or desuspensions thereof). See [ELSØ17, §2] for details.

19. We remark on the choice to work with the hypercomplete versions of motivic homotopy theory:

- Bachmann's theorems in [Bac18b] regarding étale motivic homotopy theory only hold after hypercompletion.
- In Appendix B.1, we prove that the real étale topos of a scheme of finite Krull dimension is hypercomplete. Thus, the hypercomplete and non-hypercomplete versions of real étale motivic homotopy theory coincide.

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2 Genuine stabilization of G-∞-topoi

As we recalled in the introduction, one constructs the stable ∞-category Sp^G of **genuine G-spectra** so as to have a good theory of compact and dualizable objects in equivariant stable homotopy theory. Traditionally, to do this one $\otimes$-inverts inside $\mathrm{Spc}_{G_\bullet}$ the set $\{S^V\}$ of (finite-dimensional) real representation spheres (see [Lew86] and [BH18, §9.2] for a modern treatment). More relevant for our purposes are two more recent and distinct approaches to constructing Sp^G:

1. Let $\mathrm{Mack}_G = \mathrm{Fun}^\times(\mathrm{Span}(\mathrm{Fin}_G), \mathrm{Sp})$ denote the ∞-category of **spectral Mackey functors**. A theorem of Guillou-May shows that Mack_G is equivalent to Sp^G [GM11]; see also [Nar16] for a proof of this result in the language of the current paper. This perspective was elaborated further in [Bar17] and was later shown by Nardin in [Nar16] to be an instance of a more general procedure that, given a G-∞-category, outputs a **G-stable G-∞-category** in the sense of [BDG+16].

2. Given $E \in \mathrm{Sp}^G$, the associated Mackey functor evaluates on an orbit G/H to the **categorical fixed points** spectrum E^H. The Guillou-May theorem thus says that a genuine G-spectrum can be understood diagramatically in terms of its categorical fixed points. These categorical fixed points can in turn be described via **geometric fixed points** and data coming from **Borel equivariant homotopy theory** (see [Bar17, B.7]). For example, if $G = C_p$ is the cyclic group of order p, then we have the pullback square

$$\begin{array}{ccc} E^{C_p} & \longrightarrow & E^{\Phi C_p} \\ \downarrow & & \downarrow \\ E^{hC_p} & \longrightarrow & E^{tC_p}, \end{array}$$

where E^{tC_p} is the cofiber of the additive norm map $E_{hC_p} \to E^{hC_p}$. From this point of view, one may reconstruct genuine G-spectra in terms of its "Borel pieces" (see [Gla15] and [AMGR17]).

The goal of this section is generalize this picture from G-spaces to G-∞-topoi.

Definition 2.1. Let G be a finite group. A **G-∞-topos** $\mathcal{X}$ is a G-object in $\mathscr{T}\mathrm{op}^R$, i.e., a functor

$$\mathcal{X} : BG \to \mathscr{T}\mathrm{op}^R.$$

Given a G-∞-topos $\mathcal{X}$, we will abuse notation and also refer to its underlying ∞-topos as $\mathcal{X}$.

Example 2.2. Let X be a topological space with a G-action through continuous maps. We may form the ∞-category $\mathrm{Shv}(X)$ of sheaves of spaces on X, which is naturally a G-∞-topos. For instance, if $X = *$ is the one-point space (with necessarily trivial G-action), then $\mathrm{Shv}(X) \simeq \mathrm{Spc}$ with trivial G-action, which is the terminal object in the ∞-category $(\mathscr{T}\mathrm{op}^R)^{BG} = \mathrm{Fun}(BG, \mathscr{T}\mathrm{op}^R)$ of G-∞-topoi.

In this section, we will explain how to functorially associate to a G-∞-topos $\mathcal{X}$ a certain ∞-category $\mathrm{Sp}^G(\mathcal{X})$ of "G-spectrum objects" in $\mathcal{X}$ (Definition 2.36), which we will regard as its **genuine stabilization**. For the terminal G-∞-topos Spc, this construction by design specializes to Mack_G (Example 2.37), so we are entitled to think of $\mathrm{Sp}^G(\mathcal{X})$ as a generalization of (1) wherein we adopt spectral Mackey functors as our preferred *definition* of G-spectra. Specializing to the case $G = C_p$, we then introduce the toposic generalization of the C_p-Tate construction (Definition 2.46) and prove a reconstruction theorem for $\mathrm{Sp}^{C_p}(\mathcal{X})$ (Theorem 2.47) that generalizes (2). We conclude by applying our reconstruction theorem to endow $\mathrm{Sp}^{C_p}(\mathcal{X})$ with a symmetric monoidal structure (Construction 2.60).

2.1 Toposic homotopy fixed points, homotopy orbits and genuine orbits

Suppose that C is an ∞-category that admits all (small) limits and colimits, so that we have the adjunctions

$$(-)^{\mathrm{triv}} : \mathrm{C} \rightleftarrows \mathrm{C}^{BG} : (-)^{hG} = \lim_{BG}, \qquad (-)_{hG} = \operatorname*{colim}_{BG} : \mathrm{C}^{BG} \rightleftarrows \mathrm{C} : (-)^{\mathrm{triv}}.$$

The functors $(-)^{hG}$ and $(-)_{hG}$ are called the **homotopy fixed points** and **homotopy orbits**, respectively.

Now let $\mathrm{C} = \mathscr{T}\mathrm{op}^R$, which admits all (small) limits and colimits in view of [Lur09, §6.3]. We are then interested in the homotopy fixed points and orbits as taken in $\mathscr{T}\mathrm{op}^R$, as opposed to $\widehat{\mathrm{Cat}}_\infty$. To distinguish between the two possibilities, we introduce the following modified notation/definition.

Definition 2.3. Let $\mathcal{X}$ be a G-∞-topos. Then we let

$$\mathcal{X}_{h_\diamond G} = \operatorname*{colim}_{BG} \mathcal{X}, \qquad \mathcal{X}^{h_\diamond G} = \lim_{BG} \mathcal{X}$$

denote the **toposic homotopy orbits** and **toposic homotopy fixed points** of $\mathcal{X}$, i.e., the colimit and limit over BG as formed in $\mathscr{T}\mathrm{op}^R$. We will also use the terms **homotopy orbits topos** and **homotopy fixed topos** to refer to $\mathcal{X}_{h_\diamond G}$ and $\mathcal{X}^{h_\diamond G}$.

By definition, these ∞-topoi come with the canonical geometric morphisms

$$\nu^* : \mathcal{X}^{h_\diamond G} \rightleftarrows \mathcal{X} : \nu_*, \qquad \pi^* : \mathcal{X} \rightleftarrows \mathcal{X}_{h_\diamond G} : \pi_*,$$

which are moreover G-equivariant with respect to the trivial G-actions on $\mathcal{X}_{h_\diamond G}$ and $\mathcal{X}^{h_\diamond G}$.

For brevity, we will also make use of the following notation.

Notation 2.4. Suppose $\mathcal{X}$ is a G-∞-topos. We let

$$\underline{\mathcal{X}} \to BG$$

denote the cartesian fibration associated to the functor

$$\mathcal{X}^{\mathrm{op}} : BG^{\mathrm{op}} \to \mathscr{T}\mathrm{op}^L \subset \widehat{\mathrm{Cat}}_\infty.$$

2.1.1 Toposic homotopy orbits

Recall that for any simplicial set S, we have a natural equivalence

$$(-)^\dagger : [S, \mathscr{T}\mathrm{op}^R] \xrightarrow{\simeq} [S^{\mathrm{op}}, \mathscr{T}\mathrm{op}^L] \tag{3}$$

that transports a diagram

$$\mathcal{F} : S \to \mathscr{T}\mathrm{op}^R \qquad (f : s \to t) \mapsto (f_* : \mathcal{F}(s) \to \mathcal{F}(t))$$

to the diagram

$$\mathcal{F}^\dagger : S \to \mathscr{T}\mathrm{op}^L \qquad (f : t \to s) \mapsto (f^* : \mathcal{F}(t) \to \mathcal{F}(s)),$$

where (f^*, f_*) defines a geometric morphism of ∞-topoi [Lur09, Corollary 6.3.1.8]. For clarity, we will always decorate functors into $\mathscr{T}\mathrm{op}^L$ by a dagger.

The following facts will then enable us to compute colimits in $\mathscr{T}\mathrm{op}^R$ using the equivalence (3):

1. Colimits in $\mathscr{T}\mathrm{op}^R$ are computed as limits of the corresponding diagram in $\mathscr{T}\mathrm{op}^L$.
2. The forgetful functor $\mathscr{T}\mathrm{op}^L \subset \widehat{\mathrm{Cat}}_\infty$ creates limits [Lur09, Proposition 6.3.2.3].
3. Given a diagram $f : K \to \widehat{\mathrm{Cat}}_\infty$, we have a natural equivalence [Lur09, Corollary 3.3.3.2]

$$\lim_K f \simeq \mathrm{Sect}^{\mathrm{cart}}_{K^{\mathrm{op}}}(\int f).$$

It follows that for a G-∞-topos $\mathcal{X}$, we have natural equivalences

$$\mathcal{X}_{h_\diamond G} \simeq \mathcal{X}^{hG} \simeq \mathrm{Sect}_{BG}(\underline{\mathcal{X}}).$$

Example 2.5. Suppose that $\mathcal{X}$ is a G-∞-topos where the action is trivial. Then

$$\mathcal{X}_{h_\diamond G} \simeq \mathcal{X}^{hG} \simeq \mathrm{Fun}(BG, \mathcal{X}).$$

In other words, taking toposic homotopy orbits computes the ∞-category of "Borel-equivariant" G-objects in $\mathcal{X}$.

We also introduce a construction to record the functoriality of the toposic homotopy orbits ranging over subgroups $H \leqslant G$.

Construction 2.6. Suppose $\mathcal{X} : BG \to \mathscr{T}\mathrm{op}^R$ is a G-∞-topos. Let

$$\mathcal{X}_{h_\diamond(-)} : \mathcal{O}_G \to \mathscr{T}\mathrm{op}^R$$

be the left Kan extension of $\mathcal{X}$ along the inclusion $BG \subset \mathcal{O}_G$. Under the equivalence (3), $\mathcal{X}_{h_\diamond(-)}$ corresponds to the functor

$$\mathcal{X}^\dagger_{h_\diamond(-)} : \mathcal{O}_G^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty, \qquad G/H \mapsto \mathrm{Fun}_{/BG}(BH, \underline{\mathcal{X}}),$$

with functoriality given by restriction in the source (using the action groupoid functor $\mathcal{O}_G \to \mathrm{Gpd}$ that sends G/H to BH). We denote the cocartesian fibration associated to $\mathcal{X}^\dagger_{h_\diamond(-)}$ by

$$\underline{\mathcal{X}}_{h_\diamond G} \to \mathcal{O}_G^{\mathrm{op}}.$$

In particular, $\underline{\mathcal{X}}_{h_\diamond G}$ constitutes an example of a G-∞-category.

Remark 2.7. Given a map $f : G/H \to G/K$ of G-orbits, the restriction functor

$$f^* : \mathrm{Fun}_{/BG}(BK, \underline{\mathcal{X}}) \to \mathrm{Fun}_{/BG}(BH, \underline{\mathcal{X}})$$

admits both left and right adjoints $f_!$ and f_* given by relative left and right Kan extension along $BH \to BK$ (cf. [QS19, §2.2.1] for the theory of relative Kan extensions along a general functor).

Example 2.8. Suppose that X is a scheme such that $\frac{1}{2} \in \mathcal{O}_X$. Then $X[i] \to X$ is a C_2-Galois cover and thus the ∞-topos $\widetilde{X[i]}_{\text{ét}}$ acquires a canonical C_2-action. The homotopy orbits topos of $\widetilde{X[i]}_{\text{ét}}$ is, by Galois descent, equivalent to the étale topos of X, so we have:

$$(\widetilde{X[i]}_{\text{ét}})_{h_\diamond C_2} \simeq (\widetilde{X[i]}_{\text{ét}})^{hC_2} \simeq \widetilde{X}_{\text{ét}}.$$

2.1.2 Toposic homotopy fixed points

Since limits in $\mathscr{T}\mathrm{op}^R$ are not computed at the level of the underlying ∞-categories, it is usually difficult to describe $\mathcal{X}^{h_\diamond G}$ explicitly. However, this task has been accomplished by Scheiderer in our main examples of interest at the level of 1-topoi, as we now recall. Let $\mathscr{T}\mathrm{op}_1^R$ denote the $(2,1)$-category of 1-topoi and geometric morphisms thereof, and let a G**-topos** be a G-object in $\mathscr{T}\mathrm{op}_1^R$. Note for the following two examples that Scheiderer's notion of the fixtopos of a G-topos ([Sch94, Definition 10.15]) corresponds to taking the limit over BG in $\mathscr{T}\mathrm{op}_1^R$ (cf. [Sch94, 10.15.1]; Scheiderer formulates his constructions in the setting of fibered topoi).

Example 2.9. Suppose that X is a topological space with G-action where G acts properly discontinuously on X, and consider the G-topos $\mathcal{X} = \mathrm{Shv}(X, \mathrm{Set})$. Then by [Sch94, Proposition 13.2], we have a canonical equivalence of 1-topoi

$$\mathrm{Shv}(X, \mathrm{Set})^{h_\diamond G} \simeq \mathrm{Shv}(X^G, \mathrm{Set}),$$

where X^G is the subspace of G-fixed points in X and the limit is taken in $\mathscr{T}\mathrm{op}_1^R$. Moreover, the geometric morphism (ν^*, ν_*) is the one induced by the inclusion of spaces $X^G \subset X$.

Example 2.10. Suppose that X is a scheme with $1/2 \in \mathcal{O}_X$, and consider the C_2-topos $\mathcal{X} = \mathrm{Shv}_{\text{ét}}(\text{Ét}_{X[i]}, \mathrm{Set})$. Then by [Sch94, Theorem 11.1.1], we have a canonical equivalence of 1-topoi

$$\mathrm{Shv}_{\text{ét}}(\text{Ét}_{X[i]}, \mathrm{Set})^{h_\diamond C_2} \simeq \mathrm{Shv}_{\text{rét}}(\text{Ét}_X, \mathrm{Set}),$$

where the limit is taken in $\mathscr{T}\mathrm{op}_1^R$.

We next indicate how to deduce equivalences at the level of ∞-topoi (we deal with Example 2.10; the argument for Example 2.9 is similar).

Theorem 2.11. *Suppose that X is a scheme with $1/2 \in \mathcal{O}_X$. Then there is a canonical equivalence*

$$\widetilde{X[i]}_{\text{ét}}^{h_\diamond C_2} \simeq \widetilde{X}_{\text{rét}}.$$

Proof. As we recall in Lemma B.11, there is a limit-preserving fully faithful functor $\mathscr{T}\mathrm{op}_1^R \to \mathscr{T}\mathrm{op}_\infty^R$ that sends a 1-topos to its associated 1-localic ∞-topos, such that for any 1-topos $\mathfrak{X} \simeq \mathrm{Shv}_\tau(X, \mathrm{Sets})$ presented as the 1-category of sheaves of sets on a site (X, τ) with finite limits, the associated 1-localic ∞-topos is $\mathrm{Shv}_\tau(X)$. Since the functor $\mathscr{T}\mathrm{op}_1^R \to \mathscr{T}\mathrm{op}_\infty^R$ preserves limits, the equivalence of Example 2.10 then implies the result at the level of sheaves of spaces. □

As we did with toposic homotopy orbits, we also introduce a construction that will record the functoriality of the toposic homotopy fixed points.

Construction 2.12. Suppose $\mathfrak{X} : BG \to \mathscr{T}\mathrm{op}^R$ is a G-∞-topos. Let

$$\mathfrak{X}^{h_\diamond(-)} : \mathcal{O}_G^{\mathrm{op}} \to \mathscr{T}\mathrm{op}^R$$

be the right Kan extension of $\mathfrak{X}$ along the inclusion $BG \to \mathcal{O}_G^{\mathrm{op}}$. Note that by the formula for right Kan extension, we have that $\mathfrak{X}^{h_\diamond(-)}$ evaluates to $\mathfrak{X}^{h_\diamond H} = \lim_{BH} \mathfrak{X}$ on G/H. We denote the cartesian fibration associated to $(\mathfrak{X}^{h_\diamond(-)})^\dagger$ by

$$\underline{\mathfrak{X}}^{h_\diamond G} \to \mathcal{O}_G^{\mathrm{op}}.$$

Remark 2.13. We caution the reader that $\underline{\mathfrak{X}}^{h_\diamond G}$ is not usually a cocartesian fibration over $\mathcal{O}_G^{\mathrm{op}}$, as its *cartesian* functoriality is that given by the *left* adjoints of a geometric morphism.

2.1.3 Toposic genuine orbits

In addition to the toposic homotopy orbits and homotopy fixed points, we will need the topos-theoretic counterpart of "genuine orbits". This is the most subtle of the three constructions and is one of the key innovations of this paper. First, we need a technical lemma.

Lemma 2.14. *Let S be a small ∞-category and $\mathcal{F} : S \to \mathscr{T}\mathrm{op}^R$ a functor.*

1. *The ∞-category of sections*
$$\mathrm{Sect}_S(\int \mathcal{F}^\dagger)$$
is an ∞-topos.

Now suppose $\phi : T \to S$ is a functor of small ∞-categories.

2. *The restriction functor*
$$\phi^* : \mathrm{Sect}_S(\int \mathcal{F}^\dagger) \to \mathrm{Sect}_T((\int \mathcal{F}^\dagger) \times_S T) \simeq \mathrm{Sect}_T(\int (\mathcal{F}\phi)^\dagger)$$
preserves finite limits and (small) colimits. In other words, ϕ^* is the left adjoint of a geometric morphism of ∞-topoi.

3. *Suppose that ϕ satisfies the following assumption: for any $x \in S$, the simplicial set $T \times_S S^{x/}$ is finite. Then the right adjoint ϕ_* is computed as relative right Kan extension[5] along ϕ.*

4. *Suppose that $\phi : T \to S$ is (equivalent to) a cocartesian fibration. Then ϕ^* admits a left adjoint $\phi_!$, computed as relative left Kan extension along ϕ. Moreover, given an object $x : T \to \int \mathcal{F}^\dagger$ in $\mathrm{Sect}_T(\int (\mathcal{F}\phi)^\dagger)$, $\phi_! x$ is computed by taking colimits fiberwise, i.e., for all $s \in S$, we have an equivalence*
$$(\phi_! x)(s) \simeq \mathrm{colim}(x|_{T_s} : T_s \to \mathcal{F}(s)). \tag{4}$$

[5] For us, the term "Kan extension" always refers to the concept of *pointwise* Kan extension.

Proof. 1. We have a topos fibration (in the sense of [Lur09, Definition 6.3.1.6])

$$\int \mathcal{F} \simeq (\int \mathcal{F}^{\dagger})^{\vee} \to S^{\mathrm{op}}.$$

By [QS19, (1.12)], we thus have an equivalence

$$\mathrm{Sect}_S(\int \mathcal{F}^{\dagger}) \simeq \mathrm{Fun}^{\mathrm{cocart}}_{/S^{\mathrm{op}}}(\mathrm{Tw}(S^{\mathrm{op}}), (\int \mathcal{F}^{\dagger})^{\vee}) \simeq \mathrm{Fun}^{\mathrm{cocart}}_{/S^{\mathrm{op}}}(\mathrm{Tw}(S^{\mathrm{op}}), \int \mathcal{F}).$$

The right hand side is then an ∞-topos by the criterion of [Lur09, Proposition 5.4.7.11].

2. Since the cartesian pullback functors of $\int \mathcal{F}^{\dagger}$ preserve finite limits and all colimits, finite limits and all colimits in $\mathrm{Sect}_S(\int \mathcal{F}^{\dagger})$ are computed pointwise. It follows that ϕ^* preserves finite limits and all colimits.

3. The content of the claim is that ϕ_* is computed by the pointwise formula for the relative right Kan extension. For this, it suffices to show that given a diagram

$$\begin{array}{ccccc} T & \xrightarrow{x} & \int(\mathcal{F}\phi)^{\dagger} & \xrightarrow{\rho} & \int \mathcal{F}^{\dagger} \\ \downarrow{\scriptstyle \phi} & & {\scriptstyle \phi_* x} \nearrow & & \downarrow{\scriptstyle p} \\ S & & \xrightarrow{=} & & S, \end{array}$$

there exists a dotted lift $\phi_* x$ that is a p-right Kan extension of ρx along ϕ in the sense of [QS19, Definition 2.8]. Given our hypotheses on the functor ϕ, this follows by [QS19, Remark 2.9].

4. This is immediate from the dual of [QS19, Corollary 2.11].

□

Definition 2.15. Suppose we have a diagram $\mathcal{F} : S \to \mathscr{T}\mathrm{op}^R$. The **toposic lax colimit** of $\mathcal{F}$ is the ∞-topos

$$\operatorname{lax.colim}_S \mathcal{F} = \mathrm{Sect}_S(\int \mathcal{F}^{\dagger}).$$

Remark 2.16. By [GHN17, Proposition 7.1], $\mathrm{Sect}_S(\int \mathcal{F}^{\dagger})$ computes the oplax limit of $\mathcal{F}^{\dagger}$ (as a functor into $\widehat{\mathrm{Cat}}_\infty$). This justifies the terminology of Definition 2.15.

Lemma 2.17. *1. The toposic lax colimit construction assembles into a functor*

$$\operatorname{lax.colim} : (\mathrm{Cat})^{\mathrm{op}}_{\infty/\mathscr{T}\mathrm{op}^R} \to \widehat{\mathrm{Cat}}_\infty.$$

2. Suppose that $F : K \to (\mathrm{Cat})^{\mathrm{op}}_{\infty/\mathscr{T}\mathrm{op}^R}$ is a functor such that for any $i \to j \in K$ and $x \in F(j)$, the simplicial set $F(i) \times_{F(j)} F(j)^{x/}$ is finite. Then there exists a canonical lift:

$$\begin{array}{ccc} K & \xrightarrow{\operatorname{lax.colim}} & \mathscr{T}\mathrm{op}^L \\ \downarrow{\scriptstyle F} & & \downarrow \\ (\mathrm{Cat})^{\mathrm{op}}_{\infty/\mathscr{T}\mathrm{op}^R} & \xrightarrow{\operatorname{lax.colim}} & \widehat{\mathrm{Cat}}_\infty. \end{array}$$

Proof. This follows immediately from Lemma 2.14. □

Construction 2.18. Let $\mathcal{X} : BG \to \mathscr{T}\mathrm{op}^R$ be a G-∞-topos. Consider the functor

$$\mathcal{O}_G \to (\mathrm{Cat}_\infty)_{/\mathscr{T}\mathrm{op}^R}, \qquad G/H \mapsto \left(\mathcal{X}^{h_\diamond(-)}|_{(\mathcal{O}_G^{\mathrm{op}})_{(G/H)/}} : (\mathcal{O}_G^{\mathrm{op}})_{(G/H)/} \to \mathscr{T}\mathrm{op}^R\right),$$

satisfying the hypotheses of Lemma 2.17. Then the formation of toposic lax colimits yields a functor

$$\mathcal{X}^{\dagger}_{(-)} : \mathcal{O}_G^{\mathrm{op}} \to \mathscr{T}\mathrm{op}^L,$$

whose value

$$\mathcal{X}_G = \mathrm{Sect}_{\mathcal{O}_G^{\mathrm{op}}}(\underline{\mathcal{X}}^{h_{\diamond}G}),$$

on G/G is the **toposic genuine orbits** of $\mathcal{X}$. We denote the associated *bicartesian* fibration over $\mathcal{O}_G^{\mathrm{op}}$ by

$$\underline{\mathcal{X}}_G \to \mathcal{O}_G^{\mathrm{op}}.$$

Given a G-equivariant geometric morphism $f_* : \mathcal{Y} \to \mathcal{X}$, we have an induced functor $f^* : \underline{\mathcal{X}}^{h_{\diamond}G} \to \underline{\mathcal{Y}}^{h_{\diamond}G}$ over $\mathcal{O}_G^{\mathrm{op}}$ that preserves cartesian edges, and hence postcomposition induces a functor upon taking sections

$$f^* : \mathcal{X}_G \to \mathcal{Y}_G.$$

Moreover, since colimits and finite limits are computed fiberwise in toposic lax colimits, f^* preserves colimits and finite limits and is thus the left adjoint of a geometric morphism.

As f^* is given by postcomposition of sections, it is compatible with pullback of sections, and hence we obtain an induced functor $f^* : \underline{\mathcal{X}}_G \to \underline{\mathcal{Y}}_G$. The assignment $\mathcal{X} \mapsto \underline{\mathcal{X}}_G$ thereby assembles to a functor valued in G-∞-categories

$$\mathrm{Fun}(BG, \mathscr{T}\mathrm{op}^L) \to \widehat{\mathrm{Cat}}_{\infty, \mathcal{O}_G}.$$

Remark 2.19. Suppose that X is a CW complex equipped with a G-action (in the 1-categorical sense) with at least one fixed point. Then by a result of Dror-Farjoun, for any subgroup $H \leqslant G$ we have an equivalence

$$\mathrm{Sing}_{\bullet}(X/H) \simeq \operatorname*{colim}_{K \in \mathcal{O}_{\mathrm{H}}} \mathrm{Sing}_{\bullet}(X^K),$$

where $X^K \subset X$ is the subspace of K-fixed points [Far96, Chapter 4, Lemma A.3]. In this light, we may view Construction 2.18 as a categorification of this formula for the genuine orbits of a G-space.

In the context of S-∞-categories (i.e., cocartesian fibrations over S), the theory of parametrized limits and colimits over S was comprehensively studied in the second author's thesis [Sha18]. When specialized to $S = \mathcal{O}_G^{\mathrm{op}}$, the resulting notions of admitting all **finite G-(co)products** or all **(finite) G-(co)limits** unwind to the following more explicit definition [Sha18, Proposition 5.11 and Corollary 12.15].[6]

Definition 2.20. A functor

$$\mathcal{F} : \mathrm{Fin}_G^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_{\infty}, \qquad (U \to V) \mapsto (f^* : \mathcal{F}(V) \to \mathcal{F}(U)),$$

is said to **admit finite G-products** (resp. **G-coproducts**) if

1. $\mathcal{F}$ is a right Kan extension of its restriction to $\mathcal{O}_G^{\mathrm{op}}$. In other words, given an orbit decomposition $U \simeq \coprod_i U_i$ of a finite G-set U, the canonical map $\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i)$ is an equivalence.

2. For any map $f : U \to V$ of finite G-sets, the functor $f^* : \mathcal{F}(V) \to \mathcal{F}(U)$ admits a right (resp. left) adjoint

$$f_* = \prod_f : \mathcal{F}(U) \to \mathcal{F}(V) \qquad (\text{resp. } f_! = \coprod_f : \mathcal{F}(U) \to \mathcal{F}(V)).$$

3. $\mathcal{F}$ satisfies the **Beck-Chevalley condition**: for any pullback diagram

$$\begin{array}{ccc} U' & \xrightarrow{g'} & U \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ V' & \xrightarrow{g} & V \end{array}$$

[6] The assertion that the existence of all *finite* G-(co)limits in the sense of [Sha18] (where the finiteness condition is on the diagram G-∞-category) is equivalent to the below formulation follows as a consequence of the technique of reduction to the Grothendieck construction as explained in [QS19, A.12].

of finite G-sets, the commutative diagram

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{g^*} & \mathcal{F}(V') \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f'^*} \\ \mathcal{F}(U) & \xrightarrow{g'^*} & \mathcal{F}(U') \end{array}$$

is right (resp. left) adjointable, i.e., the exchange transformation

$$f^* g_* \Rightarrow g'_* f'^*, \quad \text{resp.} \quad f'_! g'^* \Rightarrow g^* f_!$$

is an equivalence.

We then say that $\mathcal{F}$ **admits all (finite) G-limits** (resp. **admits all (finite) G-colimits**) if

1. $\mathcal{F}$ admits finite G-products (resp. finite G-coproducts).
2. For every finite G-set U, $\mathcal{F}(U)$ admits (finite) limits (resp. (finite) colimits).
3. For every map $f : U \to V$ of finite G-sets, the functor f^* is preserves all (finite) limits (resp. preserves all (finite) colimits).

Finally, we also say the same for the G-∞-category corresponding to $\mathcal{F}|_{\mathcal{O}_G^{\mathrm{op}}}$.

Remark 2.21. In Definition 2.20, if $\mathcal{F}$ admits *both* finite G-coproducts and finite G-products and $\mathcal{F}(U)$ is cocomplete and complete for all G-orbits U (e.g., presentable), then $\mathcal{F}$ admits all G-colimits and G-limits.

Remark 2.22. The functors $\mathcal{X}^\dagger_{h_\diamond(-)}$ and $\mathcal{X}^\dagger_{(-)}$ extend in a natural way from $\mathcal{O}_G^{\mathrm{op}}$ to $\mathrm{Fin}_G^{\mathrm{op}}$. For $\mathcal{X}^\dagger_{h_\diamond(-)}$, let FinGpd be the 1-category of finite groupoids, let $\omega : \mathrm{Fin}_G \to \mathrm{FinGpd}_{/BG}$ be the action groupoid functor $U \mapsto U//G$, and define the extension

$$\mathcal{X}^\dagger_{h_\diamond(-)} : \mathrm{Fin}_G^{\mathrm{op}} \xrightarrow{\omega} \mathrm{FinGpd}^{\mathrm{op}}_{/BG} \to \mathscr{T}\mathrm{op}^L, \quad U \mapsto \mathcal{X}_{h_\diamond U} = \mathrm{Fun}_{/BG}(U//G, \underline{\mathcal{X}}).$$

For $\mathcal{X}^\dagger_{(-)}$, let $\underline{U} \to \mathcal{O}_G^{\mathrm{op}}$ be the left fibration classified by $\mathrm{Hom}_{\mathrm{Fin}_G}(-, U)$ (covariantly functorial in maps of finite G-sets) and define the extension

$$\mathcal{X}^\dagger_{(-)} : \mathrm{Fin}_G^{\mathrm{op}} \to \mathscr{T}\mathrm{op}^L, \quad U \mapsto \mathcal{X}_U = \mathrm{Fun}_{/\mathcal{O}_G^{\mathrm{op}}}(\underline{U}, \underline{\mathcal{X}}^{h_\diamond G}).$$

Clearly, these extensions are right Kan extensions of their restrictions to $\mathcal{O}_G^{\mathrm{op}}$.

To proceed, we need to show that the G-∞-categories $\underline{\mathcal{X}}_{h_\diamond G}$ and $\underline{\mathcal{X}}_G$ admit all G-colimits and G-limits. By Remark 2.21, it suffices to show the existence of finite G-coproducts and finite G-products.

Proposition 2.23. *Let $\mathcal{X}$ be a G-∞-topos. Then the functors*

$$\mathcal{X}^\dagger_{h_\diamond(-)} : \mathrm{Fin}_G^{\mathrm{op}} \to \mathscr{T}\mathrm{op}^L \subset \widehat{\mathrm{Cat}}_\infty$$

and

$$\mathcal{X}^\dagger_{(-)} : \mathrm{Fin}_G^{\mathrm{op}} \to \mathscr{T}\mathrm{op}^L \subset \widehat{\mathrm{Cat}}_\infty$$

defined as in Remark 2.22 admit finite G-coproducts and finite G-products.

To prove this result, we need a few categorical preliminaries on the Beck-Chevalley condition.

Lemma 2.24. *Suppose that we have a diagram of ∞-categories*

$$\begin{array}{ccccc} I \times_J K & \xrightarrow{\psi'} & I & \xrightarrow{F} & C \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle p} \\ K & \xrightarrow{\psi} & J & \longrightarrow & S \end{array}$$

in which p is a cartesian fibration and the lefthand square is homotopy cartesian. Consider the class of simplicial sets

$$\mathcal{A} = \{I \times_J J^{y/} : y \in J\},$$

and suppose that the fibers of C admit $\mathcal{A}$-indexed limits and the cartesian pullback functors of C preserve $\mathcal{A}$-indexed limits. Then the p-right Kan extensions $\phi_* F$ and $\phi'_*(F \circ \psi')$ exist, and we may consider the exchange transformation

$$\chi : \psi^* \phi_* F \to \phi'_* \psi'^* F.$$

Furthermore, if the map θ_x of Lemma 2.25 is right cofinal for all $x \in K$, then χ is an equivalence.

Proof. The first claim follows from the pointwise existence criterion for p-right Kan extensions [QS19, Remark 2.9]. The second then follow immediately from the definitions. □

Lemma 2.25. *Suppose we have a homotopy pullback square of ∞-categories*

$$\begin{array}{ccc} I \times_J K & \xrightarrow{\psi'} & I \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} \\ K & \xrightarrow{\psi} & J. \end{array}$$

For $x \in K$, consider the induced map

$$\theta_x : I \times_J K \times_K K_{x/} \to I \times_J J_{\psi(x)/}.$$

1. *Suppose that ψ is equivalent to a left fibration and the induced functor $K_{x/} \to J_{\psi(x)/}$ has weakly contractible fibers. Then θ_x is right cofinal. In particular, if $\psi : K \to J$ is equivalent to a map of corepresentable left fibrations $f^* : S_{v/} \simeq (S_{v/})_{f/} \to S_{u/}$ for a morphism $f : u \to v$ in an ∞-category S, then θ_x is right cofinal.*

2. *Suppose that $\phi^{\mathrm{op}} : I^{\mathrm{op}} \to J^{\mathrm{op}}$ is smooth* [Lur09, Definition 4.1.2.9] *(for example, if ϕ is equivalent to a cartesian fibration* [Lur09, Proposition 4.1.2.15]*). Then θ_x is right cofinal.*

Proof. 1. By (the dual of) Joyal's version of Quillen's theorem A [Lur09, Theorem 4.1.3.1], it suffices to check that for any object $(a, \alpha) \in I \times_J J_{\psi(x)/}$, the pullback

$$(I \times_J K \times_K K_{x/}) \times_{I \times_J J_{\psi(x)/}} (I \times_J J_{\psi(x)/})_{/(a,\alpha)}$$

is weakly contractible. Note that we have a pullback square

$$\begin{array}{ccc} I \times_J K \times_K K_{x/} & \xrightarrow{\theta_x} & I \times_J J_{\psi(x)/} \\ \downarrow & & \downarrow \\ K_{x/} & \longrightarrow & J_{\psi(x)/}. \end{array} \tag{5}$$

Since ψ was assumed to be (equivalent to) a left fibration, the same holds for θ_x. Now, left fibrations are smooth by [Lur09, Proposition 4.1.2.15]. Since the inclusion of the final object

$$\{(a, \alpha)\} \hookrightarrow (I \times_J J_{\psi(x)/})_{/(a,\alpha)}$$

is a left cofinal inclusion and hence right anodyne [Lur09, Proposition 4.1.1.3(4)], we have that the induced map

$$(I \times_J K \times_K K_{x/}) \times_{I \times_J J_{\psi(x)/}} \{(a, \alpha)\} \to (I \times_J K \times_K K_{x/}) \times_{I \times_J J_{\psi(x)/}} (I \times_J J_{\psi(x)/})_{/(a,\alpha)}$$

continues to be left cofinal since right anodyne maps are preserved under base change along a smooth morphism [Lur09, Proposition 4.1.2.8].

But now, from the cartesian diagram (5), we have an equivalence

$$(K_{x/})_\alpha \simeq (I \times_J K \times_K K_{x/}) \times_{I \times_J J_{\psi(x)/}} \{(a, \alpha)\}.$$

Thus, it further suffices to check that for any object $\alpha : \psi(x) \to y$ in $J_{\psi(x)/}$, the fiber $(K_{x/})_\alpha$ is weakly contractible, which is true by assumption.

Finally, suppose that ψ is equivalent to a functor $f^* : S_{v/} \to S_{u/}$ for a morphism $f : u \to v$ in S, and let $[g : v \to w] \in S_{v/}$ be any object. Then the induced functor

$$(S_{v/})_{g/} \xrightarrow{\simeq} (S_{x/})_{(g\circ f)/} \simeq S_{w/}$$

is an equivalence, verifying the criterion of (1).

2. We may factor θ_x as

$$\begin{array}{ccc} I \times_J K \times_K \{x\} & \xrightarrow{\cong} & I \times_J \{\psi(x)\} \\ \downarrow \iota & & \downarrow \iota' \\ I \times_J K \times_K K_{x/} & \xrightarrow{\theta_x} & I \times_J J_{\psi(x)/}. \end{array}$$

Since ϕ^{op} is smooth and $\{x\} \to K_{x/}$ is a right cofinal inclusion, we deduce that ι is right cofinal, using now the dual of [Lur09, Proposition 4.1.2.8]. Likewise, ι' is right cofinal. The right cancellative property of right cofinal maps [Lur09, Proposition 4.1.1.3(2)] then shows that θ_x is right cofinal.

□

Proof of Proposition 2.23. We first verify the claim for $\mathcal{X}^\dagger_{(-)}$. Let $f : U \to V$ be any map of finite G-sets and also denote by f the induced map of G-∞-categories $\underline{U} \to \underline{V}$. By Lemma 2.14.3 and Lemma 2.14.4, the coinduction and induction functors $f_*, f_! : \mathcal{X}_U \to \mathcal{X}_V$ exist and are computed by relative right and left Kan extension along f. It remains to check the Beck-Chevalley condition. For this, suppose we have a pullback square of finite G-sets

$$\begin{array}{ccc} W \times_V U & \xrightarrow{g'} & U \\ \downarrow f' & & \downarrow f \\ W & \xrightarrow{g} & V. \end{array}$$

Without loss of generality, we may suppose that U, V, W are orbits. Then the exchange transformation $f^* g_* \Rightarrow g'_* f'^*$ is an equivalence by Lemma 2.24 and Lemma 2.25.1, and the exchange transformation $f'_! g'^* \Rightarrow g^* f_!$ is an equivalence by the dual of Lemma 2.25.2.

The claim for $\mathcal{X}_{h_\diamond(-)}$ follows by the same argument where $\mathcal{O}_G^{\mathrm{op}}$ is replaced by BG, and we use instead Lemma 2.25.2 for the coinduction functors and its dual for the induction functors. □

We are now prepared to establish functoriality of the assignment $\mathcal{X} \mapsto \underline{\mathcal{X}}_G$ in the *right* adjoint of a G-equivariant geometric morphism. For the following proposition, recall the notion of a relative adjunction [Lur17a, §7.3.2]. Given a relative adjunction $L \dashv R$ of G-∞-categories, we say that $L \dashv R$ is a *G-adjunction* if both L and R preserve cocartesian edges over $\mathcal{O}_G^{\mathrm{op}}$ [Sha18, §8].

Proposition 2.26. *Let $f_* : \mathcal{X} \to \mathcal{Y}$ be a G-equivariant geometric morphism of G-∞-topoi and consider the G-functor $f^* : \underline{\mathcal{Y}}_G \to \underline{\mathcal{X}}_G$. Then for any map of G-orbits $\alpha : U \to V$, the exchange transformation*

$$\chi : \alpha^* f_* \Rightarrow f_* \alpha^*$$

is an equivalence. Therefore, the fiberwise pushforward functors f_* assemble into a G-functor $f_* : \underline{\mathcal{X}}_G \to \underline{\mathcal{Y}}_G$ that is G-right adjoint to f^*.

Similarly, we have a G-adjunction $f^* : \underline{\mathcal{Y}}_{h_\diamond G} \rightleftarrows \underline{\mathcal{X}}_{h_\diamond G} : f_*$.

Proof. By the general theory of relative adjunctions, the functor $f_* : \underline{\mathcal{X}}_G \to \underline{\mathcal{Y}}_G$ exists as a relative right adjoint to f^* [Lur17a, Proposition 7.3.2.6], but does not necessarily preserve cocartesian edges. To check that it does amounts to verifying that χ is an equivalence for all morphisms α in $\mathcal{O}_G$.

By Proposition 2.23, the induction functor $\alpha_!$ exists. Therefore, it suffices to show the adjoint exchange transformation $\chi' : \alpha_! f^* \Rightarrow f^* \alpha_!$ is an equivalence. But this is evident in view of the fiberwise colimit description of $\alpha_!$ (4) as well as the definition of f^* as given by postcomposition of sections.

Finally, the analogous claim for $\underline{\mathcal{X}}_{h_\diamond G}$ is similar but easier. □

In this way, the assignment $\mathcal{X} \mapsto \underline{\mathcal{X}}_G$ assembles into a functor

$$\underline{(-)}_G : \mathrm{Fun}(BG, \mathscr{T}\mathrm{op}^R) \to \widehat{\mathrm{Cat}}_{\infty,\mathcal{O}_G}.$$

2.2 Genuine stabilization: the Mackey approach

The parametrized toposic genuine orbits construction is designed so as to feed into the machinery of parametrized ∞-categories. In particular, we wish to now deploy Nardin's G-stabilization construction. To explain this notion, first recall from [Lur17a, §1.4] that we have a functor

$$\mathrm{Sp} : \widehat{\mathrm{Cat}}^{\mathrm{lex}}_{\infty} \to \widehat{\mathrm{Cat}}^{\mathrm{ex}}_{\infty,\mathrm{stab}}$$

that assigns to an ∞-category C with finite limits the stable ∞-category Sp(C) of **spectrum objects** in C, which is the universal stable ∞-category equipped with a left-exact functor $\Omega^\infty : \mathrm{Sp}(\mathrm{C}) \to \mathrm{C}$.

Definition 2.27. Let $p : \mathrm{C} \to S$ be a cocartesian fibration that straightens to a functor $S \to \mathrm{Cat}^{\mathrm{lex}}_{\infty}$. We then write $\underline{\mathrm{Sp}}(\mathrm{C}) \to S$ for the **fiberwise stabilization** of p, and $\Omega^\infty : \underline{\mathrm{Sp}}(\mathrm{C}) \to \mathrm{C}$ for the induced functor over S (which preserves cocartesian edges). If p straightens to a functor valued in $\widehat{\mathrm{Cat}}^{\mathrm{ex}}_{\infty,\mathrm{stab}}$ (or equivalently, if Ω^∞ is an equivalence), then we say that C is **fiberwise stable**.

Along with fiberwise stability, we need the additional concept of **G-semiadditivity** to formulate G-stability.

Definition 2.28. Let C be a G-∞-category, and write $\mathcal{F} : \mathrm{Fin}^{\mathrm{op}}_G \to \widehat{\mathrm{Cat}}_{\infty}$ for the functor corresponding to C. Suppose that C admits finite G-products and coproducts and is pointed.[7] Then C is **G-semiadditive** if for all $f : U \to V$ in Fin_G, the canonical comparison transformation of [Nar16, Construction 5.2]

$$\coprod_f (-) \Rightarrow \prod_f (-)$$

is an equivalence.

Definition 2.29. Let C be a pointed G-∞-category that admits finite G-products and G-coproducts. We say that C is **G-stable** if it is fiberwise stable and G-semiadditive.

Let $\widehat{\mathrm{Cat}}^{\mathrm{lex}}_{\infty,\mathcal{O}_G}$ be the ∞-category of G-∞-categories that admit finite G-limits and G-left-exact G-functors, and let $\widehat{\mathrm{Cat}}^{\mathrm{ex,stab}}_{\infty,\mathcal{O}_G}$ be the ∞-category of G-stable G-∞-categories and G-exact G-functors. We now recall Nardin's construction of the G-stabilization functor

$$\underline{\mathrm{Sp}}^G : \widehat{\mathrm{Cat}}^{\mathrm{lex}}_{\infty,\mathcal{O}_G} \to \widehat{\mathrm{Cat}}^{\mathrm{ex,stab}}_{\infty,\mathcal{O}_G}.$$

Construction 2.30. Suppose that C is a G-∞-category that admits finite G-limits, and let $\mathrm{C}_H = \mathrm{C} \times_{\mathcal{O}^{\mathrm{op}}_G} \mathcal{O}^{\mathrm{op}}_H$ denote the restriction of C to a H-∞-category.[8] Let $\underline{\mathrm{Fin}}_{G*}$ denote the G-category of finite pointed G-sets [Nar16, Definition 4.12], whose fiber over G/H is equivalent to Fin_{H*}. Define the G-∞-category of **G-commutative monoids** in C to be the full G-subcategory

$$\underline{\mathrm{CMon}}^G(\mathrm{C}) \subset \underline{\mathrm{Fun}}_G(\underline{\mathrm{Fin}}_{G*}, \mathrm{C})$$

whose fiber $\mathrm{CMon}^H(\mathrm{C}_H)$ over G/H is the full subcategory on those H-functors $\underline{\mathrm{Fin}}_{H*} \to \mathrm{C}_H$ that send finite H-coproducts to finite H-products [Nar16, Definition 5.9]. $\underline{\mathrm{CMon}}^G(\mathrm{C})$ is then G-semiadditive [Nar16, Proposition 5.8].

We also have an alternative model for $\underline{\mathrm{CMon}}^G(\mathrm{C})$, given by a parametrized variant of the technology of Mackey functors. To define this, we need the **effective Burnside G-∞-category** $\underline{\mathrm{A}}^{\mathrm{eff}}(\mathrm{Fin}_G)$ [Nar16,

[7] In other words, for each $U \in \mathrm{Fin}_G$, the canonical comparison map from the initial to the terminal object in $\mathcal{F}(U)$ is an equivalence.

[8] We implicitly use the identification $\mathcal{O}^{\mathrm{op}}_H \simeq (\mathcal{O}^{\mathrm{op}}_G)^{(G/H)/}$.

Definition 4.10], whose fiber over G/H is equivalent to the effective Burnside ∞-category[9] $\mathrm{A}^{\mathrm{eff}}(\mathrm{Fin}_H)$ of finite H-sets (formed by taking spans in Fin_H). One may check that $\underline{\mathrm{A}}^{\mathrm{eff}}(\mathrm{Fin}_G)$ admits finite G-products and G-coproducts and is moreover G-semiadditive. Define the G-∞-category of **G-Mackey functors** in C to be the full G-subcategory

$$\underline{\mathrm{Mack}}^G(\mathrm{C}) = \underline{\mathrm{Fun}}^{\times}_G(\underline{\mathrm{A}}^{\mathrm{eff}}(\mathrm{Fin}_G), \mathrm{C}) \subset \underline{\mathrm{Fun}}_G(\underline{\mathrm{A}}^{\mathrm{eff}}(\mathrm{Fin}_G), \mathrm{C})$$

whose fiber $\mathrm{Mack}^H(\mathrm{C}_H)$ over G/H is the full subcategory on those H-functors $\mathrm{A}^{\mathrm{eff}}(\mathrm{Fin}_H) \to \mathrm{C}_H$ that preserve finite H-products. We then have an equivalence [Nar16, Theorem 6.5]

$$\underline{\mathrm{Mack}}^G(\mathrm{C}) \xrightarrow{\simeq} \underline{\mathrm{CMon}}^G(\mathrm{C})$$

implemented by restriction along a certain G-functor $\underline{\mathrm{Fin}}_{G*} \to \underline{\mathrm{A}}^{\mathrm{eff}}(\mathrm{Fin}_G)$.

In either case, we write U for the forgetful G-functor given by evaluation along the cocartesian section $\mathcal{O}_G^{\mathrm{op}}$ that selects either G/G_+ in Fin_{G*} or G/G in $\mathrm{A}^{\mathrm{eff}}(\mathrm{Fin}_G)$ (so U is compatible with the stated equivalence). $U : \underline{\mathrm{CMon}}^G(\mathrm{C}) \to \mathrm{C}$ is then universal among G-left-exact functors from a G-semiadditive G-∞-category [Nar16, Corollary 5.11.1], and likewise for $U : \underline{\mathrm{Mack}}^G(\mathrm{C}) \to \mathrm{C}$.[10] In particular, if C is G-semiadditive to begin with, then U is an equivalence [Nar16, Proposition 5.11].

Finally, define the **G-stabilization** $\underline{\mathrm{Sp}}^G(\mathrm{C})$ to be [Nar16, Definition 7.3]

$$\underline{\mathrm{Sp}}^G(\mathrm{C}) = \underline{\mathrm{Sp}}(\underline{\mathrm{CMon}}^G(\mathrm{C})).$$

Since $\underline{\mathrm{Sp}}$ preserves the property of G-semiadditivity [Nar16, Lemma 7.2], the G-∞-category $\underline{\mathrm{Sp}}^G(\mathrm{C})$ is G-stable. Define $\Omega^\infty : \underline{\mathrm{Sp}}^G(\mathrm{C}) \to \mathrm{C}$ to be the composite

$$\underline{\mathrm{Sp}}^G(\mathrm{C}) = \underline{\mathrm{Sp}}(\underline{\mathrm{CMon}}^G(\mathrm{C})) \xrightarrow{\Omega^\infty} \underline{\mathrm{CMon}}^G(\mathrm{C}) \xrightarrow{U} \mathrm{C}.$$

Then Ω^∞ is universal among G-left-exact G-functors to C from G-stable G-∞-categories [Nar16, Theorem 7.4]. For later use, we also observe by the above facts that the forgetful G-functor implements an equivalence

$$U : \underline{\mathrm{CMon}}^G(\underline{\mathrm{Sp}}^G(\mathrm{C})) \xrightarrow{\simeq} \underline{\mathrm{Sp}}^G(\mathrm{C}). \tag{6}$$

We now elaborate on the functoriality of genuine stabilization. Suppose D is another G-∞-category that admits finite G-limits, and let $F : \mathrm{C} \to \mathrm{D}$ be a G-left-exact G-functor. Then via postcomposition we have induced G-left-exact G-functors

$$F : \underline{\mathrm{CMon}}^G(\mathrm{C}) \to \underline{\mathrm{CMon}}^G(\mathrm{D}), \qquad F : \underline{\mathrm{Mack}}^G(\mathrm{C}) \to \underline{\mathrm{Mack}}^G(\mathrm{D}),$$
$$F : \underline{\mathrm{Sp}}(\mathrm{C}) \to \underline{\mathrm{Sp}}(\mathrm{D}), \qquad F : \underline{\mathrm{Sp}}^G(\mathrm{C}) \to \underline{\mathrm{Sp}}^G(\mathrm{D})$$

that commute with U and Ω^∞.

Nardin constructs $\underline{\mathrm{Sp}}^G$ via the two-step procedure of G-semiadditivization and fiberwise stabilization. The next lemma shows that we may perform these two operations in either order. Though simple, it will be crucial at various points of the paper.

Lemma 2.31. *Let* C *be a G-∞-category that admits finite G-limits. Then there is a canonical equivalence of G-∞-categories*

$$\underline{\mathrm{CMon}}^G(\underline{\mathrm{Sp}}(\mathrm{C})) \simeq \underline{\mathrm{Sp}}(\underline{\mathrm{CMon}}^G(\mathrm{C})).$$

Proof. We already know that $\underline{\mathrm{Sp}}$ of a G-semiadditive G-∞-category is again G-semiadditive (hence G-stable). It remains to show that given a fiberwise stable G-∞-category D that admits finite G-limits, $\mathrm{CMon}^G(\mathrm{D})$ is again stable (and thus $\underline{\mathrm{CMon}}^G(\mathrm{D})$ is fiberwise stable). To see this, we use the fully faithful embedding

$$\mathrm{CMon}^G(\mathrm{D}) \simeq \mathrm{Fun}^{\times}_G(\underline{\mathrm{A}}^{\mathrm{eff}}(\mathrm{Fin}_G), \mathrm{D}) \subset \mathrm{Fun}_G(\underline{\mathrm{A}}^{\mathrm{eff}}(\mathrm{Fin}_G), \mathrm{D}).$$

[9] In fact, $\mathrm{A}^{\mathrm{eff}}(\mathrm{Fin}_H)$ is a $(2,1)$-category.

[10] In the cited reference, Nardin more generally proves that U is universal among G-semiadditive functors $\mathrm{D} \to \mathrm{C}$ from pointed G-∞-categories D that admit finite G-coproducts.

Given any G-∞-category K, in view of our assumption on D, finite limits and colimits in $\mathrm{Fun}_G(K, \mathrm{D})$ exist and are computed fiberwise, hence $\mathrm{Fun}_G(K, \mathrm{D})$ is stable. It thus suffices to show that given a finite diagram p of G-functors $\underline{\mathrm{A}}^{\mathrm{eff}}(\mathrm{Fin}_G) \to \mathrm{D}$ that preserve finite G-products, both the limit and colimit of p as computed in $\mathrm{Fun}_G(\underline{A}^{\mathrm{eff}}(\mathrm{Fin}_G), \mathrm{D})$ is again a G-functor that preserves finite G-products. But for any map $f : U \to V$ of G-orbits, our assumption is that $f_* : \mathrm{D}_U \to \mathrm{D}_V$ is an exact functor, so the claim follows.

We then have comparison functors

$$\alpha : \underline{\mathrm{CMon}}^G(\underline{\mathrm{Sp}}(\mathrm{C})) \to \underline{\mathrm{Sp}}(\underline{\mathrm{CMon}}^G(\mathrm{C})), \quad \beta : \underline{\mathrm{Sp}}(\underline{\mathrm{CMon}}^G(\mathrm{C})) \to \underline{\mathrm{CMon}}^G(\underline{\mathrm{Sp}}(\mathrm{C}))$$

obtained via the universal properties of $\underline{\mathrm{CMon}}^G$ as G-semiadditivization and $\underline{\mathrm{Sp}}$ as fiberwise stabilization. These are easily checked to be mutually inverse equivalences, using again the universal properties. □

We next consider the presentable situation.

Lemma 2.32. *Let* C *be a* G-∞*-category that admits finite* G*-products and is fiberwise presentable. Then* $\mathrm{CMon}^G(\mathrm{C})$*, resp.* $\mathrm{Mack}^G(\mathrm{C})$ *is an accessible localization of* $\mathrm{Fun}_G(\mathrm{Fin}_{G*}, \mathrm{C})$*, resp.* $\mathrm{Fun}_G(\underline{\mathrm{A}}^{\mathrm{eff}}(\mathrm{Fin}_G), \mathrm{C})$*. Therefore,* $\mathrm{CMon}^G(\mathrm{C})$ *and* $\mathrm{Sp}^G(\mathrm{C})$ *are presentable.*

Proof. Note that under our hypotheses, $\mathrm{C} \to \mathcal{O}_G^{\mathrm{op}}$ is a presentable fibration [Lur09, Definition 5.5.3.2]. We prove the first claim; the second will follow by an identical argument (or one may use [Nar16, Theorem 6.5]). By [Lur09, Proposition 5.4.7.11], for any small G-∞-category K, $\mathrm{Fun}_G(K, \mathrm{C})$ is presentable. Since right adjoints preserve all limits, it is clear that $i : \mathrm{CMon}^G(\mathrm{C}) \subset \mathrm{Fun}_G(\underline{\mathrm{Fin}}_{G*}, \mathrm{C})$ is closed under limits. Now let κ be a regular cardinal such that for all maps $f : U \to V$ in $\mathcal{O}_G$, $f_* : \mathrm{C}_U \to \mathrm{C}_V$ is κ-accessible. Then i is likewise κ-accessible. We conclude that i is the inclusion of an accessible localization of a presentable ∞-category, and thus $\mathrm{CMon}^G(\mathrm{C})$ is presentable [Lur09, Remark 5.5.1.6]. Since Sp of a presentable ∞-category is again presentable [Lur17a, Proposition 1.4.4.4], we further conclude that $\mathrm{Sp}^G(\mathrm{C})$ is presentable. □

Notation 2.33. In Lemma 2.32, denote the localization functor by L_{sad}.

Remark 2.34. Suppose C admits finite G-products and is fiberwise presentable, so $\underline{\mathrm{Sp}}^G(\mathrm{C})$ is fiberwise presentable by Lemma 2.32. Then the G-functor $\Omega^\infty : \underline{\mathrm{Sp}}^G(\mathrm{C}) \to \mathrm{C}$ admits a G-left adjoint $\Sigma^\infty_+ : \mathrm{C} \to \underline{\mathrm{Sp}}^G(\mathrm{C})$ by [Lur17a, Proposition 7.3.2.11]. Moreover, if C admits finite G-coproducts (and thus all G-colimits), then Σ^∞_+ preserves finite G-coproducts (and any G-colimit) as a G-left adjoint [Sha18, Corollary 8.7]. Similarly, the forgetful G-functor $U : \underline{\mathrm{CMon}}^G(\mathrm{C}) \to \mathrm{C}$ admits a G-left adjoint Fr.

We obtain additional functoriality for $\underline{\mathrm{Sp}}^G$, etc., in the presentable setting.

Proposition 2.35. *Suppose* C *and* D *be* G-∞*-categories that admit finite* G*-products and are fiberwise presentable, and let* $F : \mathrm{C} \rightleftarrows \mathrm{D} : R$ *be a* G*-adjunction. Then we obtain induced* G*-adjunctions*

$$F : \underline{\mathrm{CMon}}^G(\mathrm{C}) \rightleftarrows \underline{\mathrm{CMon}}^G(\mathrm{D}) : R, \qquad F : \underline{\mathrm{Mack}}^G(\mathrm{C}) \rightleftarrows \underline{\mathrm{Mack}}^G(\mathrm{D}) : R$$
$$F : \underline{\mathrm{Sp}}(\mathrm{C}) \rightleftarrows \underline{\mathrm{Sp}}(\mathrm{D}) : R, \qquad F : \underline{\mathrm{Sp}}^G(\mathrm{C}) \rightleftarrows \underline{\mathrm{Sp}}^G(\mathrm{D}) : R$$

such that F *commutes with* Σ^∞_+ *and* Fr*, and* R *commutes with* Ω^∞ *and* U*.*

Proof. Since Sp is given as a functor $\mathrm{Sp} : \mathrm{Pr}^{\mathrm{L}}_\infty \to \mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}}$, we already know the assertion regarding fiberwise stabilization. By [Sha18, Corollary 8.3], for any G-∞-category K (and not using our hypotheses on C and D), we obtain an induced G-adjunction via postcomposition

$$F : \underline{\mathrm{Fun}}_G(K, \mathrm{C}) \rightleftarrows \underline{\mathrm{Fun}}_G(K, \mathrm{D}) : R.$$

Since the right G-adjoint R preserves G-commutative monoids (or G-Mackey functors), we may define the left G-adjoint F at the level of $\underline{\mathrm{CMon}}^G$ as postcomposition followed by L_{sad}. In this manner, we obtain the desired G-adjunctions. Finally, we already noted the claim about R and Ω^∞, U above, and the other one follows by adjunction. □

Let us now return to our study of G-∞-topoi.

Definition 2.36. Let $\mathcal{X}$ be a G-∞-topos. Then by Proposition 2.23, $\underline{\mathcal{X}}_G$ admits finite G-limits, so we may define its **parametrized genuine stabilization** to be the G-stable G-∞-category

$$\underline{\mathrm{Sp}}^G(\mathcal{X}) = \underline{\mathrm{Sp}}^G(\underline{\mathcal{X}}_G),$$

and its **genuine stabilization** to be the fiber $\mathrm{Sp}^G(\mathcal{X}) = \underline{\mathrm{Sp}}^G(\mathcal{X})_{G/G}$. We also refer to objects in $\mathrm{Sp}^G(\mathcal{X})$ as **G-spectrum objects** in $\mathcal{X}$.

Note that given a G-equivariant geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$ of G-∞-topoi, by Proposition 2.26 and Proposition 2.35 we have an induced G-adjunction

$$f^* : \underline{\mathrm{Sp}}^G(\mathcal{Y}) \rightleftarrows \underline{\mathrm{Sp}}^G(\mathcal{X}) : f_*.$$

Example 2.37. Consider the terminal G-∞-topos Spc. Then the right Kan extension $\mathrm{Spc}^{h_\diamond(-)} : \mathcal{O}_G^{\mathrm{op}} \to \mathscr{T}\mathrm{op}^R$ is the constant functor at Spc because limits of terminal objects are again terminal objects. The toposic genuine orbits are then given by

$$\mathrm{Spc}_H = \mathrm{lax.colim}(\mathrm{Spc}^{h_\diamond(-)}|_{(\mathcal{O}_G^{\mathrm{op}})_{G/H/}}) = \mathrm{Sect}_{(\mathcal{O}_G^{\mathrm{op}})_{G/H/}}(\mathrm{Spc} \times (\mathcal{O}_G^{\mathrm{op}})_{G/H/}) \simeq \mathrm{Fun}(\mathcal{O}_H^{\mathrm{op}}, \mathrm{Spc}).$$

Thus, Construction 2.18 produces the usual G-∞-category $\underline{\mathrm{Spc}}_G$ of G-spaces [Nar16, Definition 2.7]. Definition 2.36 then yields the G-∞-category $\underline{\mathrm{Sp}}^G$ of G-spectra, whose fiber over G/H is the usual ∞-category Sp^H of genuine H-spectra.

The next lemma shows that the genuine stabilization of $\underline{\mathcal{X}}_{h_\diamond G}$ is given already by fiberwise stabilization.

Lemma 2.38. *Let $\mathcal{X}$ be a G-∞-topos. Then*

1. *The fiberwise stabilization of $\underline{\mathcal{X}}_{h_\diamond G}$ is in addition G-stable, i.e., it is G-semiadditive.*
2. *There are canonical equivalences of stable ∞-categories*

$$\mathrm{Sp}^G(\underline{\mathcal{X}}_{h_\diamond G}) \simeq \mathrm{Sp}(\mathcal{X}_{h_\diamond G}) \simeq \mathrm{Sp}(\mathcal{X}^{hG}) \simeq \mathrm{Sp}(\mathcal{X})^{hG}.$$

Proof. The second claim is equivalent to saying that the genuine stabilization of $\underline{\mathcal{X}}_{h_\diamond G}$ is given by the fiberwise stabilization. Therefore, it suffices to verify the first claim. To verify G-semiadditivity of $\underline{\mathrm{Sp}}(\underline{\mathcal{X}}_{h_\diamond G})$, note first that the functors

$$\pi_*, \pi_! : \mathcal{X} \to \mathcal{X}_{h_\diamond G} \simeq \mathrm{Sect}_{BG}(\underline{\mathcal{X}})$$

that are right and left adjoint to the restriction functor π^* are computed by the pointwise formula for relative right and left Kan extension as the G-indexed product and coproduct (for instance, given $x \in \mathcal{X}$, we have $(\pi_* x)(*) \simeq \prod_G g^* x$). Since finite products and coproducts coincide in a stable ∞-category, we deduce that the canonical comparison map $\pi_! \to \pi_*$ is an equivalence after stabilization. The same reasoning holds for the left and right adjoints to the restriction f^* induced by any map $f : U \to V$ of finite G-sets, so we may conclude. □

Example 2.39. Let $\mathcal{X}$ be the terminal G-∞-topos Spc. Then $\underline{\mathrm{Sp}}^G(\underline{\mathcal{X}}_{h_\diamond G})$ is the G-∞-category of Borel G-spectra, whose fiber over G/H is equivalent to $\mathrm{Fun}(BH, \mathrm{Sp})$.

2.3 A Borel approach for cyclic groups of prime order

In this subsection, we undertake a *Borel* approach to genuine stabilization for G-∞-topoi in the sense of [Gla15] or [AMGR17] (see also [QS19, Theorem 3.44]). However, we will not fully generalize the stratification theory of [Gla15] to accommodate G-∞-topoi for an arbitrary finite group G, instead restricting our attention to the simplest case of the cyclic group C_p of prime order p. In fact, the case $p = 2$ is the only case of interest for our applications in algebraic geometry. We begin with the unstable situation.

Consider the G-equivariant geometric morphism $\nu_* : \mathcal{X}^{h_\diamond G} \to \mathcal{X}$ of Construction 2.12. Taking its toposic homotopy orbits, we obtain the geometric morphism

$$\nu(G)_* : (\mathcal{X}^{h_\diamond G})_{h_\diamond G} \to \mathcal{X}_{h_\diamond G}.$$

Since $\mathcal{X}^{h_\diamond G}$ has trivial G-action, by Example 2.5 we have a canonical equivalence of ∞-topoi

$$(\mathcal{X}^{h_\diamond G})_{h_\diamond G} \simeq \mathrm{Fun}(BG, \mathcal{X}^{h_\diamond G}).$$

Definition 2.40. Let $\mathcal{X}$ be a C_p-∞-topos. The **unstable gluing functor** is

$$\theta = (-)^{hC_p} \circ \nu(C_p)^* : \mathcal{X}_{h_\diamond C_p} \to \mathrm{Fun}(BC_p, \mathcal{X}^{h_\diamond C_p}) \to \mathcal{X}^{h_\diamond C_p}.$$

Note that θ is in general only a left-exact functor of ∞-topoi.

Example 2.41. Suppose that X is a scheme with $\frac{1}{2} \in \mathcal{O}_X$. Using the identification of Theorem 2.11 and Example 2.8, the unstable gluing functor takes the form

$$\theta : \widetilde{X}_{\text{ét}} \to \widetilde{X}_{\text{rét}}.$$

This functor is in turn computed to be [Sch94, Lemma 6.4.2(b)]

$$\widetilde{X}_{\text{ét}} \xrightarrow{i_{\text{ét}}} \widetilde{X}_{\text{pre}} \xrightarrow{L_{\text{rét}}} \widetilde{X}_{\text{rét}}.$$

Define the b**-topology** on the small site Ét_X to be the intersection of the étale and real étale topologies [Sch94, Definition 2.3]. Scheiderer identifies the right-lax limit of $L_{\text{rét}} i_{\text{ét}}$ (which he calls ρ) with the ∞-topos[11] $\widetilde{X}_b$ [Sch94, Proposition 2.6.1]. As we recall in Appendix A, the key point is that the other composite $L_{\text{ét}} i_{\text{rét}}$ is equivalent to the constant functor at the terminal sheaf (Example A.14 and Lemma A.3 with $C = \text{Ét}_X$ and $\tau = b$, $\tau_U = \text{ét}$, $\tau_Z = \text{rét}$).

The next proposition justifies the terminology of Definition 2.40.

Proposition 2.42. *Let $\mathcal{X}$ be a C_p-∞-topos. Then the right-lax limit of the unstable gluing functor θ is equivalent to $\mathcal{X}_{C_p}$. Equivalently, we have a recollement*

$$\mathcal{X}_{h_\diamond C_p} \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathcal{X}_{C_p} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{X}^{h_\diamond C_p}$$

in which the gluing functor $i^* j_*$ is equivalent to θ.

Proof. Let $\pi : \mathcal{O}_{C_p}^{\text{op}} \to \Delta^1$ be the unique functor that sends C_p/C_p to 0 and $C_p/1$ to 1. Let $p : \underline{\mathcal{X}}^{h_\diamond C_p} \to \mathcal{O}_{C_p}^{\text{op}}$ denote the cartesian fibration of Construction 2.12, so that $\mathcal{X}_{C_p} = \mathrm{Sect}(\underline{\mathcal{X}}^{h_\diamond C_p})$. Note that $(\underline{\mathcal{X}}^{h_\diamond C_p})_0 \simeq \mathcal{X}^{h_\diamond C_p}$ and $(\underline{\mathcal{X}}^{h_\diamond C_p})_1 \simeq \underline{\mathcal{X}}$. We may readily verify the existence hypotheses of [QS19, Proposition 2.4], so we have an induced recollement decomposing $\mathcal{X}_{C_p}$ in terms of $\mathcal{X}_{h_\diamond C_p}$ and $\mathcal{X}^{h_\diamond C_p}$. It remains to identify the gluing functor $i^* j_*$ of this recollement as θ. For this, we observe that by the discussion prior to [QS19, Proposition 2.4], given an object $f : BC_p \to \underline{\mathcal{X}} \subset \underline{\mathcal{X}}^{h_\diamond C_p}$ of $\mathcal{X}_{h_\diamond C_p}$, $i^* j_*(f)$ is computed as value of the p-limit diagram $\overline{f}$ on the cone point v:

$$\begin{array}{ccc} BC_p & \xrightarrow{f} & \underline{\mathcal{X}}^{h_\diamond C_p} \\ \downarrow & \overset{\overline{f}}{\nearrow} & \downarrow p \\ \mathcal{O}_{C_p}^{\text{op}} \cong (BC_p)^{\triangleleft} & \longrightarrow & \mathcal{O}_{C_p}^{\text{op}}. \end{array}$$

We now digress to explain how to compute such p-limits. In general, suppose given a diagram

$$\begin{array}{ccc} K & \xrightarrow{f} & C \\ \downarrow & \overset{\overline{f}}{\nearrow} & \downarrow p \\ K^{\triangleleft} & \xrightarrow{\overline{pf}} & S \end{array}$$

in which p is a cartesian fibration, and let $x = \overline{pf}(v)$ for v the cone point. Supposing that C_x admits K-indexed limits, we then have the dotted functor $\overline{f}$ given by sending v to the limit of $g : K \to C_x$, where g is the p-cartesian pullback of f to the fiber over x. In more detail, we may construct $\overline{f}$ via the following recipe:

[11] Scheiderer works at the level of 1-topoi, but the statement for ∞-topoi follows by the same reasoning, as we show in Appendix A.

- Let $\mathrm{Fun}(\Delta^1, C)_{\mathrm{cart}} \to \mathrm{Fun}(\Delta^1, S) \times_S C$ be the trivial fibration given by the dual of [Sha18, Lemma 2.22] that sends a p-cartesian edge $[a \to b]$ to $([p(a) \to p(b)], b)$, let σ be a choice of a section and let

$$P = \mathrm{ev}_0 \circ \sigma : \mathrm{Fun}(\Delta^1, S) \times_S C \to C$$

be the resulting choice of cartesian pullback functor. Let $r : K \times \Delta^1 \to K^{\triangleleft}$ be the unique functor that restricts to the trivial functor $K \times \{0\} \to *$ and to the identity on $K \times \{1\}$, let $q : K \to \mathrm{Fun}(\Delta^1, S)$ be the functor adjoint to $K \times \Delta^1 \xrightarrow{r} K^{\triangleleft} \xrightarrow{\overline{pf}} S$, and let

$$F = (q, f) : K \to \mathrm{Fun}(\Delta^1, S) \times_S C.$$

Then $P \circ F : K \to C$ factors through C_x. Let $g : K \to C_x$ denote this functor, which is the p-cartesian pullback of f to the fiber over x. Using that C_x admits K-indexed limits, let $\overline{g} : K^{\triangleleft} \to C_x$ denote the extension of g to a limit diagram.

Next, let $h : K \times \Delta^1 \to C$ be the functor adjoint to $\sigma \circ F : K \to \mathrm{Fun}(\Delta^1, C)_{\mathrm{cart}}$ and consider the diagram

$$\begin{array}{ccc} K \times \Delta^1 \cup_{K \times \{0\}} K^{\triangleleft} \times \{0\} & \xrightarrow{(h, \overline{g})} & C \\ \downarrow & \overset{\overline{h}}{\nearrow} & \downarrow p \\ (K \times \Delta^1)^{\triangleleft} & \longrightarrow & S. \end{array}$$

By [Lur09, Lemma 2.1.2.3], the lefthand inclusion is inner anodyne, so the dotted lift $\overline{h}$ exists. Finally, we set:

$$\overline{f} = \overline{h}|_{(K \times \{1\})^{\triangleleft}}.$$

By the proof of [Lur09, Corollary 4.3.1.11], $\overline{f}$ is then a p-limit diagram if and only if for all morphisms $(\alpha : y \to x = \overline{pf}(v))$ in S (defining the pullback functor $\alpha^* : C_x \to C_y$), $\alpha^* \overline{g}$ is a limit diagram in C_y. In particular, if x is an initial object in S, then the latter condition is vacuous and $\overline{f}$ is necessarily a p-limit diagram. Now suppose that $S \simeq S_1^{\triangleleft}$ and C_x admits S_1-indexed limits. Let $K = S_1$ and $\overline{pf} = \mathrm{id}$. Then the content of [QS19, Proposition 2.4] in this case is that $\mathrm{Fun}_{/S}(S, C)$ is the right-lax limit of the gluing functor

$$\mathrm{Fun}_{/S_1}(S_1, C \times_S S_1) \simeq \mathrm{Fun}_{/S}(S_1, C) \xrightarrow{P_*} \mathrm{Fun}(S_1, C_x) \xrightarrow{\lim} C_x,$$

where P_* denotes the functorial assignment $f \mapsto g$ above that is determined by the choice of P (which is only ambiguous up to contractible choice).

Returning to our situation of interest, observe that

$$P : \mathrm{Fun}(\Delta^1, \mathcal{O}_{C_p}^{\mathrm{op}}) \times_{\mathcal{O}_{C_p}^{\mathrm{op}}} \underline{\mathcal{X}}^{h_\diamond C_p} \to \underline{\mathcal{X}}^{h_\diamond C_p}$$

restricts to

$$P' : \underline{\mathcal{X}} \simeq BC_p \times_{\mathcal{O}_{C_p}^{op}} \underline{\mathcal{X}}^{h_\diamond C_p} \to \mathcal{X}^{h_\diamond C_p},$$

where we identify the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{O}_{C_p}^{\mathrm{op}})$ on those arrows with source C_p/C_p and target $C_p/1$ with BC_p. We may then identify P_* with postcomposition by P'.

Note that the C_p-equivariant functor $\nu^* : \mathcal{X} \to \mathcal{X}^{h_\diamond C_p}$ passes under unstraightening to the functor

$$(P', p|_{\underline{\mathcal{X}}}) : \underline{\mathcal{X}} \to \mathcal{X}^{h_\diamond C_p} \times BC_p$$

over BC_p. Therefore, the C_p-homotopy fixed points[12] $\nu(C_p)^* : \mathcal{X}_{h_\diamond C_p} \to \mathrm{Fun}(BC_p, \mathcal{X}^{h_\diamond C_p})$ is computed by postcomposition by P' in terms of the descriptions of the domain and codomain ∞-categories as sections. We conclude that $(-)^{hC_p} \circ P_*$ identifies with $\theta = (-)^{hC_p} \circ \nu(C_p)^*$, which shows that $i^* j_* \simeq \theta$. □

Remark 2.43. In view of Proposition 2.42, $\mathcal{X}_{C_p}$ corresponds to Scheiderer's notion of **quotient topos** [Sch94, Definition 14.2], which he only defines for $G = C_p$. The toposic genuine orbits construction may therefore be viewed as a generalization of Scheiderer's construction to the case of an arbitrary finite group.

[12] Recall here that the left adjoint of the geometric morphism given under formation of toposic homotopy orbits in $\mathscr{T}\mathrm{op}^R$ is computed by taking homotopy fixed points in $\widehat{\mathrm{Cat}}_\infty$.

Example 2.44. Suppose that X is a scheme with $\frac{1}{2} \in \mathcal{O}_X$. Then in view of Example 2.41 and Proposition 2.42, we have an equivalence

$$\widetilde{X}_b \simeq (\widetilde{X[i]_{\text{ét}}})_{C_2}.$$

We can also understand the functoriality of toposic genuine orbits in terms of the unstable gluing functor. For the next proposition, recall from [QS19, 1.7] that given a lax commutative square (with ϕ left-exact)

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\phi} & \mathcal{Z} \\ {\scriptstyle f^*}\downarrow & \swarrow & \downarrow{\scriptstyle f^*} \\ \mathcal{U}' & \xrightarrow[\phi]{} & \mathcal{Z}', \end{array}$$

we obtain an induced functor $f^* : \mathcal{X} = \mathcal{U} \overrightarrow{\times} \mathcal{Z} \to \mathcal{X}' = \mathcal{U}' \overrightarrow{\times} \mathcal{Z}'$ upon taking right-lax limits horizontally. Moreover, f^* is a (lax) **morphism of recollements** $(\mathcal{U}, \mathcal{Z}) \to (\mathcal{U}', \mathcal{Z}')$, that is, f^* sends j^*, resp. i^*-equivalences to j'^*, resp. i'^*-equivalences [QS19, Definition 1.1]. Conversely, if f^* is a morphism of recollements, then f^* induces functors $f^* : \mathcal{U} \to \mathcal{U}'$ and $f^* : \mathcal{Z} \to \mathcal{Z}'$ that commute with $f^* : \mathcal{X} \to \mathcal{X}'$ and the recollement left adjoints, and we thereby obtain an exchange transformation

$$\chi : f^* \phi \simeq f^* i^* j_* \Rightarrow \phi f^* \simeq i'^* j'_* f^*.$$

These two constructions are inverse equivalent to each other [Lur17a, Proposition A.8.8].

Proposition 2.45. *Let $f_* : \mathcal{X} \to \mathcal{Y}$ be a C_p-equivariant geometric morphism of C_p-∞-topoi. Then the induced functor $f^* : \mathcal{Y}_{C_p} \to \mathcal{X}_{C_p}$ is a morphism of the recollements as defined in Proposition 2.42. Therefore, we obtain an exchange transformation $\chi : f^* \theta \Rightarrow \theta f^*$ such that f^* is canonically equivalent to the right-lax limit of χ. Moreover, χ factors as*

$$\begin{array}{ccccc} \mathcal{Y}_{h_\diamond C_p} & \xrightarrow{\nu(C_p)^*} & \mathrm{Fun}(BC_p, \mathcal{Y}^{h_\diamond C_p}) & \xrightarrow{(-)^{hC_p}} & \mathcal{Y}^{h_\diamond C_p} \\ {\scriptstyle (f^*)_{h_\diamond C_p}}\downarrow & \simeq \swarrow & \downarrow{\scriptstyle f^*} & \swarrow & \downarrow{\scriptstyle (f^*)^{h_\diamond C_p}} \\ \mathcal{X}_{h_\diamond C_p} & \xrightarrow[\nu(C_p)^*]{} & \mathrm{Fun}(BC_p, \mathcal{X}^{h_\diamond C_p}) & \xrightarrow[(-)^{hC_p}]{} & \mathcal{X}^{h_\diamond C_p}, \end{array}$$

where the lefthand invertible transformation is toposic homotopy C_p-orbits of $(f^*)^{h_\diamond C_p} \nu^* \simeq \nu^* f^*$ and the righthand transformation is adjoint to $(f^*)^{h_\diamond C_p} \delta \simeq \delta (f^*)^{h_\diamond C_p}$ for the constant diagram functor δ.

Proof. The first assertion is clear since f^* is given by postcomposition of sections and j^*, i^* are given by restriction of sections. We thus obtain χ as just explained. The factorization of χ then follows from a straightforward diagram chase upon unpacking the factorization $i^* j_* \simeq (-)^{hC_p} \nu(C_p)^*$ of Proposition 2.42. □

We next aim to construct a *stable* gluing functor that recovers the genuine stabilization $\mathrm{Sp}^{C_p}(\mathcal{X})$. To begin with, let us denote the stabilization of the functor θ by

$$\Theta = \mathrm{Sp}(\theta) \simeq (-)^{hC_p} \circ \nu(C_p)^* : \mathrm{Sp}(\mathcal{X}_{h_\diamond C_p}) \to \mathrm{Fun}(BC_p, \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})) \to \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p}). \tag{7}$$

Given Proposition 2.42, it is easy to see that the right-lax limit of Θ obtains $\mathrm{Sp}(\mathcal{X}_{C_p})$ (and not $\mathrm{Sp}^{C_p}(\mathcal{X})$ in general) – we record this observation as Lemma A.7. To instead obtain the gluing functor for $\mathrm{Sp}^{C_p}(\mathcal{X})$, we must replace the homotopy fixed points by the Tate construction, as in the following definition.

Definition 2.46. Let $\mathcal{X}$ be a C_p-∞-topos. The **stable gluing functor** is

$$\Theta^{\mathrm{Tate}} = (-)^{tC_p} \circ \nu(C_p)^* : \mathrm{Sp}(\mathcal{X}_{h_\diamond C_p}) \to \mathrm{Fun}(BC_p, \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})) \to \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p}).$$

Here is the first main theorem of the paper.

Theorem 2.47. *Let $\mathcal{X}$ be a C_p-∞-topos and let $\mathrm{Sp}^{\mathrm{Tate}}_{C_p}(\mathcal{X})$ denote the right-lax limit of the stable gluing functor Θ^{Tate}. We have an equivalence of ∞-categories*

$$\mathrm{Sp}^{\mathrm{Tate}}_{C_p}(\mathcal{X}) \simeq \mathrm{Sp}^{C_p}(\mathcal{X}).$$

Question 2.48. What is the analogue of Theorem 2.47 for an arbitrary finite group?

The rest of this section will be devoted to a proof of Theorem 2.47. To begin with, we need a better understanding of how $\underline{\mathrm{Sp}}^{C_p}(-)$ interacts with recollements. Fixing notation, consider the following functors obtained from Constructions 2.6 and 2.18 respectively:

$$\mathcal{X}_{h_\diamond(-)} : \mathcal{O}^{\mathrm{op}}_{C_p} \to \mathscr{T}\mathrm{op}^L, \qquad (C_p/1 \to C_p/C_p) \mapsto (\mathcal{X} \xleftarrow{\pi^*} \mathcal{X}_{h_\diamond C_p}),$$

$$\mathcal{X}_{(-)} : \mathcal{O}^{\mathrm{op}}_{C_p} \to \mathscr{T}\mathrm{op}^L, \qquad (C_p/1 \to C_p/C_p) \mapsto (\mathcal{X} \xleftarrow{\overline{\pi}^*} \mathcal{X}_{C_p}).$$

We also have the following functor:

$$\mathcal{X}^c : \mathcal{O}^{\mathrm{op}}_{C_p} \to \mathscr{T}\mathrm{op}^L, \qquad (C_p/1 \to C_p/C_p) \mapsto (* \leftarrow \mathcal{X}^{h_\diamond C_p}),$$

where $*$ indicates the initial ∞-topos, i.e., the category Δ^0. The "c" decoration indicates that $\mathcal{X}^c$ wants to be the closed part of a fiberwise recollement on $\mathcal{X}_{(-)}$. To justify this, we need the following lemmas.

Lemma 2.49. *For any finite group G, there is a canonical adjunction*

$$j^* : \mathcal{X}_G \rightleftarrows \mathcal{X}_{h_\diamond G} : j_*,$$

such that:

1. *j^* is implemented by restriction of sections along the inclusion $BG \subset \mathcal{O}^{\mathrm{op}}_G$.*
2. *We have a factorization*

$$\mathcal{X}_G \xrightarrow[j^*]{} \mathcal{X}_{h_\diamond G} \xrightarrow[\pi^*]{} \mathcal{X}, \qquad \overline{\pi}^* : \mathcal{X}_G \to \mathcal{X}.$$

3. *If $G = C_p$, then the right adjoint j_* is implemented by (pointwise) relative right Kan extension along $BG \subset \mathcal{O}^{\mathrm{op}}_G$, hence j_* is fully faithful and we have the factorization*

$$\mathcal{X}_{h_\diamond G} \xrightarrow[j_*]{} \mathcal{X}_G \xrightarrow[\overline{\pi}^*]{} \mathcal{X}, \qquad \pi^* : \mathcal{X}_{h_\diamond G} \to \mathcal{X}.$$

Proof. By definition, $\overline{\pi}^*$ is given by restriction of sections (valued in $\underline{\mathcal{X}}^{h_\diamond G}$) along $\{G/1\} \subset \mathcal{O}^{\mathrm{op}}_G$, and π^* is given by restriction of sections (valued in $\underline{\mathcal{X}}$) along $\{G/1\} \subset BG$. If we define $j^* : \mathcal{X}_G \to \mathcal{X}^{hG} \simeq \mathcal{X}_{h_\diamond G}$ as restriction of sections along $BG \subset \mathcal{O}^{\mathrm{op}}_G$, then we have defined a functor j^* satisfying (1) and (2).

Note that the right adjoint j_* exists by presentability considerations. However, we observe that the hypotheses of (the dual of) [Lur09, Proposition 4.3.2.15] and [Lur09, Proposition 4.3.1.10] for the existence of a dotted lift in the commutative diagram

$$\begin{array}{ccc} BG & \longrightarrow & \underline{\mathcal{X}}^{h_\diamond} \\ \downarrow & \nearrow & \downarrow \\ \mathcal{O}^{\mathrm{op}}_G & \xrightarrow[=]{} & \mathcal{O}^{\mathrm{op}}_G \end{array}$$

are not satisfied *unless* $G = C_p$, since the left adjoint in a geometric morphism need not preserve infinite limits (if $G = C_p$, then condition (2) in [Lur09, Proposition 4.3.1.10] is vacuous; we already used this in the proof of Proposition 2.42). Therefore, we do not know if j_* is computed by the relative right Kan extension in general; in particular, we do not know that j_* is fully faithful unless $G = C_p$. However, if $G = C_p$, then the final assertions of the lemma follow immediately. ∎

Lemma 2.50. *For any finite group G, there is a canonical adjunction*

$$i^* : \mathcal{X}_G \rightleftarrows \mathcal{X}^{h_\diamond G} : i_*,$$

where i^* is implemented by restriction of sections along $\{G/G\} \subset \mathcal{O}_G^{\mathrm{op}}$, such that the diagrams

$$\begin{array}{ccc} \mathcal{X}_G & \xrightarrow{i^*} & \mathcal{X}^{h_\diamond G} \\ \downarrow{\scriptstyle \pi^*} & & \downarrow \\ \mathcal{X} & \longrightarrow & * \end{array} \quad , \quad \begin{array}{ccc} \mathcal{X}_G & \xleftarrow{i_*} & \mathcal{X}^{h_\diamond G} \\ \downarrow{\scriptstyle \pi^*} & & \downarrow \\ \mathcal{X} & \longleftarrow & * \end{array}$$

commute.

Proof. Since the left adjoint of a geometric morphism preserves the terminal object, we see that the right adjoint i_* to the given functor i^* exists and is computed by relative right Kan extension along $\{G/G\} \subset \mathcal{O}_G^{\mathrm{op}}$. The commutativity of the first square is trivial, and the commutativity of the second square follows from the formula for relative right Kan extension. □

To proceed further, it will be convenient to refer to the ∞-category of recollements. If a functor $f^* : \mathcal{X} \to \mathcal{X}'$ is a morphism of recollements $(\mathcal{U}, \mathcal{Z}) \to (\mathcal{U}', \mathcal{Z}')$, then we say the morphism is **strict** if f^* also commutes with the gluing functors [QS19, Definition 1.4]. The collection of recollements and strict morphisms forms an ∞-category Recoll_0. We shall also consider the ∞-category $\mathrm{Recoll}_0^{\mathrm{stab}}$ of **strict stable recollements** where the underlying ∞-categories are stable and functors are exact. If S is a small ∞-category, we say that a functor $S \to \mathrm{Recoll}_0^{(\mathrm{stab})}$ is a **(stable) fiberwise recollement**.

Corollary 2.51. *Let $\mathcal{X}$ be a C_p-∞-topos. Then the functor $\mathcal{X}_{(-)}$ lifts to a fiberwise recollement given by*

$$(\mathcal{X}_{h_\diamond(-)}, \mathcal{X}^c) : \mathcal{O}_{C_p}^{\mathrm{op}} \to \mathrm{Recoll}_0.$$

Proof. Lemmas 2.50 and 2.49 demonstrate that the diagram

$$\begin{array}{ccccc} \mathcal{X}_{h_\diamond C_p} & \xleftarrow{j^*} & \mathcal{X}_{C_p} & \xrightarrow{i^*} & \mathcal{X}^{h_\diamond C_p} \\ {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle \pi^*} & & \downarrow \\ \mathcal{X} & \xleftarrow[\mathrm{id}]{} & \mathcal{X} & \longrightarrow & * \end{array}$$

specifies a strict morphism of recollements, whence we obtain a functor as desired. □

Given a stable fiberwise recollement $\mathcal{F} : \mathcal{O}_G^{\mathrm{op}} \to \mathrm{Recoll}_0^{\mathrm{stab}}$, we say that $\mathcal{F}$ is a ***G*-stable recollement** if the underlying G-∞-category of $\mathcal{F}$ is G-stable [QS19, Definition 1.39]. This implies that the open and closed parts of $\mathcal{F}$ are G-stable G-∞-categories and the adjunctions are G-exact G-adjunctions [QS19, Corollary 1.38].

Proposition 2.52. *Let $\mathcal{X}$ be a C_p-∞-topos. Then $\underline{\mathrm{Sp}}^{C_p}(\mathcal{X})$ admits a C_p-stable recollement given by*

$$(\underline{\mathrm{Sp}}^{C_p}(\underline{\mathcal{X}}_{h_\diamond C_p}), \underline{\mathrm{Sp}}^{C_p}(\underline{\mathcal{X}}^c)).$$

Proof. Recall (Construction 2.30) that genuine stabilization is obtained by taking G-commutative monoids and fiberwise stabilization, where by Lemma 2.31 we may perform these two operations in either order. Fiberwise stabilization preserves recollements by Lemma A.7. Since $\underline{\mathrm{Sp}}^{C_p}(\mathcal{X})$ is C_p-stable, it suffices to check that taking C_p-commutative monoids on the recollement of Corollary 2.51 gives us a fiberwise recollement

$$\left(\underline{\mathrm{CMon}}^{C_p}(\underline{\mathrm{Sp}}(\underline{\mathcal{X}}_{h_\diamond C_p})), \underline{\mathrm{CMon}}^{C_p}(\underline{\mathrm{Sp}}(\underline{\mathcal{X}}^c)\right).$$

Since $\mathcal{X}^c(C_p/1) \simeq *$, this is obvious over the orbit $C_p/1$, so it suffices to check the recollement conditions over the orbit C_p/C_p. For this, first note that if we take $K = \underline{\mathrm{A}}^{\mathrm{eff}}(\mathrm{Fin}_{C_p})$ in [QS19, Lemma 1.33], we obtain a recollement

$$\mathrm{Fun}_{C_p}(\underline{\mathrm{A}}^{\mathrm{eff}}(\mathrm{Fin}_{C_p}), \underline{\mathrm{Sp}}(\underline{\mathcal{X}}_{h_\diamond C_p})) \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathrm{Fun}_{C_p}(\underline{\mathrm{A}}^{\mathrm{eff}}(\mathrm{Fin}_{C_p}), \underline{\mathrm{Sp}}(\underline{\mathcal{X}}_{C_p})) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Fun}_{C_p}(\underline{\mathrm{A}}^{\mathrm{eff}}(\mathrm{Fin}_{C_p}), \underline{\mathrm{Sp}}(\underline{\mathcal{X}}^c)).$$

Here, we denote by j_* etc. the functors induced by postcomposition by the same denoted C_p-functors for the fiberwise recollement $(\underline{\mathrm{Sp}}(\underline{\mathcal{X}}_{h_\diamond C_p}), \underline{\mathrm{Sp}}(\underline{\mathcal{X}}^c))$ on $\underline{\mathrm{Sp}}(\underline{\mathcal{X}}_{C_p})$. Since the ∞-category of C_p-Mackey functors in a fiberwise stable presentable C_p-∞-category C is a Bousfield localization of $\mathrm{Fun}_{C_p}(\underline{\mathrm{A}}^{\mathrm{eff}}(\mathrm{Fin}_{C_p}), \mathrm{C})$, and postcomposition by the C_p-left-exact C_p-functors j_*, i_* preserves C_p-commutative monoid objects, we have induced adjunctions

$$\mathrm{CMon}^{C_p}(\underline{\mathrm{Sp}}(\underline{\mathcal{X}}_{h_\diamond C_p})) \underset{\overline{j}_*}{\overset{\overline{j}^*}{\rightleftarrows}} \mathrm{CMon}^{C_p}(\underline{\mathrm{Sp}}(\underline{\mathcal{X}}_{C_p})) \underset{\overline{i}_*}{\overset{\overline{i}^*}{\rightleftarrows}} \mathrm{CMon}^{C_p}(\underline{\mathrm{Sp}}(\underline{\mathcal{X}}^c)),$$

in which $\overline{j}_*, \overline{i}_*$ are defined by postcomposition by j_*, i_* and their left adjoints $\overline{j}^*, \overline{i}^*$ are defined by postcomposition by j^*, i^* followed by L_{sad} (Notation 2.33). It is thus clear that $\overline{j}_*, \overline{i}_*$ remain fully faithful.

Furthermore, we have that $j^*\overline{\pi}_* \simeq \pi_*$, as this may be checked after application of j_*, in which case it follows from Lemma 2.49(2). Therefore, $\overline{j}^*$ is computed already by postcomposition by j^*. We deduce that $\overline{j}^*\overline{i}_* \simeq 0$ by the same property for $j^* i_*$.

Next, we observe that we have a canonical equivalence $L_{\mathrm{sad}} i_* \simeq \overline{i}_* L_{\mathrm{sad}}$. Indeed, it suffices to show that i_* of an acyclic object N is again acyclic. But for this, note that

$$i^* L_{\mathrm{sad}} i_* N \simeq L_{\mathrm{sad}} i^* i_* N \simeq L_{\mathrm{sad}} N \simeq 0 \quad \text{and} \quad j^* L_{\mathrm{sad}} i_* N \simeq L_{\mathrm{sad}} j^* i_* N \simeq 0,$$

and hence $L_{\mathrm{sad}} N \simeq 0$ by joint conservativity of j^*, i^*. It follows that the *right adjoint* $i^!$ of i_* (which exists since we are in the situation of a *stable* recollement) descends to

$$\overline{i}^{\,!} : \mathrm{CMon}^{C_p}(\underline{\mathrm{Sp}}(\underline{\mathcal{X}}_{C_p})) \to \mathrm{CMon}^{C_p}(\underline{\mathrm{Sp}}(\underline{\mathcal{X}}^c)),$$

where $\overline{i}^{\,!}$ is induced by postcomposition by the C_p-right adjoint $i^! : \underline{\mathrm{Sp}}(\underline{\mathcal{X}}_{C_p}) \to \underline{\mathrm{Sp}}(\underline{\mathcal{X}}^c)$.

Finally, note that instead of showing joint conservativity of $\overline{j}^*, \overline{i}^*$, in the stable setting it suffices instead to show joint conservativity of $\overline{j}^*, \overline{i}^{\,!}$. But this follows immediately from the same property for $j^*, i^!$. ☐

In particular, evaluating at C_p/C_p and using the identification of Lemma 2.38, we have the gluing functor[13]

$$i^* j_* : \mathrm{Sp}^{C_p}(\underline{\mathcal{X}}_{h_\diamond C_p}) \simeq \mathrm{Sp}(\mathcal{X}^{hC_p}) \to \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p}). \tag{8}$$

Lemma 2.53. *The functor $i^* j_*$ vanishes on the stabilization of $\pi_* : \mathcal{X} \to \mathcal{X}_{h_\diamond C_p}$, i.e., the composite*

$$\mathrm{Sp}(\mathcal{X}) \xrightarrow{\pi_*} \mathrm{Sp}(\mathcal{X}^{hC_p}) \xrightarrow{i^* j_*} \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})$$

is nullhomotopic.

Proof. By Proposition 2.52 and the G-exactness of the G-functors in a G-stable recollement, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Sp}(\mathcal{X}^{hC_p}) & \xleftarrow{\pi_*} & \mathrm{Sp}(\mathcal{X}) \\
{\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle =} \\
\mathrm{Sp}^{C_p}(\mathcal{X}_{C_p}) & \xleftarrow[\overline{\pi}_*]{} & \mathrm{Sp}(\mathcal{X}) \\
{\scriptstyle i^*}\downarrow & & \downarrow \\
\mathrm{Sp}(\mathcal{X}^{h_\diamond C_p}) & \longleftarrow & 0.
\end{array}$$

☐

Lemma 2.53 highlights the relevance of the following definition.

Definition 2.54. Suppose that $\mathcal{X}$ is a G-∞-topos. Then the subcategory of **induced objects** in $\mathrm{Sp}(\mathcal{X}_{h_\diamond G})$ is the essential image of the functor $\pi_* : \mathrm{Sp}(\mathcal{X}) \to \mathrm{Sp}(\mathcal{X}_{h_\diamond G})$.

[13] Note that we now drop the temporary 'overline' decoration for the recollement functors used in the proof of Proposition 2.52.

Lemma 2.55. *The adjunction*

$$\pi^* : \mathrm{Sp}(\mathcal{X}_{h_\diamond G}) \rightleftarrows \mathrm{Sp}(\mathcal{X}) : \pi_*$$

is ambidextrous, and π^* *exhibits* $\mathrm{Sp}(\mathcal{X}_{h_\diamond G})$ *as monadic over* $\mathrm{Sp}(\mathcal{X})$*. In particular,* $\mathrm{Sp}(\mathcal{X}_{h_\diamond G})$ *is generated under colimits by the induced objects.*

Proof. We already produced the ambidexterity equivalence $\pi_! \xrightarrow{\simeq} \pi_*$ in Lemma 2.38(1). We then note that π^* is conservative as it is given by evaluation of sections on $* \in BG$. This confirms the hypotheses of the Barr-Beck-Lurie theorem and thereby shows monadicity of the adjunction $\pi_! \dashv \pi^*$. The last statement then follows from the existence of monadic resolutions. □

In order to complete the proof of Theorem 2.47, we need to identify the gluing functor of (8) with the functor Θ^{Tate} of Definition 2.46. To facilitate this comparison, we will construct the analogue of **categorical fixed points** in our setting. Recall that if G is a finite group and we identify Sp_G with the ∞-category of spectral Mackey functors $\mathrm{Fun}^\times(\mathrm{Span}(\mathrm{Fin}_G), \mathrm{Sp})$, then the categorical fixed points functor Ψ^G is given by evaluation at the object G/G. Our construction will follow a similar pattern.

Theorem 2.56. *There is an exact functor*

$$\Psi^{C_p} : \mathrm{Sp}^{C_p}(\mathcal{X}) \to \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})$$

with the following three properties:

1. Ψ^{C_p} *preserves colimits.*
2. $\Psi^{C_p} j_*(-) \simeq (-)^{hC_p} \circ \nu(C_p)^*$ *as functors* $\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p}) \to \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})$.
3. $\Psi^{C_p} i_* \simeq \mathrm{id}$ *as functors* $\mathrm{Sp}(\mathcal{X}^{h_\diamond C_p}) \to \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})$.

Here, j_* *and* i_* *are the functors defined by the recollement on* $\mathrm{Sp}^{C_p}(\mathcal{X})$ *of Proposition 2.52.*

Proof. In this proof only, let us distinguish the recollement functors for $(\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p}, \mathrm{Sp}(X^{h_\diamond C_p}))$ on $\mathrm{Sp}(X_{C_p})$ (as well as the fiberwise variant for $\underline{\mathrm{Sp}}(\underline{X}_{C_p})$) by the subscript 'pre'. First, using Lemma 2.31 to describe $\mathrm{Sp}^{C_p}(\mathcal{X})$, we define the functor Ψ^{C_p} by the formula

$$\Psi^{C_p} : \mathrm{Sp}^{C_p}(\mathcal{X}) \simeq \mathrm{CMon}^{C_p}(\underline{\mathrm{Sp}}(\underline{\mathcal{X}}_{C_p})) \to \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p}),$$

$$\Big(X : \underline{\mathrm{Fin}}_{C_p*} \to \underline{\mathrm{Sp}}(\underline{\mathcal{X}}_{C_p})\Big) \mapsto \Big(\{C_p/C_{p+}\} \subset \underline{\mathrm{Fin}}_{C_p*} \xrightarrow{X} \underline{\mathrm{Sp}}(\underline{\mathcal{X}}_{C_p}) \xrightarrow{i^*_{\mathrm{pre}}} \underline{\mathrm{Sp}}(\underline{\mathcal{X}}^c)\Big).$$

In other words, Ψ^{C_p} sends a C_p-commutative monoid X to $(i^*_{\mathrm{pre}}X)(C_p/C_{p+})$, an object in the fiber $\underline{\mathrm{Sp}}(\underline{\mathcal{X}}^c)_{C_p/C_p} \simeq \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})$. It is clear that Ψ^{C_p} is exact. We now verify properties (1)-(3) in turn:

1. If we regard the above formula as a functor from $\mathrm{Fun}_{C_p}(\underline{\mathrm{Fin}}_{C_p*}, \underline{\mathrm{Sp}}(\underline{\mathcal{X}}_{C_p}))$ to $\mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})$, it obviously preserves colimits as an evaluation functor. Hence, it suffices to prove that the inclusion of the full subcategory

$$\mathrm{CMon}^{C_p}(\underline{\mathrm{Sp}}(\underline{\mathcal{X}}_{C_p})) \subset \mathrm{Fun}_{C_p}(\underline{\mathrm{Fin}}_{C_p*}, \underline{\mathrm{Sp}}(\underline{\mathcal{X}}_{C_p}),$$

preserves colimits. To this end, suppose that we have a diagram $f : K \to \mathrm{CMon}^{C_p}(\underline{\mathrm{Sp}}(\underline{\mathcal{X}}_{C_p}))$ and let X be the colimit of f as computed in $\mathrm{Fun}_{C_p}(\underline{\mathrm{Fin}}_{C_p*}, \underline{\mathrm{Sp}}(\underline{\mathcal{X}}_{C_p}))$. We need to show that X is C_p-semiadditive. For this, it suffices to check that the canonical maps

$$X(C_p/C_{p+} \coprod C_p/C_{p+}) \to X(C_p/C_{p+}) \oplus X(C_p/C_{p+}), \text{ and}$$

$$X(C_p/1_+) \to \overline{\pi}_*(X(1_+))$$

are equivalences. The first equivalence follows from stability. For the second equivalence, it suffices to prove that $\overline{\pi}_*$ preserves colimits. But to do so, in view of the conservativity of the recollement functors

$$(i^*_{\mathrm{pre}}, j^*_{\mathrm{pre}}) : \mathrm{Sp}(\mathcal{X}_{C_p}) \to \mathrm{Sp}(X^{h_\diamond C_p}) \times \mathrm{Sp}(\mathcal{X}_{h_\diamond C_p}),$$

it suffices to check that $i^*_{\mathrm{pre}}\overline{\pi}_*$ and $j^*_{\mathrm{pre}}\overline{\pi}_*$ are colimit-preserving. For the first functor, we have that $i^*_{\mathrm{pre}}\overline{\pi}_* \simeq \nu^*$, which preserves colimits since ν^* is a left adjoint. For the second functor, we have that $j^*_{\mathrm{pre}}\overline{\pi}_* \simeq \pi_*$, but $\pi_* \simeq \pi_!$ by Lemma 2.38 and $\pi_!$ preserves colimits as a left adjoint.

2. By Lemma 2.38, fiberwise stabilization for $\underline{\mathfrak{X}}_{h_\diamond C_p}$ is already C_p-stabilization, so $\mathrm{Sp}^{C_p}(\underline{\mathfrak{X}}_{h_\diamond C_p}) \simeq \mathrm{Sp}(\mathfrak{X}_{h_\diamond C_p})$. Hence, applying the equivalence (6), we may canonically regard any $X \in \mathrm{Sp}(\mathfrak{X}_{h_\diamond C_p})$ as a C_p-commutative monoid $X : \underline{\mathrm{Fin}}_{C_p*} \to \underline{\mathrm{Sp}}(\underline{\mathfrak{X}}_{h_\diamond C_p})$. The functor j_* is then computed on X as the composite
$$\underline{\mathrm{Fin}}_{C_p*} \xrightarrow{X} \underline{\mathrm{Sp}}(\underline{X}_{h_\diamond C_p}) \xrightarrow{j_*^{\mathrm{pre}}} \underline{\mathrm{Sp}}(\underline{\mathfrak{X}}_{C_p}),$$
since j_*^{pre} preserves C_p-limits (so that the composite remains C_p-semiadditive). Therefore, $\Psi^{C_p} j_* X$ is given by
$$\{C_p/C_{p+}\} \subset \underline{\mathrm{Fin}}_{C_p*} \xrightarrow{X} \underline{\mathrm{Sp}}(\underline{X}_{h_\diamond C_p}) \xrightarrow{j_*^{\mathrm{pre}}} \underline{\mathrm{Sp}}(\underline{\mathfrak{X}}_{C_p}) \xrightarrow{i^*_{\mathrm{pre}}} \underline{\mathrm{Sp}}(\underline{\mathfrak{X}}^c).$$
$\Psi^{C_p} j_*$ is thus equivalent to $(-)^{hC_p} \circ \nu(C_p)^*$ by the same formula on the unstable level (Proposition 2.42).
3. Suppose that $X \in \mathrm{Sp}(\mathfrak{X}^{h_\diamond C_p})$. By (6) and Lemma 2.31, we may canonically regard X as a C_p-commutative monoid $X : \underline{\mathrm{Fin}}_{C_p*} \to \underline{\mathrm{Sp}}(\underline{\mathfrak{X}}^c)$. The functor i_* is then computed on X as the composite
$$\underline{\mathrm{Fin}}_{C_p*} \xrightarrow{X} \underline{\mathrm{Sp}}(\underline{\mathfrak{X}}^c) \xrightarrow{i_*^{\mathrm{pre}}} \underline{\mathrm{Sp}}(\underline{\mathfrak{X}}^{C_p}),$$
since i_*^{pre} preserves C_p-limits (so that the composite remains C_p-semiadditive). Therefore, $\Psi^{C_p} i_* X$ is given by
$$\{C_p/C_{p+}\} \subset \underline{\mathrm{Fin}}_{C_p*} \xrightarrow{X} \underline{\mathrm{Sp}}(\underline{\mathfrak{X}}^c) \xrightarrow{i_*^{\mathrm{pre}}} \underline{\mathrm{Sp}}(\underline{\mathfrak{X}}_{C_p}) \xrightarrow{i^*_{\mathrm{pre}}} \underline{\mathrm{Sp}}(\underline{\mathfrak{X}}^c),$$
which is evidently the identity on X since i_*^{pre} is fully faithful.

□

Corollary 2.57. *The forgetful functor* $U : \mathrm{Sp}^{C_p}(\mathfrak{X}) \simeq \mathrm{CMon}^{C_p}(\underline{\mathfrak{X}}_{C_p}) \to \mathrm{Sp}(\mathfrak{X}_{C_p})$ *preserves all limit and colimits and is conservative. In particular, the adjunction*
$$\mathrm{Fr} : \mathrm{Sp}(\mathfrak{X}_{C_p}) \rightleftarrows \mathrm{Sp}^{C_p}(\mathfrak{X}) : U$$
is monadic.

Proof. We already have that U preserves limits as a right adjoint, and it is clearly conservative. It remains to check that U preserves colimits; monadicity of the adjunction $\mathrm{Fr} \dashv U$ will then follow from the Barr-Beck-Lurie theorem. Note that the functor $\Psi^{C_p} : \mathrm{Sp}^{C_p}(\mathfrak{X}) \to \mathrm{Sp}(\mathfrak{X}^{h_\diamond C_p})$ of Theorem 2.56 factors as
$$\Psi^{C_p} : \mathrm{Sp}^{C_p}(\mathfrak{X}) \xrightarrow{U} \mathrm{Sp}(\mathfrak{X}_{C_p}) \xrightarrow{i^*_{\mathrm{pre}}} \mathrm{Sp}(\mathfrak{X}^{h_\diamond C_p}).$$
Also, the functor $j^* : \mathrm{Sp}^{C_p}(\mathfrak{X}) \to \mathrm{Sp}(\mathfrak{X}_{h_\diamond C_p})$ factors as
$$\mathrm{Sp}^{C_p}(\mathfrak{X}) \xrightarrow{U} \mathrm{Sp}(\mathfrak{X}_{C_p}) \xrightarrow{j^*_{\mathrm{pre}}} \mathrm{Sp}(\mathfrak{X}_{h_\diamond C_p})$$
as we saw in the proof of Proposition 2.52. Since $(j^*_{\mathrm{pre}}, i^*_{\mathrm{pre}})$ are jointly conservative and colimit-preserving and (j^*, Ψ^{C_p}) are colimit-preserving, the claim follows. □

Remark 2.58. For any finite group G and G-∞-topos $\mathfrak{X}$, we expect $U : \mathrm{Sp}^G(\mathfrak{X}) \to \mathrm{Sp}(\mathfrak{X}_G)$ to preserve all colimits. For this, it would suffice to show that for every subgroup $H \leqslant G$, the composite functor
$$\Psi^H = \mathrm{ev}_{G/H} \circ U : \mathrm{Sp}^G(\mathfrak{X}) \to \mathrm{Sp}(\mathfrak{X}_G) \to \mathrm{Sp}(\mathfrak{X}^{h_\diamond H})$$
preserves colimits, or equivalently that either inclusion
$$\mathrm{CMon}^G(\underline{\mathrm{Sp}}(\underline{\mathfrak{X}}_G)) \subset \mathrm{Fun}_G(\underline{\mathrm{Fin}}_{G*}, \underline{\mathrm{Sp}}(\underline{\mathfrak{X}}_G)), \qquad \mathrm{Mack}^G(\underline{\mathrm{Sp}}(\underline{\mathfrak{X}}_G)) \subset \mathrm{Fun}_G(\underline{\mathrm{A}}^{\mathrm{eff}}(\mathrm{Fin}_G), \underline{\mathrm{Sp}}(\underline{\mathfrak{X}}_G))$$
preserves colimits. For instance, this holds for the base case of genuine G-spectra using [Bar17, Proposition 6.5] or the known compactness of the orbits $\Sigma^\infty_+ G/H$ in Sp^G. As the general case is of less significance to us in this paper, we leave the details to the reader.

Proof of Theorem 2.47. Consider the following commutative diagram of functors and transformations between them

$$\begin{array}{ccccc} (-)_{hC_p}\nu(C_p)^* & \longrightarrow & (-)^{hC_p}\nu(C_p)^* & \longrightarrow & (-)^{tC_p}\nu(C_p)^* \\ \downarrow & & \downarrow \simeq & & \downarrow \\ \Psi^{C_p}j_! & \longrightarrow & \Psi^{C_p}j_* & \longrightarrow & \Psi^{C_p}i_*i^*j_*, \end{array}$$

in which the bottom row comes from the exact functor Ψ^{C_p} of Theorem 2.56 applied to the fiber sequence $j_! \to j_* \to i_*i^*j_*$ associated to the recollement of Proposition 2.52.

We first explain how to obtain the vertical arrows. The middle equivalence holds by Theorem 2.56(2). Observe then that the resulting composite from the top left corner to the bottom right corner is nullhomotopic, since i^*j_* vanishes on induced objects by Lemma 2.53, the functor $(-)_{hC_p}\nu(C_p)^*$ preserves colimits, and $\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p})$ is generated under colimits by induced objects by Lemma 2.55. Hence, the right vertical arrow exists by taking cofibers and the left vertical arrow is in turn its fiber.

By Theorem 2.56(3) the bottom right corner is equivalent to i^*j_*, and thus we get a natural transformation

$$\alpha : \Theta^{\mathrm{Tate}} = (-)^{tC_p}\nu(C_p)^* \Rightarrow i^*j_*.$$

To prove that α is an equivalence, it suffices to check that the natural transformation on fibers

$$\beta : \mathrm{fib}(\Theta \to \Theta^{\mathrm{Tate}}) \simeq (-)_{hC_p}\nu(C_p)^* \Rightarrow \mathrm{fib}(\Theta \to i^*j_*) \simeq \Psi^{C_p}j_!$$

is an equivalence. For this, note that $(-)_{hC_p}\nu(C_p)^*$ preserves colimits, and since Ψ preserves colimits by Theorem 2.56(1), the functor $\Psi j_!$ preserves colimits as well. Moreover, β is an equivalence on induced objects because $(-)^{tC_p}\nu(C_p)^*$ and i^*j_* both vanish on induced objects: for the former claim, we use the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Sp}(\mathcal{X}) & \xrightarrow{\nu^*} & \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p}) \\ \downarrow \pi_* & & \downarrow \pi_* \\ \mathrm{Sp}(\mathcal{X}_{h_\diamond C_p}) & \xrightarrow{\nu(C_p)^*} & \mathrm{Fun}(BC_p, \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})), \end{array}$$

which holds in view of the calculation of the vertical functors as C_p-indexed products, the left-exactness of the unstable ν^*, and the definition of $\nu(C_p)^*$. The conclusion then follows by invoking Lemma 2.55 again. ☐

Remark 2.59. Let us examine the functoriality of the recollement on $\mathrm{Sp}^{C_p}(\mathcal{X})$ of Proposition 2.52 along C_p-equivariant geometric morphisms. First note that $\Omega^\infty : \mathrm{Sp}^{C_p}(\mathcal{X}) \to \mathcal{X}_{C_p}$ clearly commutes with the recollement right adjoints, since they are induced via postcomposition from their unstable counterparts (as we saw in the proof of 2.52). Therefore, $\Sigma^\infty_+ : \mathcal{X}_{C_p} \to \mathrm{Sp}^{C_p}(\mathcal{X})$ is a morphism of recollements

$$\Sigma^\infty_+ : (\mathcal{X}_{h_\diamond C_p}, \mathcal{X}^{h_\diamond C_p}) \to (\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p}), \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})).$$

Likewise, we observe the same pattern for the functor $f_* : \mathrm{Sp}^{C_p}(\mathcal{X}) \to \mathrm{Sp}^{C_p}(\mathcal{Y})$ induced by a G-equivariant geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$, hence its left adjoint f^* is a morphism of recollements

$$f^* : (\mathrm{Sp}(\mathcal{Y}_{h_\diamond C_p}), \mathrm{Sp}(\mathcal{Y}^{h_\diamond C_p})) \to (\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p}), \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})).$$

Under Proposition 2.42 and Theorem 2.47, we thus obtain exchange transformations

$$\alpha : \Sigma^\infty_+\theta \Rightarrow \Theta^{\mathrm{Tate}}\Sigma^\infty_+, \qquad \xi : f^*\Theta^{\mathrm{Tate}} \Rightarrow \Theta^{\mathrm{Tate}} f^*$$

that induce the functors Σ^∞_+ and f^* above upon taking right-lax limits. We can also understand how α and ξ interact with the factorizations $\theta \simeq (-)^{hC_p}\nu(C_p)^*$ and $\Theta^{\mathrm{Tate}} \simeq (-)^{tC_p}\nu(C_p)^*$. If we factor Σ^∞_+ as

$$\mathcal{X}_{C_p} \xrightarrow{\Sigma^\infty_+} \mathrm{Sp}(\mathcal{X}_{C_p}) \xrightarrow{\mathrm{Fr}} \mathrm{Sp}^{C_p}(\mathcal{X})$$

(so the middle term is given by the right-lax limit of Θ), we obtain a factorization of α as

$$\Sigma^\infty_+ \theta \simeq \Sigma^\infty_+ (-)^{hC_p} \nu(C_p)^* \overset{\beta}{\Rightarrow} \Theta \Sigma^\infty_+ \simeq (-)^{hC_p} \nu(C_p)^* \Sigma^\infty_+ \overset{\gamma}{\Rightarrow} \Theta^{\mathrm{Tate}} \Sigma^\infty_+ (-)^{tC_p} \nu(C_p)^* \Sigma^\infty_+,$$

where β is induced by the adjoint to $\Omega^\infty \delta \simeq \delta \Omega^\infty$ and γ is induced by the usual map $\mu : (-)^{hC_p} \to (-)^{tC_p}$. This may be depicted diagrammatically as follows:

$$\begin{array}{ccccc}
\mathcal{X}_{h_\diamond C_p} & \xrightarrow{\nu(C_p)^*} & \mathrm{Fun}(BC_p, \mathcal{X}^{h_\diamond C_p}) & \xrightarrow{(-)^{hC_p}} & \mathcal{X}^{h_\diamond C_p} \\
\downarrow{\scriptstyle \Sigma^\infty_+} & \simeq \Swarrow & \downarrow{\scriptstyle \Sigma^\infty_+} & \Swarrow & \downarrow{\scriptstyle \Sigma^\infty_+} \\
\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p}) & \xrightarrow[\nu(C_p)^*]{} & \mathrm{Fun}(BC_p, \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})) & \xrightarrow[(-)^{hC_p}]{} & \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p}) \\
\downarrow{\scriptstyle \simeq} & \simeq \Swarrow & \downarrow{\scriptstyle \simeq} & \mu \Swarrow & \downarrow{\scriptstyle \simeq} \\
\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p}) & \xrightarrow[\nu(C_p)^*]{} & \mathrm{Fun}(BC_p, \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})) & \xrightarrow[(-)^{tC_p}]{} & \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p}).
\end{array}$$

Similarly, we have a factorization of ξ as

$$\begin{array}{ccccc}
\mathrm{Sp}(\mathcal{Y}_{h_\diamond C_p}) & \xrightarrow{\nu(C_p)^*} & \mathrm{Fun}(BC_p, \mathrm{Sp}(\mathcal{Y}^{h_\diamond C_p})) & \xrightarrow{(-)^{tC_p}} & \mathrm{Sp}(\mathcal{Y}^{h_\diamond C_p}) \\
\downarrow{\scriptstyle f^*} & \simeq \Swarrow & \downarrow{\scriptstyle f^*} & \lambda \Swarrow & \downarrow{\scriptstyle f^*} \\
\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p}) & \xrightarrow[\nu(C_p)^*]{} & \mathrm{Fun}(BC_p, \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})) & \xrightarrow[(-)^{tC_p}]{} & \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})
\end{array}$$

in which λ is induced by vertically taking cofibers in the commutative square

$$\begin{array}{ccc}
f^*(-)_{hC_p} & \xrightarrow{\simeq} & (-)_{hC_p} f^* \\
\downarrow{\scriptstyle f^* \mathrm{Nm}} & & \downarrow{\scriptstyle \mathrm{Nm} f^*} \\
f^*(-)^{hC_p} & \longrightarrow & (-)^{hC_p} f^*.
\end{array}$$

Here, the desired naturality property of the additive norm follows from an elementary diagram chase using its inductive construction (see [Lur17a, Construction 6.1.6.4] or [NS18, Construction I.1.7]).

Given Proposition 2.45, both of these factorization assertions hold in view of the naturality of the comparison map $(-)^{tC_p} \nu(C_p)^* \xrightarrow{\simeq} i^* j_*$ as constructed in the proof of Theorem 2.47.

2.4 Symmetric monoidal structures

In this subsection, we apply Theorem 2.47 to endow $\mathrm{Sp}^{C_p}(\mathcal{X})$ with a symmetric monoidal structure. Interpreting dualizability as a type of "finiteness" for an object in a symmetric monoidal ∞-category, we then flesh out the way in which the genuine stabilization may be thought of as a "compactification" of a C_p-∞-topos as discussed in the introduction (Corollary 2.67 and Proposition 2.72). When $\mathcal{X}$ is taken to be the C_2-∞-topos $\widehat{X[i]}_{\text{ét}}$, we will later see how the good dualizability properties of $\mathrm{Sp}^{C_2}(\mathcal{X})$ (as opposed to $\mathrm{Sp}(\mathcal{X}_{C_2})$) intercede in the proof of Thom stability (Lemma 3.25) and thus in establishing the formalism of six operations.

Recall that given a left-exact lax symmetric monoidal functor $\phi : \mathcal{U} \to \mathcal{Z}$ of symmetric monoidal ∞-categories, the right-lax limit $\mathcal{X} = \mathcal{U} \overrightarrow{\times} \mathcal{Z}$ may be endowed with the **canonical symmetric monoidal structure** [QS19, Definition 1.21] that makes $(\mathcal{U}, \mathcal{Z})$ into a monoidal recollement [QS19, Definition 1.19]. Explicitly, given the data of a morphism of ∞-operads $\phi^\otimes : \mathcal{U}^\otimes \to \mathcal{Z}^\otimes$ lifting ϕ, we define (cf. Remark 3.8)

$$\mathcal{X}^\otimes = \mathcal{U}^\otimes \times_{\mathcal{Z}^\otimes} (\mathcal{Z}^\otimes)^{\Delta^1}.$$

Construction 2.60. Let $\mathcal{X}$ be a C_p-∞-topos. To construct the symmetric monoidal structure on $\mathrm{Sp}^{C_p}(\mathcal{X})$, we note the following two facts:

1. The stabilization functor $\mathrm{Sp} : \mathrm{Pr}^{\mathrm{L}}_{\infty} \to \mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}}$ is symmetric monoidal with respect to the Lurie tensor product of presentable ∞-categories [Lur17a, §4.8.2]. In particular, given an ∞-topos $\mathcal{X}$ equipped with the cartesian symmetric monoidal structure, we obtain a symmetric monoidal structure on its stabilization $\mathrm{Sp}(\mathcal{X})$. Moreover, for the left-exact left adjoint $f^* : \mathcal{X} \to \mathcal{Y}$ of a geometric morphism, we obtain a symmetric monoidal functor $f^* : \mathrm{Sp}(\mathcal{X}) \to \mathrm{Sp}(\mathcal{Y})$.

2. Given a stable presentable symmetric monoidal ∞-category C and finite group G, the Tate construction

$$(-)^{tG} : \mathrm{Fun}(BG, \mathrm{C}) \to \mathrm{C}$$

is lax symmetric monoidal such that the usual map $\mu : (-)^{hG} \to (-)^{tG}$ is the *universal* lax symmetric monoidal natural transformation among those to lax symmetric monoidal functors that annihilate induced objects [NS18, Theorem I.3.1].[14]

We deduce that the stable gluing functor $\Theta^{\mathrm{Tate}} = (-)^{tC_p} \circ \nu(C_p)^*$ is lax symmetric monoidal. We then regard the right-lax limit $\mathrm{Sp}^{C_p}(\mathcal{X})$ of Θ^{Tate} as a symmetric monoidal ∞-category via the canonical symmetric monoidal structure, and we call the resulting tensor product on $\mathrm{Sp}^{C_p}(\mathcal{X})$ the **smash product**.

Remark 2.61. Since the jointly conservative recollement left adjoints of a monoidal recollement are strong symmetric monoidal, we see that the smash product on $\mathrm{Sp}^{C_p}(\mathcal{X})$ commutes with colimits separately in each variable. Therefore, $\mathrm{Sp}^{C_p}(\mathcal{X})$ is presentably symmetric monoidal and lifts to an object in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}})$.

Remark 2.62. Lurie's construction of the smash product in spectra arises as a "decategorification" of the fact that Sp is an idempotent object in $\mathrm{Pr}^{\mathrm{L}}_{\infty}$ with respect to the Lurie tensor product of presentable ∞-categories [Lur17a, §4.8.2]. The analogous picture for $\underline{\mathrm{Sp}}^G$ and its G-symmetric monoidal structure (which by design also incorporates the Hill-Hopkins-Ravenel norm functors [HHR16]) has been worked out by Nardin in his thesis [Nar17, §3]. We expect that similar methods may be used to construct a G-symmetric monoidal structure on $\underline{\mathrm{Sp}}^G(\mathcal{X})$ for any G-∞-topos $\mathcal{X}$. Since the details of such a construction would take us too far afield, we will be content with the simpler and more explicit Construction 2.60, which relies on the recollement presentation of $\mathrm{Sp}^{C_p}(\mathcal{X})$.

As with the Tate construction, Θ^{Tate} has a monoidal universal property. Let $\mu : \Theta \to \Theta^{\mathrm{Tate}}$ also denote the lax symmetric monoidal transformation obtained from $\mu : (-)^{hC_p} \to (-)^{tC_p}$ via precomposition by $\nu(C_p)^*$, and recall our notion of induced objects in $\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p})$ (Definition 2.54).

Proposition 2.63. *$\mu : \Theta \to \Theta^{\mathrm{Tate}}$ is the universal lax symmetric monoidal transformation among those to lax symmetric monoidal functors that annihilate induced objects of $\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p})$.*

Proof. By the theory of the Verdier quotient [NS18, §I.3], there exists a lax symmetric monoidal transformation $\mu' : \Theta \to \widehat{\Theta}$ with the indicated universal property [NS18, Theorem I.3.6]. Moreover, by the formula of [NS18, Theorem I.3.3(ii)] (or rather, its fiber), we see that by construction $\mathrm{fib}(\mu')$ commutes with the canonical monadic resolutions in $\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p})$ exhibiting objects as colimits of induced objects. Since we also have that

$$\Theta \circ \pi_* \simeq \nu^* : \mathrm{Sp}(\mathcal{X}) \to \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p}),$$

and in particular, $\Theta \circ \pi_*$ preserves colimits, we deduce that $\mathrm{fib}(\mu')$ commutes with all colimits.

As we saw in the proof of Theorem 2.47, Θ^{Tate} annihilates induced objects, and μ is lax symmetric monoidal via Construction 2.60. By the universal propery of μ', there exists a lax symmetric monoidal transformation $\alpha : \widehat{\Theta} \to \Theta^{\mathrm{Tate}}$ factoring μ through μ'. Forgetting monoidal structure (as we may do to check that α is an equivalence), we then take fibers to obtain a commutative diagram

$$\begin{array}{ccccc} \mathrm{fib}(\mu') & \xrightarrow{\mathrm{Nm}'} & \Theta & \xrightarrow{\mu'} & \widehat{\Theta} \\ {\scriptstyle\beta}\downarrow & \nearrow_{\mathrm{Nm}} & & \searrow^{\mu} & \downarrow{\scriptstyle\alpha} \\ \mathrm{fib}(\mu) & & & & \Theta^{\mathrm{Tate}}. \end{array}$$

[14]The cited reference states this fact for $\mathrm{C} = \mathrm{Sp}$, but their proof clearly extends to accommodate the indicated generality as it only uses abstract multiplicative properties of the Verdier quotient.

Since Nm and Nm$'$ are equivalences on induced objects, β is also an equivalence on induced objects by the two-out-of-three property. Since both $\mathrm{fib}(\mu')$ and $\mathrm{fib}(\mu) \simeq (-)_{hC_p}\nu(C_p)^*$ commute with all colimits and induced objects generate $\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p})$ under colimits, we deduce that β and thus α is an equivalence. □

We will need to perform Construction 2.60 in families. Intuitively, given a lax commutative square of lax symmetric monoidal functors (with ϕ left-exact)

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\phi} & \mathcal{Z} \\ {\scriptstyle f^*}\downarrow & \eta \Swarrow & \downarrow{\scriptstyle f^*} \\ \mathcal{U}' & \xrightarrow[\phi]{} & \mathcal{Z}', \end{array}$$

the induced functor $f^* : \mathcal{X} = \mathcal{U}\overrightarrow{\times}\mathcal{Z} \to \mathcal{X}' = \mathcal{U}'\overrightarrow{\times}\mathcal{Z}'$ canonically lifts to a lax symmetric monoidal functor via the following formula: let $\gamma_F : F(-) \otimes F(-) \to F(- \otimes -)$ generically denote the lax comparison map for a lax symmetric functor F and suppose $x_i = (u_i, z_i, \alpha_i : u_i \to \phi z_i) \in \mathcal{X}$, $i = 1, 2$. Then we have equivalences

$$f^*(x_1) \otimes f^*(x_2) \simeq (f^*u_1 \otimes f^*u_2, f^*z_1 \otimes f^*z_2, \gamma_\phi \circ (\eta(u_1) \otimes \eta(u_2)) \circ (f^*\alpha_1 \otimes f^*\alpha_2)),$$
$$f^*(x_1 \otimes x_2) \simeq (f^*(u_1 \otimes u_2), f^*(z_1 \otimes z_2), \eta(u_1 \otimes u_2) \circ f^*\gamma_\phi \circ f^*(\alpha_1 \otimes \alpha_2)),$$

with respect to which the lax comparison map

$$\gamma_{f^*} : f^*(x_1) \otimes f^*(x_2) \to f^*(x_1 \otimes x_2)$$

is given by the maps γ_{f^*} on components and the commutative diagram

$$\begin{array}{ccccccc} f^*z_1 \otimes f^*z_2 & \xrightarrow{f^*\alpha_1 \otimes f^*\alpha_2} & f^*\phi u_1 \otimes f^*\phi u_2 & \xrightarrow{\eta(u_1)\otimes\eta(u_2)} & \phi f^*u_1 \otimes \phi f^*u_2 & \xrightarrow{\gamma_\phi} & \phi(f^*u_1 \otimes f^*u_2) \\ \downarrow{\scriptstyle \gamma_{f^*}} & & \downarrow{\scriptstyle \gamma_{f^*}} & \searrow & \simeq & \searrow & \downarrow{\scriptstyle \phi\gamma_{f^*}} \\ f^*(z_1 \otimes z_2) & \xrightarrow{f^*(\alpha_1\otimes\alpha_2)} & f^*(\phi u_1 \otimes \phi u_2) & \xrightarrow{f^*\gamma_\phi} & f^*\phi(u_1 \otimes u_2) & \xrightarrow{\eta(u_1\otimes u_2)} & \phi f^*(u_1 \otimes u_2). \end{array}$$

Furthermore, if $f^* : \mathcal{U} \to \mathcal{U}'$ and $f^* : \mathcal{Z} \to \mathcal{Z}'$ are in addition strong symmetric monoidal, then we see that $f^* : \mathcal{X} \to \mathcal{X}'$ is strong symmetric monoidal.

Using the fibrational perspective, we can make this intuition precise, at least if the vertical functors f^* are in addition symmetric monoidal. Recall that a diagram

$$\mathrm{C}_{(-)} : S \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)$$

of symmetric monoidal ∞-categories and strong symmetric monoidal functors unstraightens to a **cocartesian S-family of symmetric monoidal ∞-categories** [Lur17a, Definition 4.8.3.1]

$$\mathrm{C}^\otimes \to S \times \mathrm{Fin}_*.$$

A lax natural transformation $\phi : \mathrm{C}_{(-)} \Rightarrow \mathrm{D}_{(-)}$ through lax symmetric monoidal functors then corresponds to a functor $\phi : \mathrm{C}^\otimes \to \mathrm{D}^\otimes$ over $S \times \mathrm{Fin}_*$ such that for every $s \in S$, the restriction $\phi_s : \mathrm{C}^\otimes_s \to \mathrm{D}^\otimes_s$ is a morphism of ∞-operads. By Lemma 3.9 (with $B = S \times \mathrm{Fin}_*$) and [QS19, Lemma B.2], the fiberwise right-lax limit $\mathrm{C}^\otimes \overrightarrow{\times} \mathrm{D}^\otimes$ of ϕ yields again a cocartesian S-family of symmetric monoidal ∞-categories. Moreover, unpacking the description of the pushforward functor in Lemma 3.9 yields the formula just described. Using the analysis in Remark 2.59, we may then deduce:

Theorem 2.64. *The functor* $\mathrm{Sp}^{C_p} : \mathrm{Fun}(BC_p, \mathscr{T}\mathrm{op}^L) \to \mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}}$ *lifts to a functor*

$$(\mathrm{Sp}^{C_p})^\otimes : \mathrm{Fun}(BC_p, \mathscr{T}\mathrm{op}^L) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}}).$$

Proof. By a Yoneda argument and the naturality of unstraightening, it will suffice to consider a diagram

$$\mathcal{X}_{(-)} : S \to \mathrm{Fun}(BC_p, \mathscr{T}\mathrm{op}^L)$$

of C_p-∞-topoi and construct the corresponding diagram

$$\mathrm{Sp}^{C_p}(\mathcal{X})^{\otimes} : S \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}}).$$

Let $\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p})^{\otimes} \to S \times \mathrm{Fin}_*$ and $\mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})^{\otimes} \to S \times \mathrm{Fin}_*$ be the two cocartesian S-families of symmetric monoidal ∞-categories obtained via unstraightening. As we have just indicated, we need to lift the stable gluing functor $\Theta^{\mathrm{Tate}} = (-)^{tC_p} \circ \nu(C_p)^*$ to a map

$$\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p})^{\otimes} \xrightarrow{\nu(C_p)^*} (\mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})^{\otimes})^{BC_p} \xrightarrow{(-)^{tC_p}} \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})^{\otimes}$$

over $S \times \mathrm{Fin}_*$ that preserves inert edges. For $\nu(C_p)^*$, this is easy; since $\nu(C_p)^*$ is already strong symmetric monoidal, we may obtain it at the level of cocartesian S-families via unstraightening. We now invoke Lemma 2.65 to handle the more difficult case of $(-)^{tC_p}$. □

Lemma 2.65. *Let $\mathrm{C}_{(-)} : S \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}})$ be a functor and let $\mathrm{C}^{\otimes} \to S \times \mathrm{Fin}_*$ be the corresponding cocartesian S-family of symmetric monoidal ∞-categories. The collection of lax symmetric monoidal transformations*

$$\left\{ \left(\mu_s : (-)^{hG} \Rightarrow (-)^{tG}\right) : \mathrm{C}_s^{BG} \to \mathrm{C}_s \mid s \in S \right\}$$

of [NS18, Theorem I.3.1] *assemble to a natural transformation*

$$\left(\mu : (-)^{hG} \Rightarrow (-)^{tG}\right) : (\mathrm{C}^{\otimes})^{BG} \to \mathrm{C}^{\otimes}$$

of functors over $S \times \mathrm{Fin}_*$, which restrict to maps of ∞-operads over $\{s\} \times \mathrm{Fin}_*$ for all $s \in S$.

Proof. The diagonal $\delta : \mathrm{C}_{(-)} \Rightarrow (\mathrm{C}_{(-)})^{BG}$ is a natural transformation valued in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}})$, and hence unstraightens to a functor $\delta : \mathrm{C}^{\otimes} \to (\mathrm{C}^{\otimes})^{BG}$ over $S \times \mathrm{Fin}_*$ that preserves cocartesian edges (where $(-)^{BG}$ now indicates the cotensor in (marked) simplicial sets over $S \times \mathrm{Fin}_*$). By [Lur17a, Corollary 7.3.2.7], δ admits a relative right adjoint $(-)^{hG}$ over $S \times \mathrm{Fin}_*$, which restricts to a map of ∞-operads over $\{s\} \times \mathrm{Fin}_*$ for all $s \in S$.

To further assemble the lax symmetric monoidal transformations μ_s (and, along the way, construct $(-)^{tG}$), we apply an ∞-operadic variant of the pairing construction of [Lur09, Corollary 3.2.2.13] (also see [Sha18, Example 2.24]) in order to speak of a "family of lax monoidal functors and lax monoidal transformations." Namely, given cocartesian S-families of symmetric monoidal ∞-categories $A^{\otimes}, B^{\otimes} \to S \times \mathrm{Fin}_*$, let ${}_\natural A^{\otimes}$ indicate the marking given by the inert edges and consider the span of marked simplicial sets

$$S^\sharp \xleftarrow{q} {}_\natural A^{\otimes} \xrightarrow{p} (S \times \mathrm{Fin}_*)^\sharp,$$

where q is the composition of the structure map p and the projection to S. Then we have the functor

$$q_* p^* : s\mathrm{Set}^+_{/(S \times \mathrm{Fin}_*)} \to s\mathrm{Set}^+_{/S}, \qquad {}_\natural B^{\otimes} \mapsto \widetilde{\mathrm{Fun}}^{\otimes,\mathrm{lax}}(A,B) = q_* p^*({}_\natural B^{\otimes})$$

that carries cocartesian fibrations over $S \times \mathrm{Fin}_*$ to categorical fibrations over S by [Lur17a, Proposition B.4.5], using that q is a flat categorical fibration [Lur17a, Definition B.3.8] since it is a cocartesian fibration by [Lur17a, Example B.3.11]. Unwinding the definition of the simplicial set $\widetilde{\mathrm{Fun}}^{\otimes,\mathrm{lax}}(A,B)$, we see that the fiber over $s \in S$ is isomorphic to the functor ∞-category $\mathrm{Fun}^{\otimes,\mathrm{lax}}_{/\{s\}\times\mathrm{Fin}_*}(A_s^{\otimes}, B_s^{\otimes})$ of maps of ∞-operads $f : A_s^{\otimes} \to B_s^{\otimes}$, and a morphism $f \to g$ in $\widetilde{\mathrm{Fun}}^{\otimes,\mathrm{lax}}(A,B)$ covering $\alpha : s \to t$ in S is given by the data of a lax commutative square of lax symmetric monoidal functors

$$\begin{array}{ccc} A_s & \xrightarrow{f} & B_s \\ {\scriptstyle \alpha_!^A}\downarrow & \swarrow & \downarrow{\scriptstyle \alpha_!^B} \\ A_t & \xrightarrow[g]{} & B_t. \end{array}$$

Moreover, if $B^{\otimes}$ is classified by a functor to $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\infty})$ (so that the underlying ∞-category is presentable and the tensor product distributes over colimits), then the existence of operadic left Kan extensions[15] [Lur17a, Theorem 3.1.2.3] ensures that $\widetilde{\mathrm{Fun}}^{\otimes,\mathrm{lax}}(A,B)$ is a *locally cocartesian* fibration[16] over S, with the indicated square corresponding to a locally cocartesian edge if and only if g is a operadic left Kan extension of $\alpha^B_! \circ f$ along $\alpha^A_!$. Finally, we note that sections of $\widetilde{\mathrm{Fun}}^{\otimes,\mathrm{lax}}(A,B)$ over S correspond to functors $A^{\otimes} \to B^{\otimes}$ over $S \times \mathrm{Fin}_*$ that preserve inert edges.

Returning to the task at hand, consider $\widetilde{\mathrm{Fun}}^{\otimes,\mathrm{lax}}(\mathrm{C}^{BG},\mathrm{C})$ and the full subcategory $\widetilde{\mathrm{Fun}}_0^{\otimes,\mathrm{lax}}(\mathrm{C}^{BG},\mathrm{C})$ on those lax symmetric monoidal functors $\mathrm{C}_s^{BG} \to \mathrm{C}_s$ that annihilate induced objects. Note that since the ∞-category of lax symmetric monoidal functors $\mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathrm{C},\mathrm{D})$ is equivalent to $\mathrm{CAlg}(\mathrm{Fun}(\mathrm{C},\mathrm{D}))$ for the Day convolution symmetric monoidal structure on $\mathrm{Fun}(\mathrm{C},\mathrm{D})$ [Gla16, Proposition 2.12], it follows that $\mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathrm{C},\mathrm{D})$ is presentable if D is presentable symmetric monoidal by [Gla16, Lemma 2.13] and [Lur17a, Corollary 3.2.3.5], and limits in $\mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathrm{C},\mathrm{D})$ are created in $\mathrm{Fun}(\mathrm{C},\mathrm{D})$ by [Lur17a, Corollary 3.2.2.5]. Since the condition that functors annihilate induced objects is stable under arbitrary limits, it then follows by the adjoint functor theorem that the inclusion

$$\mathrm{Fun}_0^{\otimes,\mathrm{lax}}(\mathrm{C}_s^{BG},\mathrm{C}_s) \subset \mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathrm{C}_s^{BG},\mathrm{C}_s)$$

admits a left adjoint, which sends $(-)^{hG}$ to $(-)^{tG}$ by [NS18, Theorem I.3.1]. We also see that $\widetilde{\mathrm{Fun}}_0^{\otimes,\mathrm{lax}}(\mathrm{C}^{BG},\mathrm{C})$ inherits the property of being locally cocartesian over S by applying the (fiberwise in the target) localization functor to the operadic left Kan extension (so the inclusion of $\widetilde{\mathrm{Fun}}_0$ into $\widetilde{\mathrm{Fun}}$ need not preserve locally cocartesian edges). Invoking [Lur17a, Proposition 7.3.2.11], we deduce that these fiberwise left adjoints assemble to a localization functor

$$L : \widetilde{\mathrm{Fun}}^{\otimes,\mathrm{lax}}(\mathrm{C}^{BG},\mathrm{C}) \to \widetilde{\mathrm{Fun}}_0^{\otimes,\mathrm{lax}}(\mathrm{C}^{BG},\mathrm{C}).$$

The desired functor $(-)^{tG} : (\mathrm{C}^{\otimes})^{BG} \to \mathrm{C}^{\otimes}$ is now obtained by postcomposition of $(-)^{hG}$ as a section of $\widetilde{\mathrm{Fun}}^{\otimes,\mathrm{lax}}(\mathrm{C}^{BG},\mathrm{C})$ with L, and the natural transformation μ similarly corresponds to the unit for the localization applied to $(-)^{hG}$. □

Remark 2.66. The exchange transformation $\alpha : \Sigma^{\infty}_{+}\theta \Rightarrow \Theta^{\mathrm{Tate}}\Sigma^{\infty}_{+}$ of Remark 2.59 is adjoint to an equivalence of lax symmetric monoidal functors, hence is a lax symmetric monoidal transformation. We deduce that its right-lax limit $\Sigma^{\infty}_{+} : \mathcal{X}_{C_p} \to \mathrm{Sp}^{C_p}(\mathcal{X})$ is strong symmetric monoidal with respect to the cartesian product on the source and the smash product on the target. Moreover, with respect to the smash product on $\mathrm{Sp}(\mathcal{X}_{C_p})$, Σ^{∞}_{+} factors as the composite of strong symmetric monoidal functors

$$\mathcal{X}_{C_p} \xrightarrow{\Sigma^{\infty}_{+}} \mathrm{Sp}(\mathcal{X}_{C_p}) \xrightarrow{\mathrm{Fr}} \mathrm{Sp}^{C_p}(\mathcal{X}).$$

Corollary 2.67. *If the unit in $\mathrm{Sp}(\mathcal{X}_{C_p})$ is compact, then the unit in $\mathrm{Sp}^{C_p}(\mathcal{X})$ is compact (and hence any dualizable object in $\mathrm{Sp}^{C_p}(\mathcal{X})$ is compact).*

Proof. By Remark 2.66 and Corollary 2.57, $\mathrm{Fr} : \mathrm{Sp}(\mathcal{X}_{C_p}) \to \mathrm{Sp}^{C_p}(\mathcal{X})$ is a strong symmetric monoidal functor whose right adjoint U preserves colimits. The claim follows. □

We can also produce dualizable objects in $\mathrm{Sp}(\mathcal{X}_{C_p})$ from $\mathrm{Sp}(\mathcal{X})$. To show such a result, we first need to establish the projection formula for the monoidal adjunction

$$\overline{\pi}^* : \mathrm{Sp}^{C_p}(\mathcal{X}) \rightleftarrows \mathrm{Sp}(\mathcal{X}) : \overline{\pi}_*,$$

where $\overline{\pi}^*$ comes endowed with the structure of a strong symmetric monoidal functor as the composition

$$\pi^* j^* : \mathrm{Sp}^{C_p}(\mathcal{X}) \to \mathrm{Sp}(\mathcal{X}_{h_\diamond C_p}) \to \mathrm{Sp}(\mathcal{X}).$$

Proposition 2.68. *Let $\mathcal{X}$ be a C_p-∞-topos and consider the monoidal adjunction $\overline{\pi}^* \dashv \overline{\pi}_*$. Then:*

[15] To invoke the existence theorem (and later on, the adjoint functor theorem), we note that if the fibers of $A^{\otimes}$ are presentable, we also implicitly restrict to the full subcategory of accessible functors.

[16] Note that this sort of functoriality is separate from that articulated in [Lur09, Corollary 3.2.2.13].

1. *$\overline{\pi}_*$ preserves colimits.*
2. *$\overline{\pi}_*$ is conservative.*
3. *For any $A \in \mathrm{Sp}^{C_p}(\mathcal{X})$ and $B \in \mathrm{Sp}(\mathcal{X})$, the natural map*

$$\gamma : \overline{\pi}_*(B) \otimes A \to \overline{\pi}_*(B \otimes \overline{\pi}^*(A))$$

is an equivalence.

Therefore, we have an equivalence $\mathrm{Sp}(\mathcal{X}) \simeq \mathrm{Mod}_{\overline{\pi}_*(\mathbb{1})}(\mathrm{Sp}^{C_p}(\mathcal{X}))$ of symmetric monoidal ∞-categories.

Proof. The conclusion will follow from [MNN17, Theorem 5.29] once we establish the three properties. (1) holds in view of the ambidexterity equivalence $\overline{\pi}_* \simeq \overline{\pi}_!$. For (2) and (3), we first note that for any $X \in \mathrm{Sp}(\mathcal{X})$, we have the formula

$$\overline{\pi}^*\overline{\pi}_*(X) \simeq X \oplus \sigma^*(X),$$

using that $\underline{\mathrm{Sp}}^{C_p}(\mathcal{X})$ admits finite C_p-products and the Beck-Chevalley condition (here, σ^* denotes the C_2-action on $\mathrm{Sp}(\mathcal{X})$). It is thus clear that $\overline{\pi}_*$ is conservative. As for the projection formula, since $i^* : \mathrm{Sp}^{C_p}(\mathcal{X}) \to \mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})$ is symmetric monoidal and annihilates induced objects, it suffices to check that $\overline{\pi}^*(\gamma)$ is an equivalence. But for this, under the given formula $\overline{\pi}^*(\gamma)$ becomes

$$(B \oplus \sigma^* B) \otimes \overline{\pi}^*(A) \to (B \otimes \overline{\pi}^*(A)) \oplus \sigma^*(B \otimes \overline{\pi}^* A),$$

which is an equivalence in view of $\sigma^*\overline{\pi}^* \simeq \overline{\pi}^*$ and distributivity of $\otimes$ over $\oplus$. □

Let $\underline{\mathrm{Hom}}(-,-)$ denote the internal hom in a presentable symmetric monoidal ∞-category.

Corollary 2.69. *Let $\mathcal{X}$ be a C_p-∞-topos. Then:*

1. *For all $A \in \mathrm{Sp}^{C_p}(\mathcal{X})$ and $B \in \mathrm{Sp}(\mathcal{X})$, the natural map*

$$\overline{\pi}_*\underline{\mathrm{Hom}}(B, \overline{\pi}^* A) \to \underline{\mathrm{Hom}}(\overline{\pi}_* B, A)$$

is an equivalence.

2. *Suppose $E \in \mathrm{Sp}(\mathcal{X})$ is a dualizable object (with dual $E^\vee \simeq \underline{\mathrm{Hom}}(E, \mathbb{1})$). Then $\overline{\pi}_*(E)$ is dualizable with dual $\overline{\pi}_*(E)^\vee \simeq \overline{\pi}_*(E^\vee)$. In particular, $\overline{\pi}_*(\mathbb{1})$ is a self-dual $\mathbb{E}_\infty$-algebra.*

Proof. (1) is an easy consequence of the $(\overline{\pi}^*, \overline{\pi}_*)$ projection formula and ambidexterity equivalence $\overline{\pi}^* \simeq \overline{\pi}_!$. For (2), it suffices to show that for any $X \in \mathrm{Sp}^{C_p}(\mathcal{X})$, the natural map

$$\underline{\mathrm{Hom}}(\overline{\pi}_*(E), \mathbb{1}) \otimes X \to \underline{\mathrm{Hom}}(\overline{\pi}_*(E), X)$$

is an equivalence. But under (1) and the projection formula, this map is equivalent to

$$\overline{\pi}_*(\underline{\mathrm{Hom}}(E, \mathbb{1}) \otimes \overline{\pi}^* X) \to \overline{\pi}_*\underline{\mathrm{Hom}}(E, \overline{\pi}^* X).$$

Since E is dualizable, we have that $E^\vee \otimes \overline{\pi}^* X \simeq \underline{\mathrm{Hom}}(E, \overline{\pi}^* X)$, and the claim follows. □

Remark 2.70. Suppose $\mathcal{X}$ is a presentable symmetric monoidal ∞-category and $(\mathcal{U}, \mathcal{Z})$ is a monoidal recollement of $\mathcal{X}$. Let $X \in \mathcal{X}$. Then the endofunctor $(-) \otimes X : \mathcal{X} \to \mathcal{X}$ is a morphism of recollements that induces $(-) \otimes j^*(X)$ on $\mathcal{U}$ and $(-) \otimes i^*(X)$ on $\mathcal{Z}$. Moreover, if X is dualizable, then $(-) \otimes X$ is a *strict* morphism of recollements. This amounts to showing that the (j^*, j_*)-projection formula holds, i.e., for every $U \in \mathcal{U}$, the natural map

$$j_*(U) \otimes X \to j_*(U \otimes j^* X).$$

is an equivalence. Indeed, if we let $Y \in \mathcal{X}$ be any object, then using dualizability of X we see that

$$\mathrm{Map}(Y, j_*(U) \otimes X) \simeq \mathrm{Map}(j^* Y \otimes j^*(X)^\vee, U) \simeq \mathrm{Map}(Y, j_*(U \otimes j^* X)).$$

Remark 2.71. We can use Corollary 2.69 to control the smashing localization i_*i^* on $\mathrm{Sp}^{C_p}(\mathcal{X})$ as a finite localization (in the sense of Miller [Mil92]), by presenting the idempotent $\mathbb{E}_\infty$-algebra $i_*i^*(\mathbb{1})$ as a filtered colimit of dualizable objects. Namely, we observe that the dualizable $\mathbb{E}_\infty$-algebra object $A = \overline{\pi}_*\overline{\pi}^*(\mathbb{1})$ fits into the general theory of A-complete, A-torsion, and A^{-1}-local objects in the sense of Mathew-Naumann-Noel [MNN17, Part 1]. In particular, using that $\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p})$ is monadic over $\mathrm{Sp}(\mathcal{X})$ (Lemma 2.55), we see that the full subcategory of A-complete objects [MNN17, Definition 2.15] and the functor L_A of A-completion [MNN17, Definition 2.19] identify with $\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p})$ embedded via j_* and j_*j^*. Moreover, the recollement cofiber sequence

$$j_!j^*(\mathbb{1}) \to \mathbb{1} \to i_*i^*(\mathbb{1})$$

identifies with the cofiber sequence of [MNN17, Construction 3.4]

$$V_A \to \mathbb{1} \to U_A,$$

where U_A is defined to be the filtered colimit of the sequence of maps

$$\mathbb{1} = U_0 \to U_1 \to U_2 \to U_3 \to \cdots$$

dual to the A-based Adams tower for $\mathbb{1}$ (where $I = \mathrm{fib}(\mathbb{1} \to A)$ and $U_i := (I^{\otimes i})^\vee$)

$$\mathbb{1} = I^{\otimes 0} \leftarrow I \leftarrow I^{\otimes 2} \leftarrow I^{\otimes 3} \leftarrow \cdots .$$

Thus, the full subcategory of A-torsion objects [MNN17, Definition 3.1] and the A-acyclization functor AC_A [MNN17, Construction 3.2] identify with $\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p})$ embedded via $j_!$ and the colocalization $j_!j^*$, while the full subcategory of A^{-1}-local objects [MNN17, Definition 3.10] and the A^{-1}-localization functor identify with $\mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})$ embedded via i_* and the smashing localization i_*i^*.

We further observe that if $p = 2$, I and hence U_k are all *invertible* objects. Indeed, since I is dualizable, it suffices to show that $i^*(I)$ and $\overline{\pi}^*(I)$ are invertible. But $i^*(I) \simeq \mathbb{1}$ since $i^*(A) \simeq 0$, and $\overline{\pi}^*(I) \simeq \Sigma^{-1}\mathbb{1}$ using that the map $\overline{\pi}^*(\mathbb{1} \to A)$ identifes with the summand inclusion $\mathbb{1} \to \mathbb{1} \oplus \mathbb{1}$. This reflects the fact that the cofiber sequence

$$A^\vee \simeq A \to \mathbb{1} \to U_1$$

is a categorical avatar of the cofiber sequence (for σ the sign C_2-real representation)

$$(C_2)_+ = S(\sigma)_+ \to S^0 \to S^\sigma$$

in equivariant homotopy theory, which recovers it when $\mathcal{X} = \mathrm{Spc}$ with trivial C_2-action. We will see another perspective on this in the algebro-geometric setting in Lemma 3.25.

We end this subsection by applying Remark 2.71 to prove a partial converse to Corollary 2.67. For a presentable symmetric monoidal ∞-category C, call an object $X \in \mathrm{C}$ **internally compact** if the endofunctor $\underline{\mathrm{Hom}}(X, -) : \mathrm{C} \to \mathrm{C}$ preserves filtered colimits. Note that if X is dualizable, then X is internally compact, since $X^\vee \otimes (-) \simeq \underline{\mathrm{Hom}}(X, -)$.

Proposition 2.72. *Let $\mathcal{X}$ be a C_p-∞-topos and $X \in \mathrm{Sp}^{C_p}(\mathcal{X})$ an internally compact object. Suppose that $j^*(X)$ is dualizable in $\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p})$ (equivalently, $\overline{\pi}^*(X)$ is dualizable in $\mathrm{Sp}(\mathcal{X})$)[17] and $i^*(X)$ is dualizable in $\mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})$. Then X is dualizable.*

Proof. Let $Y \in \mathrm{Sp}^{C_p}(\mathcal{X})$ be any object. Without any assumptions on X or Y, we note that the natural map

$$j^*\underline{\mathrm{Hom}}(X, Y) \to \underline{\mathrm{Hom}}(j^*X, j^*Y)$$

is an equivalence by [QS19, Proposition 1.30.5]. Given that X is internally compact, we further note that the natural map

$$i^*\underline{\mathrm{Hom}}(X, Y) \to \underline{\mathrm{Hom}}(i^*X, i^*Y)$$

[17] Note that this latter condition is sometimes easier to check. For instance, if (internally) compact objects are known to be dualizable in $\mathrm{Sp}(\mathcal{X})$, then we may use that $\overline{\pi}^*$ preserves (internally) compact objects as it participates in an ambidextrous adjunction. However, j^* does not preserve all compact objects, and correspondingly, dualizable objects are almost never compact in $\mathrm{Sp}(\mathcal{X}_{h_\diamond C_p})$ unless they are also induced.

is an equivalence. Indeed, it suffices to check the map is an equivalence upon applying i_*, after which it identifies as

$$\underline{\mathrm{Hom}}(X,Y)\otimes i_*i^*(\mathbb{1})\to\underline{\mathrm{Hom}}(X,Y\otimes i_*i^*(\mathbb{1})),$$

where on the righthand side, we have traded the internal hom in $\mathrm{Sp}(\mathcal{X}^{h_\diamond C_p})$ for that in $\mathrm{Sp}^{C_p}(\mathcal{X})$ (cf. [QS19, 1.29]). Now by Remark 2.71, we have that $i_*i^*(\mathbb{1})\simeq\mathrm{colim}_{k\in\mathbb{N}}U_k$, and using that X is internally compact, the map is equivalent to

$$\operatorname*{colim}_{k\in\mathbb{N}}(f_k):\operatorname*{colim}_{k\in\mathbb{N}}\underline{\mathrm{Hom}}(X,Y)\otimes U_k\to\operatorname*{colim}_{k\in\mathbb{N}}\underline{\mathrm{Hom}}(X,Y\otimes U_k).$$

It thus suffices to check that each map f_k is an equivalence, which follows readily from the dualizability of each of the U_k. Indeed, we note

$$\underline{\mathrm{Hom}}(X,Y)\otimes U_k\simeq\underline{\mathrm{Hom}}(X\otimes U_k^\vee,Y)\simeq\underline{\mathrm{Hom}}(X,Y\otimes U_k).$$

We now generically write $(-)^\vee=\underline{\mathrm{Hom}}(-,\mathbb{1})$, so that $(j^*X)^\vee$ and $(i^*X)^\vee$ are the respective duals of j^*X and i^*X, and claim that $X^\vee$ is dual to X. For this, first note that the above equivalences specialize to show that $j^*(X^\vee)\simeq(j^*X)^\vee$ and $i^*(X^\vee)\simeq(i^*X)^\vee$. Next, note that the natural maps

$$X^\vee\otimes Y\to\underline{\mathrm{Hom}}(X,Y),\qquad X\otimes Y\to\underline{\mathrm{Hom}}(X^\vee,Y)$$

are sent under j^* and i^* to the maps

$$\begin{aligned}(j^*X)^\vee\otimes j^*Y&\to\underline{\mathrm{Hom}}(j^*X,j^*Y),\qquad & j^*X\otimes j^*Y&\to\underline{\mathrm{Hom}}((j^*X)^\vee,j^*Y),\\(i^*X)^\vee\otimes i^*Y&\to\underline{\mathrm{Hom}}(i^*X,i^*Y),\qquad & i^*X\otimes i^*Y&\to\underline{\mathrm{Hom}}((i^*X)^\vee,i^*Y),\end{aligned}$$

which are equivalences by our assumption that j^*X and i^*X are dualizable. We then invoke the joint conservativity of (j^*,i^*) to conclude that X is dualizable. □

Remark 2.73. Suppose $X\in\mathrm{Sp}^G$ is a compact object. Then X is internally compact. We can show this fact even without assuming the implication that compact objects in Sp^G are dualizable. Indeed, using the joint conservativity of the categorical fixed point functors (that preserve colimits in view of the reverse implication that dualizable objects are compact in Sp^G), it suffices to check that for all subgroups $H\leqslant G$, the composite

$$\mathrm{Sp}^G\xrightarrow{\underline{\mathrm{Hom}}(X,-)}\mathrm{Sp}^G\xrightarrow{\mathrm{res}^G_H}\mathrm{Sp}^H\xrightarrow{\mathrm{Hom}(\mathbb{1},-)}\mathrm{Sp}$$

preserves colimits (where $\mathrm{Hom}(-,-)$ denotes the mapping spectrum). But a diagram chase shows this composite to be equivalent to

$$\mathrm{Sp}^G\xrightarrow{\mathrm{res}^G_H}\mathrm{Sp}^H\xrightarrow{\mathrm{Hom}(\mathrm{res}^G_HX,-)}\mathrm{Sp},$$

and since $\mathrm{res}^G_H(X)$ is compact, this functor does preserve colimits. However, we do not expect a similar implication to hold for $\mathrm{Sp}^G(\mathcal{X})$ in general.

3 The six functors formalism for b-sheaves with transfers

To set the stage for our work, we first recall the formalism of six operations after the work of Ayoub [Ayo07] and as further elaborated upon by Cisinski-Déglise [CD19] (an ∞-categorical framework is explained in Khan's thesis [Kha]). Let Sch' be a subcategory of schemes such that for any smooth morphism $f:T\to S$ in Sch', the pullback of f along any morphism $g:S'\to S$ in Sch' again exists in Sch' (for example, we may suppose that Sch' is an "adequate category of schemes" as defined in [CD19, 2.0.1]).

Definition 3.1 ([Kha, Chapter 2, Definition 3.1.2]). A **premotivic category of coefficients** (or **premotivic functor**) is a functor[18]

$$\mathrm{D}^*:(\mathrm{Sch}')^{\mathrm{op}}\to\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_\infty)$$

that satisfies the following axioms:

[18] Some authors do not suppose presentability hypotheses.

1. For any morphism $f : T \to S$, we have an adjunction[19]

$$f^* : \mathrm{D}^*(S) \rightleftarrows \mathrm{D}^*(T) : f_*.$$

2. For any smooth morphism $f : T \to S$, we have an adjunction

$$f_\sharp : \mathrm{D}^*(T) \rightleftarrows \mathrm{D}^*(S) : f^*.$$

3. ($\sharp$-base change) For a cartesian square

$$\begin{array}{ccc} T' & \xrightarrow{g'} & T \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array} \tag{9}$$

in which f is smooth, the exchange transformation

$$f_\sharp g^* \to g'^* f'_\sharp \tag{10}$$

is an equivalence.

4. ($\sharp$-projection formula) For any smooth morphism $f : T \to S$ and objects $X \in \mathrm{D}^*(T), Y \in \mathrm{D}^*(S)$, the canonical map

$$f_\sharp(X \otimes f^*Y) \to f_\sharp X \otimes Y \tag{11}$$

is an equivalence.

We say that D^* is **stable** if D^* is valued in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}})$.

Remark 3.2. We note that the functor f^* is always strong monoidal. Therefore, for any morphism $f : T \to S$ in Sch', we have that $\mathrm{D}^*(T)$ is endowed with a canonical $\mathrm{D}^*(S)$-algebra structure, from which the natural transformation (11) is obtained [Kha, Chapter 0, Lemma 2.7.7].

We next introduce three properties that can be asked of a premotivic functor.

Definition 3.3. Suppose $\mathrm{D}^* : (\mathrm{Sch}')^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}})$ is a stable[20] premotivic functor.

1. Let $f : \mathcal{E} \to T$ be a vector bundle in Sch'. Then D^* satisfies **homotopy invariance with respect to** $\mathcal{E}$ if the functor

$$f^* : \mathrm{D}^*(T) \to \mathrm{D}^*(\mathcal{E})$$

is fully faithful. We say that D^* satisifies **homotopy invariance** if it satisfies homotopy invariance with respect to all vector bundles in Sch'.

2. Let $i : Z \hookrightarrow T$ be a closed immersion in Sch' with open complement $j : U \hookrightarrow T$. The counit and unit of the adjunctions $j_\sharp \dashv j^*$ and $i^* \dashv i_*$ give a sequence of natural transformations

$$j_\sharp j^* \to \mathrm{id} \to i_* i^*. \tag{12}$$

We say that D^* satisfies **localization with respect to** i if $\mathrm{D}^*(\emptyset)$ is the trivial category, i_* is fully faithful, and (12) is a cofiber sequence (or equivalently, (i^*, j^*) are jointly conservative [Kha, Lemma 3.3.11]).[21] We say that D^* satisfies **localization** if it satisfies localization with respect to all closed immersions in Sch'.

[19] Of course, this axiom is already implicit since D^* is valued in $\mathrm{Pr}^{\mathrm{L}}_\infty$.
[20] The stability assumption simplifies our discussion of the localization property – see [Kha, 3.3.10]. Note that even if we do not assume D^* is stable to begin with, it follows as a corollary of the three properties [Kha, Corollary 3.4.20].
[21] It is a consequence of $\sharp$-base change that $j^* i_* \simeq 0$ and that $j_\sharp$ is fully faithful. Therefore j_* is fully faithful and $(\mathrm{D}^*(U), \mathrm{D}^*(Z))$ defines a recollement of $\mathrm{D}^*(T)$. This holds once we know that D^* satisfies the full six functors formalism.

If D^* satisfies localization, then for any closed immersion $g : S' \hookrightarrow S$ we have an adjunction [Kha, Lemma 3.3.13]

$$g_* : \mathrm{D}^*(S') \rightleftarrows \mathrm{D}^*(S) : g^!.$$

In the notation of (9), we then have the exchange transformation

$$g^! f_* \to g'_* f'^!,$$

which is necessarily an equivalence.

3. Let $p : \mathcal{E} \to T$ be a vector bundle in Sch' and let $s : T \to \mathcal{E}$ be its zero section. Then we have an adjunction

$$p_\sharp s_* : \mathrm{D}^*(T) \rightleftarrows \mathrm{D}^*(T) : s^! p^*,$$

and we define the **Thom transformation** associated to p to be

$$\mathrm{Th}(\mathcal{E}) = p_\sharp s_*, \qquad \mathrm{Th}(\mathcal{E})^{-1} = s^! p^*.$$

We say that D^* has **Thom stability with respect to** $\mathcal{E}$ if the transformation $\mathrm{Th}(\mathcal{E})$ is an equivalence, and D^* has **Thom stability** if it has Thom stability with respect to all vector bundles in Sch'.

The next theorem is due originally to Ayoub [Ayo07, Scholie 1.4.2] and was expanded upon by Cisinski-Déglise [CD19, Theorem 2.4.50]. We follow the version found in [Kha, Chapter 2, Theorem 3.5.4 and Theorem 4.2.2].

Theorem 3.4. *[Ayoub, Cisinski-Déglise] Let* Sch' *be the category of noetherian schemes of finite dimension or, more generally, an adequate category of schemes as in* [CD19, 2.0.1]. *Suppose* D^* *is a premotivic functor that satisfies homotopy invariance, localization, and Thom stability. Then* D^* *satisfies the full six functors formalism as in* [Kha, Chapter 2, Theorem 4.2.2].

In particular, given a separated morphism $f : T \to S$ in Sch', we have a transformation

$$f_! \to f_*$$

that is an equivalence whenever f is a proper morphism. The functor $f_!$ participates in an adjunction

$$f_! : \mathrm{D}^*(T) \rightleftarrows \mathrm{D}^*(S) : f^!.$$

If f is moreover smooth, then we have the **purity equivalences**

$$f_\sharp \xrightarrow{\simeq} f_! \mathrm{Th}(\Omega_f), \qquad f^* \xrightarrow{\simeq} \mathrm{Th}(\Omega_f)^{-1} f^!. \tag{13}$$

Remark 3.5. Using the results in [Hoy14, Appendix C], one may eliminate the noetherian hypotheses appearing in Theorem 3.4; see, in particular, [Hoy14, Remark C.14]. We also refer to [Hoy17] for a construction of the six functors formalism in optimal generality, within the more general setting of quotient stacks. We leave it to the reader to formulate the results of this paper in the non-noetherian setting.

3.1 Recollement of six functors formalisms

Suppose D^* and C^* are two premotivic functors and let $\phi : \mathrm{D}^* \Rightarrow \mathrm{C}^*$ be a natural transformation valued in $\widehat{\mathrm{Cat}}_\infty$. Then for each $T \in \mathrm{Sch}'$, we have a functor $\phi_T : \mathrm{D}^*(T) \to \mathrm{C}^*(T)$, and for every morphism $f : T \to S$ in Sch', we have a canonical equivalence $f^* \phi_S \simeq \phi_T f^*$. By adjunction, we obtain exchange transformations

$$\phi_S f_* \to f_* \phi_T \tag{14}$$

for all f, and also

$$f_\sharp \phi_T \to \phi_S f_\sharp \tag{15}$$

whenever f is smooth. We further say that ϕ is **(lax) symmetric monoidal** if we have the data of a lift of ϕ to a functor $(\mathrm{Sch}')^{\mathrm{op}} \times \Delta^1 \to \mathrm{CAlg}_{(\mathrm{lax})}(\widehat{\mathrm{Cat}}_\infty)$ that restricts to the given functors D^* and C^*.

Theorem 3.6. *Suppose $\phi : \mathrm{D}^* \Rightarrow \mathrm{C}^*$ is left-exact, accessible, and lax symmetric monoidal. Then its right-lax limit $\mathrm{D}^*\overrightarrow{\times}\mathrm{C}^*$ is canonically a premotivic functor that comes equipped with adjoint natural transformations*

$$j^* : \mathrm{D}^*\overrightarrow{\times}\mathrm{C}^* \rightleftarrows \mathrm{D}^* : j_*, \qquad i^* : \mathrm{D}^*\overrightarrow{\times}\mathrm{C}^* \rightleftarrows \mathrm{C}^* : i_*$$

in which j^*, i^* are strong symmetric monoidal and j_*, i_* are lax symmetric monoidal (and the naturality assertion for i_* only holds if the pullback functors of D^* preserve the terminal object). Moreover, if D^* and C^* are stable, then $\mathrm{D}^*\overrightarrow{\times}\mathrm{C}^*$ is stable.

Remark 3.7. Most authors [CD19], [Hoy18, Remark 18] insist that a morphism ϕ of premotivic functors commutes with $f_\sharp$ for f smooth. We note that we do not assume this in our definition — nonetheless, the functor $f_\sharp$ is still defined on the recollement for f smooth and is left adjoint to f^*. We also note that ϕ does not in general commute with f_*, and thus f_* does not in general commute with $i^* : \mathrm{D}^*\overrightarrow{\times}\mathrm{C}^* \Rightarrow \mathrm{C}^*$ (but does with $j^* : \mathrm{D}^*\overrightarrow{\times}\mathrm{C}^* \Rightarrow \mathrm{D}^*$).

To prove Theorem 3.6, we first record a few general facts concerning the functoriality of right-lax limits.

Remark 3.8. Let B be a ∞-category, $\mathrm{D}^*, \mathrm{C}^* : B^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ be two functors, and $\phi : \mathrm{D}^* \Rightarrow \mathrm{C}^*$ a left-exact natural transformation. It is often useful to consider the right-lax limit $\mathrm{D}^*\overrightarrow{\times}\mathrm{C}^*$ from the "parametrized" point of view. Namely, let $\mathrm{D} = \int \mathrm{D}^*$, $\mathrm{C} = \int \mathrm{C}^*$ be the corresponding cartesian fibrations over B, and let $\phi = \int \phi$ (abusing notation). Recall that given any (co)cartesian fibration $X \to B$, we have the **fiberwise arrow category** $X^{\Delta^1} \to B$ defined as the cotensor of X with Δ^1 in (marked) simplicial sets over B; this is again a (co)cartesian fibration whose fiber over $b \in B$ is isomorphic as a simplicial set to $\mathrm{Fun}(\Delta^1, X_b)$. Let

$$\mathrm{D}\overrightarrow{\times}\mathrm{C} = \mathrm{D} \times_{\phi, \mathrm{C}, \mathrm{ev}_1} \mathrm{C}^{\Delta^1}$$

be the fiberwise right-lax limit of $\phi : \mathrm{D} \to \mathrm{C}$ as a map over B. Using that the unstraightening functor commutes with pullbacks and cotensors, we then have an equivalence

$$\int (\mathrm{D}^*\overrightarrow{\times}\mathrm{C}^*) \simeq \mathrm{D}\overrightarrow{\times}\mathrm{C}.$$

We also have a similar result if we take cocartesian fibrations over B^{op} instead.

We have the following pair of lemmas that give componentwise formulas for $f_\sharp$ resp. f_* on the right-lax limit. Note that although the two statements could be combined, we separate them out so as to clarify the situation with adjoints $f_\sharp \dashv f^* \dashv f_*$.

Lemma 3.9. *Let B be an ∞-category and suppose that C, D are cocartesian fibrations over B and $\phi : \mathrm{D} \to \mathrm{C}$ is a functor over B (not necessarily preserving cocartesian edges). Then the fiberwise right-lax limit $\mathrm{D}\overrightarrow{\times}\mathrm{C}$ of ϕ is again a cocartesian fibration, and given a morphism $f : S \to T$ in B, the induced functor $f_\sharp$ is computed by the formula*

$$f_\sharp(X, Y, Y \to \phi X) \simeq (f_\sharp X, f_\sharp Y, f_\sharp Y \to f_\sharp \phi X \to \phi f_\sharp X).$$

In particular, if C, D are also cartesian fibrations over B and ϕ preserves cartesian edges, then $\mathrm{D}\overrightarrow{\times}\mathrm{C} \to B$ is a bicartesian fibration with left adjoints as described.

Proof. The functor $\mathrm{ev}_1 : \mathrm{C}^{\Delta^1} \to \mathrm{C}$ is clearly a B-cocartesian fibration in the sense of [Sha18, Definition 7.1], hence is itself a cocartesian fibration [Sha18, Remark 7.3]. Thus, the functor $\mathrm{D}\overrightarrow{\times}\mathrm{C} \to \mathrm{D}$ obtained via pullback is a cocartesian fibration. Composing this by the structure map of D, we conclude that $\mathrm{D}\overrightarrow{\times}\mathrm{C} \to B$ is a cocartesian fibration. The desired formula is then easily seen upon unwinding the definitions. □

Lemma 3.10. *Let B^{op} be an ∞-category, let C, D be bicartesian fibrations over B^{op}, and let $\phi : \mathrm{D} \to \mathrm{C}$ be a functor over B^{op} (not necessarily preserving cocartesian edges). Suppose that D admits fiberwise pullbacks. Then the fiberwise right-lax limit $\mathrm{D}\overrightarrow{\times}\mathrm{C}$ of ϕ is a bicartesian fibration over B^{op}, and given a morphism $f : S \to T$ in B, the induced functor f_* (encoded by the cartesian functoriality) is computed by*

$$f_*(X, Y, Y \to \phi X) \simeq (f_* X, f_* Y \times_{f_* \phi X} \phi f_* X, f_* Y \times_{f_* \phi X} \phi f_* X \to \phi f_* X).$$

Proof. By Lemma. 3.9 with the base taken to be B^{op}, we have that $\mathrm{D}\overrightarrow{\times}\mathrm{C} \to B^{\mathrm{op}}$ is a cocartesian fibration such that for a morphism $f : S \to T$ in B, the induced functor f^* is computed by

$$f^*(X, Y, Y \to \phi X) \simeq (f^*X, f^*Y, f^*Y \to f^*\phi X \to \phi f^*X).$$

It remains to check is that $\mathrm{D}\overrightarrow{\times}\mathrm{C}$ is a cartesian fibration over B^{op}. Suppose that $f : S \to T$ is a morphism in B and we have an object $Z = (X, Y, Y \to \phi X) \in (\mathrm{D}\overrightarrow{\times}\mathrm{C})_S$. On D and C, denote the cocartesian functoriality by f^* and the cartesian functoriality by f_*. Let

$$f_*(Z) = (f_*X, f_*Y \times_{f_*\phi X} \phi f_*X, f_*Y \times_{f_*\phi X} \phi f_*X \to \phi f_*X).$$

To specify the morphism $\gamma : f_*Z \to Z$ lifting f, we may equivalently define the morphism $\epsilon : f^*f_*Z \to Z$ in the fiber over S (which will be the counit of the adjunction $f^* \dashv f_*$). To this end, we take ϵ to be

$$(f^*f_*X \to X, f^*(f_*Y \times_{f_*\phi X} \phi f_*X) \to f^*f_*Y \to Y, \square)$$

where $\square$ is the diagram

$$\begin{array}{ccccc}
f^*(f_*Y \times_{f_*\phi X} \phi f_*X) & \longrightarrow & f^*\phi f_*(X) & \longrightarrow & \phi f^* f_*(X) \\
\downarrow & & \downarrow & & \\
f^*f_*Y & \longrightarrow & f^*f_*\phi(X) & & \downarrow \\
\downarrow & & \downarrow & & \\
Y & \longrightarrow & \phi(X) & \xrightarrow{=} & \phi(X).
\end{array}$$

To complete the proof, it now suffices to check that γ is a *locally* cartesian edge over f, since we already know that $\mathrm{D}\overrightarrow{\times}\mathrm{C}$ is a cocartesian fibration over B^{op}. This follows by a simple diagram chase. ☐

Corollary 3.11. *Suppose we are in the situation of Lemma 3.10 and let $f : S \to T$ be a morphism in B.*

1. *If $f_* : \mathrm{D}_S \to \mathrm{D}_T$ and $f_* : \mathrm{C}_S \to \mathrm{C}_T$ are fully faithful, then $f_* : (\mathrm{D}\overrightarrow{\times}\mathrm{C})_S \to (\mathrm{D}\overrightarrow{\times}\mathrm{C})_T$ is fully faithful.*
2. *If $f^* : \mathrm{D}_T \to \mathrm{D}_S$ and $f^* : \mathrm{C}_T \to \mathrm{C}_S$ are fully faithful, then $f^* : (\mathrm{D}\overrightarrow{\times}\mathrm{C})_T \to (\mathrm{D}\overrightarrow{\times}\mathrm{C})_S$ is fully faithful.*

Proof. This follows immediately from Lemma 3.10 in view of the componentwise formulas for f^* and f_*. ☐

Proof of Theorem 3.6. Let $\mathrm{rlax.lim} : \mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty) \to \widehat{\mathrm{Cat}}_\infty$ denote the right-lax limit functor. First note that $\mathrm{rlax.lim}(F)$ of an accessible left-exact functor $F : \mathrm{A} \to \mathrm{B}$ of presentable ∞-categories is again presentable [QS19, Corollary 1.36], and $\mathrm{rlax.lim}(\gamma)$ of a natural transformation $\gamma : F \Rightarrow F'$ through colimit-preserving functors again preserves colimits, using that $\mathrm{rlax.lim}(\gamma)$ is a morphism of recollements [QS19, 1.7] and we have on the target $\mathrm{rlax.lim}(F')$ the jointly conservative colimit-preserving functors (i'^*, j'^*). Also, it is clear that rlax. lim of an exact functor of stable ∞-categories is again stable, and rlax. lim of an exact natural transformation thereof is an exact functor.

As for the symmetric monoidal structure, in [QS19, 1.28] we constructed a lift of rlax. lim to a functor[22]

$$\mathrm{rlax.lim} : \mathrm{Fun}(\Delta^1, \mathrm{CAlg}_{\mathrm{lax}}(\widehat{\mathrm{Cat}}_\infty)) \to \mathrm{CAlg}_{\mathrm{lax}}(\widehat{\mathrm{Cat}}_\infty),$$

where we endow $\mathrm{rlax.lim}(F)$ with its canonical symmetric monoidal structure (recalled in §2.4). Moreover, rlax. lim sends natural transformations through strong symmetric monoidal functors to strong symmetric monoidal functors, and the tensor product on $\mathrm{rlax.lim}(F)$ commutes with colimits separately in each variable if the same is true for the domain and codomain of F (Remark 2.61).

Combining the above facts, we thus obtain a functor

$$\mathrm{D}^*\overrightarrow{\times}\mathrm{C}^* : (\mathrm{Sch}')^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_\infty), \qquad T \mapsto \mathrm{D}^*(T)\overrightarrow{\times}\mathrm{C}^*(T),$$

[22] We could also use the construction described before Theorem 2.64 (note that the cited reference only establishes functoriality through strictly commutative squares, but this is all that we need here).

which lands in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}})$ if $\mathrm{D}^*, \mathrm{C}^*$ are stable. Furthermore, if we write objects of $\mathrm{D}^*(T)\overrightarrow{\times}\mathrm{C}^*(T)$ as tuples $(X, Y, Y \to \phi_T X)$ for $X \in \mathrm{D}^*(T)$ and $Y \in \mathrm{C}^*(T)$, then for a morphism $f : T \to S$ in Sch', we may express the pullback functor f^* as given by

$$f^*(X, Y, Y \to \phi_S X) \simeq (f^*X, f^*Y, f^*Y \to f^*\phi_S X \simeq \phi_T f^*X), \tag{16}$$

whereas its right adjoint f_* is given by

$$f_*(X, Y, Y \to \phi_T X) \simeq (f_*X, f_*Y \times_{f_*\phi_T X} \phi_S f_*X, f_*Y \times_{f_*\phi_T X} \phi_S f_*X \to \phi_S f_*X) \tag{17}$$

in view of Remark 3.8 and Lemma 3.10 with $B = \mathrm{Sch}'$, $\mathrm{C} = (\int \mathrm{C}^*)^\vee$, $\mathrm{D} = (\int \mathrm{D}^*)^\vee$, and $\phi = (\int \phi)^\vee$. This reflects the fact that rlax. lim carries natural transformations through symmetric monoidal functors to *strict* morphisms of monoidal recollements [QS19, 1.7 and Lemma 1.23], and thus $\mathrm{D}^*\overrightarrow{\times}\mathrm{C}^*$ comes equipped with j^*, j_*, i^* as indicated. Moreover, i_* commutes with f^* if f^* preserves the terminal object [QS19, 1.3].

Now suppose $f : T \to S$ is a smooth morphism. By Remark 3.8 and Lemma 3.9 with $B = \mathrm{Sch}'$, $\mathrm{C} = \int \mathrm{C}^*$, $\mathrm{D} = \int \mathrm{D}^*$, and $\phi = \int \phi$, we have an adjunction

$$f_\sharp : \mathrm{D}^*(T)\overrightarrow{\times}\mathrm{C}^*(T) \rightleftarrows \mathrm{D}^*(S)\overrightarrow{\times}\mathrm{C}^*(S) : f^*$$

in which the left adjoint $f_\sharp$ is given by the formula

$$f_\sharp(X, Y, Y \to \phi_T X) \simeq (f_\sharp X, f_\sharp Y, f_\sharp Y \to f_\sharp \phi_T X \to \phi_S f_\sharp X). \tag{18}$$

We now prove the $\sharp$-base change formula. Suppose that we have a cartesian square of schemes as in (9) with f smooth. Since we have constructed the appropriate adjoints for $\mathrm{D}^*\overrightarrow{\times}\mathrm{C}^*$, we get an exchange transformation as in (10). Using the formula for f^* (16) and $f_\sharp$ (18) and the joint conservativity of (j^*, i^*), we get that this transformation is invertible since the corresponding exchange transformations for D^* and C^* are. The argument for the $\sharp$-projection formula is similar, using also that j^*, i^* are strong symmetric monoidal. □

Now suppose that $\phi : \mathrm{D}^* \Rightarrow \mathrm{C}^*$ is an exact accessible lax symmetric monoidal morphism of stable premotivic functors, so that $\mathrm{D}^*\overrightarrow{\times}\mathrm{C}^*$ is a stable premotivic functor by Theorem 3.6. We next show that, under certain hypotheses, $\mathrm{D}^*\overrightarrow{\times}\mathrm{C}^*$ inherits the properties enumerated in Definition 3.3 from D^* and C^*.

Lemma 3.12. *Let $\mathcal{E} \to T$ be a vector bundle and suppose that D^* and C^* satisfy homotopy invariance with respect to $\mathcal{E}$. Then $\mathrm{D}^*\overrightarrow{\times}\mathrm{C}^*$ satisfies homotopy invariance with respect to $\mathcal{E}$. Therefore, if D^* and C^* satisfy homotopy invariance, then $\mathrm{D}^*\overrightarrow{\times}\mathrm{C}^*$ satisfies homotopy invariance.*

Proof. This follows immediately from Corollary 3.11.2. □

Lemma 3.13. *Let $f : Z \hookrightarrow T$ be a closed immersion and suppose that D^* and C^* satisfy localization with respect to f. Then:*

1. *The exchange transformation*
$$\chi : \phi_T f_* \to f_* \phi_Z$$
is an equivalence.
2. *$\mathrm{D}^*\overrightarrow{\times}\mathrm{C}^*$ satisfies localization with respect to f.*

Therefore, if D^* and C^* satisfy localization, then $\mathrm{D}^*\overrightarrow{\times}\mathrm{C}^*$ satisfies localization.

Proof. Let $g : U \hookrightarrow T$ be the open immersion complementary to f. For (1), by the joint conservativity of (g^*, f^*) it suffices to check that $g^*\chi$ and $f^*\chi$ are equivalences. For the former, using that $g^* f_* \simeq 0$, we have

$$g^*\chi \simeq 0 : g^*\phi_T f_* \simeq \phi_U g^* f_* \simeq 0 \to g^* f_* \phi_Z \simeq 0.$$

For the latter, using that f_* is fully faithful (so $f^* f_* \simeq \mathrm{id}$), we have

$$f^*\chi \simeq \mathrm{id} : f^*\phi_T f_* \simeq \phi_Z f^* f_* \simeq \phi_Z \to f^* f_* \phi_Z \simeq \phi_Z.$$

For (2), we verify the three defining conditions in turn. It is obvious that $\mathrm{D}^* \overrightarrow{\times} \mathrm{C}^*(\emptyset) \simeq *$. The full faithfulness of f_* follows from Corollary 3.11.1. Finally, it remains to check that the sequence

$$g_\sharp g^* \to \mathrm{id} \to f_* f^*$$

is a cofiber sequence, for which we wish to apply the joint conservativity of (j^*, i^*) to reduce to the known cofiber sequences on $\mathrm{D}^*(T)$ and $\mathrm{C}^*(T)$. But we have generically that both j^* and i^* commute with $g_\sharp$, g^*, f^*, and j^* commutes with f_*, whereas (1) implies that i^* commutes with f_*. The claim then follows. □

The most subtle property is that of Thom stability. Let us begin with a warning:

Warning 3.14. If a premotivic functor D^* satisfies the full six functors formalism, then we have the purity equivalences (13). In particular, if S is a scheme and $p : X \to S$ is a smooth proper morphism, then we have the **ambidexterity equivalence** [Hoy17, Theorem 6.9]

$$p_\sharp \Sigma^{-\Omega_p} \simeq p_*$$

that leads to **Atiyah duality**: the dual of $p_! p^! \mathbb{1}_S$ is $p_\sharp \Sigma^{-\Omega_p} \mathbb{1}_X$ [Hoy17, Corollary 6.13]. Hence, any premotivic functor satisfying the full six functors formalism has a large supply of dualizable objects whose duals are computed by a shift of the conormal bundle. We emphasize that Thom stability is essential here; indeed, one needs Thom stability to even formulate the negative shifts occuring in the statement of Atiyah duality.

In light of this, we can produce explicit counterexamples to the naive expectation that if D^* and C^* both satisfy the full six functors formalism, then the right-lax limit of $\phi : \mathrm{D}^* \Rightarrow \mathrm{C}^*$ again satisfies the full six functors formalism. Indeed, we can use the gluing functor Θ from (7) to construct a premotivic functor that assigns to $\operatorname{Spec} \mathbb{R}$ the ∞-category of naive C_2-spectra $\mathrm{Fun}(\mathcal{O}_{C_2}^{\mathrm{op}}, \mathrm{Sp})$. As we have already mentioned, Atiyah duality does not hold in this ∞-category; for instance, if V is the regular real C_2-representation, then $\Sigma^\infty_+ S^V$ fails to be dualizable in $\mathrm{Fun}(\mathcal{O}_{C_2}^{\mathrm{op}}, \mathrm{Sp})$.

The next lemma gives a sufficient condition for Thom stability to be inherited by the right-lax limit. Assume that for any $X \in \mathrm{Sch}'$, we have that $\mathbb{G}_m \times X \in \mathrm{Sch}'$. For a stable premotivic functor D^* and any scheme $S \in \mathrm{Sch}'$, let $\pi_S : \mathbb{G}_m \times S \to S$ denote the projection map and consider the counit $\pi_{S\sharp} \pi_S^* \mathbb{1} \to \mathbb{1}$ in $\mathrm{D}^*(S)$. Define the **Tate motive** in $\mathrm{D}^*(S)$ to be

$$\mathbb{1}_S^{\mathrm{D}^*}(1) = \mathrm{fib}(\pi_{S\sharp} \pi_S^* \mathbb{1} \to \mathbb{1})[-1].$$

Definition 3.15. We say that D^* is **Tate-dualizable over** S if $\mathbb{1}_S^{\mathrm{D}^*}(1) \in \mathrm{D}^*(S)$ is a dualizable object. If D^* is Tate-dualizable over S for all $S \in \mathrm{Sch}'$, we say that D^* is **Tate-dualizable**.

Lemma 3.16. *Suppose D^* and C^* satisfy localization and assume that $\mathrm{D}^* \overrightarrow{\times} \mathrm{C}^*$ is Tate-dualizable. Then if D^* and C^* have Thom stability, $\mathrm{D}^* \overrightarrow{\times} \mathrm{C}^*$ has Thom stability.*

Proof. By Lemma 3.13, $\mathrm{D}^* \overrightarrow{\times} \mathrm{C}^*$ has localization, so the lemma is well-posed. By [CD19, Corollary 2.4.14],[23] Thom stability holds for $\mathrm{D}^* \overrightarrow{\times} \mathrm{C}^*$ if and only if for any scheme S, the Tate motive $\mathbb{1}_S^{\mathrm{D}^* \overrightarrow{\times} \mathrm{C}^*}(1)$ is invertible. Since we have assumed that $\mathrm{D}^* \overrightarrow{\times} \mathrm{C}^*$ is Tate-dualizable and j^*, i^* are jointly conservative and symmetric monoidal, it suffices to prove that the Tate motive is invertible after applying j^* and i^*. But since j^*, i^* commute with $\pi_{S\sharp}$ by Lemma 3.9, we have

$$i^* \mathbb{1}_S^{\mathrm{D}^* \overrightarrow{\times} \mathrm{C}^*}(1) \simeq \mathbb{1}_S^{\mathrm{D}^*}(1), \qquad j^* \mathbb{1}_S^{\mathrm{D}^* \overrightarrow{\times} \mathrm{C}^*}(1) \simeq \mathbb{1}_S^{\mathrm{D}^*}(1),$$

and these are invertible objects by assumption (and appealing once more to [CD19, Corollary 2.4.14]). □

We may now invoke Theorem 3.4 to conclude:

Theorem 3.17. *Let Sch' be an adequate category of schemes, and let $\mathrm{D}^*, \mathrm{C}^*$ be stable premotivic functors, each of which satisfy the full six functors formalism (or equivalently, homotopy invariance, localization, and Thom stability). Let $\phi : \mathrm{D}^* \Rightarrow \mathrm{C}^*$ be an exact accessible lax symmetric monoidal morphism. Then its right-lax*

[23] For the reference, we note that localization implies (wLoc) and (Zar-sep).

limit $\mathrm{D}^*\overrightarrow{\times}\mathrm{C}^*$ *is canonically a stable premotivic functor that satisfies homotopy invariance and localization. Moreover, if* $\mathrm{D}^*\overrightarrow{\times}\mathrm{C}^*$ *is Tate-dualizable, then* $\mathrm{D}^*\overrightarrow{\times}\mathrm{C}^*$ *also satisfies Thom stability and thus has the full six functors formalism.*

In addition, $\mathrm{D}^*\overrightarrow{\times}\mathrm{C}^*$ *comes equipped with symmetric monoidal morphisms*

$$j^* : \mathrm{D}^*\overrightarrow{\times}\mathrm{C}^* \Rightarrow \mathrm{D}^*, \qquad i^* : \mathrm{D}^*\overrightarrow{\times}\mathrm{C}^* \Rightarrow \mathrm{C}^*$$

and lax symmetric monoidal morphisms

$$j_* : \mathrm{D}^* \Rightarrow \mathrm{D}^*\overrightarrow{\times}\mathrm{C}^*, \qquad i_* : \mathrm{C}^* \Rightarrow \mathrm{D}^*\overrightarrow{\times}\mathrm{C}^*$$

that are objectwise the recollement adjunctions.

3.2 Genuine stabilization and the six functors formalism

Let X be a scheme with $\frac{1}{2} \in \mathcal{O}_X$, and consider the C_2-∞-topos $\mathcal{X} = \widetilde{X[i]}_{\text{ét}}$ with C_2-action as induced by the involution $\sigma : X[i] \to X[i]$.

Definition 3.18. The ∞-category of **(hypercomplete) b-sheaves of spectra with transfers** on X is

$$\mathrm{Sp}^{C_2}(\widetilde{X}_b) = \mathrm{Sp}^{C_2}(\mathcal{X}), \quad \text{resp.} \quad \mathrm{Sp}^{C_2}_b(X) = \mathrm{Sp}^{C_2}(\widehat{X}_b) = \mathrm{Sp}^{C_2}(\widehat{\mathcal{X}}).$$

Remark 3.19. By Example 2.44, we have that $\widetilde{X}_b \simeq \mathcal{X}_{C_2}$. Moreover, using the C_2-Galois cover $\pi : X[i] \to X$ and the involution σ, we may naturally extend $\widetilde{X}_b$ to a C_2-∞-category

$$\underline{\widetilde{X}}_b = \left(\widetilde{X}_b \xrightarrow{\pi^*} \widetilde{X[i]}_b \simeq \widetilde{X[i]}_{\text{ét}}\right),$$

such that $\underline{\widetilde{X}}_b \simeq \underline{\mathcal{X}}_{C_2}$. In this light, Definition 3.18 is thus sensible.

Furthermore, the stable gluing functor of Definition 2.46 has the form

$$\Theta^{\mathrm{Tate}}_X : \mathrm{Sp}(\mathcal{X}_{h_\diamond C_2}) \simeq \mathrm{Sp}(\widetilde{X}_{\text{ét}}) \to \mathrm{Sp}(\mathcal{X}^{h_\diamond C_2}) \simeq \mathrm{Sp}(\widetilde{X}_{\text{rét}}),$$

and by Theorem 2.47, the right-lax limit of Θ^{Tate}_X is equivalent to $\mathrm{Sp}^{C_2}(\widetilde{X}_b)$.

Remark 3.20. We feel justified in referring to objects of $\mathrm{Sp}^{C_2}(\widetilde{X}_b)$ as "b-sheaves of spectra with transfers" in view of the Mackey description of the C_2-stabilization. Roughly speaking, to specify an object of $\mathrm{Sp}^{C_2}(\widetilde{X}_b)$ is to give a b-sheaf of spectra on Ét_X together with a single transfer map along $\pi : X[i] \to X$, subject to certain compatibilities; see §6.1 for more details.

We will use results of Bachmann together with those in §3.1 to promote the assignment $X \mapsto \mathrm{Sp}^{C_2}_b(X)^\wedge_p$ to a premotivic functor admitting the full six functors formalism (on a suitable category of schemes).

Theorem 3.21. *Let S be a noetherian scheme of finite Krull dimension. The functor*

$$\mathrm{Sp}(\widetilde{(-)}_{\text{rét}}) : (\mathrm{Sch}_S^{\mathrm{fin.dim,noeth}})^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}})$$

is a premotivic functor satisfying the full six functors formalism.

Proof. This follows from [Bac18a, Theorem 35]. □

Let $\mathrm{Sch}^{p\text{-fin}}_S$ denote the subcategory of Sch_S on the **locally p-étale finite** S-schemes in the sense of [Bac18b]. These are the S-schemes that admit an étale cover by **p-étale finite S-schemes**, which in turn are those S-schemes X such that for any finite type X-scheme Y, there exists an n such that for every finitely presented, qcqs étale Y-scheme Z, we have that $\mathrm{cd}_p(Z_{\text{ét}}) \leqslant n$. Let $\mathrm{Sch}_S[\frac{1}{p}]$ be the subcategory of S-schemes X for which p is invertible in $\mathcal{O}_X$, and let

$$\mathrm{Sch}^{p\text{-fin}}_S[\tfrac{1}{p}] = \mathrm{Sch}^{p\text{-fin}}_S \cap \mathrm{Sch}_S[\tfrac{1}{p}].$$

Theorem 3.22. *Let p be a prime and suppose that S is a locally p-étale finite scheme. Then the functor*

$$\mathrm{Sp}(\widehat{(-)}_{\text{ét}})^{\wedge}_p : (\mathrm{Sch}_S^{p\text{-fin,noeth}}[\tfrac{1}{p}])^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}})$$

is a premotivic functor satisfying the full six functors formalism.

Proof. It is a theorem of Ayoub that $\mathrm{SH}_{\text{ét}}$ satisfies the hypotheses of Theorem 3.4 (cf. [Bac18b, Theorem 5.1]). The theorem now follows from [Bac18b, Theorem 6.6]. □

Now let θ_X be the unstable gluing functor for $\widetilde{X[i]}_{\text{ét}}$ (Definition 2.40), and recall by Example 2.41 that

$$\theta_X \simeq L_{\text{rét}} i_{\text{ét}} : \widetilde{X}_{\text{ét}} \to \widetilde{X}_{\text{rét}}.$$

Proposition 3.23. *For any morphism of schemes $f : X \to Y$, the canonical exchange transformation of Proposition 2.45*

$$\chi : f^*_{\text{rét}}\theta_Y \Rightarrow \theta_X f^*_{\text{ét}}$$

is ∞-connective. Therefore, the hypercompletion $\widehat{\chi}$ is an equivalence, or if X is of finite Krull dimension, then χ is an equivalence.

Proof. Let $\mathcal{F} \in \widetilde{Y}_{\text{ét}}$ and consider the map

$$\chi_{\mathcal{F}} : f^*_{\text{rét}}\theta\mathcal{F} \to \theta f^*_{\text{ét}}\mathcal{F}$$

as a morphism in $\widetilde{X}_{\text{rét}}$. We claim that $\chi_{\mathcal{F}}$ is an equivalence on real étale points. To verify this, let $\alpha : \operatorname{Spec} k \to X$ be a morphism for k a real closed field. Then on the stalk α we have

$$(\chi_{\mathcal{F}})_\alpha : (f^*_{\text{rét}}\theta\mathcal{F})_\alpha \simeq (\theta\mathcal{F})_{\alpha\circ f} \simeq \Gamma(\alpha^*_{\text{ét}} f^*_{\text{ét}}\mathcal{F}) \simeq (\theta f^*_{\text{ét}}\mathcal{F})_\alpha,$$

where the second and third equivalences follow from Lemma B.14.

To conclude, we note that the real étale site is bounded (since it is 1-localic) and coherent (since the schemes in question are quasicompact). By Deligne's completeness theorem [Lur17b, Theorem A.4.0.5], it follows that $\alpha_{\mathcal{F}}$ is ∞-connective and thus an equivalence after hypercompletion. For the last statement, we note that if X is of finite Krull dimension, then $\widetilde{X}_{\text{rét}}$ is hypercomplete by Theorem B.13. □

Since we are considering the hypercomplete version $\mathrm{Sp}_b^{C_2}(X)$ of b-sheaves with transfers, we note that Θ_X^{Tate} in the next statement refers to the stable gluing functor with respect to $\mathcal{X} = \widehat{X[i]}_{\text{ét}}$.

Corollary 3.24. *Let $f : X \to Y$ be a morphism of schemes. Then the canonical exchange transformation of Remark 2.59*

$$\xi : f^*\Theta_Y^{\mathrm{Tate}} \Rightarrow \Theta_X^{\mathrm{Tate}} f^*$$

is an equivalence.

Proof. As we saw in Remark 2.59, we have a commutative diagram

$$\begin{array}{ccccc} f^*(-)_{hC_2}\nu(C_2)^* & \longrightarrow & f^*\Theta & \longrightarrow & f^*\Theta^{\mathrm{Tate}} \\ \downarrow & & \downarrow{\scriptstyle \chi'} & & \downarrow{\scriptstyle \xi} \\ (-)_{hC_2}\nu(C_2)^* f^* & \longrightarrow & \Theta f^* & \longrightarrow & \Theta^{\mathrm{Tate}} f^* \end{array}$$

in which the rows are cofiber sequences and χ' is the exchange transformation induced by stabilizing $\widehat{\chi}$ in Proposition 3.23. Since $\widehat{\chi}$ is an equivalence and the unstable f^* is left-exact, we deduce that χ' is an equivalence. We also note that the lefthand vertical arrow is always an equivalence. Indeed, we may check commutation of the right adjoints, in which case the claim is obvious. It follows that ξ is an equivalence. □

We will also need that $(\mathrm{Sp}_b^{C_2})_p^\wedge$ is Tate-dualizable. To this end, given a premotivic functor D^*, we first recall the functoriality of the assignment

$$((p_X : X \to S) \in \mathrm{Sm}_S) \mapsto (p_{X\sharp}\mathbb{1} \in \mathrm{D}^*(S)).$$

Suppose that $p_X : X \to S$ and $p_Y : Y \to S$ are two smooth S-schemes and $f : X \to Y$ is an S-morphism (which is not necessarily smooth). Then f induces a map in $\mathrm{D}^*(S)$

$$[f] : p_{X\sharp}\mathbb{1} \to p_{Y\sharp}\mathbb{1},$$

defined to be the adjoint of the map $\mathbb{1} \simeq f^*\mathbb{1} \xrightarrow{f^*\eta} f^*p_Y^*p_{Y\sharp}\mathbb{1} \simeq p_X^*p_{Y\sharp}\mathbb{1}$. One may check that this construction assembles to a functor

$$\mathrm{Sm}_S \to \mathrm{D}^*(S), \qquad (p_X : X \to S) \mapsto (p_{X\sharp}\mathbb{1}),$$

as a special case of a "geometric section" in the sense of [CD19, §1.1.34].

Let $1 : S \to \mathbb{G}_m \times S$ be the map that classifies the unit 1, so we obtain the map $[1] : \mathbb{1} \to \pi_{S\sharp}\mathbb{1}$ in $\mathrm{D}^*(S)$. The counit map $\epsilon : \pi_{S\sharp}\mathbb{1} \to \mathbb{1}$ coincides with the map induced by the projection $\mathbb{G}_m \times S \to S$, and thus the composite $\mathbb{1} \xrightarrow{[1]} \pi_{S\sharp}\mathbb{1} \xrightarrow{\epsilon} \mathbb{1}$ is homotopic to the identity. Therefore, we get a splitting

$$\pi_{S\sharp}\mathbb{1} \simeq \mathbb{1} \oplus \mathbb{1}_S^{\mathrm{D}^*}(1)[1]. \tag{19}$$

On the other hand, the map $-1 : S \to \mathbb{G}_m \times S$ classifying the unit -1 induces a map $[-1] : \mathbb{1} \to \pi_{S\sharp}\mathbb{1}$. Projecting to $\mathbb{1}_S^{\mathrm{D}^*}(1)[1]$ under the splitting (19) then yields a map

$$\rho : \mathbb{1} \to \mathbb{1}_S^{\mathrm{D}^*}(1)[1]$$

that we think of as a premotivic avatar of the usual map ρ in $\mathrm{SH}(S)$ (and which recovers it if $\mathrm{D}^* = \mathrm{SH}$).

Lemma 3.25. *Let p be a prime and suppose that S is a locally p-étale finite scheme on which 2 and p are invertible. Then $(\mathrm{Sp}_b^{C_2})_p^\wedge$ is Tate-dualizable over S.*

Proof. Consider the C_2-Galois cover $p_S : S[i] \to S$. We also write $\mathbb{1}_S(1)_p^\wedge$, $\mathbb{1}_S^{\text{ét}}(1)_p^\wedge$, and $\mathbb{1}_S^{\text{rét}}(1)$ for $\mathbb{1}_S^{\mathrm{Sp}_b^{C_2}(-)_p^\wedge}(1)$, $\mathbb{1}_S^{\mathrm{Sp}(\widehat{-}_{\text{ét}})_p^\wedge}(1)$, and $\mathbb{1}_S^{\mathrm{Sp}(\widetilde{-}_{\text{rét}})}(1)$, respectively. From the first part of Theorem 3.17, $(\mathrm{Sp}_b^{C_2})_p^\wedge$ is a premotivic functor, at least whenever $\mathrm{Sp}(\widehat{-}_{\text{ét}})_p^\wedge$ is. From the discussion above, we have the sequence of maps in $\mathrm{Sp}_b^{C_2}(S)_p^\wedge$

$$p_{S\sharp}p_S^*(\mathbb{1}_S)_p^\wedge \to (\mathbb{1}_S)_p^\wedge \xrightarrow{\rho} \mathbb{1}_S(1)[1]_p^\wedge. \tag{20}$$

We claim that this is a cofiber sequence. Given this, by Corollary 2.69 (and noting that the adjunction $p_S^* \dashv p_{S*} \simeq p_{S\sharp}$ is the same as the adjunction written there as $\overline{\pi}^* \dashv \overline{\pi}_*$ for $\mathfrak{X} = \widehat{S[i]}_{\text{ét}}$; see Remark 3.19), it will then follow that $\mathbb{1}_S(1)[1]_p^\wedge$ (and thus $\mathbb{1}_S(1)_p^\wedge$) is dualizable. To prove the claim, it suffices to show that (20) is a cofiber sequence after applying i^* and j^*.

Upon applying i^* and using that i^* annihilates induced objects, we get the p-completion of the sequence of maps in $\mathrm{Sp}(\widetilde{S}_{\text{rét}})$:

$$0 \longrightarrow \mathbb{1}_S \longrightarrow \mathbb{1}_S^{\text{rét}}(1)[1].$$

But the second map is invertible even before p-completion, since it corresponds to the motivic ρ under the equivalence (1), and ρ is invertible in $\mathrm{SH}_{\text{rét}}(S) \simeq \mathrm{SH}(S)[\rho^{-1}]$ by Bachmann's theorem. This proves exactness after applying i^*.

On the other hand, we claim that (20) is exact after applying j^*. Since all functors in sight commutes with base change and we are working with étale sheaves, it suffice to check this after base change along the Galois cover $S[i] \to S$. Since we have an isomorphism of S-schemes $S[i] \times_S S[i] \simeq S[i] \coprod S[i]$, the sequence (20) then becomes

$$(\mathbb{1}_{S[i]} \oplus \mathbb{1}_{S[i]})_p^\wedge \xrightarrow{\nabla} (\mathbb{1}_{S[i]})_p^\wedge \longrightarrow (\mathbb{1}_{S[i]}^{\text{ét}}(1)[1])_p^\wedge,$$

in $\mathrm{Sp}(\widehat{S[i]}_{\text{ét}})_p^\wedge$, where ∇ is the fold map.

To proceed, recall that for any scheme X with $\frac{1}{p} \in \mathcal{O}_X$ we have Bachmann's "twisting spectrum"

$$\hat{\mathbb{1}}_p(1)_X \in \mathrm{Sp}(\widehat{X}_{\text{ét}})^{\wedge}_p.$$

More precisely, this is the object that maps under the change-of-site functor $\nu^* : \mathrm{Sp}(\widehat{X}_{\text{ét}})^{\wedge}_p \to \mathrm{Sp}(\widehat{X}_{\text{proét}})^{\wedge}_p$ to the spectrum denoted by $\hat{\mathbb{1}}_p(1)_X$ in [Bac18b] that satisfies [Bac18b, Theorem 3.6.1-3] and exists by [Bac18b, Theorem 3.6.4]. Whenever X is also locally p-étale finite, we have a canonical equivalence $\hat{\mathbb{1}}_p(1)_X \simeq (\mathbb{1}^{\text{ét}}_X(1))^{\wedge}_p$ [Bac18b, Proposition 4.5]. If X contains all p^n-th roots of unity for all n, we furthermore have a canonical equivalence $\hat{\mathbb{1}}_p(1)_X \simeq (\mathbb{1}_X)^{\wedge}_p$ by [Bac18b, Theorem 3.6].

To conclude, we note that by [Bac18b, Corollary 5.12] the pullback functor to algebraically closed fields form a conservative family. Under the assumption that p is invertible in $\mathcal{O}_S$, we need only pullback to those fields with characteristics prime to p hence, since they are furthermore algebraically closed, contains all p^n-th roots of unity. In this case, when S is the spectrum of such a field, $\mathbb{1}_{S[i]}(1)[1]^{\wedge}_p$ is equivalent to $\mathbb{1}_S[1]^{\wedge}_p$, which is exactly the cofiber of the fold map. □

Remark 3.26. The strategy of the proof of Lemma 3.25 is to show that the "premotivic" $\rho : \mathbb{1} \to \mathbb{1}(1)[1]$ and the "categorical" $\rho : \mathbb{1} \to U_1$ of Remark 2.71 coincide after p-completion (in effect, also dispensing with the Tate twist). We do not expect such a result to hold in $\mathrm{SH}(-)$ itself. For example, in [BS20], we observed that in $\mathrm{SH}(\mathbb{R})$ the two maps are *not* equivalent even after p-completion [BS20, Remark 8.4], but *are* equivalent after p-completion and cellularization [BS20, Proposition 8.3].

We are now ready to prove the main theorem of this section.

Theorem 3.27. *Let p be a prime and suppose that S is a noetherian locally p-étale finite scheme. Then*

$$(\mathrm{Sp}^{C_2}_b)^{\wedge}_p : (\mathrm{Sch}^{p\text{-fin,noeth}}_S[\tfrac{1}{p}, \tfrac{1}{2}])^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}})$$

is a premotivic functor that satisfies the full six functors formalism.

Proof. We note that we have a functor valued in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}})$ by Theorem 2.64. Under Corollary 3.24, the theorem then follows from Bachmann's theorems 3.21 and 3.22 along with Theorem 3.17 and the fact that $(\mathrm{Sp}^{C_2}_b)^{\wedge}_p$ is Tate-dualizable by Lemma 3.25. □

Example 3.28. Fix a prime p. By [Bac18b, Corollary 2.13], if S is locally p-étale finite, then so is any finite type S-scheme. Therefore, the premotivic functor $(\mathrm{Sp}^{C_2}_b)^{\wedge}_p$ satisfies the full six functors formalism when restricted to $\mathrm{Sch}^{\mathrm{ft}}_{\mathbb{Z}[\frac{1}{p},\frac{1}{2}]}$. However, we note that the ∞-category $\mathrm{Sp}(\widetilde{X}_{\text{rét}})$ is nonzero only if one of its residue fields is an orderable field, which necessitates characteristic zero. Hence, the premotivic functor $(\mathrm{Sp}^{C_2}_b)^{\wedge}_p$ is most interesting when restricted to schemes whose residue fields are characteristic zero (otherwise, it coincides with the ∞-category of p-completed hypercomplete étale sheaves).

In light of this, Theorem 3.27 produces a genuinely new six functors formalism when $(\mathrm{Sp}^{C_2}_b)^{\wedge}_p$ is restricted to $\mathrm{Sch}^{\mathrm{ft}}_{\mathbb{Q}}$. In fact, letting p vary across all primes and using the fact that we are in characteristic zero, the profinite-completed version of $\mathrm{Sp}^{C_2}_b$ assembles into a premotivic functor

$$(\mathrm{Sp}^{C_2}_b)^{\wedge} : (\mathrm{Sch}^{\mathrm{ft}}_{\mathbb{Q}})^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}})$$

that satisfies the full six functors formalism.

4 Motivic spectra over real closed fields and genuine C_2-spectra

Let S be a scheme. We define the ***b*-topology** on the large site Sm_S to be the intersection of the real étale and étale topologies on Sm_S. The b-topology is finer than the Nisnevich topology, so we may consider the full subcategory $\mathrm{SH}_b(S) \subset \mathrm{SH}(S)$ of b-local motivic spectra as a Bousfield localization. We find it apposite to then make the following definition.

Definition 4.1. The ∞-category of **Scheiderer motivic spectra** over S is $\mathrm{SH}_b(S)$.

Our primary aim for the remainder of this paper is to relate Scheiderer motivic spectra and b-sheaves of spectra with transfers via a parametrized and semialgebraic analogue of the C_2-Betti realization functor $\mathrm{Be}^{C_2} : \mathrm{SH}(\mathbb{R}) \to \mathrm{Sp}^{C_2}$ of Heller-Ormsby [HO16]. We first consider the simplest nontrivial example with S taken to be the spectrum of a real closed field. To this end, fix a real closed field k and let $C = k[i]$ be an algebraic closure of k, so that C/k is a Galois extension of degree 2. We will define a C_2**-semialgebraic** or **Delfs-Knebusch realization** functor

$$\mathrm{DK}^{C_2} : \mathrm{SH}(k) \to \mathrm{Sp}^{C_2}$$

and then show its right adjoint Sing^{C_2} is fully faithful after p-completion with essential image $\mathrm{SH}_b(k)^{\wedge}_p$ (Theorem 4.20), thereby locating C_2-equivariant homotopy theory as a concept in motivic homotopy theory. The relationship with C_2-Betti realization of [HO16] is addressed in Remark 4.9. For the proof, we imitate the strategy of [BS20], in which Behrens and the second author studied a similar question concerning C_2-Betti realization and cellular real motivic spectra; apart from semialgebraic topology, the only novel aspect to our proof here is the identification of étale motivic spectra over k with Borel C_2-spectra under Sing^{C_2} after p-completion (Corollary 4.15), which permits us to eliminate cellularity hypotheses (but also see Remark 4.21). Along the way, we additionally establish the symmetric monoidal monadicity of the adjunction $\mathrm{DK}^{C_2} \dashv \mathrm{Sing}^{C_2}$ (Theorem 4.7), and hence of $(L_b)^{\wedge}_p \dashv (i_b)^{\wedge}_p$ as well under Theorem 4.20.

4.1 C_2-equivariant semialgebraic realization

Suppose that X is a smooth k-variety. Recall that we may endow the set $X(k)$ of k-points of X with the **semialgebraic topology**, thereby making it into a semialgebraic space (cf. [DK82] for the relevant definitions and [Sch94, Chapter 15], [BCR87] for textbook references). This procedure assembles to a functor

$$\mathrm{Sm}_{k+} \to \mathrm{Top}_\bullet, \qquad X_+ \mapsto X(k)_+.$$

Furthermore, if we suppose instead that Y is a smooth quasiprojective C-variety, then we may endow $Y(C)$ with the semialgebraic topology under the isomorphism

$$R_{C/k}Y(k) \cong Y(C)$$

that identifies $Y(C)$ with the k-points of its Weil restriction $R_{C/k}Y$ (cf. [Sch94, 15.2]).[24] Then for a smooth quasiprojective k-variety X, the induced conjugation C_2-action on $X(C)$ is seen to be continuous. We thereby obtain a functor

$$\mathrm{SmQP}_{k+} \to \mathrm{Top}_\bullet^{BC_2}, \qquad X_+ \mapsto X(C)_+$$

into the category of pointed topological spaces with C_2-action, such that the natural map $X(k)_+ \to X(C)_+$ induces a homeomorphism $X(k)_+ \cong X(C)_+^{C_2}$. Passing to the ∞-category of C_2-spaces, we get a functor

$$\mathrm{DK}^{C_2} : \mathrm{SmQP}_{k+} \to \mathrm{Spc}_{C_2 \bullet} \simeq \mathrm{PShv}(\mathcal{O}_{C_2})_\bullet, \tag{21}$$

such that under Elmendorf's theorem,[25] DK^{C_2} is given by[26]

$$X_+ \mapsto (\mathrm{Sing}(X(C))_+ \leftarrow \mathrm{Sing}(X(k))_+).$$

Similarly, we have the functor

$$\mathrm{DK}_C : \mathrm{SmQP}_{C+} \to \mathrm{Spc}_\bullet, \qquad Y_+ \mapsto \mathrm{Sing}(Y(C))_+.$$

With respect to the symmetric monoidal structure on the presheaf category $\mathrm{PShv}(\mathcal{O}_{C_2})_\bullet$ that is given by the pointwise smash product, the next lemma is immediate.

[24] In fact, as Scheiderer notes, given any C-variety Y, we may obtain the semialgebraic topology on $Y(C)$ from the above procedure by gluing on affine open pieces.

[25] Note that we could define DK^{C_2} into pointed presheaves $\mathrm{PShv}(\mathcal{O}_{C_2})_\bullet$ directly by means of the formula and avoid Elmendorf's theorem altogether.

[26] To ward off any potential confusion, we note here a clash of terminology between the total singular complex functor Sing and the right adjoint to (C_2-)Betti or semialgebraic realization, also denoted $\mathrm{Sing}^{(C_2)}$.

Lemma 4.2. *The functors* DK^{C_2} *and* DK_C *are strong symmetric monoidal.*

The symmetric monoidal structure on $\mathrm{PShv}_\Sigma(\mathrm{SmQP}_{k+})$ in which we will eventually invert $\mathbb{T}$ to obtain $\mathrm{SH}(k)$ is given by Day convolution (and likewise for C). By the universal property of PShv_Σ and Day convolution [Lur17a, Proposition 4.8.1.10], we get sifted colimit-preserving strong symmetric monoidal extensions

$$\mathrm{DK}^{C_2} : \mathrm{PShv}_\Sigma(\mathrm{SmQP}_{k+}) \simeq \mathrm{PShv}_\Sigma(\mathrm{SmQP}_k)_\bullet \to \mathrm{Spc}_{C_2\bullet},$$
$$\mathrm{DK}_C : \mathrm{PShv}_\Sigma(\mathrm{SmQP}_{C+}) \simeq \mathrm{PShv}_\Sigma(\mathrm{SmQP}_C)_\bullet \to \mathrm{Spc}_\bullet.$$

Lemma 4.3. *The functors* DK^{C_2} *and* DK_C *convert Nisnevich sieves and* $\mathbb{A}^1$*-homotopy equivalences to equivalences.*

Proof. We prove the claim for DK^{C_2}, since the claim for DK_C is both similar and easier. The assertion regarding Nisnevich sheaves is essentially [DI04, Theorem 5.5] and proceeds as follows. By standard arguments, we may reduce to the case of a Nisnevich sieve generated by a single map $X' \to X$. In this case, it suffices to prove that the Čech nerves of the maps

$$\mathrm{Sing}(X'[i](C))_+ \to \mathrm{Sing}(X(C))_+, \qquad \mathrm{Sing}(X')(k)_+ \to \mathrm{Sing}(X)(k)_+$$

induce equivalences

$$|\check{C}_\bullet(\mathrm{Sing}(X'[i](C)))_+| \to (\mathrm{Sing}(X(C)))_+, \qquad |\check{C}_\bullet(\mathrm{Sing}(X')(k))_+| \to \mathrm{Sing}(X)(k)_+.$$

We note that since equivalences are detected pointwise in presheaf categories, we do not have to consider the C_2-action on $\mathrm{Sing}(X'[i](C))_+$. The claim then follows from the fact that étale morphisms are converted to generalized space covers in the sense of [DI04, 4.8] under taking semialgebraization by [DK84, Example 5.1] and [DI04, Proposition 4.10].

The claim about $\mathbb{A}^1$-homotopy equivalences follows from the fact that DK^{C_2} is strong symmetric monoidal and that $\mathrm{DK}^{C_2}(\mathbb{A}^1)$ defines homotopies in the semialgebraic category (cf. [Del85]). □

Recall that for any scheme S, $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{SmQP}_S) \simeq \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S)$ since any smooth S-scheme admits a Zariski cover by smooth quasiprojective ones. Thus, by Lemma 4.3 we further get sifted colimit-preserving strong symmetric monoidal functors

$$\mathrm{DK}^{C_2} : \mathrm{H}(k)_\bullet \to \mathrm{Spc}_{C_2\bullet}, \qquad \mathrm{DK}_C : \mathrm{H}(C)_\bullet \to \mathrm{Spc}_\bullet.$$

Lemma 4.4. *The functors* DK^{C_2} *and* DK_C *preserve compact objects.*

Proof. It suffices to prove that $\mathrm{Sing}(X(C))$ and $\mathrm{Sing}(X(k))$ have the homotopy type of finite CW-complexes for a k-variety X. This follows from the fact that both X and $X[i]$ are, by assumption, finite type and hence the semialgebraic spaces $X(k)$ and $X(C)$ may be covered by finitely many semialgebraic subsets. A theorem of Delfs [DK82, Theorem 4.1] then asserts the existence of a finite triangulation of these semialgebraic spaces and thus they have the homotopy types of finite CW-complexes. □

Let V be a real C_2-representation. Then we can view S^V, the 1-point compactification of V, as an object in $\mathrm{PShv}(\mathcal{O}_{C_2})_\bullet$.

Lemma 4.5. *We have the following natural equivalences in* $\mathrm{Spc}_{C_2\bullet}$*, resp.* $\mathrm{Spc}_\bullet$

$$\mathrm{DK}^{C_2}((\mathbb{G}_m, 1)) \simeq S^\sigma, \qquad \mathrm{DK}_C((\mathbb{G}_m, 1)) \simeq S^1,$$
$$\mathrm{DK}^{C_2}((S^1, 1)) \simeq S^1, \qquad \mathrm{DK}_C((S^1, 1)) \simeq S^1,$$

and

$$\mathrm{DK}^{C_2}((\mathbb{P}^1, \infty)) \simeq \Sigma S^\sigma \simeq S^{\sigma+1}, \qquad \mathrm{DK}_C((\mathbb{P}^1, \infty)) \simeq \Sigma S^1 \simeq S^2.$$

Proof. The first two pairs of equivalences are standard. The last pair of equivalences holds in view of the Zariski cover $\{\mathbb{A}^1 \to \mathbb{P}^1, \mathbb{A}^1 \to \mathbb{P}^1\}$ and Lemma 4.3. □

Stabilization is now standard procedure. The representation sphere S^σ is invertible in Sp^{C_2}; this is either true by fiat if Sp^{C_2} is constructed out of $\mathrm{Spc}_{C_2\bullet}$ from inverting representation spheres (see [BH18, §9.2] for a treatment in our language) or a consequence of the comparison between C_2-spectra and spectral Mackey functors (a theorem of Guillou-May [GM11]). We also refer the reader to Nardin's proof of the comparison [Nar16, Theorem A.4] which is native to the language of this paper (also see [BH18, Proposition 9.10]).

Using [BH18, Lemma 4.1], which is a variant of Robalo's theorem [Rob15], we get the colimit-preserving symmetric monoidal functors of C_2 (resp. C)-**semialgebraic realization**

$$\mathrm{DK}^{C_2} : \mathrm{SH}(k) \to \mathrm{Sp}^{C_2}, \qquad \mathrm{DK}_C : \mathrm{SH}(C) \to \mathrm{Sp}$$

that commute with the suspension functors and the unstable DK^{C_2}, resp. DK_C. We denote their right adjoints by

$$\mathrm{Sing}^{C_2} : \mathrm{Sp}^{C_2} \to \mathrm{SH}(k), \qquad \mathrm{Sing}_C : \mathrm{Sp} \to \mathrm{SH}(C).$$

Note that the following diagram commutes by construction:

$$\begin{array}{ccc} \mathrm{SH}(k) & \xrightarrow{\mathrm{DK}^{C_2}} & \mathrm{Sp}^{C_2} \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \mathrm{res}^{C_2}} \\ \mathrm{SH}(C) & \xrightarrow{\mathrm{DK}_C} & \mathrm{Sp}, \end{array}$$

where $\pi : \mathrm{Spec}\, C \to \mathrm{Spec}\, k$ denotes the Galois cover.

Remark 4.6. There is yet another adjunction that connects C_2-equivariant homotopy theory and motivic spectra over real closed fields that was studied by Heller-Ormsby [HO16]. It takes the form

$$c : \mathrm{Sp}^{C_2} \rightleftarrows \mathrm{SH}(k) : u.$$

By definition, the functor DK^{C_2} left splits the constant functor $c : \mathrm{Sp}^{C_2} \to \mathrm{SH}(k)$ of [HO16, §4], so that the composite $\mathrm{Sp}^{C_2} \xrightarrow{c} \mathrm{SH}(k) \xrightarrow{\mathrm{DK}^{C_2}} \mathrm{Sp}^{C_2}$ is equivalent to the identity functor (cf. [BH18, Proposition 10.6]). It follows that c is a faithful functor and we have an equivalence

$$u \circ \mathrm{Sing}^{C_2} \simeq \mathrm{id} \tag{22}$$

by uniqueness of adjoints. The functor c is further proved to be fully faithful in [HO18].

Our next result is the semialgebraic version of [BS20, Lemma 8.13].

Theorem 4.7. *The symmetric monoidal adjunctions*

$$\mathrm{DK}^{C_2} : \mathrm{SH}(k) \rightleftarrows \mathrm{Sp}^{C_2} : \mathrm{Sing}^{C_2}, \qquad \mathrm{DK}_C : \mathrm{SH}(C) \rightleftarrows \mathrm{Sp} : \mathrm{Sing}_C$$

satisfy the following conditions:

1. *The functors DK^{C_2} and DK_C preserve compact objects.*
2. *The functors Sing^{C_2} and Sing_C preserve colimits.*
3. *The projection formula holds, i.e., the canonical maps*

$$\mathrm{Sing}^{C_2}(K) \otimes E \to \mathrm{Sing}^{C_2}(K \otimes \mathrm{DK}^{C_2}(E)),$$
$$\mathrm{Sing}_C(L) \otimes F \to \mathrm{Sing}_C(L \otimes \mathrm{DK}_C(F)),$$

are equivalences for all $K \in \mathrm{Sp}^{C_2}, E \in \mathrm{SH}(k)$, resp. $L \in \mathrm{Sp}, F \in \mathrm{SH}(C)$.

4. *The functors Sing^{C_2} and Sing_C are conservative.*

Consequently, we have equivalences of symmetric monoidal ∞-categories

$$\mathrm{Sp}^{C_2} \simeq \mathrm{Mod}_{\mathrm{Sing}^{C_2}\mathrm{DK}^{C_2}(\mathbb{1})}(\mathrm{SH}(k)), \qquad \mathrm{Sp} \simeq \mathrm{Mod}_{\mathrm{Sing}_C \mathrm{DK}_C(\mathbb{1})}(\mathrm{SH}(C))$$

as $\mathrm{SH}(k)$-algebras, resp. $\mathrm{SH}(C)$-algebras.

Proof. We give the proof for $\mathrm{DK}^{C_2} \dashv \mathrm{Sing}^{C_2}$ as the proof for $\mathrm{DK}_C \dashv \mathrm{Sing}_C$ is the same. We check:

1. This follows from the unstable statement Lemma 4.4 and the fact that DK^{C_2} commutes with the appropriate suspension functors.

2. Since the functor Sing^{C_2} is exact, we need only prove that Sing^{C_2} preserves filtered colimits. Let $K_{(-)} : I \to \mathrm{Sp}^{C_2}$ be a filtered diagram with colimit $K = \mathrm{colim}_i K_i$. We have a comparison map

$$\operatorname*{colim}_i \mathrm{Sing}^{C_2}(K_i) \to \mathrm{Sing}^{C_2}(K)$$

that we claim is an equivalence. It suffices to prove this after applying $[\Sigma^{p,q}X_+, -]$ where $X \in \mathrm{Sm}_k$ and $p, q \in \mathbb{Z}$. The result now follows from the fact that DK^{C_2} is strong symmetric monoidal, and that X_+ and $\mathrm{DK}^{C_2}(X_+)$ are compact objects in $\mathrm{SH}(k)$ and Sp^{C_2} respectively.

3. In this case, since k is characteristic zero, $\mathrm{SH}(k)$ is generated by dualizable objects [LYZ19, Proposition B.1]. The argument follows by [EK20, Lemma 5.1].

4. This is a consequence of the faithfulness of the Sing^{C_2} functor thanks to (22).

The consequence now follows from the monoidal Barr-Beck theorem [MNN17, Theorem 5.29] (also see [EK20, Corollary 5.3]). □

Remark 4.8. We also have a ***k*-semialgebraic realization** functor, specified by extending the functor

$$\mathrm{DK}_k : \mathrm{SmQP}_{k\bullet} \to \mathrm{Spc}_\bullet, \qquad X_+ \mapsto \mathrm{Sing}(X(k))_+$$

to $\mathrm{DK}_k : \mathrm{SH}(k) \to \mathrm{Sp}$ by the same methods as above. However, this realization functor is already present in the above picture. Namely, if we let $i^* : \mathrm{Sp}^{C_2} \to \mathrm{Sp}$ denote the functor of geometric C_2-fixed points, then by construction we have an equivalence $\mathrm{DK}_k \simeq i^*\mathrm{DK}^{C_2}$.

Remark 4.9. When $k = \mathbb{R}$, we note that C_2-semialgebraic realization DK^{C_2} is equivalent to C_2-Betti realization Be^{C_2}. To see this, we first compare the two as functors $\mathrm{SmQP}_{\mathbb{R}} \to \mathrm{Spc}_{C_2}$. Recall that Be^{C_2} is defined by sending a smooth quasiprojective $\mathbb{R}$-scheme X to the C_2-space $X(\mathbb{C})^{\mathrm{an}}$, where we endow $X(\mathbb{C})$ with the analytic topology. Since an analytic space is locally homeomorphic to $\mathbb{C}^n$ with the standard topology, the space $X(\mathbb{C})^{\mathrm{an}}$ is given by endowing $X(\mathbb{C}) \cong R_{\mathbb{C}/\mathbb{R}}X_{\mathbb{C}}(\mathbb{R})$ with the Euclidean or strong topology on the $\mathbb{R}$-points of an $\mathbb{R}$-variety, noting that $R_{\mathbb{C}/\mathbb{R}}\mathbb{A}^n \simeq \mathbb{A}^{2n}$. On the other hand, the semialgebraic topology on $X(\mathbb{C})$ is finer than the analytic topology, so if we denote the set $X(\mathbb{C})$ equipped with the semialgebraic topology as $X(\mathbb{C})^{\mathrm{salg}}$, then the identity map on sets defines a continuous map of spaces $\alpha_X^{\mathrm{pre}} : X(\mathbb{C})^{\mathrm{salg}} \to X(\mathbb{C})^{\mathrm{an}}$, which is clearly C_2-equivariant. Upon passing to Spc_{C_2}, we obtain a natural transformation $\alpha : \mathrm{DK}^{C_2} \Rightarrow \mathrm{Be}^{C_2}$. From the existence of triangulations for semialgebraic spaces [DK82, Theorem 4.1] together with [Kne92, §10, Second Comparison Theorem], we get that α_X^{pre} and $(\alpha_X^{\mathrm{pre}})^{C_2}$ induce isomorphisms of homotopy groups, whence α_X^{pre} is a weak homotopy equivalence of C_2-topological spaces. We conclude that α is an equivalence. It follows that all successive extensions of these functors are equivalent. Similarly, we have that $\mathrm{DK}_{\mathbb{C}}$ is equivalent to complex Betti realization $\mathrm{Be}_{\mathbb{C}}$ and $\mathrm{DK}_{\mathbb{R}}$ is equivalent to real Betti realization $\mathrm{Be}_{\mathbb{R}}$.

4.2 The real étale part

In motivic stable homotopy theory, we have a map $\rho : S^{0,0} \to S^{1,1}$ in $\mathrm{SH}(S)$ for any scheme S, defined as the stabilization of the map of schemes $S \to \mathbb{G}_m$ that classifies the unit -1. We also have a map ρ in Sp^{C_2} that is the Euler class for the C_2-sign representation σ, given by the stabilization of the unique C_2-equivariant inclusion of C_2-spaces $S^0 \to S^\sigma$. By the construction of DK^{C_2} and the identification of Lemma 4.5, we note:

Lemma 4.10. *The C_2-semialgebraic realization functor $\mathrm{DK}^{C_2} : \mathrm{SH}(k) \to \mathrm{Sp}^{C_2}$ sends ρ to ρ.*

Since $\mathrm{Sper}(k) = \mathrm{pt}$, we have that $\mathrm{SH}_{\mathrm{rét}}(k) \simeq \mathrm{Sp}(\widetilde{k_{\mathrm{rét}}}) = \mathrm{Sp}$ by Bachmann's theorem (1), which moreover identifies the localization endofunctor $i_{\mathrm{rét}}L_{\mathrm{rét}}$ on $\mathrm{SH}(k)$ with ρ-inversion (thereby embedding Sp as ρ-inverted objects). We wish to now identify $L_{\mathrm{rét}}$ and $i^*\mathrm{DK}^{C_2}$. To this end, we first make a simple observation.

Lemma 4.11. *Suppose* C *is a stable presentable symmetric monoidal* ∞*-category and* $F : \mathrm{C} \to \mathrm{Sp}$ *is a colimit-preserving symmetric monoidal functor. Suppose* $\rho : \mathbb{1} \to E$ *is a map in* C *such that we have an abstract equivalence* $\gamma : \mathrm{Sp} \xrightarrow{\simeq} \mathrm{C}[\rho^{-1}]$ *of symmetric monoidal* ∞*-categories. Then if* $F(\rho)$ *is an equivalence in* Sp*, the functor* F *descends to an equivalence* $\overline{F} : \mathrm{C}[\rho^{-1}] \xrightarrow{\simeq} \mathrm{Sp}$ *of symmetric monoidal* ∞*-categories. Therefore, if we let* R *denote the right adjoint to* F*, the adjunction* $F \dashv R$ *is a smashing localization with the essential image of* R *given by the* ρ*-inverted objects.*

Proof. By assumption and using the universal property of ρ-inversion, F descends to a colimit-preserving symmetric monoidal functor $\overline{F} : \mathrm{C}[\rho^{-1}] \to \mathrm{Sp}$. Because the ∞-category of colimit-preserving, symmetric monoidal endofunctors of Sp is contractible (e.g., by [Nik16, Corollary 6.9]), we see that $\overline{F} \circ \gamma$ and id are homotopic. By the two-out-of-three property of equivalences, we deduce that $\overline{F}$ is an equivalence. Therefore, the unit map $\eta_{\mathbb{1}} : \mathbb{1} \to RF(\mathbb{1})$ is homotopic to the idempotent object $\mathbb{1} \to \mathbb{1}[\rho^{-1}]$ that determines the smashing localization $\mathrm{C}[\rho^{-1}] \subset \mathrm{C}$ (cf. [Lur17a, Proposition 4.8.2.10]), so the conclusion follows. □

Proposition 4.12. *The adjunction*

$$i^*\mathrm{DK}^{C_2} : \mathrm{SH}(k) \rightleftarrows \mathrm{Sp} : \mathrm{Sing}^{C_2} i_*$$

is a smashing localization with the essential image of $\mathrm{Sing}^{C_2} i_*$ *given by the* ρ*-inverted objects (or equivalently, the* rét*-local objects).*

Proof. We note that DK^{C_2} is symmetric monoidal by Theorem 4.7 and carries the motivic ρ to the C_2-equivariant ρ by Lemma 4.10, while i^* is the functor of geometric C_2-fixed points and is thus symmetric monoidal and given by inverting the C_2-equivariant ρ (see [GM95, Proposition 3.20] for a classical reference). By Lemma 4.11 and under Bachmann's theorem (1), the conclusion follows. □

4.3 The étale part

Since the Galois group of a real closed field is C_2, we have that $\widetilde{k_{\text{ét}}} \simeq \mathrm{Spc}^{BC_2}$. In particular, the ∞-topos of étale sheaves on $\operatorname{Spec} k$ is already hypercomplete. Bachmann's theorem (2) thus furnishes equivalences

$$\mathrm{Sp}(\widetilde{k_{\text{ét}}})^{\wedge}_p \simeq \mathrm{SH}_{\text{ét}}(k)^{\wedge}_p, \qquad \mathrm{Sp}(\widetilde{C_{\text{ét}}})^{\wedge}_p \simeq \mathrm{SH}_{\text{ét}}(C)^{\wedge}_p.$$

Proposition 4.13. *For any prime* p*, the following diagrams commute*

$$\begin{array}{ccc} \mathrm{SH}(k) & \xrightarrow{\mathrm{DK}^{C_2}} & \mathrm{Sp}^{C_2} \\ \downarrow{\scriptstyle (L_{\text{ét}})^{\wedge}_p} & & \downarrow{\scriptstyle (j^*)^{\wedge}_p} \\ \mathrm{SH}_{\text{ét}}(k)^{\wedge}_p & \xrightarrow{\simeq} & (\mathrm{Sp}^{BC_2})^{\wedge}_p \end{array}, \qquad \begin{array}{ccc} \mathrm{SH}(C) & \xrightarrow{\mathrm{DK}_C} & \mathrm{Sp} \\ \downarrow{\scriptstyle (L_{\text{ét}})^{\wedge}_p} & & \downarrow{\scriptstyle (-)^{\wedge}_p} \\ \mathrm{SH}_{\text{ét}}(C)^{\wedge}_p & \xrightarrow{\simeq} & \mathrm{Sp}^{\wedge}_p. \end{array}$$

Proof. Note that the equivalences $\widetilde{k_{\text{ét}}} \simeq \mathrm{Spc}^{BC_2}$ and $\widetilde{C_{\text{ét}}} \simeq \mathrm{Spc}$ are implemented by sending an étale sheaf $\mathcal{F}$ over k, resp. C to the C_2-space $\mathcal{F}(C)$, resp. space $\mathcal{F}(C)$; we implicitly make these identifications for the rest of the proof. Let γ^* denote the change of site functor. We first define lax commutative squares

$$\begin{array}{ccc} \mathrm{Sm}_{k+} & \xrightarrow{\mathrm{DK}^{C_2}} & \mathrm{Spc}^{C_2}_{\bullet} \\ \downarrow{\scriptstyle y} & \nearrow & \downarrow{\scriptstyle j^*} \\ \mathrm{Shv}_{\text{ét}}(\mathrm{Sm}_k)_{\bullet} & \xrightarrow{\gamma^*} & \mathrm{Spc}^{BC_2}_{\bullet} \end{array}, \qquad \begin{array}{ccc} \mathrm{Sm}_{C+} & \xrightarrow{\mathrm{DK}_C} & \mathrm{Spc}_{\bullet} \\ \downarrow{\scriptstyle y} & \nearrow & \downarrow{\scriptstyle =} \\ \mathrm{Shv}_{\text{ét}}(\mathrm{Sm}_C)_{\bullet} & \xrightarrow{\gamma^*} & \mathrm{Spc}_{\bullet} \end{array}$$

as specified by natural transformations

$$\zeta_k : \gamma^* y \Rightarrow j^*\mathrm{DK}^{C_2}, \qquad \zeta_C : \gamma^* y \Rightarrow \mathrm{DK}_C.$$

For ζ_k, we take the transformation that on pointed k-varieties X_+ evaluates to the canonical map $X(C)^{\delta}_+ \to X(C)_+$ of topological spaces that forgets to the identity map on sets (where the decoration $(-)^{\delta}$ indicates that we take the discrete topology), and ζ_C is defined similarly.

We further note that ζ_k and ζ_C are strong monoidal transformations between strong monoidal functors. By the universal properties of the functors involved, we thereby obtain lax commutative squares

$$\begin{array}{ccc} \mathrm{SH}(k) & \xrightarrow{\mathrm{DK}^{C_2}} & \mathrm{Sp}^{C_2} \\ \downarrow{\scriptstyle L_{\text{ét}}} & \zeta_k \nearrow & \downarrow{\scriptstyle j^*} \\ \mathrm{SH}_{\text{ét}}(k) & \xrightarrow{\gamma^*} & \mathrm{Sp}^{BC_2} \end{array} \quad , \qquad \begin{array}{ccc} \mathrm{SH}(C) & \xrightarrow{\mathrm{DK}_C} & \mathrm{Sp} \\ \downarrow{\scriptstyle L_{\text{ét}}} & \zeta_C \nearrow & \downarrow{\scriptstyle =} \\ \mathrm{SH}_{\text{ét}}(k) & \xrightarrow{\gamma^*} & \mathrm{Sp}. \end{array}$$

Our goal is to prove that the homotopies of these lax squares are p-adic equivalences. For ζ_k, since its codomain lies in Borel C_2-spectra, it suffices to check that the map is an equivalence after forgetting the C_2-action. But using the compatibility of both sides with base change, we note that the underlying map of ζ_k is equivalent to

$$\zeta_C \circ \pi^* : \gamma^* L_{\text{ét}} \pi^* \Rightarrow \mathrm{DK}_C \pi^*,$$

where $\pi^* : \mathrm{SH}(k) \to \mathrm{SH}(C)$ is base change along the Galois cover. It thus suffices to show that ζ_C is a p-adic equivalence.

Since all functors in sight preserve colimits and are strong symmetric monoidal after p-completion,[27] it suffices to show that for quasiprojective $X \in \mathrm{Sm}_C$ and all $n \geqslant 0$, the induced map

$$H^n(X(C); \mathbb{F}_p) \to H^n(\gamma^* L_{\text{ét}} X; \mathbb{F}_p)$$

is an isomorphism.

Under the rigidity equivalence $\mathrm{SH}_{\text{ét}}(C)^\wedge_p \simeq \mathrm{Sp}(\widetilde{C_{\text{ét}}})^\wedge_p = \mathrm{Sp}^\wedge_p$, the spectrum $H\mathbb{F}_p \in \mathrm{Sp}^\wedge_p$ coincides with the spectrum representing étale cohomology (with coefficients in $\mu_p \simeq \mathbb{F}_p$) in $\mathrm{SH}_{\text{ét}}(C)^\wedge_p$, essentially because of [CD16, Proposition 3.2.3] and the evident compatibility of the rigidity equivalences in [Bac18b] and [CD16]. Hence, for the right hand side, we have an isomorphism

$$H^*(\gamma^* L_{\text{ét}} X; \mathbb{F}_p) \cong H^*_{\text{ét}}(X, \mathbb{Z}/p),$$

noting that all Tate twists disappear since we are working over an algebraically closed field.

On the other hand, $H^n(X(C); \mathbb{F}_p)$ computes the semialgebraic cohomology $H^n_{\text{sa}}(X(C); \mathbb{F}_p)$ for the constant sheaf $\mathbb{F}_p$, as this is just sheaf cohomology by the discussion in [Sch94, 15.2]. The claim now follows from the étale-semialgebraic comparison theorem of Roland Huber [Sch94, Theorem 15.2.1]. □

Remark 4.14. If $k = \mathbb{R}$ and $C = \mathbb{C}$, then the conclusion of Proposition 4.13 holds (with the same proof) if one instead considers C_2 and complex Betti realization and uses the Artin comparison theorem.

Passing to the right adjoints of the functors in Proposition 4.13, we deduce:

Corollary 4.15. *For any prime p, the following diagrams commute*

$$\begin{array}{ccc} \mathrm{SH}(k)^\wedge_p & \xleftarrow{\mathrm{Sing}^{C_2}} & (\mathrm{Sp}^{C_2})^\wedge_p \\ {\scriptstyle i_{\text{ét}}}\uparrow & & \uparrow{\scriptstyle j_*} \\ \mathrm{SH}_{\text{ét}}(k)^\wedge_p & \xleftarrow[\simeq]{} & (\mathrm{Sp}^{BC_2})^\wedge_p \end{array} \quad , \qquad \begin{array}{ccc} \mathrm{SH}(C)^\wedge_p & \xleftarrow{\mathrm{Sing}_C} & \mathrm{Sp}^\wedge_p \\ {\scriptstyle i_{\text{ét}}}\uparrow & & \uparrow{\scriptstyle =} \\ \mathrm{SH}_{\text{ét}}(C)^\wedge_p & \xleftarrow[\simeq]{} & \mathrm{Sp}^\wedge_p . \end{array} \qquad (23)$$

In particular, the functors $\mathrm{Sing}^{C_2} j_*$ and Sing_C are fully faithful after p-completion, with essential image given by the p-complete étale-local objects.

[27] The functor γ^* is only lax symmetric monoidal, but becomes an strongly so after p-completion since it induces an equivalence of symmetric monoidal ∞-categories.

Remark 4.16. More precisely, we have the following commutative diagram displaying the interaction with p-completion

$$\begin{array}{ccc} \mathrm{SH}(k) & \xleftarrow{\mathrm{Sing}^{C_2}} & \mathrm{Sp}^{C_2} \\ {\scriptstyle i_{\text{ét}}}\uparrow & & \uparrow{\scriptstyle j_*} \\ \mathrm{SH}_{\text{ét}}(k) & & \mathrm{Sp}^{BC_2} \\ {\scriptstyle i_p}\uparrow & & \uparrow{\scriptstyle i_p} \\ \mathrm{SH}_{\text{ét}}(k)^\wedge_p & \xleftarrow[\simeq]{} & (\mathrm{Sp}^{BC_2})^\wedge_p . \end{array} \tag{24}$$

In other words, on p-complete objects the functors $\mathrm{Sing}^{C_2} j_*$ and $i_{\text{ét}}$ coincides.

4.4 The C_2-Tate construction in algebro-geometric terms

The goal for the rest of this section is to identify the C_2-Tate construction in terms of motivic homotopy theory, and thereby identify Scheiderer motives over k with genuine C_2-spectra (all after p-completion).

Theorem 4.17. *Under the equivalences $(\mathrm{Sp}^{BC_2})^\wedge_p \xrightarrow{\simeq} \mathrm{SH}_{\text{ét}}(k)^\wedge_p$ and $\mathrm{Sp}^\wedge_p \xrightarrow{\simeq} \mathrm{SH}_{\text{rét}}(k)^\wedge_p$, we have a canonical equivalence of lax symmetric monoidal functors*

$$(-)^{tC_2} \simeq L_{\text{rét}} i_{\text{ét}} : \mathrm{SH}_{\text{ét}}(k)^\wedge_p \to \mathrm{SH}_{\text{rét}}(k)^\wedge_p .$$

To prove Theorem 4.17, we begin with some preliminary identifications that exactly parallel [BS20, Lemma 8.19-20] which was done in the context of Betti realization. For the convenience of the reader, we transcribe those proofs into our setting. Recall that the functor

$$j^* i_* : \mathrm{Sp} \to \mathrm{Sp}^{C_2} \to \mathrm{Sp}^{BC_2}$$

is nullhomotopic. This remains true after inserting $\mathrm{DK}^{C_2}\mathrm{Sing}^{C_2}$ in the middle.

Lemma 4.18. *Let k be a real closed field. The composite of functors*

$$\mathrm{Sp} \xrightarrow{i_*} \mathrm{Sp}^{C_2} \xrightarrow{\mathrm{Sing}^{C_2}} \mathrm{SH}(k) \xrightarrow{\mathrm{DK}^{C_2}} \mathrm{Sp}^{C_2} \xrightarrow{j^*} \mathrm{Sp}^{BC_2},$$

is nullhomotopic.

Proof. In Proposition 4.12, we saw that the essential image of $\mathrm{Sing}^{C_2} i_*$ was given by the ρ-inverted objects. Since $\mathrm{DK}^{C_2}(\rho) \simeq \rho$, the same holds for $\mathrm{DK}^{C_2}\mathrm{Sing}^{C_2} i_*$. Since j^* annihilates this subcategory (as it is the essential image of i_*), the claim follows. □

Lemma 4.19. *The natural transformation*

$$i^*\mathrm{DK}^{C_2}\mathrm{Sing}^{C_2} j_* \Rightarrow i^* j_*,$$

induced by the counit $\mathrm{DK}^{C_2}\mathrm{Sing}^{C_2} \Rightarrow \mathrm{id}$, is invertible.

Proof. The functor i_* is fully faithful and is thus conservative, and the endofunctor $i_* i^*$ is given by tensoring with $\widetilde{EC_2} \simeq S^0[\rho^{-1}]$. Therefore, it suffices to prove that the canonical map

$$\widetilde{EC_2} \otimes \mathrm{DK}^{C_2}\mathrm{Sing}^{C_2} j_*(X) \to \widetilde{EC_2} \otimes j_*(X)$$

is an equivalence for all $X \in \mathrm{Sp}^{BC_2}$. By Lemma 4.10, we have that $\widetilde{EC_2} \simeq \mathrm{DK}^{C_2}(\mathbb{1}[\rho^{-1}])$, and thus the projection formula from Theorem 4.7 yields the equivalence

$$\widetilde{EC_2} \otimes \mathrm{DK}^{C_2}\mathrm{Sing}^{C_2} j_*(X) \simeq \mathrm{DK}^{C_2}\mathrm{Sing}^{C_2}(j_*(X) \otimes \widetilde{EC_2})$$

under which $\widetilde{EC_2} \otimes \epsilon_{j_*(X)}$ identifies with $\epsilon_{\widetilde{EC_2} \otimes j_* X}$. Therefore, if we let $Z = \widetilde{EC_2} \otimes j_* X$, it suffices to show that for all $Z \in \mathrm{Sp}$, the counit map induces an equivalence

$$\mathrm{DK}^{C_2}\mathrm{Sing}^{C_2}(i_* Z) \to i_* Z.$$

But for this, note that we have the recollement cofiber sequence

$$j_! j^* \mathrm{DK}^{C_2}\mathrm{Sing}^{C_2}(i_* Z) \to \mathrm{DK}^{C_2}\mathrm{Sing}^{C_2}(i_* Z) \to i_* i^* \mathrm{DK}^{C_2}\mathrm{Sing}^{C_2}(i_* Z),$$

and $j_! j^* \mathrm{DK}^{C_2}\mathrm{Sing}^{C_2}(i_* Z) \simeq 0$ by Lemma 4.18, while $i_* i^* \mathrm{DK}^{C_2}\mathrm{Sing}^{C_2}(i_* Z) \simeq i_* Z$ since $\mathrm{Sing}^{C_2} i_*$ is fully faithful by Proposition 4.12. □

Proof of Theorem 4.17. We need to prove that on all p-complete objects, the functors $(-)^{tC_2} \simeq i^* j_*$ and $L_{\text{rét}} i_{\text{ét}}$ agree as lax symmetric monoidal functors. To show this, we use the sequence of equivalences

$$\begin{aligned} L_{\text{rét}} i_{\text{ét}} i_p &\simeq L_{\text{rét}} \mathrm{Sing}^{C_2} j_* i_p \\ &\simeq i^* \mathrm{DK}^{C_2}\mathrm{Sing}^{C_2} j_* i_p \\ &\simeq i^* j_* i_p. \end{aligned}$$

The first equivalence follows from Remark 4.16 and Corollary 4.15, the second from Proposition 4.12, and the last equivalence from Lemma 4.19. □

We now apply Theorem 4.17 to prove the main theorem of this section.

Theorem 4.20. *Let p be any prime. The functor $\mathrm{Sing}^{C_2} : (\mathrm{Sp}^{C_2})^\wedge_p \to \mathrm{SH}(k)^\wedge_p$ is fully faithful with essential image given by the p-complete Scheiderer motivic spectra, and we have a canonical equivalence of symmetric monoidal ∞-categories*

$$\mathrm{SH}_b(k)^\wedge_p \simeq (\mathrm{Sp}^{C_2})^\wedge_p.$$

Furthermore, $\mathrm{SH}_b(k)^\wedge_p$ is a smashing localization of $\mathrm{SH}(k)^\wedge_p$.

Proof. Following the proof of [BS20, Theorem 8.22], we need only verify the hypotheses (1)-(3) of [BS20, Lemma 5.1] for the full faithfulness assertion. (1) holds by Lemma 4.19, and (2) holds by Lemma 4.18. The second part of (3) holds by Proposition 4.12, while the first part of (3) (and the only part requiring p-completion) holds by Corollary 4.15. The identification of the essential image and the resulting equivalence then follow from Example A.14, Theorem 4.17, and the uniqueness of symmetric monoidal structures on a monoidal recollement as determined by the lax symmetric monoidal gluing functor [QS19, Proposition 1.26]. We then deduce the last assertion by combining full faithfulness with Theorem 4.7 (which persists after p-completion by [BS20, Lemma 3.7]). □

Remark 4.21. Let $k = \mathbb{R}$. We note that taking the cellularization of Sing^{C_2} still yields a fully faithful functor on p-complete objects by [BS20, Theorem 8.22]. Since there is no reason for the cellularization of a fully faithful functor to remain fully faithful, Theorem 4.20 does not imply that result, which has a substantially different proof and results in a distinct embedding of $(\mathrm{Sp}^{C_2})^\wedge_p$ into $\mathrm{SH}(\mathbb{R})$.

5 Scheiderer motives versus b-sheaves with transfers

We now aim to promote the equivalence of Theorem 4.20 to a larger class of schemes. We begin with some preliminaries. We always have the localization adjunction

$$L_b : \mathrm{SH}(X) \rightleftarrows \mathrm{SH}_b(X) : i_b.$$

By Example A.14, we also have a monoidal stable recollement

$$\mathrm{SH}_{\text{ét}}(X) \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathrm{SH}_b(X) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{SH}_{\text{rét}}(X). \tag{25}$$

We denote the gluing functor of this recollement by

$$\Theta_X^{\mathrm{mot}} = L_{\mathrm{rét}} i_{\mathrm{ét}} : \mathrm{SH}_{\mathrm{ét}}(X) \to \mathrm{SH}_{\mathrm{rét}}(X) \tag{26}$$

and call Θ^{mot} the **motivic gluing functor**. We also have the (motivic) unstable[28] and S^1-stable variants

$$\begin{aligned} \theta_X^{\mathrm{big}} &= L_{\mathrm{rét}} i_{\mathrm{ét}} : \mathrm{Shv}_{\mathrm{ét}}(\mathrm{Sm}_X) \to \mathrm{Shv}_{\mathrm{rét}}(\mathrm{Sm}_X), \\ \theta_X^{\mathrm{mot}} &= L_{\mathrm{rét}} i_{\mathrm{ét}} : \mathrm{H}_{\mathrm{ét}}(X) \to \mathrm{H}_{\mathrm{rét}}(X), \\ \Theta_X^{S^1\mathrm{mot}} &= L_{\mathrm{rét}} i_{\mathrm{ét}} : \mathrm{SH}^{S^1}_{\mathrm{ét}}(X) \to \mathrm{SH}^{S^1}_{\mathrm{rét}}(X). \end{aligned}$$

As we observe in Appendix A, these fit together into a lax commutative diagram

$$\begin{array}{ccccccc} \mathrm{Shv}_{\mathrm{ét}}(\mathrm{Sm}_Y) & \xrightarrow{L_{\mathbb{A}^1}} & \mathrm{H}_{\mathrm{ét}}(X) & \xrightarrow{\Sigma^\infty_+} & \mathrm{SH}^{S^1}_{\mathrm{ét}}(X) & \xrightarrow{\Sigma^\infty_{\mathbb{G}_m}} & \mathrm{SH}_{\mathrm{ét}}(X) \\ \downarrow{\scriptstyle \theta_X^{\mathrm{big}}} & \nearrow & \downarrow{\scriptstyle \theta_X^{\mathrm{mot}}} & \nearrow & \downarrow{\scriptstyle \Theta_X^{S^1\mathrm{mot}}} & \nearrow & \downarrow{\scriptstyle \Theta_X^{\mathrm{mot}}} \\ \mathrm{Shv}_{\mathrm{rét}}(\mathrm{Sm}_X) & \xrightarrow{L_{\mathbb{A}^1}} & \mathrm{H}_{\mathrm{rét}}(X) & \xrightarrow{\Sigma^\infty_+} & \mathrm{SH}^{S^1}_{\mathrm{rét}}(X) & \xrightarrow{\Sigma^\infty_{\mathbb{G}_m}} & \mathrm{SH}_{\mathrm{rét}}(X). \end{array}$$

In more detail, $\theta_X^{\mathrm{mot}} \simeq L_{\mathbb{A}^1}\theta_X^{\mathrm{big}}$ upon inclusion into rét-sheaves in view of the discussion at the start of §A.2 (or rather its unstable counterpart), we have that $\Theta_X^{S^1\mathrm{mot}}$ is the stabilization[29] of θ_X^{mot}, and Θ_X^{mot} is obtained via monoidal inversion of $\mathbb{G}_m$ from $\Theta_X^{S^1\mathrm{mot}}$ in the sense of Lemma A.9.

We next address the behavior of the motivic gluing functor under base change by proving the "big site" analogue of Proposition 3.23. Given a morphism $f : Y \to X$, we have the "big site" analogues of the adjunctions displayed in §B.2:

$$\begin{aligned} f^*_{\mathrm{pre}} &: \mathrm{PShv}(\mathrm{Sm}_X) \rightleftarrows \mathrm{PShv}(\mathrm{Sm}_Y) : f_{*\mathrm{pre}} \\ f^*_{\mathrm{rét}} &: \mathrm{Shv}_{\mathrm{rét}}(\mathrm{Sm}_X) \rightleftarrows \mathrm{Shv}_{\mathrm{rét}}(\mathrm{Sm}_Y) : f_{*\mathrm{rét}} \\ f^*_{\mathrm{ét}} &: \mathrm{Shv}_{\mathrm{ét}}(\mathrm{Sm}_X) \rightleftarrows \mathrm{Shv}_{\mathrm{ét}}(\mathrm{Sm}_Y) : f_{*\mathrm{ét}} \end{aligned}$$

and the induced adjunctions on the level of H, SH^{S^1}, and SH.

Proposition 5.1. *Let $f : X \to Y$ be a morphism of schemes. Then the canonical exchange transformations*

$$\begin{aligned} (\chi_0 : f^*_{\mathrm{rét}}\widehat{\theta^{\mathrm{big}}} \Rightarrow \widehat{\theta^{\mathrm{big}}} f^*_{\mathrm{ét}}) &: \widehat{\mathrm{Shv}}_{\mathrm{ét}}(\mathrm{Sm}_Y) \to \widehat{\mathrm{Shv}}_{\mathrm{rét}}(\mathrm{Sm}_X), \\ (\chi : f^*_{\mathrm{rét}}\theta^{\mathrm{mot}} \Rightarrow \theta^{\mathrm{mot}} f^*_{\mathrm{ét}}) &: \mathrm{H}_{\mathrm{ét}}(Y) \to \mathrm{H}_{\mathrm{rét}}(X), \\ (\chi' : f^*_{\mathrm{rét}}\Theta^{S^1\mathrm{mot}} \Rightarrow \Theta^{S^1\mathrm{mot}} f^*_{\mathrm{ét}}) &: \mathrm{SH}^{S^1}_{\mathrm{ét}}(Y) \to \mathrm{SH}^{S^1}_{\mathrm{rét}}(X), \\ (\chi'' : f^*_{\mathrm{rét}}\Theta^{\mathrm{mot}} \Rightarrow \Theta^{\mathrm{mot}} f^*_{\mathrm{ét}}) &: \mathrm{SH}_{\mathrm{ét}}(Y) \to \mathrm{SH}_{\mathrm{rét}}(X), \end{aligned}$$

are equivalences.

Proof. We first prove the claim for χ_0 by a stalkwise argument. For any $X' \in \mathrm{Sm}_X$ and any $\alpha : \mathrm{Spec}\, k \to X'$ where k is a real closed field, Lemma B.14 gives equivalences:

$$(\chi_{0\mathcal{F}})_\alpha : (f^*_{\mathrm{rét}}\theta^{\mathrm{big}}\mathcal{F})_\alpha \simeq (\theta^{\mathrm{big}}\mathcal{F})_{\alpha\circ f} \simeq \alpha^*_{\mathrm{ét}} f^*_{\mathrm{ét}}\mathcal{F}(1_\alpha) \simeq (\theta^{\mathrm{big}} f^*_{\mathrm{ét}}\mathcal{F})_\alpha.$$

By the same reasoning as in Proposition 3.23, χ_0 is then an equivalence.

The claim for χ follows since $L_{\mathbb{A}^1}(\chi_0) \simeq \chi$. The stable claims now follow in succession from the unstable ones. Indeed, by Remark A.10 all functors in question are computed first pointwise on the level of prespectra and then by applying a localization functor L_{sp} to get to either S^1-spectra in $\mathrm{H}_{\mathrm{rét}}(X)$ or $\mathbb{G}_m$-spectra in $\mathrm{SH}^{S^1}_{\mathrm{rét}}(X)$. Since equivalences at the level of prespectra are preserved by L_{sp}, if χ is an equivalence then so is χ', and likewise if χ' is an equivalence then so is χ''. □

[28]Note that we do not know if θ_X^{mot} is the gluing functor for a recollement on $\mathrm{H}_b(X)$, since we do not know if the unstable $L_{\mathrm{rét}} : \mathrm{H}_b(X) \to \mathrm{H}_{\mathrm{rét}}(X)$ is left-exact (Warning A.12).

[29]Note that we do not need to assume the unstable $L_{\mathrm{rét}} : \mathrm{H}_b(X) \to \mathrm{H}_{\mathrm{rét}}(X)$ is left-exact. More precisely, we have that $L_{\mathrm{rét}} \simeq \mathrm{Sp}(L_{\mathrm{rét}})$ in the sense for *left* adjoints, so that $\Sigma^\infty_+ L_{\mathrm{rét}} \simeq L_{\mathrm{rét}}\Sigma^\infty_+$, and $i_{\mathrm{ét}} \simeq \mathrm{Sp}(i_{\mathrm{ét}})$ in the sense for *left-exact* functors, so we have an exchange transformation $\Sigma^\infty_+ i_{\mathrm{ét}} \Rightarrow i_{\mathrm{ét}}\Sigma^\infty_+$.

Corollary 5.2. *Let S be a noetherian finite-dimensional base scheme and let Sch'_S be an adequate category of finite-dimensional noetherian S-schemes. Then $\mathrm{SH}_b : (\mathrm{Sch}'_S)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}})$ satisfies the full six functors formalism.*

Proof. First note that $\mathrm{SH}_{\text{ét}}$ and $\mathrm{SH}_{\text{rét}}$ are known to satisfy the full six functors formalism (the former by Ayoub's work [Bac18b, Theorem 5.1], and the latter using the finite-dimensional hypothesis and Bachmann's theorem (1)). Since $\mathrm{SH}_b(X)$ is the right-lax limit of Θ^{mot} (as a symmetric monoidal ∞-category), we may conclude by Theorem 3.17 if we know that Θ^{mot} commutes with base change and that $\mathrm{SH}_b(X)$ is Tate-dualizable for all $X \in \mathrm{Sch}'_S$. We just showed the former in Proposition 5.1, and the latter is true by design since we invert $\mathbb{G}_m$ in $\mathrm{SH}_b(X)$ and for the map $\pi_X : \mathbb{G}_m \times X \to X$, $\pi_{X\sharp}(\mathbb{1}) \simeq \Sigma^\infty_+(\mathbb{G}_m)$ by construction. ☐

We now proceed to construct a comparison functor between $\mathrm{Sp}_b^{C_2}$ and SH_b. From hereon, we suppose that our schemes X have $\frac{1}{2} \in \mathcal{O}_X$. Let $p : X[i] \to X$ denote the C_2-Galois cover.

Definition 5.3. An **étale induced object** in $\mathrm{SH}_{\text{ét}}(X)$ is an object in the essential image of the functor $p_\sharp : \mathrm{SH}_{\text{ét}}(X[i]) \to \mathrm{SH}_{\text{ét}}(X)$.

We note that since $\mathrm{Shv}_{\text{rét}}(\mathrm{Sm}_Y) \simeq *$ if $\sqrt{-1} \in Y$, we have that $\mathrm{SH}_{\text{rét}}(X[i]) \simeq *$.

Lemma 5.4. *The functor $\Theta^{\mathrm{mot}} : \mathrm{SH}_{\text{ét}}(X) \to \mathrm{SH}_{\text{rét}}(X)$ vanishes on étale induced objects.*

Proof. Since we have the ambidexterity equivalence $p_* \simeq p_\sharp$ for any premotivic functor with the full six functors formalism, we get

$$\begin{aligned} \Theta^{\mathrm{mot}} p_\sharp E &= L_{\text{rét}} i_{\text{ét}} p_\sharp E \\ &\simeq L_{\text{rét}} i_{\text{ét}} p_* E \\ &\simeq L_{\text{rét}} p_* i_{\text{ét}} E \\ &\simeq L_{\text{rét}} p_\sharp i_{\text{ét}} E. \end{aligned}$$

The last spectrum is zero. Indeed, this follows from the commutative diagram of left adjoints

$$\begin{array}{ccc} \mathrm{SH}(X[i]) & \xrightarrow{L_{\text{rét}}} & \mathrm{SH}_{\text{rét}}(X[i]) \simeq * \\ \downarrow{\scriptstyle p_\sharp} & & \downarrow{\scriptstyle p_\sharp} \\ \mathrm{SH}(X) & \xrightarrow{L_{\text{rét}}} & \mathrm{SH}_{\text{rét}}(X). \end{array}$$

☐

Remark 5.5. Stabilizing Construction 2.6 applied to Example 2.8 gives us a functor $\pi_* : \mathrm{Sp}(\widehat{X[i]}_{\text{ét}}) \to \mathrm{Sp}(\widehat{X}_{\text{ét}})$. After p-completion and applying Bachmann's theorem (2), the functor π_* agrees with the functor $p_\sharp$. In this way, the notion of étale induced objects is consistent with our earlier notion of induced objects (Definition 2.54); in fact, we have an integral compatibility as recorded by Lemma 5.6. Lemma 5.4 is then the motivic analogue of the vanishing result of Lemma 2.53.

Lemma 5.6. *If we let $\pi_* : \mathrm{Sp}(\widehat{X[i]}_{\text{ét}}) \to \mathrm{Sp}(\widehat{X}_{\text{ét}})$ be the functor of Construction (2.6), then the diagram*

$$\begin{array}{ccc} \mathrm{Sp}(\widehat{X[i]}_{\text{ét}}) & \xrightarrow{\pi_*} & \mathrm{Sp}(\widehat{X}_{\text{ét}}) \\ \downarrow{\scriptstyle \Sigma^\infty_{\mathbb{G}_m,\text{ét}} u^{\text{ét}}_{X[i]!}} & & \downarrow{\scriptstyle \Sigma^\infty_{\mathbb{G}_m,\text{ét}} u^{\text{ét}}_{X!}} \\ \mathrm{SH}_{\text{ét}}(X[i]) & \xrightarrow{p_\sharp} & \mathrm{SH}_{\text{ét}}(X) \end{array}$$

commutes. In other words, $\Sigma^\infty_{\mathbb{G}_m,\text{ét}} u^{\text{ét}}_{X!}$ carries induced objects to étale induced objects.

Proof. All functors in sight commute with colimits, and the claim is obvious on the level of the representable objects that generate under colimits. ☐

Construction 5.7. Consider the following concatenation of lax commutative squares

$$\begin{array}{ccc} \mathrm{Sp}(\widehat{X}_{\text{ét}}) & \xrightarrow{L_{\text{rét}} i_{\text{ét}}} & \mathrm{Sp}(\widehat{X}_{\text{rét}}) \\ {\scriptstyle u^{\text{ét}}_{X!}}\downarrow & \swArrow & \downarrow{\scriptstyle u^{\text{rét}}_{X!}} \\ \mathrm{SH}^{S^1}_{\text{ét}}(X) & \xrightarrow{L_{\text{rét}} i_{\text{ét}}} & \mathrm{SH}^{S^1}_{\text{rét}}(X) \\ {\scriptstyle \Sigma^\infty_{\mathbb{G}_m,\text{ét}}}\downarrow & \swArrow & \downarrow{\scriptstyle \Sigma^\infty_{\mathbb{G}_m,\text{rét}}} \\ \mathrm{SH}_{\text{ét}}(X) & \xrightarrow{L_{\text{rét}} i_{\text{ét}}} & \mathrm{SH}_{\text{rét}}(X). \end{array} \tag{27}$$

Here, the top lax square is given by stabilizing Construction C.6 and the bottom lax square is given by Lemma A.9. Let $\gamma^{\text{ét}}_X = \Sigma^\infty_{\mathbb{G}_m,\text{ét}} u^{\text{ét}}_{X!}$ and $\gamma^{\text{rét}}_X = \Sigma^\infty_{\mathbb{G}_m,\text{rét}} u^{\text{rét}}_{X!}$. We thus obtain a lax symmetric monoidal transformation (using Remark C.7 for strong monoidality of the $u_!$ functors)

$$\gamma^{\text{rét}}_X \Theta \Rightarrow \Theta^{\text{mot}} \gamma^{\text{ét}}_X. \tag{28}$$

Moreover, $\Theta^{\text{mot}}\gamma^{\text{ét}}_X$ vanishes on induced objects by Lemma 5.6 and Lemma 5.4. If X has finite Krull dimension, so that $\gamma^{\text{rét}}_X$ is an equivalence by Bachmann's theorem (1), then Proposition 2.63 directly applies to produce a comparison lax symmetric monoidal transformation $T_X : \Theta^{\text{Tate}} \Rightarrow \Theta^{\text{mot}}\gamma^{\text{ét}}_X$. In the general case, we note that the logic of the proof also applies to show preservation of the universal property upon whiskering with a colimit-preserving symmetric monoidal functor such as $\gamma^{\text{rét}}_X$. We thereby obtain a lax symmetric monoidal transformation

$$T_X : \gamma^{\text{rét}}_X \Theta^{\text{Tate}} \Rightarrow \Theta^{\text{mot}} \gamma^{\text{ét}}_X, \tag{29}$$

or equivalently, a lax commutative square

$$\begin{array}{ccc} \mathrm{Sp}(\widehat{X}_{\text{ét}}) & \xrightarrow{\Theta^{\text{Tate}}} & \mathrm{Sp}(\widehat{X}_{\text{rét}}) \\ {\scriptstyle \gamma^{\text{ét}}_X}\downarrow & \swArrow T_X & \downarrow{\scriptstyle \gamma^{\text{rét}}_X} \\ \mathrm{SH}_{\text{ét}}(X) & \xrightarrow{\Theta^{\text{mot}}} & \mathrm{SH}_{\text{rét}}(X). \end{array}$$

In view of the discussion below Proposition 2.63 and using that the vertical functors are strong symmetric monoidal, upon taking right-lax limits we obtain a strong symmetric monoidal functor

$$C_X : \mathrm{Sp}^{C_2}_b(X) \to \mathrm{SH}_b(X),$$

where we have identified the right-lax limit of Θ^{mot} by Example A.14 and the right-lax limit of Θ^{Tate} by Theorem 2.47.

Now let $f : X \to Y$ be a morphism of schemes and consider the lax commutative diagrams

$$\begin{array}{ccc} \mathrm{Sp}(\widehat{Y}_{\text{ét}}) & \xrightarrow{\Theta^{\text{Tate}}} & \mathrm{Sp}(\widehat{Y}_{\text{rét}}) \\ {\scriptstyle \gamma^{\text{ét}}_Y}\downarrow & \swArrow T_Y & \downarrow{\scriptstyle \gamma^{\text{rét}}_Y} \\ \mathrm{SH}_{\text{ét}}(Y) & \xrightarrow{\Theta^{\text{mot}}} & \mathrm{SH}_{\text{rét}}(Y) \\ {\scriptstyle f^*}\downarrow & \swArrow \simeq & \downarrow{\scriptstyle f^*} \\ \mathrm{SH}_{\text{ét}}(X) & \xrightarrow{\Theta^{\text{mot}}} & \mathrm{SH}_{\text{rét}}(X) \end{array}, \qquad \begin{array}{ccc} \mathrm{Sp}(\widehat{Y}_{\text{ét}}) & \xrightarrow{\Theta^{\text{Tate}}} & \mathrm{Sp}(\widehat{Y}_{\text{rét}}) \\ {\scriptstyle f^*}\downarrow & \swArrow \simeq & \downarrow{\scriptstyle f^*} \\ \mathrm{Sp}(\widehat{X}_{\text{ét}}) & \xrightarrow{\Theta^{\text{Tate}}} & \mathrm{Sp}(\widehat{X}_{\text{rét}}) \\ {\scriptstyle \gamma^{\text{ét}}_X}\downarrow & \swArrow T_X & \downarrow{\scriptstyle \gamma^{\text{rét}}_X} \\ \mathrm{SH}_{\text{ét}}(X) & \xrightarrow{\Theta^{\text{mot}}} & \mathrm{SH}_{\text{rét}}(X), \end{array}$$

where for the equivalences we use that Θ^{Tate} and Θ^{mot} are both stable under base change by Corollary 3.24 and Proposition 5.1 respectively. We then claim that the lax symmetric monoidal transformations

$$f^*\gamma^{\text{rét}}_Y \Theta^{\text{Tate}} \Rightarrow \Theta^{\text{mot}} f^* \gamma^{\text{ét}}_Y, \qquad \gamma^{\text{rét}}_X f^* \Theta^{\text{Tate}} \Rightarrow \Theta^{\text{mot}} \gamma^{\text{ét}}_X f^*$$

are canonically homotopic under the equivalences $f^*\gamma^{\text{rét}}_Y \simeq \gamma^{\text{rét}}_X f^*$, $f^*\gamma^{\text{ét}}_Y \simeq \gamma^{\text{ét}}_X f^*$. Indeed, this again follows from the universal property of $\Theta \Rightarrow \Theta^{\text{Tate}}$ and its stability under whiskering with a colimit-preserving strong symmetric monoidal functor, as well as the fact that $f^*: \mathrm{Sp}(\widehat{Y}_{\text{ét}}) \to \mathrm{Sp}(\widehat{X}_{\text{ét}})$, resp. $f^*: \mathrm{SH}_{\text{ét}}(Y) \to \mathrm{SH}_{\text{ét}}(X)$ preserves induced objects, resp. étale induced objects. Upon taking right-lax limits, we then obtain a commutative square of strong symmetric monoidal functors

$$\begin{array}{ccc} \mathrm{Sp}_b^{C_2}(Y) & \xrightarrow{C_Y} & \mathrm{SH}_b(Y) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{Sp}_b^{C_2}(X) & \xrightarrow{C_X} & \mathrm{SH}_b(X). \end{array}$$

Theorem 5.8. *The functors C_X of Construction 5.7 assemble to a strong symmetric monoidal transformation*

$$C: \mathrm{Sp}_b^{C_2} \Rightarrow \mathrm{SH}_b$$

of functors $(\mathrm{Sch}[\frac{1}{2}])^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\infty,\mathrm{stab}})$. Moreover, if p is a prime and X is a locally p-étale finite scheme with $1/p \in \mathcal{O}_X$, then $(C_X)_p^\wedge$ is an equivalence.

Proof. For this proof, we denote $S := (\mathrm{Sch}[\frac{1}{2}])^{\mathrm{op}}$. To assemble the transformation C, we employ the same strategy as in Lemma 2.65 and we assume the notation introduced in that proof. The pairing construction furnishes a locally cocartesian fibration

$$\widetilde{\mathrm{Fun}}^{\otimes,\mathrm{lax}}(\mathrm{Sp}(\widehat{-}_{\text{ét}}), \mathrm{SH}_{\text{rét}}(-)) \to S,$$

equipped with sections

$$\gamma^{\text{rét}}\Theta, \gamma^{\text{rét}}\Theta^{\mathrm{Tate}}, \Theta^{\mathrm{mot}}\gamma^{\text{ét}}$$

which fiberwise restricts to the functors $\mathrm{Sp}(\widehat{X}_{\text{ét}}) \to \mathrm{SH}_{\text{rét}}(X)$ of the same name. We first construct a section

$$\begin{array}{ccc} S \times \Delta^1 & \xrightarrow{q} & \widetilde{\mathrm{Fun}}^{\otimes,\mathrm{lax}}(\mathrm{Sp}(\widehat{-}_{\text{ét}}), \mathrm{SH}_{\text{rét}}(-)) \\ & \searrow^{\mathrm{pr}_S} & \downarrow \\ & & S, \end{array}$$

classifying a transformation

$$\gamma^{\text{rét}}\Theta \Rightarrow \Theta^{\mathrm{mot}}\gamma^{\text{ét}},$$

which is fiberwise given by the lax commutative square (27). We begin by noting that have the following commutative diagram of cocartesian fibrations over $S \times \mathrm{Fin}_*$

$$\begin{array}{ccc} \mathrm{Sp}(\widehat{-}_b)^{\otimes} & \xrightarrow{L_\tau} & \mathrm{Sp}(\widehat{-}_\tau)^{\otimes} \\ {\scriptstyle \gamma_b}\downarrow & & \downarrow{\scriptstyle \gamma_\tau} \\ \mathrm{SH}_b(-)^{\otimes} & \xrightarrow{L_\tau} & \mathrm{SH}_\tau(-)^{\otimes}, \end{array}$$

where $\tau = \text{rét}, \text{ét}$. The theory of relative adjunctions (specifically, using its formulation in terms of unit and counit maps [Lur17a, Proposition 7.3.2.1]) yields a lax commutative mate square

$$\begin{array}{ccc} \mathrm{Sp}(\widehat{-}_b)^{\otimes} & \xleftarrow{i_\tau} & \mathrm{Sp}(\widehat{-}_\tau)^{\otimes} \\ {\scriptstyle \gamma_b}\downarrow & \Searrow & \downarrow{\scriptstyle \gamma_\tau} \\ \mathrm{SH}_b(-)^{\otimes} & \xleftarrow[i_\tau]{} & \mathrm{SH}_\tau(-)^{\otimes}, \end{array}$$

where the maps i_τ are fiberwise only lax symmetric monoidal. Concatenating the strict commutative square for $\tau = \text{rét}$ with the lax commutative square for $\tau = \text{ét}$ yields a lax commutative square over $S \times \mathrm{Fin}_*$ that globalizes (27).

Now, we have the full subcategory $\iota : \widetilde{\mathrm{Fun}}_0^{\otimes,\mathrm{lax}}(\mathrm{Sp}(\widehat{-}_{\acute{e}t}), \mathrm{SH}_{r\acute{e}t}(-)) \subset \widetilde{\mathrm{Fun}}^{\otimes,\mathrm{lax}}(\mathrm{Sp}(\widehat{-}_{\acute{e}t}), \mathrm{SH}_{r\acute{e}t}(-))$ on those lax symmetric monoidal functors that annihilate induced objects. As in the proof of Lemma 2.65, this restricts fiberwise over $X \in S$ to a Bousfield localization (bearing in mind that we implicitly restrict to accessible functors) that admits a left adjoint

$$L_X : \mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathrm{Sp}(\widehat{X}_{\acute{e}t}), \mathrm{SH}_{r\acute{e}t}(X)) \to \mathrm{Fun}_0^{\otimes,\mathrm{lax}}(\mathrm{Sp}(\widehat{X}_{\acute{e}t}), \mathrm{SH}_{r\acute{e}t}(X)).$$

Moreover, the locally cocartesian pushforward functor for $\widetilde{\mathrm{Fun}}^{\otimes,\mathrm{lax}}(\mathrm{Sp}(\widehat{-}_{\acute{e}t}), \mathrm{SH}_{r\acute{e}t}(-))$ over $f : X \to Y$ is given by the composite of left adjoints

$$\mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathrm{Sp}(\widehat{Y}_{\acute{e}t}), \mathrm{SH}_{r\acute{e}t}(Y)) \xrightarrow{f^*_{r\acute{e}t}\circ} \mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathrm{Sp}(\widehat{Y}_{\acute{e}t}), \mathrm{SH}_{r\acute{e}t}(X)) \xrightarrow{(f^*_{\acute{e}t})_!} \mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathrm{Sp}(\widehat{X}_{\acute{e}t}), \mathrm{SH}_{r\acute{e}t}(X))$$

where $f^*_{r\acute{e}t}\circ$ is postcomposition by $f^*_{r\acute{e}t} : \mathrm{SH}_{r\acute{e}t}(Y) \to \mathrm{SH}_{r\acute{e}t}(X)$ and is left adjoint to $f_{*r\acute{e}t}\circ$, and $(f^*_{\acute{e}t})_!$ is operadic left Kan extension along $f^*_{\acute{e}t} : \mathrm{Sp}(\widehat{Y}_{\acute{e}t}) \to \mathrm{Sp}(\widehat{X}_{\acute{e}t})$ and is left adjoint to restriction $(f^*_{\acute{e}t})^*$ along $f^*_{\acute{e}t}$. Since $f^*_{\acute{e}t}$ preserves induced objects, it follows that $(f_{*r\acute{e}t}\circ) \circ (f^*_{\acute{e}t})^*$ preserves L-local objects and hence $(f^*_{\acute{e}t})_! \circ (f^*_{r\acute{e}t}\circ)$ preserves L-equivalences. Therefore, the relative adjoint functor theorem [Lur17a, Proposition 7.3.2.11] applies to show that the inclusion ι admits a relative left adjoint

$$L : \widetilde{\mathrm{Fun}}^{\otimes,\mathrm{lax}}(\mathrm{Sp}(\widehat{-}_{\acute{e}t}), \mathrm{SH}_{r\acute{e}t}(-)) \to \widetilde{\mathrm{Fun}}_0^{\otimes,\mathrm{lax}}(\mathrm{Sp}(\widehat{-}_{\acute{e}t}), \mathrm{SH}_{r\acute{e}t}(-)).$$

We then see that $L \circ q$ corresponds to the desired transformation

$$\gamma^{r\acute{e}t}\Theta^{\mathrm{Tate}} \Rightarrow \Theta^{\mathrm{mot}}\gamma^{\acute{e}t}.$$

Indeed, by definition the fiber over $X \in S$ of this transformation yields the transformation T_X of Construction 5.7. Formation of right-lax limits in the parametrized sense of Remark 3.8 now yields a functor $C^{\otimes} : \mathrm{Sp}_b^{C_2}(-)^{\otimes} \to \mathrm{SH}_b(-)^{\otimes}$ over $S \times \mathrm{Fin}_*$ that preserves cocartesian edges, and the straightening of $C^{\otimes}$ defines the transformation $C : \mathrm{Sp}_b^{C_2} \Rightarrow \mathrm{SH}_b$ of interest.

Next, suppose X is a locally p-étale finite scheme with $\frac{1}{p} \in \mathcal{O}_X$. Restricting the transformation T_X to p-complete objects, we get a diagram

$$\begin{array}{ccc} \mathrm{Sp}(\widehat{X}_{\acute{e}t})^\wedge_p & \xrightarrow{(\Theta^{\mathrm{Tate}})^\wedge_p} & \mathrm{Sp}(\widetilde{X}_{r\acute{e}t})^\wedge_p \\ \simeq\downarrow & \swarrow (T_X)^\wedge_p & \downarrow\simeq \\ \mathrm{SH}_{\acute{e}t}(X)^\wedge_p & \xrightarrow[(\Theta^{\mathrm{mot}})^\wedge_p]{} & \mathrm{SH}_{r\acute{e}t}(X)^\wedge_p, \end{array}$$

where we have applied (1) and (2) to deduce the vertical equivalences. Therefore, to prove the result, it suffices to prove that $(T_X)^\wedge_p$ is invertible. For this, let $E \in \mathrm{Sp}(\widehat{X}_{\acute{e}t})^\wedge_p$ and consider the map

$$T_X(E)^\wedge_p : (\Theta^{\mathrm{Tate}})^\wedge_p(E) \to (\Theta^{\mathrm{mot}})^\wedge_p(E).$$

Since $\mathrm{SH}_{r\acute{e}t}(X) \simeq \mathrm{Sp}(\widetilde{X_{r\acute{e}t}})$ and $\widetilde{X_{r\acute{e}t}}$ is hypercomplete, it suffices to check that $T_X(E)^\wedge_p$ is an equivalence after base change to all real closed points $\alpha : \operatorname{Spec} k \to X$. But since $T^\wedge_p$ is stable under base change, we have $\alpha^* T_X(E)^\wedge_p \simeq T_k(\alpha^* E)^\wedge_p$. Appealing once more to the universal property of T_k, it also identifies upon p-completion with the canonical equivalence $((-)^{tC_2})^\wedge_p \simeq (L_{r\acute{e}t} i_{\acute{e}t})^\wedge_p$ of Theorem 4.17, so we are done. □

Remark 5.9. We note that Theorem 5.8 in conjunction with Corollary 5.2 gives another proof that $(\mathrm{Sp}_b^{C_2})^\wedge_p$ satisfies the full six functors formalism when restricted to $(\mathrm{Sch}_S^{p\text{-fin},\mathrm{noeth}}[\frac{1}{p}, \frac{1}{2}])^{\mathrm{op}}$ for S a noetherian locally p-étale finite scheme (Theorem 3.27), which in particular bypasses the explicit verification of Tate-dualizability for $(\mathrm{Sp}_b^{C_2})^\wedge_p$ (Lemma 3.25). However, the proof of Lemma 3.25 still yields more information (cf. Remark 3.26).

6 Applications

6.1 Spectral b-rigidity

We explain how to interpret Theorem 5.8 as a spectral "rigidity" result for the b-topology, along the lines studied in the étale and real étale settings by Bachmann [Bac18b, Bac18a]. Recall the notation $\underline{\widehat{X}}_b$ from §3.2, and the equivalence $\underline{\widehat{X}}_b \simeq \underline{\mathcal{X}}_{C_2}$ for $\mathcal{X} = \widehat{X[i]}_{\text{ét}}$ (the hypercomplete version of Remark 3.19).

Theorem 6.1. *Let X be a locally p-étale finite scheme with $\frac{1}{2}, \frac{1}{p} \in \mathcal{O}_X$. Then there is a canonical equivalence of ∞-categories*[30]

$$\mathrm{Fun}^{\times}_{\mathrm{C}_2}(\underline{\mathrm{A}}^{\mathrm{eff}}(\mathrm{Fin}_{C_2}), \underline{\mathrm{Sp}}(\underline{\widehat{X}}_b))^{\wedge}_p \simeq \mathrm{SH}_b(X)^{\wedge}_p.$$

Proof. After Lemma 2.31 and Theorem 2.47, the result follows immediately from Theorem 5.8. □

The meaning of Theorem 6.1 is this: after p-completion and with mild finiteness hypotheses, the sole distinction between Scheiderer motives and b-sheaves of spectra (on the small site!) lies in adjoining certain transfer maps, which serve to repair the obvious obstruction presented by the failure of Tate-dualizability. Let us now elaborate the structure of these transfers.

Remark 6.2. An object in $\mathrm{Fun}^{\times}_{\mathrm{C}_2}(\underline{\mathrm{A}}^{\mathrm{eff}}(\mathrm{Fin}_{C_2}), \underline{\mathrm{Sp}}(\underline{\widehat{X}}_b))$ is concretely given as follows: it is the data of a hypercomplete b-sheaf of spectra $E \in \mathrm{Sp}(\widehat{X}_b)$, together with single transfer map

$$\mathrm{tr} : \overline{\pi}_* \overline{\pi}^* E \to E,$$

and the data of relations as encoded by the effective Burnside category. Here, the adjunction

$$\overline{\pi}^* : \mathrm{Sp}(\widehat{X}_b) \rightleftarrows \mathrm{Sp}(\widehat{X[i]}_b) \simeq \mathrm{Sp}(\widehat{X[i]}_{\text{ét}}) : \overline{\pi}_*$$

follows our customary notation (first introduced around Lemma 2.49), but also coincides with the pullback-pushforward adjunction for sheaves.

Remark 6.3. We can further unwind the transfer map $\mathrm{tr} : \overline{\pi}_* \overline{\pi}^* E \to E$ in terms of the real étale and the étale parts of a b-sheaf. Let us place ourselves in the context of the (stabilization of the) recollement given in Proposition 2.42. Given $E \in \mathrm{Sp}(\widehat{X}_b)$, consider the étale and real étale parts

$$F := j^* E \in \mathrm{Sp}(\widehat{X}_{\text{ét}}), \qquad G := i^* E \in \mathrm{Sp}(\widetilde{X}_{\text{rét}}).$$

We also have an ambidextrous adjunction

$$\pi^* : \mathrm{Sp}(\widehat{X}_{\text{ét}}) \rightleftarrows \mathrm{Sp}(\widehat{X[i]}_{\text{ét}}) : \pi_*.$$

Now, using the ambidexterity equivalence $\pi_! \simeq \pi_*$, the transfer restricted to the étale site coincides with the counit map

$$\epsilon : \pi_* \pi^* F \simeq \pi_! \pi^* F \to F.$$

In particular, the étale component of the transfer is no additional data. By contrast, the real étale component is specified by a map

$$\mathrm{tr}_{\text{rét}} : \Theta \pi_* \pi^* F \to G,$$

subject to the commutativity constraint

$$\begin{array}{ccc} \Theta\pi_*\pi^*F & & \\ {\scriptstyle \mathrm{tr}_{\text{rét}}}\downarrow & \searrow{\scriptstyle \Theta(\epsilon)} & \\ G & \longrightarrow & \Theta F, \end{array}$$

where the bottom arrow arises from the gluing datum of the b-sheaf E (in terms of the recollement, this is given by the transformation $i^*E \to i^* j_* j^* E$). Note also that $\Theta \pi_* \simeq \nu^*$.

In the simplest case of $X = \operatorname{Spec} k$ for a real closed field k, so that E is a spectral presheaf on $\mathcal{O}_{C_2}$ that underlies a genuine C_2-spectrum, we see that the map $\mathrm{tr}_{\text{rét}} : \nu^* \pi^* F \to G$ is precisely the usual transfer map $t : E^e \to E^{C_2}$, the map $G \to \Theta F$ is the inclusion of fixed points $r : E^{C_2} \to E^e$, and the constraint amounts to the relation $r \circ t \simeq \sigma + \mathrm{id}$ for σ the C_2-action on E^e.

[30] We have not explicated a construction of the symmetric monoidal structure on parametrized Mackey functors in this paper.

6.2 Parametrized C_2-realization functor

We append the next theorem to the long list of realization functors already constructed in stable motivic homotopy theory. Again, let X be a locally p-étale finite scheme with $\frac{1}{2}, \frac{1}{p} \in \mathcal{O}_X$.

Theorem 6.4. *There is a colimit preserving and strong symmetric monoidal* **parametrized C_2-Delfs-Knebusch realization functor**

$$\mathrm{DK}_X^{C_2}{}_p^\wedge : \mathrm{SH}(X) \to \mathrm{Sp}_b^{C_2}(X)_p^\wedge,$$

which agrees with the p-completion of the C_2-Delfs-Knebusch realization functor DK^{C_2} of §4.1 whenever X is the spectrum of a real closed field.

Proof. After Corollary 6.1, this functor is given by applying L_b and then p-completing, both of which are strong symmetric monoidal functors. □

Remark 6.5. The functor of Theorem 6.4 defines the p-complete part of a functor whose existence was conjectured by Bachmann-Hoyois [BH18, Remark 11.8]. We defer the construction of an integral version of this functor, as well as an appropriately compatible norm structure in their sense, to a future work.

Remark 6.6. In [BS20], Behrens and the second author gave an explicit formula for $\mathrm{Be}^{C_2}{}_p^\wedge$ when restricted to cellular real motivic spectra, as a certain combination of the operations of ρ-inversion and τ-inversion on the ρ-completion, where τ is the "spherical Bott element".[31] In [BEØ20], Bachmann, Østvaær, and the first author will understand étale descent in terms of τ-inversion. In a future work, we aim to unite these two perspectives to give an explicit formula for $\mathrm{DK}_X^{C_2}{}_p^\wedge$ in terms of ρ and τ. In particular, such a description would enable us to show that the right adjoint $\mathrm{Sing}_X^{C_2}{}_p^\wedge$ to $\mathrm{DK}_X^{C_2}{}_p^\wedge$ preserves colimits. Conditional on this, one can then enhance the equivalence of Theorem 4.7 after p-completion using [EK20, Theorem 5.5]: if X is a regular scheme over a field k of characteristic zero, then we expect a natural equivalence

$$\mathrm{Sp}_b^{C_2}(X)_p^\wedge \simeq \mathrm{Mod}_{\mathrm{Sing}_X^{C_2}{}_p^\wedge \mathrm{DK}_X^{C_2}{}_p^\wedge(\mathbb{1})}(\mathrm{SH}(X)_p^\wedge).$$

Since $\mathrm{Sp}_b^{C_2}(X)_p^\wedge$ also embeds fully faithfully into $\mathrm{SH}(X)_p^\wedge$ as $\mathrm{SH}_b(X)_p^\wedge$, we would then deduce that $\mathrm{SH}_b(X)_p^\wedge$ is a smashing localization of $\mathrm{SH}(X)_p^\wedge$, thereby extending Theorem 4.20.

6.3 The Segal conjecture and the Scheiderer sphere

Let us now place ourselves in the context of the recollement in Proposition 2.52 for $p = 2$; specializing to the case of real closed fields, this yields the recollement $(\mathrm{Sp}^{BC_2}, \mathrm{Sp})$ of Sp^{C_2}. Recall the following formulation of the Segal conjecture for the group C_2, which is a theorem of Lin [Lin80, LDMA80].

Theorem 6.7 (Lin). *In Sp^{C_2}, the canonical map $\mathbb{1} \to j_* j^* \mathbb{1}$ is an equivalence after 2-completion, i.e, the C_2-sphere spectrum is Borel complete after 2-completion.*

We have the following translation of the Segal conjecture into the motivic setting.

Corollary 6.8. *Suppose that k is a real closed field and let $E \in \mathrm{SH}_b(k)_2^\wedge$ be in the thick subcategory generated by $\mathbb{1}_2^\wedge$. Then E satisfies étale descent.*

Proof. After Theorem 4.20, this is a consequence of Theorem 6.7. □

Question 6.9. Can one give an independent proof of the Segal conjecture under the connection furnished by Theorem 4.20? See [HW19] for another new perspective on the Segal conjecture.

The Segal conjecture further implies the following descent result for the Scheiderer sphere over any scheme of finite Krull dimension.

Theorem 6.10. *Let X be a finite-dimensional scheme and let $E \in \mathrm{SH}_b(X)_2^\wedge$ be in the thick subcategory generated by the 2-complete Scheiderer sphere spectrum $\mathbb{1}_b{}_2^\wedge$. Then E satisfies étale descent.*

[31] More precisely, we must take an inverse limit of $C(\rho^n)[\tau_N^{-1}]$ for self-maps τ_N on $C(\rho^n)$ to define $\mathbb{1}_\rho^\wedge[\tau^{-1}]$; τ as a self-map on $\mathbb{1}_\rho^\wedge$ itself does not exist as the periodicity lengths of the self-maps τ_N go to ∞ as $n \to \infty$.

Proof. It suffices to prove the result for $E = \mathbb{1}_2^\wedge$. We are in the 2-completion of the recollement situation of (25). To prove that $\mathbb{1}_2^\wedge$ has étale descent, we need only prove that the canonical map $\mathbb{1}_2^\wedge \to j_* j^* \mathbb{1}_2^\wedge$ is an equivalence, for which it suffices to show that $i^! \mathbb{1}_2^\wedge \simeq 0$. To prove this, consider the recollement fiber sequence

$$i^! \to i^* \to i^* j_* j^* = \Theta^{\mathrm{mot}} j^*.$$

Both $j^* = L_{\text{ét}}$ and Θ^{mot} are stable under base change (the former is obvious, and the latter is Proposition 5.1), so $i^!$ is also stable under base change. Now by the finite-dimensionality assumption on X, we have that $\mathrm{SH}_{\text{rét}}(X) \simeq \mathrm{Sp}(\widetilde{X_{\text{rét}}})$ and $\widetilde{X_{\text{rét}}}$ is hypercomplete, so to prove that $i^! \mathbb{1}_2^\wedge$ is zero, it suffices to show that $i^! \mathbb{1}_2^\wedge$ is zero after base change along all $\alpha : \operatorname{Spec} k \to X$ for k a real closed field. We thereby reduce to Corollary 6.8. □

A Motivic spectra over various topologies

The goal of this appendix is to show that recollements on sheaves induced by intersecting Grothendieck topologies descend to recollements of motivic spectra.

A.1 Generalities

We first work in greater generality. Let B be an ∞-category and suppose that τ_U and τ_Z are two Grothendieck topologies on B. Let $\tau = \tau_U \cap \tau_Z$, so that τ is the finest topology on B coarser than both τ_U and τ_Z. Explicitly, for every $X \in \mathrm{B}$, a sieve $R \subset \mathrm{B}_{/X}$ is a τ-covering sieve if and only if R is both a τ_U and τ_Z-covering sieve. The recollement on the level of motivic spaces then fits into the format of the next construction:

Construction A.1. Let ω be any Grothendieck topology on B.

- We let $\mathcal{K}_\omega$ be the small set of morphisms $\{R \hookrightarrow h_X\}$ in $\mathrm{PShv}(\mathrm{B})$ ranging over $X \in \mathrm{B}$ and ω-covering sieves $R \subset \mathrm{B}_{/X}$.
- Given another small set $\mathcal{L}$ of morphisms in $\mathrm{PShv}(\mathrm{B})$, let $\mathrm{Shv}_\omega^{\mathcal{L}}(\mathrm{B}) \subset \mathrm{PShv}(\mathrm{B})$ be the full subcategory of $(\mathcal{L} \cup \mathcal{K})$-local objects.

Let $\mathrm{Shv}_\omega^{\mathcal{L}}(\mathrm{B}, \mathcal{E}) = \mathrm{Shv}_\omega^{\mathcal{L}}(\mathrm{B}) \otimes \mathcal{E}$ be the tensor product in $\mathrm{Pr}_\infty^{\mathrm{L}}$. Appealing to the theory of Bousfield localization [Lur09, Proposition 5.5.4.15], we have the two adjunctions

$$\mathrm{Shv}_{\tau_U}^{\mathcal{L}}(\mathrm{B}, \mathcal{E}) \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathrm{Shv}_{\tau}^{\mathcal{L}}(\mathrm{B}, \mathcal{E}) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Shv}_{\tau_Z}^{\mathcal{L}}(\mathrm{B}, \mathcal{E}).$$

Lemma A.2. *In the notation of Construction A.1, the functors i^* and j^* are jointly conservative.*

Proof. This follows from the observation that $\mathcal{K}_\tau \cup \mathcal{L} = (\mathcal{K}_{\tau_U} \cup \mathcal{L}) \cap (\mathcal{K}_{\tau_Z} \cup \mathcal{L})$. □

The next lemma introduces an asymmetry that distinguishes the “open” from the “closed” part of the recollement.

Lemma A.3. *Suppose for every $X \in \mathrm{B}$, there exists a τ_U-covering sieve R such that for all $Y \in R$, the empty sieve is τ_Z-covering for Y. Then the composite functor*

$$\mathrm{Shv}_{\tau_Z}(\mathrm{B}) \xrightarrow{i_*} \mathrm{Shv}_{\tau}(\mathrm{B}) \xrightarrow{j^*} \mathrm{Shv}_{\tau_U}(\mathrm{B})$$

is constant at the terminal object.

Proof. Let F be a τ_Z-sheaf. It suffices to show that the τ_U-sheafification of F is trivial. For this, we will use the formula for the τ_U-sheafification functor as a filtered colimit of τ_U-**plus constructions** $F^\dagger$ that was established by Lurie in the proof of [Lur09, Proposition 6.2.2.7]. The presheaf $F^\dagger$ is defined in [Lur09, Remark 6.2.2.12] by the formula

$$F^\dagger(X) = \operatorname*{colim}_{R \subset C_{/X}} \lim_{Y \in R} F(Y),$$

where the colimit is taken over the poset of τ_U-covering sieves of X. Let κ be the regular cardinal specified in the proof of [Lur09, Proposition 6.2.2.7], and let

- $T_0 = \mathrm{id}$,
- $T_{\beta+1}(F) = T_\beta(F)^\dagger$ for $\beta < \kappa$, and
- $T_\gamma(F) = \mathrm{colim}_{\beta<\gamma} T_\beta(F)$ for every limit ordinal $\gamma \leqslant \kappa$,

so that τ_U-sheafification j^* is computed as $j^*F \simeq T_\kappa F$. We claim:

($*$) For all $\gamma \leqslant \kappa$, $(T_\gamma F)(Y) \simeq *$ for all $Y \in \mathrm{B}$ such that the empty sieve in $\mathrm{B}_{/Y}$ is τ_Z-covering.

The proof is by ordinal induction on γ. For the base case $\gamma = 0$, this holds since F is a τ_Z-sheaf, whence $F(Y)$ is equivalent to the limit over the empty diagram. Suppose we have proven the claim for all $\beta < \gamma$. If γ is a limit ordinal, then we have

$$(T_\gamma F)(Y) \simeq \underset{\beta<\gamma}{\mathrm{colim}}(T_\beta)(Y) \simeq \underset{\beta<\gamma}{\mathrm{colim}} * \simeq *,$$

where for the last equivalence, we use that a filtered category is weakly contractible [Lur09, Lemma 5.3.1.20]. For the successor ordinal case, let us write $\gamma = \beta + 1$. Note that for every $Z \to Y$ in B, the empty sieve is also τ_Z-covering for Z by the stability of covering sieves under pullback. Therefore, we may compute

$$(T_\beta F)^\dagger(Y) \simeq \underset{R\subset \mathrm{B}_{/Y}}{\mathrm{colim}} \lim_{Z\in R}(T_\beta F)(Z) \simeq \underset{R\subset \mathrm{B}_{/Y}}{\mathrm{colim}} * \simeq *,$$

using for the last equivalence that the poset of covering sieves is filtered [Lur09, Rem. 6.2.2.11], hence weakly contractible. This proves claim ($*$).

Next, we claim:

($**$) For all infinite ordinals $\gamma \leqslant \kappa$, $(T_\gamma F)(X) \simeq *$ for all $X \in \mathrm{B}$.

It suffices to consider $\gamma = \omega$. We note that for every $X \in \mathrm{B}$, we have a factorization

$$(T_n F)(X) \to * \to (T_n F)^\dagger(X)$$

via selecting the τ_U-covering sieve $R \subset \mathrm{B}_{/X}$ given by our assumption, where we use the previous claim to see that $(T_n F)(Y) \simeq *$ for all $Y \in R$. It then follows that $(T_\omega F)(X) \simeq *$, and we are done. ☐

Given this, we deduce:

Proposition A.4. *With assumptions as in Lemma A.3, the pair*

$$(\mathrm{Shv}_{\tau_U}(\mathrm{B}), \mathrm{Shv}_{\tau_Z}(\mathrm{B}))$$

constitutes a recollement of $\mathrm{Shv}_\tau(\mathrm{B})$. *Moreover, this recollement is monoidal with respect to the cartesian monoidal structure.*

Proof. Appealing to the theory of sheafification, we have that the localization functors j^* and i^* are left-exact. Since j^*, i^* are jointly conservative by Lemma A.2 and $j^* i_* \simeq *$ by Lemma A.3, the claim follows. ☐

Remark A.5. In view of [Lur17a, Proposition A.8.15], Proposition A.4 is equivalent to the obvious ∞-categorical extension of [Sch94, Proposition 2.2] (in the (ii)⇒(i) direction), although our proof differs from his.

We now pass to stabilizations. Recall the following paradigm:

Construction A.6. Suppose that we have a left-exact functor $G : \mathrm{C} \to \mathrm{D}$, then we have have an induced functor

$$\overline{G} : \mathrm{Sp}(\mathrm{C}) = \mathrm{Exc}_*(\mathrm{Spc}_\bullet^{\mathrm{fin}}, \mathrm{C}) \to \mathrm{Sp}(\mathrm{D}) = \mathrm{Exc}_*(\mathrm{Spc}_\bullet^{\mathrm{fin}}, \mathrm{D}). \qquad (f : \mathrm{Spc}_\bullet^{\mathrm{fin}} \to \mathrm{C}) \mapsto (G \circ f).$$

Moreover, if G admits a left-exact left adjoint $F : \mathrm{D} \to \mathrm{C}$, then $\overline{G}$ admits a left adjoint

$$\overline{F} : \mathrm{Sp}(\mathrm{D}) \to \mathrm{Sp}(\mathrm{C})$$

defined by postcomposition by F.

Lemma A.7. *Suppose we have a recollement*

$$\mathcal{U} \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathcal{X} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{Z}.$$

Then the induced adjunctions

$$\mathrm{Sp}(\mathcal{U}) \underset{\overline{j}_*}{\overset{\overline{j}^*}{\leftrightarrows}} \mathrm{Sp}(\mathcal{X}) \underset{\overline{i}_*}{\overset{\overline{i}^*}{\rightleftarrows}} \mathrm{Sp}(\mathcal{Z}).$$

together yield a stable recollement.

Proof. Since j^*i_* is constant at the terminal object, we have that $\overline{j}^*\overline{i}_* \simeq 0$ in view of the description of this functor as postcomposition by j^*i_* in terms of spectrum objects. Likewise, the joint conservativity of $\overline{j}^*$ and $\overline{i}^*$ holds since equivalences of spectrum objects are detected levelwise. □

The next corollary is now immediate.

Corollary A.8. *With assumptions as in Lemma A.3, the pair*

$$(\mathrm{Shv}_{\tau_U}(\mathrm{B}, \mathrm{Sp}), \mathrm{Shv}_{\tau_Z}(\mathrm{B}, \mathrm{Sp}))$$

constitutes a stable recollement of $\mathrm{Shv}_\tau(\mathrm{B}, \mathrm{Sp})$*. Moreover, this recollement is monoidal with respect to the smash product symmetric monoidal structure.*

Finally, we explain how to pass recollements through monoidal inversion of objects. For this, recall from [Rob15, Corollary 2.22] that if C is a presentable symmetric monoidal ∞-category and $X \in \mathrm{C}$ is a symmetric object (i.e., for some $n > 1$ the cyclic permutation of $X^{\otimes n}$ is homotopic to the identity), then the presentable symmetric monoidal ∞-category $C[X^{-1}]$ is computed as the filtered colimit of presentable ∞-categories

$$\mathrm{C}[X^{-1}] \simeq \mathrm{colim}(\mathrm{C} \xrightarrow{-\otimes X} \mathrm{C} \xrightarrow{-\otimes X} \cdots).$$

Lemma A.9. *Let*

$$\mathcal{U} \; \begin{smallmatrix} \xrightarrow{j_!} \\ \xleftarrow{j^*} \\ \xrightarrow[j_*]{} \end{smallmatrix} \; \mathcal{X} \; \begin{smallmatrix} \xrightarrow{i^*} \\ \xleftarrow{i_*} \\ \xrightarrow[i^!]{} \end{smallmatrix} \; \mathcal{Z}$$

be a stable[32] *monoidal recollement of presentable stable symmetric monoidal ∞-categories and let $E \in \mathcal{X}$ be a symmetric object. Then we may descend the recollement adjunctions to obtain a stable monoidal recollement*

$$\mathcal{U}[(j^*E)^{-1}] \; \begin{smallmatrix} \xrightarrow{\overline{j}_!} \\ \xleftarrow{\overline{j}^*} \\ \xrightarrow[\overline{j}_*]{} \end{smallmatrix} \; \mathcal{X}[E^{-1}] \; \begin{smallmatrix} \xrightarrow{\overline{i}^*} \\ \xleftarrow{\overline{i}_*} \\ \xrightarrow[\overline{i}^!]{} \end{smallmatrix} \; \mathcal{Z}[(i^*E)^{-1}].$$

Proof. Let us write

$$\Sigma^\infty_E : \mathcal{X} \rightleftarrows \mathcal{X}[E^{-1}] : \Omega^\infty_E$$

for the defining adjunction, and likewise for $\mathcal{U}$ and $\mathcal{Z}$. Also let E, j^*E, i^*E denote the images of the objects in $\mathcal{X}[E^{-1}]$, etc. We first explain how the various overlined functors are defined. Since j^* and i^* are symmetric monoidal, we have the commutative diagram

$$\begin{array}{ccccc} \mathcal{U} & \xleftarrow{j^*} & \mathcal{X} & \xrightarrow{i^*} & \mathcal{Z} \\ {\scriptstyle -\otimes j^*E}\downarrow & & \downarrow{\scriptstyle -\otimes E} & & \downarrow{\scriptstyle -\otimes i^*E} \\ \mathcal{U} & \xleftarrow{j^*} & \mathcal{X} & \xrightarrow{i^*} & \mathcal{Z}, \end{array}$$

[32]Stability is used in an essential way in our proof.

hence $\bar{j}^*$ and $\bar{i}^*$ may be taken to be the filtered colimit of the system of functors $\{j^*\}$ and $\{i^*\}$, and we have the commutativity relations

$$\bar{j}^* \circ (E^{-n} \otimes \Sigma^\infty_E) \simeq ((j^*E)^{-n} \otimes \Sigma^\infty_{j^*E}) \circ j^*, \quad \bar{i}^* \circ (E^{-n} \otimes \Sigma^\infty_E) \simeq ((i^*E)^{-n} \otimes \Sigma^\infty_{i^*E}) \circ i^*.$$

By adjunction, $\bar{j}^*$ and $\bar{i}^*$ admit right adjoints $\bar{j}_*$ and $\bar{i}_*$ that satisfy adjoint commutativity relations involving Ω^∞_-. To aid understanding, it may be helpful to note that $\Sigma^\infty_E X$ is given as a E-spectrum object by $\{X \otimes E^n\}$, and $\Omega^\infty_E \{X_n\} \simeq X_0$ (cf. Remark A.10).

We next observe that the projection formulas

$$j_!(U \otimes j^*X) \simeq j_!(U) \otimes X \qquad i_*(Z) \otimes X \simeq i_*(Z \otimes i^*X)$$

established in [QS19, Proposition 1.30] imply that $\bar{j}_!$ and $\bar{i}_*$ may be defined by the same colimit procedure as was done with $\bar{j}^*$ and $\bar{i}^*$. In particular, $\bar{j}^*$ commutes with Ω^∞_- and $\bar{i}_*$ commutes with Σ^∞_- and admits a right adjoint $\bar{i}^!$ that commutes with Ω^∞_- (and the same is true for shifts thereof by E^n).

We next establish the full faithfulness of $\bar{j}_!, \bar{j}_*$, and $\bar{i}_*$. For $\bar{j}_*$, note that the counit $\bar{j}^*\bar{j}_* \to \mathrm{id}$ is sent to an equivalence by $\Omega^\infty_{j^*E}(E^n \otimes -)$, hence is an equivalence. For $\bar{i}_*$, the counit $\bar{i}^*\bar{i}_* \to \mathrm{id}$ is a natural transformation of colimit-preserving functors and is evidently an equivalence on objects in the image of $(i^*E)^{-n} \otimes \Sigma^\infty_{i^*E}$; since such objects generate $\mathcal{Z}[(i^*E)^{-1}]$ under colimits, we may conclude. The case of $\bar{j}_!$ follows by the same reasoning. Finally, also note that the $\bar{j}_! \dashv \bar{j}^*$ and $\bar{i}^* \dashv \bar{i}_*$ projection formulas follow by the same reasoning.

Now let $A = i_*i^*\mathbb{1}$, so that A is the idempotent $\mathbb{E}_\infty$-algebra in $\mathcal{X}$ that determines the smashing localization $\mathcal{Z} \simeq \mathrm{Mod}_A(\mathcal{X})$. Invoking the monoidal Barr-Beck theorem [MNN17, Theorem 5.29], we may descend this equivalence to obtain

$$\mathcal{Z}[(i^*E)^{-1}] \simeq \mathrm{Mod}_{\Sigma^\infty_E A}(\mathcal{X}[E^{-1}]).$$

We thus see that $\mathcal{Z}[(i^*E)^{-1}]$ is the smashing localization of $\mathcal{X}[E^{-1}]$ determined by the idempotent $\mathbb{E}_\infty$-algebra $\Sigma^\infty_E A$. In this situation, we recall from [QS19, 1.32] that to establish the recollement conditions, it suffices to show

- $(*)$ the essential image of $\bar{j}_*$ coincides with the full subcategory of objects X such that $\mathrm{Map}(\bar{i}_*Z, X) \simeq *$ for all $Z \in \mathcal{Z}[(i^*E)^{-1}]$.

To prove this, first note that $\bar{j}^*\bar{i}_* \simeq 0$ using that for all $n \in \mathbb{Z}$,

$$\Omega^\infty_{j^*E}((j^*E)^n \otimes \bar{j}^*\bar{i}_*(-)) \simeq j^*i_*\Omega^\infty_{i^*E}((i^*E)^n \otimes -) \simeq 0.$$

Therefore, given $U \in \mathcal{U}[(j^*E)^{-1}]$, we have that

$$\mathrm{Map}(\bar{i}_*Z, \bar{j}_*U) \simeq \mathrm{Map}(\bar{j}^*\bar{i}_*Z, U) \simeq \mathrm{Map}(0, U) \simeq *.$$

Conversely, suppose $X \in \mathcal{X}[E^{-1}]$ is such that $\mathrm{Map}(\bar{i}_*Z, X) \simeq *$ for all $Z \in \mathcal{Z}[(i^*E)^{-1}]$. By adjunction, we have that $\bar{i}^!(X) \simeq 0$. It follows that $\Omega^\infty_E(E^n \otimes X)$ lies in the essential image of j_* for all n, and thus X itself lies in the essential image of $\bar{j}_*$. This proves the claim $(*)$ and hence the lemma is proved. ☐

Remark A.10. For sake of reference, we record some basic facts about prespectrum and spectrum objects in the ∞-categorical setting, fleshing out some details in [ELSØ17, §4.0.6], [Hoy16, §3]. Let C be a presentable symmetric monoidal ∞-category, let $E \in \mathrm{C}$ be a symmetric object, and let Σ^E, resp. Ω^E denote the endofunctors $E \otimes (-)$, resp. $\underline{\mathrm{Hom}}(E, -)$ of C. Let $p : \mathbb{Z}^{\mathrm{op}}_{\geqslant 0} \to \widehat{\mathrm{Cat}}_\infty$ be the functor given by

$$\mathrm{C} \xleftarrow{\Omega^E} \mathrm{C} \xleftarrow{\Omega^E} \mathrm{C} \xleftarrow{\Omega^E} \cdots$$

Because E is a symmetric object, we have an equivalence of ∞-categories $\mathrm{C}[E^{-1}] \simeq \lim(p)$. Moreover, if we let $\widehat{\mathrm{C}} = \int p \to \mathbb{Z}_{\geqslant 0}$ be the cartesian fibration classified by p, then $\mathrm{Sect}^{\mathrm{cart}}(\widehat{\mathrm{C}}) \simeq \lim(p)$.[33] Using that the inclusion of the spine[34] of $\mathbb{Z}_{\geqslant 0}$ into $\mathbb{Z}_{\geqslant 0}$ is inner anodyne, we see that the objects of $\mathrm{Sect}^{\mathrm{cart}}(\widehat{\mathrm{C}})$ may be

[33] We could lift $\mathrm{C}[E^{-1}] \simeq \mathrm{Sect}^{\mathrm{cart}}(\widehat{\mathrm{C}})$ to an equivalence of symmetric monoidal ∞-categories by means of the Day convolution, but we will not need this.

[34] The *spine* of $\mathbb{Z}_{\geqslant 0}$ is the sub-simplicial set given by $\{0 < 1\} \cup_{\{1\}} \{1 < 2\} \cup_{\{2\}} \{2 < 3\} \cup \cdots$.

described as tuples $\{X_n \in \mathrm{C}, \alpha_n : X_n \xrightarrow{\simeq} \Omega^E X_{n+1}\}$ with further coherences determined essentially uniquely, and similarly for morphisms, so we are entitled to refer to $\mathrm{Sect}^{\mathrm{cart}}(\widehat{\mathrm{C}})$ as the ∞-category of E-spectra in C.

Moreover, the cartesian fibration $\widehat{\mathrm{C}} \to \mathbb{Z}_{\geqslant 0}$ is also a cocartesian fibration with pushforward functors the left adjoints Σ^E. Thus, if we consider $\mathrm{Sect}(\widehat{\mathrm{C}})$ instead, its objects may be described as tuples $\{X_n \in \mathrm{C}, \beta_n : \Sigma^E X_n \to X_{n+1}\}$, so $\mathrm{Sect}(\widehat{\mathrm{C}})$ is the ∞-category of E-prespecta in C. We have the inclusion functor $\mathrm{Sect}^{\mathrm{cart}}(\widehat{\mathrm{C}}) \subset \mathrm{Sect}(\widehat{\mathrm{C}})$ that preserves limits and is accessible (since the same is true for Ω^E), so admits a left adjoint L_{sp} that exhibits E-spectra as a Bousfield localization of E-prespectra. We also have the equivalence $\mathrm{Sect}^{\mathrm{cocart}}(\widehat{\mathrm{C}}) \xrightarrow{\simeq} \mathrm{C}$ implemented by evaluation at the initial object $0 \in \mathbb{Z}_{\geqslant 0}$, and the inclusion $\mathrm{Sect}^{\mathrm{cocart}}(\widehat{\mathrm{C}}) \subset \mathrm{Sect}(\widehat{\mathrm{C}})$ preserves colimits (as Σ^E does) and thus admits a right adjoint which also identifies with evaluation at 0.

Using these identifications, we then may factor the adjunction $\Sigma^\infty_E : \mathrm{C} \rightleftarrows \mathrm{C}[E^{-1}] : \Omega^\infty_E$ as

$$\mathrm{C} \underset{\mathrm{ev}_0}{\overset{}{\rightleftarrows}} \mathrm{Sect}(\widehat{\mathrm{C}}) \underset{}{\overset{L_{\mathrm{sp}}}{\rightleftarrows}} \mathrm{C}[E^{-1}].$$

Now suppose $i^* : \mathrm{C} \to \mathrm{D}$ is a strong symmetric monoidal colimit-preserving functor of presentable symmetric monoidal ∞-categories, and let $\bar{i}^* : \mathrm{C}[E^{-1}] \to \mathrm{D}[(i^*E)^{-1}]$ be the induced functor. We may then factor the equivalence $\Sigma^\infty_{i^*E} i^* \simeq \bar{i}^* \Sigma^\infty_E$ as

$$\begin{array}{ccccc}
\mathrm{C} & \hookrightarrow & \mathrm{Sect}(\widehat{\mathrm{C}}) & \xrightarrow{L_{\mathrm{sp}}} & \mathrm{C}[E^{-1}] \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*_{\mathrm{pre}}} & & \downarrow{\scriptstyle \bar{i}^*} \\
\mathrm{D} & \hookrightarrow & \mathrm{Sect}(\widehat{\mathrm{D}}) & \xrightarrow{L_{\mathrm{sp}}} & \mathrm{D}[(i^*E)^{-1}],
\end{array}$$

where i^*_{pre} is obtained by postcomposition by $\widehat{i^*} : \widehat{\mathrm{C}} \to \widehat{\mathrm{D}}$, the unstraightening of the natural transformation

$$\begin{array}{ccccccc}
\mathrm{C} & \xrightarrow{\Sigma^E} & \mathrm{C} & \xrightarrow{\Sigma^E} & \mathrm{C} & \xrightarrow{\Sigma^E} & \cdots \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \\
\mathrm{D} & \xrightarrow{\Sigma^{i^*E}} & \mathrm{D} & \xrightarrow{\Sigma^{i^*E}} & \mathrm{D} & \xrightarrow{\Sigma^{i^*E}} & \cdots
\end{array}$$

In particular, we see that $\bar{i}^*$ is computed by inclusion into E-prespectra and then $L_{\mathrm{sp}} \circ i^*_{\mathrm{pre}}$.

A.2 Motivic homotopy theory

Let us now specialize to the situation where $\mathrm{B} = \mathrm{Sm}_S$. In general, if we have two topologies τ' and τ on Sm_S with τ' finer than τ, then the localization functor $L_{\tau'} : \mathrm{SH}^{S^1}_\tau(S) \to \mathrm{SH}^{S^1}_{\tau'}(S)$ always preserves the set of morphisms $\{\Sigma^\infty_+(X \times \mathbb{A}^1) \to \Sigma^\infty_+ X\}$ that generate the $\mathbb{A}^1$-local equivalences, using that sheafification preserves products. Moreover, $L_{\tau'}$ preserves hypercovers as a sheafification. Therefore, we get a commutative diagram[35]

$$\begin{array}{ccc}
\mathrm{Shv}_\tau(\mathrm{Sm}_S, \mathrm{Sp}) & \xrightarrow{L_{\tau'}} & \mathrm{Shv}_{\tau'}(\mathrm{Sm}_S, \mathrm{Sp}) \\
\downarrow{\scriptstyle L_{\mathbb{A}^1}} & & \downarrow{\scriptstyle L_{\mathbb{A}^1}} \\
\mathrm{SH}^{S^1}_\tau(S) & \xrightarrow{L_{\tau'}} & \mathrm{SH}^{S^1}_{\tau'}(S).
\end{array}$$

Using the criterion of [Lur17a, Example 2.2.1.7], we also see that this is a diagram of localizations compatible with the various symmetric monoidal structures in the sense of [Lur17a, Definition 2.2.1.6]. Returning to the setting of two topologies τ_U and τ_Z with $\tau = \tau_U \cap \tau_Z$, we thus obtain a commutative diagram of symmetric monoidal left adjoints

$$\begin{array}{ccccc}
\mathrm{Shv}_{\tau_U}(\mathrm{Sm}_S, \mathrm{Sp}) & \xleftarrow{j^*} & \mathrm{Shv}_\tau(\mathrm{Sm}_S, \mathrm{Sp}) & \xrightarrow{i^*} & \mathrm{Shv}_{\tau_Z}(\mathrm{Sm}_S, \mathrm{Sp})) \\
\downarrow{\scriptstyle L_{\mathbb{A}^1}} & & \downarrow{\scriptstyle L_{\mathbb{A}^1}} & & \downarrow{\scriptstyle L_{\mathbb{A}^1}} \\
\mathrm{SH}^{S^1}_{\tau_U}(S) & \xleftarrow{j^*} & \mathrm{SH}^{S^1}_\tau(S) & \xrightarrow{i^*} & \mathrm{SH}^{S^1}_{\tau_Z}(S)
\end{array} \tag{30}$$

[35] Note that the localization functor $L_{\mathbb{A}^1}$ also imposes hyperdescent according to our conventions.

with fully faithful right adjoints j_*, i_*, and $\iota_{\mathbb{A}^1}$.

Proposition A.11. *The pair* $(\mathrm{SH}^{S^1}_{\tau_U}(S), \mathrm{SH}^{S^1}_{\tau_Z}(S))$ *constitute a stable monoidal recollement of* $\mathrm{SH}^{S^1}_{\tau}(S)$.

Proof. For clarity, we decorate the lower horizontal functors of (30) with an S^1 in this proof. By Lemma A.2, $j^*_{S^1}$ and $i^*_{S^1}$ are jointly conservative. As for the composite functor $j^*_{S^1} i^{S^1}_*$, this follows from Corollary A.8 via the following computation:

$$j^*_{S^1} i^{S^1}_* \simeq L_{\mathbb{A}^1} j^* \iota_{\mathbb{A}^1} i^{S^1}_* \simeq L_{\mathbb{A}^1} j^* i_* \iota_{\mathbb{A}^1} \simeq L_{\mathbb{A}^1}(0) \simeq 0.$$

□

Warning A.12. On the unstable level of (pointed) motivic spaces, one likewise has adjunctions

$$\mathrm{H}_{\tau_U}(S)_{(\bullet)} \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathrm{H}_{\tau}(S)_{(\bullet)} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{H}_{\tau_Z}(S)_{(\bullet)}.$$

such that j^*, i^* are jointly conservative and $j^* i_* \simeq *$. However, we do not know how to show that i^* is left-exact (although j^* is left-exact since we may descend the adjunction $j_! \dashv j^*$).

To pass from $\mathrm{SH}^{S^1}_{\tau}(S)$ to $\mathrm{SH}_{\tau}(S)$, we may invert $\mathbb{G}_m$. We then have:

Theorem A.13. *The pair* $(\mathrm{SH}_{\tau_U}(S), \mathrm{SH}_{\tau_Z}(S))$ *constitutes a stable monoidal recollement of* $\mathrm{SH}_{\tau}(S)$.

Proof. Since $\mathbb{G}_m$ is a symmetric object for any topology, the theorem follows from Proposition A.11 and Lemma A.9. □

Example A.14. Let τ_U be the étale topology and τ_Z be the real étale topology, so that τ is Scheiderer's b-topology. By [Sch94, 2.4], the criterion of A.3 is satisfied in this case. Therefore, Theorem A.13 applies to decompose $\mathrm{SH}_b(S)$ as a stable monoidal recollement of $\mathrm{SH}_{\text{ét}}(S)$ and $\mathrm{SH}_{\text{rét}}(S)$.

B The real étale and b-topologies

We begin with some recollections on the real étale topology.

Definition B.1. Suppose that $f : Y \to X$ is a morphism of schemes. We say that f is a **real cover** if for any morphism $\alpha : \operatorname{Spec} k \to X$ where k is a real closed field, there exists an extension of real closed fields k'/k and a morphism $\alpha' : \operatorname{Spec} k' \to Y$ such that the following diagram commutes

$$\begin{array}{ccc} \operatorname{Spec} k' & \xrightarrow{\alpha'} & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{\alpha} & X. \end{array}$$

Example B.2. For an example of a non-trivial extension of real closed fields, consider the field $\mathbb{R}_{\mathrm{alg}}$ of real algebraic numbers, which is the real closure of $\mathbb{Q}$. Then the inclusion $\mathbb{R}_{\mathrm{alg}} \subset \mathbb{R}$ is an extension of real closed fields.

Lemma B.3. *Let $f : Y \to X$ be a morphism of schemes. The following conditions on f are equivalent:*

1. *$f : Y \to X$ is a real cover.*
2. *The induced map on real spectra $f_r : Y_r \to X_r$ is surjective.*
3. *For any point $x \in X$ with an ordering on $k(x)$, there exists a point $y \in Y$ with an ordering on $k(y)$ such that $f(y) = x$ and $k(y)/k(x)$ is an extension of ordered fields.*

Proof. The equivalence of (2) and (3) follows by tracing through the definition of the real spectrum. The equivalence of (1) and (2) follows from the description of the points of X_r as classes of morphisms $\operatorname{Spec} k \to X$ where k is a real closed field up to the equivalence relation specified in [Sch94, 0.4.3]. ☐

Remark B.4. A real cover need not be surjective on any other field points. For example, the morphism $\operatorname{Spec}\mathbb{R} \to \operatorname{Spec}\mathbb{R}[x,y]/(x^2+y^2)$ classifying the $\mathbb{R}$-point "(0,0)" is a real cover. However, on $\mathbb{C}$-points, it is the inclusion of a single point into a conic and thus is not surjective.

Definition B.5. Let X be a scheme. The **real étale topology on** $\acute{\mathrm{E}}\mathrm{t}_X$ is the coarsest topology such that the sieves generated by families $\{U_\alpha \to U\}$ for which $\coprod U_\alpha \to U$ is a real cover are covering sieves. Similarly, we can also consider the **real étale topology on** Sm_X or Sch_X as the coarsest topology such that the sieves generated by families $\{U_\alpha \to U\}$ for which $U_\alpha \to U$ is étale and $\coprod U_\alpha \to U$ is a real cover are covering sieves.

Note that if $k \subset k'$ is an algebraic extension of ordered fields and k is real closed, then $k = k'$. Hence, if we assume that a real cover $f : Y \to X$ is étale (so that the singleton $\{Y \to X\}$ generates a real étale covering sieve), we have the following refinement of Lemma B.3.

Lemma B.6. *Let $f : Y \to X$ be a morphism of schemes such that for any $y \in Y$ the induced extension of residue fields $f : k(f(y)) \subset k(y)$ is algebraic (e.g., f is étale). The following are equivalent:*

1. *$f : Y \to X$ is a real cover.*
2. *For any real closed field k, the induced map $Y(k) \to X(k)$ is surjective.*

Example B.7. Recall that a morphism $f : Y \to X$ is **completely decomposed** if it is surjective on k-points. Hence, one might call condition (2) of Lemma B.6 **real completely decomposed**, since we only demand surjectivity on real closed points. Since the Nisnevich topology is given by étale morphisms that are jointly surjective on k-points, we see that any Nisnevich cover is a real étale cover.

We now discuss points of the real étale topology.

Suppose that $X = \operatorname{Spec} A$ is an affine scheme and $x \in \operatorname{Sper} A$ is a point corresponding to a pair $(\mathfrak{p}, \leqslant_\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal of A and $\leqslant_\mathfrak{p}$ is an ordering on $\kappa(\mathfrak{p})$. Let $\mathrm{Neib}_{\mathrm{r\acute{e}t}}(X,x)$ be the filtered category of tuples $(B, y \in \operatorname{Sper} B)$, where B is an étale A-algebra and $y = (\mathfrak{q}, \leqslant_\mathfrak{q})$ is a point over x, i.e., $\mathfrak{q}$ lies over $\mathfrak{p}$ and $k(\mathfrak{q})/k(\mathfrak{p})$ is an ordered extension. Unlike the situation of separably closed fields, we note that the real closures of ordered fields are unique (see, for example, [San91]), whence we do not have to make a further choice of an embedding of $k(\mathfrak{q})$ into the real closure k of $k(\mathfrak{p})$; it follows that to define $\mathrm{Neib}_{\mathrm{r\acute{e}t}}(X,x)$, we may have equivalently specified factorizations of $\operatorname{Spec} k \to \operatorname{Spec} A$ through étale A-schemes. We then have the notion of a **strictly real henselian local ring**, defined to be a henselian local ring with real closed residue field, and the **strict real henselization** of X at $x = (\mathfrak{p}, \leqslant_\mathfrak{p})$, which is the initial strictly real henselian local ring $A_x^{\mathrm{r\acute{e}t}h}$ equipped with a local homomorphism $A_\mathfrak{p} \to A_x^{\mathrm{r\acute{e}t}h}$ preserving the ordering on residue fields (see [AR87, Definition 3.1.1] or [Sch94, Definition 3.7.3]). By [AR87, Proposition 3.1.3], the strict real henselization of X at x exists and is computed by the usual formula:

$$A_x^{\mathrm{r\acute{e}t}h} = \operatorname*{colim}_{\mathrm{Neib}_{\mathrm{r\acute{e}t}}(X,x)} B. \tag{31}$$

Lemma B.8. *Let R be a commutative ring. Then the following are equivalent:*

1. *The ring R is strictly real henselian.*
2. *For any real étale cover $\{\phi_\alpha : \operatorname{Spec} R_\alpha \to \operatorname{Spec} R\}$, one of the morphisms ϕ_α admits a splitting.*

Proof. This is in [CCR81]. ☐

Lemma B.9. *Suppose that S is the limit of a cofiltered diagram of schemes*[36] *S_α. Then we have equivalences of ∞-categories*

$$\widetilde{S}_{\mathrm{r\acute{e}t}} \simeq \lim_\alpha \widetilde{S_{\alpha\,\mathrm{r\acute{e}t}}}, \qquad \mathrm{Sp}(\widetilde{S}_{\mathrm{r\acute{e}t}}) \simeq \lim_\alpha \mathrm{Sp}(\widetilde{S_{\alpha\,\mathrm{r\acute{e}t}}}).$$

[36] Recall that, by convention, schemes are all qcqs.

Proof. Since étale morphisms are Zariski-locally finitely presented [TS15, Tag 02GR] and the real étale topology is finer than Zariski, we have that $\widetilde{X}_{\text{rét}} \simeq \text{Shv}_{\text{rét}}(\text{Ét}^{\text{fp}}_X)$ in general. Following the same logic as in [Hoy14, Proposition C.7(3)], the first claim follows from the fact that a finitely presented real-étale surjection in Ét^{fp}_S is pulled back from one in $\text{Ét}^{\text{fp}}_{S_\alpha}$; this is the case for étale maps by [TS15, Tag 07RP] and clear for surjectivity on real closed points. The stable case follows from the unstable case since Sp preserves limits of ∞-categories with finite limits (using the formula for stabilization in [Lur17a, Proposition 1.4.2.24]). ☐

B.1 Hypercompleteness of the real étale site

In this section, we will prove that if X is of finite Krull dimension, then the real étale site of X is hypercomplete. One use of this result in the main body of the paper is to show that certain gluing functors are stable under base change — an argument that appeals to "checking on stalks." If X is a topological space, we write $\widetilde{X}$ for the ∞-category of sheaves on spaces on X, i.e., that of functors

$$\text{Open}(X)^{\text{op}} \to \text{Spc}$$

satisfying descent with respect to open covers. We first state a fundamental theorem in the subject of real étale cohomology, which in the setting of sheaves of sets is due originally to Coste-Roy and Coste (cf. [Sch94, 1.4]); we will follow Scheiderer's more functorial formulation, which may be readily promoted to a statement involving ∞-topoi.

Theorem B.10 (Scheiderer)**.** *Let X be a scheme. There is a canonical equivalence of ∞-topoi*

$$\widetilde{X}_{\text{rét}} \simeq \widetilde{X_r}.$$

We can deduce this theorem from [Sch94, Theorem 1.3] after some topos-theoretic preliminaries. Recall from [Lur09, Definition 6.4.1.1] that for $0 \leqslant n \leqslant \infty$, an **$n$-topos** is an ∞-category $\mathcal{X}$ that is a left-exact accessible localization of $\text{PShv}(\mathrm{C}, \text{Spc}_{\leqslant n-1})$ for C a small ∞-category. For example, a 1-topos is a left-exact accessible localization of a category of set-valued presheaves. A **geometric morphism between n-topoi $\mathcal{X}$ and $\mathcal{Y}$** is a functor $f_* : \mathcal{X} \to \mathcal{Y}$ that admits a left-exact left adjoint.

Examples of n-topoi and geometric morphisms thereof are furnished by the following construction. Suppose that $1 \leqslant m \leqslant n \leqslant \infty$. An **$m$-site** (C, τ) is an m-category C equipped with a Grothendieck topology τ. If (C, τ) and (D, τ') are m-sites with finite limits, then a **morphism of sites** is a functor $f : \mathrm{C} \to \mathrm{D}$ that preserves finite limits and enjoys the following property:

(∗) For every family of morphisms $\{U_\alpha \to X\}$ that generates a τ-covering sieve, the collection $\{f(U_\alpha) \to f(X)\}$ generates a τ'-covering sieve.

Note that this recovers the usual notion of a morphism of 1-sites.[37] Given $f : \mathrm{C} \to \mathrm{D}$ a morphism of sites, by [Lur, Lemma 2.4.7] restriction along f induces a geometric morphism of n-topoi[38]

$$f^* : \text{Shv}_{\tau'}(\mathrm{D}, \text{Spc}_{\leqslant n-1}) \to \text{Shv}_{\tau}(\mathrm{C}, \text{Spc}_{\leqslant n-1})$$

fitting into the following commutative diagram

$$\begin{array}{ccc} \text{Shv}_{\tau'}(\mathrm{D}, \text{Spc}_{\leqslant n-1}) & \xrightarrow{f^*} & \text{Shv}_{\tau}(\mathrm{C}, \text{Spc}_{\leqslant n-1}) \\ \big\downarrow & & \big\downarrow \\ \text{PShv}(\mathrm{D}, \text{Spc}_{\leqslant n-1}) & \xrightarrow{f^*_{\text{pre}}} & \text{PShv}(\mathrm{C}, \text{Spc}_{\leqslant n-1}), \end{array}$$

where f^*_{pre} is the restriction functor. In other words, f^*_{pre} preserves sheaves.

[37] In [72, III Definition 1.1], this functor is called **continuous**. A better name might be **topologically continuous**, in line with our usage of topologically cocontinuous in Appendix C.

[38] We warn the reader that f^* here is the right adjoint of the geometric morphism, in conflict with our usual notation and that in [Lur, Lemma 2.4.7]. We will also write $f_!$ for the left-exact left adjoint, which is induced by the functor of left Kan extension along f at the level of presheaves.

We write $\mathscr{T}\mathrm{op}_n^R \subset \widehat{\mathrm{Cat}}_\infty$ for the subcategory spanned by n-topoi and geometric morphisms thereof. Using the results of [Lur09, §6.4.5], we have that passage to $(n-1)$-truncated objects defines a functor

$$\tau_{\leqslant n-1} : \mathscr{T}\mathrm{op}^R \to \mathscr{T}\mathrm{op}_n^R,$$

equipped with a fully faithful right adjoint $\mathscr{T}\mathrm{op}_n^R \hookrightarrow \mathscr{T}\mathrm{op}^R$. If $\mathfrak{X}$ is an n-topos, we call its image in $\mathscr{T}\mathrm{op}^R$ the **associated n-localic ∞-topos**. We now have the following lemma, which allows us to promote equivalences of n-topoi to the level of ∞-topoi.

Lemma B.11. *Let $1 \leqslant m \leqslant n \leqslant \infty$. Suppose that $f : (\mathrm{C}, \tau) \to (\mathrm{D}, \tau')$ is a morphism of m-sites such that the induced geometric morphism $f^* : \mathrm{Shv}_{\tau'}(\mathrm{D}, \mathrm{Spc}_{\leqslant n-1}) \to \mathrm{Shv}_\tau(\mathrm{C}, \mathrm{Spc}_{\leqslant n-1})$ is an equivalence of n-topoi. Then the induced geometric morphism*

$$f^* : \mathrm{Shv}_{\tau'}(\mathrm{D}) \to \mathrm{Shv}_\tau(\mathrm{C})$$

is an equivalence of ∞-topoi.[39]

Proof. According to [Lur09, Proposition 6.4.5.6], under the hypotheses on (C, τ) and (D, τ'), the n-localic ∞-topoi associated to the n-topoi $\mathrm{Shv}_{\tau'}(\mathrm{D}, \mathrm{Spc}_{\leqslant n-1})$ and $\mathrm{Shv}_\tau(\mathrm{C}, \mathrm{Spc}_{\leqslant n-1})$ are given by passing to sheaves of spaces, whence the claim follows. □

Remark B.12. Suppose that (C, τ) is an m-site and consider the n-topos $\mathrm{Shv}_\tau(\mathrm{C}, \mathrm{Spc}_{\leqslant n-1})$. Without the assumption on the existence of finite limits, passing to sheaves of spaces need not compute the associated n-localic ∞-topos; instead, one only expects an equivalence upon hypercompletion. See the discussion in [Car] and also [AS18, Remark 2.6].

Proof of Theorem B.10. The equivalence of [Sch94, Theorem 1.3] is implemented by a zig-zag of equivalences through geometric morphisms arising from a zig-zag of morphisms of sites, which we first recall from [Sch94, 1.9]. Define a site $(\mathrm{Pairs}_X, \mathrm{aux})$ in the following way: the objects are pairs (U, W) where $U \in \acute{\mathrm{E}}\mathrm{t}_X$ and $W \subset U_r$ is an open subset of the real spectrum of U, and a morphism $f : (U, W) \to (U', W')$ is a morphism $f : U \to U'$ of X-schemes such that $f_r(W) \subset W'$. A family $\{f_\alpha : (U_\alpha, W_\alpha) \to (U, W)\}$ is an aux-cover whenever $W = \cup f_{\alpha,r}(W_\alpha)$. This category evidently has finite limits. By design, we then have a span of morphisms of sites:

$$(\acute{\mathrm{E}}\mathrm{t}_X, \mathrm{r\acute{e}t}) \xleftarrow{\phi} (\mathrm{Pairs}_X, \mathrm{aux}) \xrightarrow{\psi} (\mathrm{Open}(X_r), \mathrm{usual}).$$

It is easy to see that ϕ preserves finite limits, while ψ preserves finite limits by [Sch94, Corollary 1.7.1]. Lemma B.11 thus applies to promote [Sch94, Theorem 1.3] to the claimed theorem. □

We now prove:

Theorem B.13. *Let X be a scheme of finite Krull dimension. Then the ∞-topos $\widetilde{X}_{\mathrm{r\acute{e}t}}$ is hypercomplete.*

Proof. Let A be a commutative ring. By [BCR87, Corollaire 7.1.16], $\mathrm{Sper}\, A$ is a spectral space. Furthermore, the Krull dimension of a spectral space is given by the supremum over lengths of chains of specializations. With this, the Krull dimension of $\mathrm{Sper}\, A$ is then bounded above by the Krull dimension of $\mathrm{Spec}\, A$ as the former is computed using specializations of real prime ideals in A. In this case, [CM19, Theorem 3.12] tells us that if the Krull dimension of $\mathrm{Spec}\, A$ is d, then the homotopy dimension of $\widetilde{\mathrm{Sper}\, A}$ is less than d. Hence, for any scheme X of finite Krull dimension d, the ∞-topos $\widetilde{X}_r$ is locally of homotopy dimension $\leqslant d$, whence hypercomplete by [Lur09, Corollary 7.2.1.12]. Theorem B.10 then implies that $\widetilde{X}_{\mathrm{r\acute{e}t}}$ is hypercomplete. □

B.2 The gluing functor

In this section, we will compute the stalks of the gluing functor. To begin with, we need to introduce some notation. We have the gluing functor from Example 2.41

$$\theta = L_{\mathrm{r\acute{e}t}} i_{\acute{\mathrm{e}}\mathrm{t}} : \widetilde{X}_{\acute{\mathrm{e}}\mathrm{t}} \to \widetilde{X}_{\mathrm{r\acute{e}t}}.$$

More generally, suppose that $\mathrm{Sch}'_X \subset \mathrm{Sch}_X$ is a subcategory such that:

[39] We thank Peter Haine for alerting us to this result.

1. For any $Y \in \mathrm{Sch}'_X$, if $U \to Y$ is étale then $U \in \mathrm{Sch}'_X$.
2. $\mathrm{Sch}'_X \subset \mathrm{Sch}_X$ is closed under limits.

The only example that will concern us is $\mathrm{Sch}'_X = \mathrm{Sm}_X$. In this case, we can define the étale and real étale topology on X, and thus we can define the gluing functor in this level of generality:

$$\theta' = L_{\text{rét}} i_{\text{ét}} : \mathrm{Shv}_{\text{ét}}(\mathrm{Sch}'_X) \to \mathrm{Shv}_{\text{rét}}(\mathrm{Sch}'_X).$$

To proceed further, we now appeal to the results of Appendix C to relate θ and θ'. Namely, if

$$u^*_\tau : \mathrm{Shv}_\tau(\mathrm{Sch}'_X) \to \mathrm{Shv}_\tau(\text{Ét}_X),\ \tau \in \{\text{ét}, \text{rét}\}$$

is the restriction functor, then $u^*_{\text{rét}}\theta' \simeq \theta u^*_{\text{ét}}$ by Lemma C.5. Furthermore, if $f : Y \to X$ is a morphism, then we have adjunctions

$$f^*_{\text{pre}} : \mathrm{PShv}(\mathrm{Sch}'_X) \rightleftarrows \mathrm{PShv}(\mathrm{Sch}'_Y) : f_{*\text{pre}}$$
$$f^*_{\text{rét}} : \mathrm{Shv}_{\text{rét}}(\mathrm{Sch}'_X) \rightleftarrows \mathrm{Shv}_{\text{rét}}(\mathrm{Sch}'_Y) : f_{*\text{rét}}$$
$$f^*_{\text{ét}} : \mathrm{Shv}_{\text{ét}}(\mathrm{Sch}'_X) \rightleftarrows \mathrm{Shv}_{\text{ét}}(\mathrm{Sch}'_Y) : f_{*\text{ét}}$$

The next lemma is a simple enhancement of [Sch94, Proposition 3.7.2] to a slightly more general context.

Lemma B.14. *Let $\mathrm{Sch}'_X \subset \mathrm{Sch}_X$ be as above. Suppose that $\mathcal{F} : (\mathrm{Sch}'_X)^{\mathrm{op}} \to \mathrm{Spc}$ is an étale sheaf, $Y \in \mathrm{Sch}'_X$, and we have a morphism $\alpha : \operatorname{Spec} k \to Y$ where k is a real closed field. Then we have an equivalence*

$$(\theta'\mathcal{F})_\alpha \simeq (\alpha^*_{\text{ét}}\mathcal{F})(1_\alpha),$$

where $1_\alpha \in \mathrm{Shv}_{\text{ét}}(k)$ is the terminal object.

Proof. Without loss of generality, we may assume that $X = Y$. For this proof we follow the notation in Lemma C.5. By (31), taking stalks at α is given by the formula[40]

$$(\theta'\mathcal{F})_\alpha \simeq \operatorname*{colim}_{T \in \mathrm{Neib}_{\text{rét}}(Y,\alpha)} (\theta'\mathcal{F})(T).$$

Since each T in $\mathrm{Neib}_{\text{rét}}(Y, \alpha)$ is, in particular, an étale Y-scheme, we have the following equivalences

$$\begin{aligned}
\operatorname*{colim}_{T \in \mathrm{Neib}_{\text{rét}}(Y,\alpha)} \theta'\mathcal{F}(T) &\simeq \operatorname*{colim}_{T \in \mathrm{Neib}_{\text{rét}}(Y,\alpha)} \mathrm{Map}(L_{\text{rét}} u_{\text{pre}!} T, \theta'\mathcal{F}) \\
&\simeq \operatorname*{colim}_{T \in \mathrm{Neib}_{\text{rét}}(Y,\alpha)} \mathrm{Map}(u_{\text{rét}!} L_{\text{rét}_{\text{small}}} T, \theta'\mathcal{F}) \\
&\simeq \operatorname*{colim}_{T \in \mathrm{Neib}_{\text{rét}}(Y,\alpha)} \mathrm{Map}(L_{\text{rét}_{\text{small}}} T, u^*_{\text{rét}}\theta'\mathcal{F}) \\
&\simeq (\alpha^*_{\text{rét}_{\text{small}}} u^*_{\text{rét}} \theta'\mathcal{F})(1_\alpha),
\end{aligned}$$

where $\alpha^*_{\text{rét}_{\text{small}}} : \widetilde{X}_{\text{rét}} \to \widetilde{\operatorname{Spec} k}_{\text{rét}} \simeq \mathrm{Spc}$ is the pullback functor on the small site and we use Lemma C.5.A.

By definition of the gluing functor and the fact that sheafifications always commute with pullbacks and Lemma C.5.B, we have equivalences

$$\begin{aligned}
(\alpha^*_{\text{rét}_{\text{small}}} u^*_{\text{rét}} \theta'\mathcal{F})(1_\alpha) &= (\alpha^*_{\text{rét}_{\text{small}}} u^*_{\text{rét}} L_{\text{rét}} i_{\text{ét}}\mathcal{F})(1_\alpha) \\
&\simeq (\alpha^*_{\text{rét}_{\text{small}}} L_{\text{rét}_{\text{small}}} u^*_{\text{pre}} i_{\text{ét}}\mathcal{F})(1_\alpha) \\
&\simeq (L_{\text{rét}_{\text{small}}} \alpha^*_{\text{pre}} u^*_{\text{pre}} i_{\text{ét}}\mathcal{F})(1_\alpha).
\end{aligned}$$

But now, we note that there are no nontrivial real étale covers of a real closed field, so since $\mathcal{F}$ was an étale sheaf and hence, in particular, transforms finite coproducts to finite products, the real étale sheafification of $\alpha^*_{\text{pre}} u^*_{\text{pre}} i_{\text{ét}}\mathcal{F}$ does nothing to 1_α:

$$(L_{\text{rét}_{\text{small}}} \alpha^*_{\text{pre}} u^*_{\text{pre}} i_{\text{ét}}\mathcal{F})(1_\alpha) \simeq (\alpha^*_{\text{pre}} u^*_{\text{pre}} i_{\text{ét}}\mathcal{F})(1_\alpha).$$

[40] The strict real henselization may not exist as an object in Sch'_X although it is an object in $\mathrm{Pro}(\mathrm{Sch}'_X)$, hence the stalk is defined via left Kan extension along the functor $(\mathrm{Sch}'X)^{\mathrm{op}} \to \mathrm{Pro}(\mathrm{Sch}'_X)^{\mathrm{op}}$.

To finish the proof, it suffices to check that the canonical map, induced by sheafification,

$$(\alpha^*_{\mathrm{pre}} u^*_{\mathrm{pre}} i_{\text{ét}} \mathcal{F})(1_\alpha) \simeq (\alpha^*_{\mathrm{pre}} i_{\text{ét}_{\mathrm{small}}} u^*_{\text{ét}} \mathcal{F})(1_\alpha) \to (\alpha^*_{\text{ét}} u^*_{\text{ét}} \mathcal{F})(1_\alpha),$$

is an equivalence. This follows as in the proof of [Sch94, Lemma 3.5.1]; for completeness, we now recall the argument. Let $\mathrm{Neib}_{\text{ét}}(Y,\alpha)$ be the filtered category whose objects are factorizations $\operatorname{Spec} k \to U \to Y$ of α with U étale over Y, and let $Y^\alpha = \lim_{\mathrm{Neib}_{\text{ét}}(Y,\alpha)} U$, so that Y^α is affine and its ring R of global sections is the strict real henselization of Y at α. Let k' be the residue field of the local ring R, so k' is a real closed field (but we need not have that the inclusion $k' \subset k$ is an equality; cf. Example B.2). We then have a factorization of α as

$$\begin{array}{ccccc} \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k' & \longrightarrow & Y^\alpha \\ & {\scriptstyle\alpha}\searrow & \downarrow{\scriptstyle\alpha''} & \swarrow{\scriptstyle\alpha'} & \\ & & Y & & \end{array}$$

The claim now follows from the chain of equivalences:

$$\begin{aligned} (\alpha^*_{\mathrm{pre}} i_{\text{ét}_{\mathrm{small}}} u^*_{\text{ét}} \mathcal{F})(1_\alpha) &\simeq \operatorname*{colim}_{U \in \mathrm{Neib}_{\text{ét}}(Y,\alpha)} u^*_{\text{ét}} \mathcal{F}(U) \\ &\simeq \alpha'^*_{\text{ét}} u^*_{\text{ét}} \mathcal{F}(R) \\ &\simeq \alpha''^*_{\text{ét}} u^*_{\text{ét}} \mathcal{F}(k') \\ &\simeq \alpha^*_{\text{ét}} u^*_{\text{ét}} \mathcal{F}(k) = \alpha^*_{\text{ét}} u^*_{\text{ét}} \mathcal{F}(1_\alpha). \end{aligned}$$

Here, the first equivalence follows by the left Kan extension formula for α^*_{pre}. The second equivalence follows from the analogue of Lemma B.9 for the small étale site. The third equivalence follows from the fact that the category of finite étale morphisms $V \to Y^\alpha = \operatorname{Spec} R$ is equivalent to the category of finite étale morphisms $V' \to \operatorname{Spec} k'$ ([TS15, Tag 04GK]). The fourth equivalence follows from the fact that the morphism $\operatorname{Spec} k \to \operatorname{Spec} k'$ induces an equivalence $\widetilde{k'}_{\text{ét}} \xrightarrow{\simeq} \widetilde{k}_{\text{ét}}$.

□

C Big versus small sites

In this appendix, we record some facts about big and small sites which are surely well-known to experts. First, let us fix some notation that we have already used above. Suppose that $\mathcal{E}$ is a presentable ∞-category and $f : \mathrm{C} \to \mathrm{D}$ is a functor of small ∞-categories. Then we have adjunctions

$$\mathrm{PShv}(\mathrm{C}, \mathcal{E}) \;\; \begin{array}{c} \xrightarrow{f_!^{\mathrm{pre}}} \\ \xleftarrow{f^*_{\mathrm{pre}}} \\ \xrightarrow[f_*^{\mathrm{pre}}]{} \end{array} \;\; \mathrm{PShv}(\mathrm{D}, \mathcal{E}).$$

Warning C.1. This convention differs from the one adopted in [Lur], but agrees with the conventions in [Lur17b, Definition 20.6.1].

We have already recalled the definition of a morphism of sites in §B.1. The next notion is usually called "cocontinuous", but we adopt the terminology in [Kha] to avoid conflict with the notion of a functor that preserves colimits.

Definition C.2. Suppose that $(C, \tau), (\mathrm{D}, \tau')$ are sites. A functor $f : \mathrm{C} \to \mathrm{D}$ is said to be **topologically cocontinuous (with respect to τ and τ')** if for each $Y \in \mathrm{C}$ and τ'-covering sieve $R' \hookrightarrow h_{f(Y)}$, the sieve $f^*R \times_{f^* h_{f(Y)}} h_Y \hookrightarrow h_Y$ is a τ-covering sieve.

Unpacking the above definition, this is equivalent to the following condition:

($*$) For each τ'-covering sieve $R' \hookrightarrow h_{f(Y)}$, the sieve of h_Y generated by morphisms $Y' \to Y$ such that $h_{f(Y')} \to h_{f(Y)}$ factors through R' is a τ-covering sieve.

Lemma C.3. *Suppose that $(C,\tau),(\mathrm{D},\tau')$ are sites and $f:\mathrm{C}\to\mathrm{D}$ is a topologically cocontinuous functor. Let $\mathcal{E}$ be a presentable ∞-category. Then $f^*_{\mathrm{pre}}:\mathrm{PShv}(\mathrm{D},\mathcal{E})\to\mathrm{PShv}(\mathrm{C},\mathcal{E})$ sends τ'-local equivalences to τ-local equivalences. Therefore, there exists a functor $f^*:\mathrm{Shv}_\tau(\mathrm{D},\mathcal{E})\to\mathrm{Shv}_\tau(\mathrm{C},\mathcal{E})$ rendering the following diagram commutative*

$$\begin{array}{ccc} \mathrm{Shv}_{\tau'}(\mathrm{D},\mathcal{E}) & \xrightarrow{f^*} & \mathrm{Shv}_\tau(\mathrm{C},\mathcal{E}) \\ {\scriptstyle L_{\tau'}}\uparrow & & \uparrow{\scriptstyle L_\tau} \\ \mathrm{PShv}(\mathrm{D},\mathcal{E}) & \xrightarrow{f^*_{\mathrm{pre}}} & \mathrm{PShv}(\mathrm{C},\mathcal{E}). \end{array}$$

Proof. In case $\mathcal{E}=\mathrm{Spc}$, the first statement is [GH19, Lemma 2.23], and the proof for a general $\mathcal{E}$ is the same. The conclusion follows since the ∞-category of sheaves is obtained by inverting the local equivalences. □

We now specialize to the comparison between small and big sites. Let X be a scheme and suppose that we have $\mathrm{Sch}'_X\subset\mathrm{Sch}_X$ a full subcategory of X-schemes which is stable under pullbacks and contains étale X-schemes; the most prominent example will be $\mathrm{Sch}'_X=\mathrm{Sm}_X$. Given a Grothendieck topology τ, we also have the following condition:

Small For each τ-covering sieve $R\hookrightarrow h_Y$, there exists a collection of étale morphisms $\{U_\alpha\to Y\}$ which belongs to R.

Remark C.4. This condition is inspired by the theory of **geometric sites** in the sense of [Lur17b, §20.6], tailormade for our purposes. Examples of Grothendieck sites satisfying Small include the étale, real étale, and Nisnevich sites on Sm_X.

Consider the inclusion $u:\mathrm{\acute{E}t}_X\subset\mathrm{Sch}'_X$.

Lemma C.5. *In the situation above, suppose that τ is a Grothendieck topology on Sch'_X. There exists a Grothendieck topology τ_{small} on $\mathrm{\acute{E}t}_X$ satisfying the condition* Small *such that*

1. *The inclusion $u:\mathrm{\acute{E}t}_X\subset\mathrm{Sch}'_X$ determines a morphism of sites*
$$(\mathrm{\acute{E}t}_X,\tau_{\mathrm{small}})\to(\mathrm{Sch}'_X,\tau).$$
2. *τ_{small} is the coarsest topology on $\mathrm{\acute{E}t}_X$ for which the inclusion functor u induces a morphism of sites.*
3. *If τ further satisfies the condition* Small*, then the functor $\mathrm{\acute{E}t}_X\subset\mathrm{Sm}_X$ is topologically cocontinuous.*

Furthermore, let $\mathcal{E}$ be a presentable ∞-category and denote by $\iota_\tau:\mathrm{Shv}_\tau(\mathrm{Sch}'_X,\mathcal{E})\subset\mathrm{PShv}(\mathrm{Sch}'_X,\mathcal{E})$ and $\iota_{\tau_{\mathrm{small}}}:\mathrm{Shv}_{\tau_{\mathrm{small}}}(\mathrm{\acute{E}t}_X,\mathcal{E})\subset\mathrm{PShv}(\mathrm{\acute{E}t}_X,\mathcal{E})$ the obvious inclusions. Then:

(A) *We have an adjunction*
$$u_!:\mathrm{Shv}_{\tau_{\mathrm{small}}}(\mathrm{\acute{E}t}_X,\mathcal{E})\rightleftarrows\mathrm{Shv}_\tau(\mathrm{Sch}'_X,\mathcal{E}):u^*$$
satisfying:
$$u^*_{\mathrm{pre}}\iota_\tau\simeq\iota_{\tau_{\mathrm{small}}}u^*,$$
and
$$L_\tau u^{\mathrm{pre}}_!\simeq u_!L_{\tau_{\mathrm{small}}}.$$

(B) *We have an adjunction*
$$u^*:\mathrm{Shv}_\tau(\mathrm{Sch}'_X,\mathcal{E})\rightleftarrows\mathrm{Shv}_{\tau_{\mathrm{small}}}(\mathrm{\acute{E}t}_X,\mathcal{E}):u_*$$
satisfying:
$$L_{\tau_{\mathrm{small}}}u^*_{\mathrm{pre}}\simeq u^*L_\tau,$$
and
$$u^{\mathrm{pre}}_*\iota_{\tau_{\mathrm{small}}}\simeq\iota_\tau u_*.$$

(C) *The functors u_* and $u_!$ are fully faithful.*

Proof. The Grothendieck topology τ_{small} is the one given by [Lur17b, Proposition 20.6.1.1]. Explictly, for each étale morphism $Y \to X$, a sieve $R \hookrightarrow h_Y$ is a τ_{small}-covering sieve if and only if $u_!R \hookrightarrow h_{u(Y)}$ is a τ-covering sieve. (A) then follows from [Lur17b, Proposition 20.6.1.3]. Now, Small ensures that for Ét_X equipped with the Grothendieck topology τ_{small}, the inclusion functor is topologically cocontinuous (which is (3)). Hence, Lemma C.3 gives (B).

We now prove (C). Clearly, $u_!^{\text{pre}}$ (resp. u_*^{pre}) is fully faithful as it is computed as left (resp. right) Kan extension along a fully faithful functor. To see that $u_!$ is fully faithful, it suffices to show that the unit map $\text{id} \to u^*u_!$ is an equivalence. But this follows from the computation:

$$u^*u_! \simeq u^*L_\tau u_!^{\text{pre}}\iota_{\text{small}} \simeq L_{\tau_{\text{small}}}u^*_{\text{pre}}u_!^{\text{pre}}\iota_{\text{small}} \simeq \text{id},$$

where the last equivalence follows from the presheaf-level statement. The argument for u_* is similar. □

Construction C.6. Let X be a scheme. Then we have the following lax commutative square

$$\begin{array}{ccc} \widetilde{X}_{\text{ét}} & \xrightarrow{i_{\text{ét}_{\text{small}}}} & \widetilde{X}_{\text{Nis}} \\ {\scriptstyle u_{\text{ét}!}}\downarrow & \Swarrow & \downarrow{\scriptstyle u_{\text{Nis}!}} \\ \text{Shv}_{\text{ét}}(\text{Sm}_X) & \xrightarrow{i_{\text{ét}}} & \text{Shv}_{\text{Nis}}(\text{Sm}_X), \end{array}$$

obtained as

$$u_{\text{Nis}!}i_{\text{ét}_{\text{small}}} \Rightarrow u_{\text{Nis}!}i_{\text{ét}_{\text{small}}}u^*_{\text{ét}}u_{!\text{ét}} \simeq u_{\text{Nis}!}u^*_{\text{Nis}}i_{\text{ét}}u_{!\text{ét}} \Rightarrow i_{\text{ét}}u_{!\text{ét}},$$

where the middle equivalence is furnished by Lemma C.5. Composing with equivalence $L_{\text{rét}}u_!^{\text{Nis}} \simeq u_{\text{Nis}!}L_{\text{rét}_{\text{small}}}$ from the same lemma then gives us a lax commutative square

$$\begin{array}{ccc} \widetilde{X}_{\text{ét}} & \xrightarrow{L_{\text{rét}}i_{\text{ét}_{\text{small}}}} & \widetilde{X}_{\text{rét}} \\ {\scriptstyle u_{\text{ét}!}}\downarrow & \Swarrow & \downarrow{\scriptstyle u_{\text{rét}!}} \\ \text{Shv}_{\text{ét}}(\text{Sm}_X) & \xrightarrow{L_{\text{rét}}i_{\text{ét}}} & \text{Shv}_{\text{rét}}(\text{Sm}_X), \end{array}$$

comparing big versus small sites.

Remark C.7. Since u is a morphism of sites, the functor $u_{\tau!} : \text{Shv}_{\tau_{\text{small}}}(\text{Ét}_X) \to \text{Shv}_\tau(\text{Sch}'_X)$ is the left-exact left adjoint of a geometric morphism of ∞-topoi. Therefore, its stabilization is strong symmetric monoidal with respect to the induced smash product symmetric monoidal structure on the stabilization of an ∞-topos.

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