Classifying spaces & coniveau filtrations

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§ Recap:

X smooth complex variety, $\alpha \in H^l(X;\mathbb{Z})$ $(:= H^l(X(\mathbb{C});\mathbb{Z}))$

- α coniveau $\geq c$: $\exists\, Z \subseteq X$ closed subv. of cod. $\geq c$, $\alpha|_{X \setminus Z} = 0$
- α strong coniveau $\geq c$: $\exists\, Y \xrightarrow{f} X$ proper map, Y smooth, $\alpha \in \operatorname{Im} f_*$, $\dim Y \leq \dim X - c$
- $N^c H^l(X;\mathbb{Z})$ = classes of coniveau $\geq c$ $\qquad \tilde{N}^c \subseteq N^c$
- $\tilde{N}^c H^l(X;\mathbb{Z})$ = classes of strong coniveau $\geq c$

Prop: $\dim X = n$, if $l \leq 2c$ or $n \leq l - c$: $\tilde{N}^c H^l = N^c H^l$

Prop: $\alpha \in H^l(X;\mathbb{Z})$ torsion, then it has coniveau ≥ 1.

Prop: $\alpha \in H^l(X;\mathbb{Z})$, $j \geq 0$ s.t. $l \leq 2c + j$, $S_j \bar{\alpha} \neq 0 \Rightarrow \alpha$ has strong coniveau $< c$.

$\left(S_j = Sq^{2^j - 1} Sq^{2^{j-1} - 1} \cdots Sq^3\right)$

Lemma (Serre): $\forall m \geq 1$ $\exists\, Y$ smooth proj variety $/\mathbb{C}$ w/ torsion canonical bundle & a $(m-1)$-connected map $Y \to BC_2 \times \mathbb{CP}^\infty$

Proof: $\mathbb{P}^{2m+1} \circlearrowleft C_2$ via the involution

$$[x_0 : \cdots x_{2m+1}] \longmapsto [x_0 : \cdots : x_m : -x_{m+1} : \cdots : -x_{2m+1}]$$

its fixed points are the union of two m-planes: it is $\boxed{m}$-dimensional

Let Z be a generic intersection of $(m+1)$ invariant quadric hypersurfaces

$\Rightarrow Z$ has trivial canonical bundle, $Z \cap \{\text{fixed pt locus}\} = \emptyset$

$\Rightarrow Y = Z/C_2$ smooth projective variety, Y has torsion canonical bundle

Step 1: $\boxed{Z \hookrightarrow \mathbb{P}^{2m+1}}$ $(m-1)$-connected

$$Z \hookrightarrow \mathbb{P}^{2m+1} \hookrightarrow \mathbb{P}^{\binom{2m+1}{2}} \quad \text{(Veronese embedding)}$$

Z is obtained by $\cap$ $\mathbb{P}^{2m+1}$ by hyperplanes in $\mathbb{P}^{\binom{2m+1}{2}}$, up to small deformations we can ensure that these hyperplanes are all transverse $\Rightarrow$ we can apply the Lefschetz hyperplane thm

Step: Pass to the quotient

$$\begin{matrix} Z & \hookrightarrow & \mathbb{P}^{2m+1} & \hookrightarrow & \mathbb{P}^\infty \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ C_2 & & C_2 & & C_2 \end{matrix} \quad (m-1)\text{-connected}$$

Claim: $C_2 \circlearrowright \mathbb{P}^\infty$ is homotopically trivial $\Rightarrow (\mathbb{P}^\infty)_{hC_2} = \mathbb{P}^\infty \times BC_2$ $\qquad \mathbb{CP}^n = \overbrace{S^{2n+1}}^{\text{unit sphere in } \mathbb{C}^{n+1}}/S^1$

$Y = Z/C_2 = Z_{hC_2} \longrightarrow \mathbb{P}^\infty_{hC_2} = \mathbb{P}^\infty \times BC_2$ is $(m-1)$-connected

In fact $C_2 \circlearrowright \mathbb{P}^\infty$ lifts to an action $C_2 \circlearrowright S^\infty$ (contractible) $\qquad (\mathbb{CP}^\infty = \overbrace{S^\infty}^{\text{unit vectors in } \mathbb{C}^\infty}/S^1)$. □

Lemma: $\forall s \geq 1$ $\forall j$ $1 \leq j \leq s$ $\exists\, \zeta \in H^s(BC_2^s;\mathbb{Z}/2)$ w/ $S_j Sq^1 \zeta \neq 0$.

Proof: $H^*(BC_2;\mathbb{Z}/2) = \mathbb{Z}/2[x]$ $\quad |x| = 1$

$H^*(BC_2^s;\mathbb{Z}/2) \simeq \mathbb{Z}/2[x_1, \ldots, x_s]$ $\quad |x_i| = 1$. $\qquad B(C_2^s) = (BC_2)^s$

$\zeta = x_1 \cdots x_s$

Recall: $Sq^j x^k = \binom{k}{j} x^{j+k}$ in $H^*(BC_2;\mathbb{Z}/2)$

$\Rightarrow$ By induction $S_j Sq^1(x_1 \cdots x_s)$ has leading term in the lexicographic order given by $x_1^{2^j} x_2^{2^{j-1}} \cdots x_j^2 x_{j-1} \cdots x_n$. □

Thm $\forall c \geq 1$, $l \geq 2c+1$ $\exists$ smooth proj variety $X/\mathbb{C}$ w/ torsion can. bundle s.t. $\tilde{N}^c H^l \neq N^c H^l$.

Proof: $m = 2^{j+1} - j - 1$, let Y as in the first lemma

$Y \to BC_2 \times \mathbb{CP}^\infty$ $(m-1)$-connected. $\Rightarrow Y \to BC_2$ s.t. $H^* BC_2 \to H^* Y$ is injective for $* \leq m-1$. Therefore we can pullback ζ from $H^*(BC_2;\mathbb{Z}/2)$ to a class $\zeta \in H^*(Y;\mathbb{Z}/2)$ s.t. $S_j Sq^1 \zeta \neq 0$.

Let now T be any smooth proj variety of dim $c-1$ w/ torsion can. bundle.

$$\begin{tikzcd} & X = Y^m \times T \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Y^m & & T \end{tikzcd}$$

$\lambda \in H^{2c-2}(T;\mathbb{Z})$ be the orientation class.

$$\alpha := p^* \beta_{\mathbb{Z}} \zeta \smile q^* \lambda \quad (= \beta_{\mathbb{Z}} \zeta \otimes \lambda)$$

Ex: $|\alpha| = l$

moreover α is, by definition, the pushforward of the 2-torsion class $\beta_{\mathbb{Z}} \zeta \in H^*(Y^m;\mathbb{Z})$ along the inclusion $Y^m \times \{t\} \hookrightarrow Y^m \times T = X$ $\quad$ ($t \in T$ any)

$\beta_{\mathbb{Z}}$ bound hom of $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ $\Rightarrow \beta_{\mathbb{Z}} \zeta$ has coniveau ≥ 1 in Y^m

$\Rightarrow \alpha$ push of $\beta_{\mathbb{Z}} \zeta$ has coniveau $\geq c$. $\quad$ (i.e. $\alpha \in N^c H^l$)

If we show $S_{l-2c+1} \bar{\alpha} \neq 0$ we win (i.e. $\alpha \notin \tilde{N}^c H^l$)

$\alpha = p^* \beta_{\mathbb{Z}} \zeta \smile q^* \lambda \Rightarrow \bar{\alpha} = p^* Sq^1 \zeta \smile q^* \bar{\lambda}$

Note $Sq^i \bar{\lambda} = 0$ $\forall i > 0$ $\Rightarrow$ By the Cartan formula $S_j \bar{\alpha} = p^* \overset{\neq 0}{S_j Sq^1 \zeta} \smile q^* \bar{\lambda}$, $j = l - 2c + 1$ $\Rightarrow$

$S_{l-2c+1} \bar{\alpha} \neq 0$. □

Prop: If $c \geq 2$, $l \geq 2c+1$ X can be chosen to be rational

Proof: W smooth proj variety $\sigma \in N^{c-1} H^{l-2}(W;\mathbb{Z})$ s.t. $S_{l-2c+1} \bar{\sigma} \neq 0$.

$n = \dim W$. Pick a generic map $W \to \mathbb{P}^{n+2}$ (W birational to the image)

Embedded res. of singularities $\rightsquigarrow \tilde{W} \subseteq Y$ $\qquad Y \to \mathbb{P}^{n+2}$ birational, $\tilde{W} \to W$ birational

$X = Bl_{\tilde{W}} Y$, E exceptional divisor $\xrightarrow{\pi} \tilde{W} \to W$ (proj bundle)

$i : E \hookrightarrow X$ $\qquad \alpha = i_* \pi^* \sigma$

Claim α coniveau $\geq c$, $S_{l-2c+1} \bar{\alpha} \neq 0$.

$$S_{l-2c+1} = i_* \underbrace{\pi^* S_{l-2c+1} \sigma}_{\neq 0}$$

Claim: $\pi^* \sigma$ coniv $\geq c-1$ in E because σ is in coniveau $\geq c-1$ in W

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[d, "\mathbb{P}^1"] \arrow[dd, bend right, "\pi"'] & X \arrow[d, "bir"] \\ \tilde{W} \arrow[r, hook] \arrow[d] & Y \arrow[d, "bir"] \\ W \arrow[r] & \mathbb{P}^{n+2} \end{tikzcd}$$

. □

Torsion-free classes: IDEA replace C_2^s w/ G_2 reductive alg. group

$$H^*(BG_2;\mathbb{Z}) \simeq \mathbb{Z}[y_4, y_{12}] \oplus \mathbb{Z}/2[y_6, y_{10}] \qquad |y_i| = i$$

$Sq^3 \bar{y}_4 = Sq^1 \bar{y}_6 \neq 0$ $\qquad$ (Alekhandrut-Kalafat, Borel)

Thm (Ekedahl): H red. group $/\mathbb{C}$ w/ positive dim center, $\forall k$ find X smooth projective an alg map $X \to BH$ which is iso in cohomology up to degree k.

Thm (Edidin-Graham): G affine alg group, the Hodge conj holds for BG.

Prop: $\exists\, X$ smooth proj variety s.t. $H^4(X;\mathbb{Z})$ torsion free, $\tilde{N}^1 H^4 \neq N^1 H^4$.

Proof: $H = G_2 \times \mathbb{G}_m$, $k = 8$ apply Ekedahl's thm to get X, α pullback of $y_4 \in H^*(BG_2)$.

$S_1 \bar{\alpha} \neq 0 \Rightarrow \alpha$ has strong coniveau 0.

$\Rightarrow$ by the Hodge conj some multiple of α is algebraic $\rightsquigarrow$ some multiple of α has coniveau 1

$\Rightarrow$ on some U open dense α is torsion $\Rightarrow \alpha$ has coniveau 1. □

$$CH^2_{\mathbb{Q}} \xrightarrow{\simeq} \underset{\alpha}{\overset{}{H^4_{\mathbb{Q}}}} \ni \alpha \qquad \zeta \in CH^2_{\mathbb{Q}} = CH^2 \otimes \mathbb{Q} \quad \frac{\tilde{\zeta}}{n} \quad \tilde{\zeta} \in CH^2$$

$$\tilde{\zeta} \mapsto n\alpha$$