Hermitian Talk 1: Mind your $\mathcal{C}$'s and Ϙ's

Kaif Hilman

University of Copenhagen

September 23, 2021

Contents

Abstract

In this note we introduce the basic structure in hermitian K-theory, namely that of (spectral) quadratic functors. These can in turn be decomposed into their bilinear and linear parts, and the high point of the note is a stable recollement cleanly packaging these decompositions. Lastly, we will introduce the fundamental notion of a Poincare category which will be the fuel in all that is to follow in the modern treatment of hermitian K-theory.

1 Quadratics, bilinears, linears

Recollections 1.1. Let $\mathcal{C}, \mathcal{D}$ be stable categories. Let $\mathcal{P}(S)$ denote the power poset of a set S. An n-cube in $\mathcal{C}$ is just a diagram $X : \mathcal{P}([n]) \to \mathcal{C}$. We then say that:

- An n-cube $X : \mathcal{P}([n]) \to \mathcal{C}$ is *strongly cocartesian* if it is a left Kan extension of $X|_{\mathcal{P}_{\leq 1}([n])}$. In other words, all its 2-faces are cocartesian.
- An n-cube $Y : \mathcal{P}([n]) \to \mathcal{D}$ is *cartesian* if it is a right Kan extension of its restriction $\mathcal{P}([n])\backslash\{\varnothing\} \to \mathcal{D}$.
- A (not necessarily exact) functor $F : \mathcal{C} \to \mathcal{D}$ is *n-excisive* if it sends all strongly cocartesian n-cubes to cartesian n-cubes.
- A (not necessarily exact) functor $F : \mathcal{C} \to \mathcal{D}$ is *reduced* if it preserves zero objects.

Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ between stables is exact iff it is reduced 1-excisive. Moreover, recall that if $m \geq n$, then an n-excisive functor is always m-excisive, and so in particular, 1-excisive functors are automatically 2-excisive.

Definition 1.2 (I.1.1.11)**.** Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be stable categories.

- A functor $B : \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ is *bireduced* if $B(x, y) \simeq 0$ when $x \simeq 0$ or $y \simeq 0$.
- A functor $b : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is *bilinear* if it is exact in each variable.

Definition 1.3 (I.1.1.14)**.** Let $\mathcal{C}$ be stable and $\Omega : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ a functor. We say that Ω is:

- *quadratic* if it is reduced and 2-excisive,
- *linear* if it is reduced and 1-excisive (in other words, exact).

We'll need to establish a bunch of notations for the kinds of functor categories of interest.

Notation 1.4. Let $\mathcal{C}$ be stable. Then we denote by

- $\mathrm{Fun}_*(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ the category of reduced functors. Note that this has a canonical C_2-action given by swapping the two copies of $\mathcal{C}^{\mathrm{op}}$,
- $\mathrm{BiFun}(\mathcal{C}) \subseteq \mathrm{Fun}_*(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ the full subcategory of bireduced functors,
- $\mathrm{Fun}^b(\mathcal{C}) \subseteq \mathrm{BiFun}(\mathcal{C})$ the full subcategory of bilinear functors. This clearly inherits the C_2-action,
- $\mathrm{Fun}^s(\mathcal{C}) := \mathrm{Fun}^b(\mathcal{C})^{hC_2}$ which we call the category of *symmetric bilinear functors*,
- $\mathrm{Fun}^q(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ the full subcategory of quadratic functors.

Note in particular that $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ is a full subcategory of $\mathrm{Fun}^q(\mathcal{C})$.

Remark 1.5 (Quadratic (co)completeness)**.** Note that since being cartesian in the stable category Sp is preserved under arbitrary (co)limits, the full subcategory $\mathrm{Fun}^q(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ is closed under arbitrary (co)limits, and so is in particular stable also.

One of the first basic insights of this story is that much of the Goodwillie calculus framework can be simplified in the presence of stability and 2-excision. For example, there are many adjunctions between the categories we've introduced, culminating in a stable recollement fo quadratic functors. This will be the high point of this talk, and for that we will need many constructions.

Construction 1.6 (Bireduction, I.1.1.3)**.** Note that we have a retraction

$$B(x, 0) \oplus B(0, y) \to B(x, y) \to B(x, 0) \oplus B(0, y)$$

so taking the cofibre of the first map (or equivalently fibre of the second map since it's a retraction) gives a bireduced form which we denote by $B(-,-)^{\mathrm{red}}$. Note that this commutes with restriction along pairs of reduced functors. This also commutes with the flip functor and so the bireduction refines to a functor

$$(-)^{\mathrm{red}} : \mathrm{Fun}_*(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathrm{Sp})^{hC_2} \to \mathrm{BiFun}(\mathcal{C})^{hC_2}$$

Note that since we're in the stable situation, taking bireduction clearly commutes with arbitrary (co)limits: in fact, it participates in a biadjunction as we'll state in the next section.

Construction 1.7 (Cross-effects, I.1.1.6)**.** Let $\mathcal{C}$ be stable and $\Omega : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ be a reduced functor. We define the *cross-effect* (or *polarisation*) to be

$$B_{\Omega} := \Omega(- \oplus -)^{\mathrm{red}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$$

yielding a functor

$$B_{(-)} : \mathrm{Fun}_*(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{BiFun}(\mathcal{C})$$

This functor commutes with restrictions along direct sum preserving reduced functors, ie. for $f : \mathcal{C} \to \mathcal{D}$, we have $(f \times f)^* B^{\mathcal{D}}_{\Qoppa} \simeq B^{\mathcal{C}}_{f^*\Qoppa}$. We can also define

$$B^{\Delta}_{(-)} := \Delta^* \circ B_{(-)} : \mathrm{Fun}_*(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{BiFun}(\mathcal{C}) \to \mathrm{Fun}_*(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$$

Recall that B^{red} is a retract of B, and so we obtain the following canonical natural transformations

$$\left(B^{\Delta}_{\Qoppa} \Rightarrow \Qoppa\right) = \Big(X \mapsto \left(\Qoppa(X \oplus X)^{\mathrm{red}} \to \Qoppa(X \oplus X) \xrightarrow{\Delta^*} \Qoppa(X)\right)\Big)$$

$$\left(\Qoppa \Rightarrow B^{\Delta}_{\Qoppa}\right) = \Big(X \mapsto \left(\Qoppa(X) \xrightarrow{\nabla^*} \Qoppa(X \oplus X) \to \Qoppa(X \oplus X)^{\mathrm{red}}\right)\Big)$$

Lemma 1.8 (C_2-equivariance, I.1.1.9, I.1.1.10)**.** *We collect all the available C_2-equivariance here.*

1. *The cross effect $B_{\Qoppa}$ is symmetric, ie. it's in the image of the forgetful functor $Fun_*(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, Sp)^{hC_2} \to Fun_*(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, Sp)$.*
2. *The functor $\Delta^* : Fun_*(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, Sp) \to Fun_*(\mathcal{C}^{\mathrm{op}}, Sp)$ is C_2-equivariant.*
3. *And so the natural transformations constructed above descend to*

$$(B^{\Delta}_{\Qoppa})_{hC_2} \Rightarrow \Qoppa \Rightarrow (B^{\Delta}_{\Qoppa})^{hC_2}$$

Proof. Since the bi-reduction functor was C_2-equivariant, we just have to show that $\Qoppa \circ \bigoplus \in \mathrm{Fun}_*(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ lives in the image from $\mathrm{Fun}_*(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathrm{Sp})^{hC_2}$. And for this it suffices to note that $\bigoplus : \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ inherits a C_2-equivariant structure from the cartesian symmetric monoidal structure on $\mathcal{C}^{\mathrm{op}}$. ☐

Observation 1.9. For a bilinear $B : \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$, the functor $B^{\Delta}(x) = B(x, x)$ is a quadratic functor by HA.6.1.3.5. Unwinding the bireduction process, the symmetric bilinear part is $(x, y) \mapsto B(x, y) \oplus B(y, x)$.

Construction 1.10 (Associated quadratics)**.** For $B \in \mathrm{Fun}^s(\mathcal{C})$ we can define

$$\Qoppa^q_B(x) := B^{\Delta}_{hC_2}(x) = B(x,x)_{hC_2} \qquad \Qoppa^s_B(x) := (B^{\Delta})^{hC_2}(x) = B(x,x)^{hC_2}$$

and these are both quadratic: this is just because $B(x, x)$ is quadratic by the previous example and $\mathrm{Fun}^q(\mathcal{C})$ is closed under limits and colimits in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ by (1.5). Note that we have

$$B_{\Qoppa^q_B} \simeq B \simeq B_{\Qoppa^s_B}$$

since taking cross-effects commute with limits and colimits and so by the previous observation we get

$$B_{\Qoppa^q_B}(x,y) \simeq (B(x,y) \oplus B(y,x))_{hC_2} \simeq B(x,y) \simeq (B(x,y) \oplus B(y,x))^{hC_2} \simeq B_{\Qoppa^s_B}(x,y)$$

Definition 1.11 ((Symmetric) bilinear parts)**.** By (1.8) and (2.4) we see that $B_{(-)} : \mathrm{Fun}^q(\mathcal{C}) \to \mathrm{Fun}_*(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ lifts to $B_{(-)} : \mathrm{Fun}^q(\mathcal{C}) \to \mathrm{Fun}^b(\mathcal{C})$. We call $B_{\Qoppa}$ the *symmetric bilinear part* of $\Qoppa$, and the underlying bilinear functor $B_{\Qoppa}$ as the *bilinear part* of $\Qoppa$.

Construction 1.12 ((Co)linear parts and (co)homogeneity, I.1.1.22, I.1.3.1)**.** Let $\Qoppa : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ be a quadratic functor. We define the linear (resp. colinear) part to be the cofibre (resp. fibre)

$$(B^{\Delta}_{\Qoppa})_{hC_2} \Rightarrow \Qoppa \Rightarrow L_{\Qoppa} \quad \text{or} \quad cL_{\Qoppa} \Rightarrow \Qoppa \Rightarrow (B^{\Delta}_{\Qoppa})^{hC_2}$$

Note that by (1.7) this process commutes with restrictions along exact functors. These will be shown to be exact in (2.4), hence justifying the names. If $L_{\Qoppa} \simeq *$ (resp. $cL_{\Qoppa} \simeq *$) then we say that $\Qoppa$ is homogeneous (resp. cohomogeneous).

2 Recognition criteria

Lemma 2.1 (Total fibre yoga)**.** *Suppose we have a square*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

in a stable category $\mathcal{C}$*. Then there is a canonical identification of* $\mathrm{fib}(A \to B \times_D C)$ *with the total fibre of the square.*

Proof. Write $F := \mathrm{fib}(A \to B)$ and $G := \mathrm{fib}(C \to D)$. Recall that by definition the total fibre is given by $X := \mathrm{fib}(F \to G)$. On the other hand, note that $\mathrm{fib}(B \times_D C \to B \times_D D) \simeq G$ since

$$(B \times_D C) \times_{B \times_D D} * \simeq (B \times_D C) \times_{B \times_D D} (* \times_* *) \simeq (B \times_B *) \times_{D \times_D *} (C \times_D *) \simeq G$$

Now just consider the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & G \arrow[d] \\
\mathrm{fib}(A \to B \times_D C) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B \times_D C \arrow[d] \\
* \arrow[r] & B \arrow[r, equal] & B \times_D D
\end{tikzcd}$$

and we're done. □

The following result is quite important: it shows that while quadratic functors are not in general 1-excisive, their failure to be so is totally controlled by its associated symmetric bilinear part.

Lemma 2.2 (Quadratic failure of exactness, I.1.1.19)**.** *Let* $Ϙ : \mathcal{C}^{\mathrm{op}} \to Sp$ *be a quadratic functor with bilinear part* B*. Let*

$$\begin{tikzcd}
x \arrow[r, "a'"] \arrow[d, "b'"'] & y \arrow[d, "b"] \\
z \arrow[r, "a"] & w
\end{tikzcd}$$

be an exact square in $\mathcal{C}$*. Then the two squares in the diagram*

$$\begin{tikzcd}
Ϙ(w) \arrow[r] \arrow[d] & B(z,y) \arrow[d] & B(\mathrm{cofib}(b'), \mathrm{cofib}(a')) \arrow[l] \arrow[d] \\
Ϙ(z) \times_{Ϙ(x)} Ϙ(y) \arrow[r] & B(z,x) \times_{B(x,x)} B(x,y) & 0 \arrow[l]
\end{tikzcd} \tag{1}$$

are exact. In particular there is a natural equivalence

$$\mathrm{fib}\left(Ϙ(w) \to Ϙ(z) \times_{Ϙ(x)} Ϙ(y)\right) \simeq B(\mathrm{cofib}(b'), \mathrm{cofib}(a'))$$

We similarly have dual statements for pushouts.

Proof. We consider the maps of squares

$$\begin{tikzcd}
\Qoppa(w) \arrow[r] \arrow[d] & \Qoppa(y) \arrow[d] & & \Qoppa(z\oplus y) \arrow[r] \arrow[d] & \Qoppa(x\oplus y) \arrow[d] & & B(z,y) \arrow[r] \arrow[d] & B(x,y) \arrow[d] \\
\Qoppa(z) \arrow[r] & \Qoppa(x) \arrow[r, phantom, "\Rightarrow" very near end, shift left=4ex] & & \Qoppa(z\oplus x) \arrow[r] & \Qoppa(x\oplus x) \arrow[r, phantom, "\Rightarrow" very near end, shift left=4ex] & & B(z,x) \arrow[r] & B(x,x)
\end{tikzcd} \tag{2}$$

where the first map is induced by the strongly cocartesian cube

$$\begin{tikzcd}
& x\oplus x \arrow[rr] \arrow[dl, "(\mathrm{id}\quad \mathrm{id})"'] \arrow[dd] & & x\oplus y \arrow[dl, "(a'\quad \mathrm{id})"'] \arrow[dd] \\
x \arrow[rr] \arrow[dd] & & y \arrow[dd] & \\
& z\oplus x \arrow[rr] \arrow[dl, "(\mathrm{id}\quad b')"] & & z\oplus y \arrow[dl, "(a\quad b)"] \\
z \arrow[rr] & & w &
\end{tikzcd}$$

If we can show that each of the two maps induce an equivalence of total fibres then we would have shown that the first square in (1) is cartesian. Since $\Qoppa$ was 2-excisive, $\Qoppa$ sends this cube to a cartesian cube in Sp, that is, the square

$$\begin{tikzcd}
\Qoppa(w) \arrow[r] \arrow[d] & \Qoppa(z) \times_{\Qoppa(x)} \Qoppa(y) \arrow[d] \\
\Qoppa(z\oplus y) \arrow[r] & \Qoppa(z\oplus x) \times_{\Qoppa(x\oplus x)} \Qoppa(x\oplus y)
\end{tikzcd}$$

is cartesian. By considering the horizontal fibres of this square and invoking (2.1) we get that the first map of squares in (2) induces an equivalence of total fibres.

Now for the second map of squares in (2), we know by definition of $B(-,-) = B_{\Qoppa}(-,-)$ that the fibre is

$$\begin{tikzcd}
\Qoppa(z)\oplus\Qoppa(y) \arrow[r] \arrow[d] & \Qoppa(x)\oplus\Qoppa(y) \arrow[d] \\
\Qoppa(z)\oplus\Qoppa(x) \arrow[r] & \Qoppa(x)\oplus\Qoppa(x)
\end{tikzcd}$$

which has trivial total fibre. And hence this map of squares also induces an equivalence on total fibres. This completes the proof that the first square in (1) is exact.

To see that the right hand square in (1) is cartesian, just observe that by design $B(-,-)$ is exact in each variable and so we compute the total fibre of the right most square in (2) to be

$$\begin{tikzcd}
B(\mathrm{cofib}(b'),\mathrm{cofib}(a')) \arrow[d, dashed] & & \\
B(\mathrm{cofib}(b'),y) \arrow[r, dashed] \arrow[d, dashed] & B(z,y) \arrow[r] \arrow[d] & B(x,y) \arrow[d] \\
B(\mathrm{cofib}(b'),x) \arrow[r, dashed] & B(z,x) \arrow[r] & B(x,x)
\end{tikzcd}$$

as required. ☐

Corollary 2.3 (2-out-of-3 formula for $\Qoppa$, I.1.1.21)**.** *Furthermore, applying to the case of $z = 0$ we get that for an exact sequence $x \to y \to w$ in $\mathcal{C}$ the natural map from $\Qoppa(w)$ to the total fibre*

$$\Qoppa(w) \to \mathrm{fib}\left(\mathrm{fib}(\Qoppa(y) \to \Qoppa(x)) \to \mathrm{fib}(B(x,y) \to B(x,x))\right)$$

is an equivalence.

Proof. This is just an easy consequence of (2.2) by setting $z = *$ in the left hand square of (1) which yields

$$\begin{tikzcd}
\Qoppa(w) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B(z,y) \arrow[d] & \simeq & \Qoppa(w) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
\Qoppa(z) \times_{\Qoppa(x)} \Qoppa(y) \arrow[r] & B(z,x) \times_{B(x,x)} B(x,y) & & \mathrm{fib}(\Qoppa(y) \to \Qoppa(x)) \arrow[r] & \mathrm{fib}(B(x,y) \to B(x,x))
\end{tikzcd}$$

as required. ☐

We now come to one of the most important basic results that will be the bread-and-butter of this story.

Proposition 2.4 (Characterisations of quadratics, I.1.1.13)**.** *Let $\Qoppa : \mathcal{C}^{\mathrm{op}} \to Sp$ be a functor. The following are equivalent and functors satisfying such are called quadratic.*

(1) $\Qoppa$ is reduced 2-excisive.

(2) $B_\Qoppa$ is bilinear and $\mathrm{fib}\left(\Qoppa(-) \Rightarrow (B_\Qoppa \circ \Delta)^{hC_2}\right) : \mathcal{C}^{\mathrm{op}} \to Sp$ *is exact.*

(3) $B_\Qoppa$ is bilinear and $\mathrm{cofib}\left((B_\Qoppa \circ \Delta)_{hC_2} \Rightarrow \Qoppa(-)\right) : \mathcal{C}^{\mathrm{op}} \to Sp$ *is exact.*

Proof. Since Sp is stable the property of being reduced and 2-excisive is closed under limits and colimits. Hence (2) and (3) implies (1) since exact functors and diagonal restrictions of bilinears are reduced and 2-excisive by HA.6.1.3.5 (additivity of excisiveness). For the other direction, we invoke HA.6.1.3.22 to say that the cross-effect is bilinear. On the other hand, applying cross-effects and noting that it preserves (co)limits we see that the cross-effect on $\mathrm{fib}\left(\Qoppa(-) \Rightarrow (B_\Qoppa \circ \Delta)^{hC_2}\right)$ and $\mathrm{cofib}\left((B_\Qoppa \circ \Delta)_{hC_2} \Rightarrow \Qoppa(-)\right)$ are trivial, and so by the characterisation (2.2) we see that the fibre and cofibre are reduced 2-excisive with trivial cross-effects, and so exact. ☐

Remark 2.5. In light of this, we should think of 2-excisives as quadratic affine maps and 1-excisives as affine maps. If furthermore reduced then linear. Note that 1-excisive implies 2-excisive.

Proposition 2.6 (Characterisations of (co)homogeneity, I.1.3.1)**.** *Let $\Qoppa : \mathcal{C}^{\mathrm{op}} \to Sp$ be a quadratic functor. Then the following conditions are equivalent for being homogeneous.*

(1) The map $B_\Qoppa(x,x)_{hC_2} \to \Qoppa(x)$ is an equivalence for every $x \in \mathcal{C}$.

(2) $\Qoppa$ is equivalent to a quadratic functor of form $\Qoppa^q_B$ for $B \in Fun^s(\mathcal{C})$

(3) The spectrum $Nat(\Qoppa, L)$ is trivial for any linear L.

Dually we have the characterisations for cohomogeneous functors.

Proof. That (1) implies (2) is immediate. That (2) implies (3) is an immediate consequence of (3.4), since $L_\Qoppa \simeq *$ by definition of the linear part. That (3) implies (1) is again a consequence of (3.4) since by definition (1) is saying precisely that $L_\Qoppa \simeq *$. ☐

3 Adjunctions galore

Lemma 3.1 (Bireduction adjunction, I.1.1.3)**.** *We have a biadjunction*

$$\begin{tikzcd} BiFun(\mathcal{C}) \arrow[r, hook] & Fun_*(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, Sp) \arrow[l, bend right=40, "(-)^{red}"'] \arrow[l, bend left=40, "(-)^{red}"] \end{tikzcd}$$

Proof. Immediate from the retraction and inclusion maps. □

Corollary 3.2 (Cross-effect adjunction, I.1.1.8)**.** *The biadjunction* $\Delta : \mathcal{C}^{\mathrm{op}} \rightleftarrows \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} : \bigoplus$ *together with the bireduction biadjunction induces a biadjunction*

$$\begin{tikzcd} Fun_*(\mathcal{C}^{\mathrm{op}}, Sp) \arrow[r, shift left, "\bigoplus^*"] & Fun_*(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, Sp) \arrow[l, shift left, "\Delta^*"] \arrow[r, shift left, "(-)^{red}"] & BiFun(\mathcal{C}) \arrow[l, hook, shift left] \end{tikzcd}$$

where the top composite is precisely $B_{(-)}$ *by definition. The diagonal* $\Delta_x : x \to x \oplus x$ *and fold* $\nabla_x : x \oplus x \to x$ *induce the counit and unit*

$$B^{\Delta}_{\Qoppa} \Rightarrow \Qoppa \Rightarrow B^{\Delta}_{\Qoppa}$$

respectively.

Corollary 3.3 (Quadratic-bilinear biadjunction, I.1.1.18)**.** *We have a biadjunction*

$$\begin{tikzcd} Fun^q(\mathcal{C}) \arrow[r, "B_{(-)}"] & Fun^b(\mathcal{C}) \arrow[l, bend right=50, "\Delta^*"'] \arrow[l, bend left=50, "\Delta^*"] \end{tikzcd}$$

with unit and counit given by the natural maps

$$B_{\Qoppa}(x,x) \to \Qoppa(x) \to B_{\Qoppa}(x,x)$$

Proof. This is just by applying (3.2): consider the diagram

$$\begin{tikzcd} \mathrm{Fun}_*(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \arrow[r, "B_{(-)}"] & \mathrm{BiFun}(\mathcal{C}) \arrow[l, bend right=50, "\Delta^*"'] \arrow[l, bend left=50, "\Delta^*"] \\ \mathrm{Fun}^q(\mathcal{C}) \arrow[u, hook] \arrow[r, "B_{(-)}"] & \mathrm{Fun}^b(\mathcal{C}) \arrow[u, hook] \arrow[l, bend right=50, "\Delta^*"'] \arrow[l, bend left=50, "\Delta^*"] \end{tikzcd}$$

where the $B_{(-)}$ square commutes by (2.4) and the Δ^* squares commute by (1.9). □

Proposition 3.4 (Quadratic-(co)linear adjunctions, I.1.1.24)**.** *The natural* $\Qoppa \Rightarrow L_{\Qoppa}$ *and* $cL_{\Qoppa} \Rightarrow \Qoppa$ *exhibits the unit (resp. counit) of the adjunctions*

$$\begin{tikzcd}
\mathit{Fun}^q(\mathcal{C}) \arrow[r, bend left, "L_{(-)}"] \arrow[r, bend right, "cL_{(-)}"'] & \mathit{Fun}^{ex}(\mathcal{C}^{\mathrm{op}}, \mathit{Sp}) \arrow[l, hook']
\end{tikzcd}$$

Proof. We show the linear part. We just need to show that the mapping spectrum from the fibre $(B^{\Delta}_{\Qoppa})_{hC_2}$ of $\Qoppa \Rightarrow L_{\Qoppa}$ to any exact functor is zero. So let f be an exact functor.

$$\mathrm{map}((B^{\Delta}_{\Qoppa})_{hC_2}, f) \simeq \mathrm{map}(\Delta^* B_{\Qoppa}, f)^{hC_2} \simeq \mathrm{map}(B_{\Qoppa}, B_f)^{hC_2} \simeq 0$$

where the second equivalence is by (3.3) and $B_f \simeq 0$ for f exact by (2.2). □

Corollary 3.5 (Quadratic-symmetric bilinear adjunction, I.1.3.3)**.** *We have an adjunction*

$$\begin{tikzcd}
\mathit{Fun}^q(\mathcal{C}) \arrow[r, "B_{(-)}"] & \mathit{Fun}^s(\mathcal{C}) \arrow[l, bend right, "\Qoppa^q_{(-)}"'] \arrow[l, bend left, "\Qoppa^s_{(-)}"]
\end{tikzcd}$$

where both $\Qoppa^q$ and $\Qoppa^s$ are fully faithful, and their essential images are precisely the homogeneous and cohomogeneous functors, respectively.

Proof. We will argue in the homogeneous case, and the other will then be similar. We will show two things in turn: (a) that $\Qoppa^q_{(-)} : \mathrm{Fun}^s(\mathcal{C}) \to \mathrm{Fun}^q(\mathcal{C})$ is fully faithful with the prescribed essential image; (b) that we have an adjunction $\Qoppa^q_{(-)} \dashv B_{(-)}$. To see (a), we factor it as

$$\Qoppa^q_{(-)} : \mathrm{Fun}^s(\mathcal{C}) \xrightarrow{\varphi} \mathrm{Fun}^{\mathrm{hom}}(\mathcal{C}) \subseteq \mathrm{Fun}^q(\mathcal{C})$$

where $\mathrm{Fun}^{\mathrm{hom}}(\mathcal{C})$ is the full subcategory spanned by homogeneous quadratics. We have this factorisation by the characterisation of homogeneity (2.6). On the other hand, the formation of cross-effects

$$\psi : \mathrm{Fun}^{\mathrm{hom}}(\mathcal{C}) \subseteq \mathrm{Fun}^q(\mathcal{C}) \xrightarrow{B_{(-)}} \mathrm{Fun}^s(\mathcal{C})$$

gives a right inverse $\varphi \circ \psi \simeq \mathrm{id}$ by (2.6), whereas (1.10) gives that $\psi \circ \varphi \simeq \mathrm{id}$, as required.

Finally to see (b), standard adjunction yoga says that we need to show that the natural comparison $\varepsilon : \Qoppa^q_{B_{\Qoppa}} \Longrightarrow \Qoppa$ induces an equivalence

$$\mathrm{Nat}_b(\beta, B_{\Qoppa}) \xrightarrow{\Qoppa^q_{(-)}} \mathrm{Nat}_q(\Qoppa^q_B, \Qoppa^q_{B_{\Qoppa}}) \xrightarrow{\varepsilon_*} \mathrm{Nat}_q(\Qoppa^q_B, \Qoppa)$$

for all $\beta \in \mathrm{Fun}^s(\mathcal{C})$. Now the first map is an equivalence by (a). On the other hand, the second map is also an equivalence since $\mathrm{cofib}(\Qoppa^q_{B_{\Qoppa}} \Rightarrow \Qoppa) \simeq L_{\Qoppa}$, and $\mathrm{Nat}_q(\Qoppa^q_B, L_{\Qoppa}) \simeq *$ by (2.6). And so we're done. □

4 The quadratic stable recollement

Definition 4.1. A Bousfield localisation is a left adjoint whose right adjoint is fully faithful.

Fact 4.2 (II.A.1)**.** Let $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \widehat{\mathrm{Cat}}_\infty^{\mathrm{ex}}$ and $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ are functors with trivial composite. Then

(1) It is a fibre sequence in $\widehat{\mathrm{Cat}}_\infty^{\mathrm{ex}}$ iff f was fully faithful with essential image the kernel of p.

(2) If $p : \mathcal{D} \to \mathcal{E}$ is a Bousfield localisation and the inclusion $\mathrm{Im}(f) \subseteq \ker(p)$ is an equivalence, then the sequence is a cofibre sequence in $\widehat{\mathrm{Cat}}_\infty^{\mathrm{ex}}$.

Definition 4.3 (II.A.2.10)**.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be functors between stable categories with trivial composite. Then we say that it is a *stable recollement* if the following conditions hold:

(A) It is a fibre sequence

(B) f admits a left adjoint (that is, it participates in a Bousfield localisation)

(C) p admits a fully faithful right adjoint (that is, it is a Bousfield localisation).

Remark 4.4. By II.A.2.5, a stable recollement in fact always complete automatically to a diagram of adjunctions

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "f"] & \mathcal{D} \arrow[l, bend right, "\bar{g}"'] \arrow[l, dashed, bend left, "g"] \arrow[r, "p"] & \mathcal{E} \arrow[l, hook, dashed, bend right, "\bar{q}"'] \arrow[l, hook, bend left, "q"]
\end{tikzcd}$$

where each layer is a bifibre sequence of stable categories. These things are called split Verdier sequences in the papers, and so "stable recollement = split Verdier sequences." The following lemma clarifies the different guises of stable recollement in different parts of the literature and each description has their uses.

Lemma 4.5 (Characterisations of stable recollement)**.** *Suppose we have Bousfield localisations in $\widehat{Cat}_\infty^{ex}$*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "f"] & \mathcal{D} \arrow[l, bend right, "\bar{g}"'] \arrow[r, "p"] & \mathcal{E} \arrow[l, hook, bend left, "q"]
\end{tikzcd}$$

such that $p \circ f \simeq *$. Then the following conditions are equivalent:

(1) This data is a stable recollement, that is, $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ is a fibre sequence.

(2) The diagram is a pullback

$$\begin{tikzcd}
id_{\mathcal{D}} \arrow[r] \arrow[d] & qp \arrow[d] \\
f\bar{g} \arrow[r] & f\bar{g}qp
\end{tikzcd}$$

(3) p and $\bar{g}$ are jointly conservative, that is, if $d \in \mathcal{D}$ is such that $p(d) \simeq 0 \simeq \bar{g}(d)$, then $d \simeq 0$.

Moreover in this case, we get a canonical identification

$$\mathrm{cofib}\left(\bar{q}p \Rightarrow id \Rightarrow qp\right) \xrightarrow{\simeq} f\bar{g}qp$$

Proof. We omit the proof of the last statement: it is not hard, but tedious. We will show (3) ⇔ (1) and (2) ⇔ (3). To see (3) ⇒ (1), let $d \in \ker(p)$ and consider its unit map $d \to f\bar{g}(d)$. Note that this becomes an equivalence upon applying p and $\bar{g}$, and so by (3), we get that it is an equivalence, and so $d \in \mathrm{Im}(f)$ as required. For (1) implies (3), if $p(d) \simeq 0$ then (1) says that $d \simeq f(c)$ some $c \in \mathcal{C}$. On the other hand, we have $0 \simeq \bar{g}(d) \simeq \bar{g}f(c) \simeq c$, and so $d \simeq 0$ as required. For (2) ⇒ (3), let $d \in \mathcal{D}$ be such that $p(d) \simeq 0 \simeq \bar{g}(d)$. Then applying the pullback square above to d shows that the bottom right corners of the square are zero, and so $d \simeq 0$ also. Finally, to see (3) implies (2), consider the completed diagram by taking horizontal fibres

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & id_{\mathcal{D}} \arrow[r] \arrow[d] & qp \arrow[d] \\
f\bar{g}F \arrow[r] & f\bar{g} \arrow[r] & f\bar{g}qp
\end{tikzcd}$$

We want to show that the left vertical is an equivalence by using joint conservativity. Now $pF \simeq 0$ by definition of F being the fibre of $\mathrm{id}_{\mathcal{D}} \Rightarrow qp$ so that $pF \to pFf\bar{g}$ is an equivalence. On the other hand, $\bar{g}F \to \bar{g}f\bar{g}F$ is of course an equivalence, and so we're done. ☐

Example 4.6 (Arithmetic square at p)**.** We have Bousfield localisations $(-)[1/p] : \mathrm{Sp} \to \mathrm{Sp}[1/p]$ and $(-)^{\wedge}_{p} : \mathrm{Sp} \to \mathrm{Sp}^{\wedge}_{p}$. Then the kernel of $(-)^{\wedge}_{p}$ is precisely the image of $(-)[1/p]$ since if $X^{\wedge}_{p} \simeq 0$ then $X/p \simeq 0$ since p-completeness is tested by smashing with $\mathbb{S}/p$. On the other hand, we have the cofibre sequence $X \xrightarrow{p} X \to X/p$, and so $X \xrightarrow{p} X$ is an equivalence. But $\mathrm{Sp}[1/p] \subseteq \mathrm{Sp}$ is precisely the subcategory of spectra on which p acts invertibly. This gives us the well-known arithmetic square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{\wedge}_{p} \arrow[d] \\
X[1/p] \arrow[r] & X^{\wedge}_{p}[1/p]
\end{tikzcd}$$

Example 4.7 (Genuine C_p-equivariance)**.** We know that we have Bousfield localisations $\Phi^{C_p} : \mathrm{Sp}^{C_p} \to \mathrm{Sp}$ and $\mathrm{fgt} : \mathrm{Sp}^{C_p} \to \mathrm{Fun}(BC_p, \mathrm{Sp})$. By viewing these as spectral Mackey functors, we know that $\mathrm{Im}(\Phi^{C_p}) \subseteq \mathrm{Sp}^{C_p}$ are precisely those spectral Mackey functors with trivial value at the orbit C_p/e. On the other hand, fgt is given by evaluating at the orbit C_p/e, and so clearly the inclusion $\mathrm{Im}(\Phi^{C_p}) \subseteq \ker(\mathrm{fgt})$ is an equivalence. This gives us the equivariant fracture square

$$\begin{tikzcd}
X^{C_p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Phi^{C_p}X \arrow[d] \\
X^{hC_p} \arrow[r] & X^{tC_p}
\end{tikzcd}$$

Theorem 4.8 (Quadratic stable recollement, I.1.3.12)**.** *We have the stable recollement*

$$\begin{tikzcd}
Fun^{ex}(\mathcal{C}^{\mathrm{op}}, Sp) \arrow[r, hook] & Fun^{q}(\mathcal{C}) \arrow[l, bend right=40, "L_{(-)}"'] \arrow[l, bend left=40, "cL_{(-)}"] \arrow[r, "B_{(-)}"] & Fun^{s}(\mathcal{C}) \arrow[l, bend right=40, "\Qoppa^{q}_{(-)}"'] \arrow[l, bend left=40, hook, "\Qoppa^{s}_{(-)}"]
\end{tikzcd}$$

In particular by standard recollement fractures we have the cartesian square for any $\Qoppa \in Fun^{q}(\mathcal{C})$

$$\begin{tikzcd}
\Qoppa \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{\Qoppa} \arrow[d] \\
\Qoppa^s_{B_{\Qoppa}} \arrow[r] & L_{\Qoppa^s_{B_{\Qoppa}}}
\end{tikzcd}$$

where the right vertical is the linearisation of the left. Moreover the bottom map is equivalent to

$$B_{\Qoppa}(X,X)^{hC_2} \to B_{\Qoppa}(X,X)^{tC_2}$$

the usual Tate map.

Proof. To see the stable recollement, just note that (2.2) gives axiom (A); (3.4) gives axiom (B); (3.5) gives axiom (C). We only need to prove the last assertion. By general principles of stable recollement we know that $L_{\Qoppa^s_{B_{\Qoppa}}}$ is computed as the cofibre of the adjunction counit

$$\Qoppa^q_B \simeq \Qoppa^q_{B_{\Qoppa^s_B}} \Rightarrow \Qoppa^s_B$$

Now, by unwinding adjunctions, we have

$$B_{(-)} : \mathrm{Nat}(\Qoppa^q_B, \Qoppa^s_B) \xrightarrow{\simeq} \mathrm{Nat}(B,B)$$

hence since $B_{(-)}$ preserve all limits and colimits, it preserves norm maps and so to show that the natural transformation in question is given by norm map it's enough to show that it has the same image as the norm map under $B_{(-)}$. On the one hand, again by an easy unwinding of adjunctions, the image of the map of interest under the functor $B_{(-)}$ is the identity natural transformation $B \Rightarrow B$. On the other hand, applying $B_{(-)}$ to the norm $(B^{\Delta}_{\Qoppa})_{hC_2} \Rightarrow (B^{\Delta}_{\Qoppa})^{hC_2}$ and commuting it with $(-)_{hC_2}$ and $(-)^{hC_2}$ we obtain

$$(B \oplus B)_{hC_2} \to (B \oplus B)^{hC_2}$$

which *is* the identity on B by the general theory on norms, and so we're done. □

Remark 4.9. Just to sum up the situation, we have the schematic diagram

$$\begin{tikzcd}
 & cL_{\Qoppa}(x) \arrow[r, equal] \arrow[d, "\simeq"', "\text{linear}"] & cL_{\Qoppa}(x) \arrow[d] \\
B_{\Qoppa}(x,x)_{hC_2} \arrow[r, "\simeq", "\text{homogeneous}"'] \arrow[d, equal] & \Qoppa(x) \arrow[r] \arrow[d, "\simeq"', "\text{cohomogeneous}"] & L_{\Qoppa}(x) \arrow[d] \\
B_{\Qoppa}(x,x)_{hC_2} \arrow[r, "\text{norm}"'] & B_{\Qoppa}(x,x)^{hC_2} \arrow[r] & B_{\Qoppa}(x,x)^{tC_2}
\end{tikzcd}$$

5 Hermitian and Poincare structures

Definition 5.1 (I.1.2.1)**.** A *hermitian* category is a pair $(\mathcal{C}, \Qoppa)$ where $\mathcal{C}$ is small stable and $\Qoppa$ is quadratic. This can be organised into a large category Cat^h_∞ given by unstraightening $\mathrm{Cat}^{\mathrm{ex}}_\infty \to \widehat{\mathrm{Cat}}_\infty :: \mathcal{C} \mapsto \mathrm{Fun}^q(\mathcal{C})$. Unwinding definitions, we see that a *hermitian functor* $(\mathcal{C}, \Qoppa) \to (\mathcal{C}', \Qoppa')$ consists of a functor $f : \mathcal{C} \to \mathcal{C}'$ and a natural transformation $\eta : \Qoppa \Rightarrow f^*\Qoppa'$.

We now explore some categorified notions of non-degeneracies that will lead to the notion of Poincare categories.

Construction 5.2 (The duality functor)**.** Let $B \in \mathrm{Fun}^b(\mathcal{C})$ be bilinear. Suppose the curried functor

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \quad :: \quad y \mapsto B(-,y)$$

has the property that it lands in the representables. This functor then can be lifted to a functor

$$D_B^R : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$$

so that we have

$$B(x,y) \simeq \mathrm{map}_{\mathcal{C}}(x, D_B^R y)$$

Similarly for when the functor $B(x,-) \simeq \mathrm{map}_{\mathcal{C}}(-, D_B^L x)$. Clearly if B was symmetric then it is right non-degenerate iff left non-degenerate.

Definition 5.3 (Perfectness, I.1.2.8)**.** If $B \in \mathrm{Fun}^{ns}(\mathcal{C})$ then writing $D : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ and D^{op} for the opposite, we see that

$$\mathrm{Map}_{\mathcal{C}}(x, Dy) \simeq B(x,y) \simeq B(y,x) \simeq \mathrm{Map}_{\mathcal{C}}(y, Dx) \simeq \mathrm{Map}_{\mathcal{C}^{\mathrm{op}}}(D^{\mathrm{op}}x, y)$$

and so D^{op} *is* the left adjoint to D. We define the *duality evaluation* to be the adjunction unit

$$\mathrm{ev} : id \Rightarrow DD^{\mathrm{op}} : \mathcal{C} \to \mathcal{C}$$

A symmetric bilinear functor is called *perfect* if ev is an equivalence, and this implies $D_B : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ is an equivalence.

Definition 5.4 (I.1.2.2)**.** We say that a bilinear functor B is right (resp. left) non-degenerate if $B(-,y)$ (resp. $B(x,-)$) are representables. If it's both left and right non-degenerate, we say it's *non-degenerate*. In this case the resulting dualities are of course adjoint to each other as $D_B^L : \mathcal{C} \rightleftarrows \mathcal{C}^{\mathrm{op}} : D_B^R$. We say that a quadratic functor $\Qoppa : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ is *non-degenerate* if the bilinear part $B_{\Qoppa}$ is non-degenerate. We denote by

$$\mathrm{Fun}^{nb}(\mathcal{C}) \subseteq \mathrm{Fun}^{b}(\mathcal{C}) \qquad \mathrm{Fun}^{ns}(\mathcal{C}) \subseteq \mathrm{Fun}^{s}(\mathcal{C}) \qquad \mathrm{Fun}^{nq}(\mathcal{C}) \subseteq \mathrm{Fun}^{q}(\mathcal{C})$$

for the full subcategories spanned by non-degenerates.

Lemma 5.5 (I.1.2.4)**.** *Let $(\mathcal{C}, \Qoppa), (\mathcal{C}', \Qoppa')$ be two non-degenerate hermitian categories with associated dualities $D_{\Qoppa} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ and $D_{\Qoppa'} : \mathcal{C}'^{\mathrm{op}} \to \mathcal{C}'$. Let $f, g : \mathcal{C} \to \mathcal{C}'$ be exact functors. Then there is an equivalence*

$$Nat(B_{\Qoppa}, (f \times g)^* B_{\Qoppa'}) \simeq Nat(fD_{\Qoppa}, D_{\Qoppa'} g^{\mathrm{op}})$$

of spectra of natural equivalences.

Proof. We have

$$\mathrm{Nat}(B_{\Qoppa}, (f \times g)^* B_{\Qoppa'}) \simeq \mathrm{Nat}((f \times 1)_! B_{\Qoppa}, (1 \times g)^* B_{\Qoppa'})$$

By hypothesis, for fixed $y \in \mathcal{C}$ we have

$$B_{\Qoppa}(-,y) \simeq \mathrm{map}_{\mathcal{C}}(-, Dy)$$

On the other hand, by easy adjunction yoga we see that left Kan extensions commute with representables and so we have

$$(f \times 1)_! \mathrm{map}_{\mathcal{C}}(-, Dy) \simeq \mathrm{map}_{\mathcal{C}'}(-, fDy)$$

Hence in total we have

$$\begin{aligned}
\mathrm{Nat}(B_{\Qoppa}, (f \times g)^* B_{\Qoppa'}) &\simeq \mathrm{map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathrm{Sp})}((f \times 1)_! B_{\Qoppa}, (1 \times g)^* B_{\Qoppa'}) \\
&\simeq \mathrm{map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}))}((f \times 1)_! B_{\Qoppa}, (1 \times g)^* B_{\Qoppa'}) \\
&\simeq \lim_{(x \to y) \in \mathrm{TwAr}(\mathcal{C}^{\mathrm{op}})} \mathrm{map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})}((f \times 1)_! B_{\Qoppa}(-,x), (1 \times g)^* B_{\Qoppa'}(-,y)) \\
&\simeq \lim_{(x \to y) \in \mathrm{TwAr}(\mathcal{C}^{\mathrm{op}})} \mathrm{map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})}(\mathrm{map}_{\mathcal{C}'}(-, fDx), \mathrm{map}_{\mathcal{C}'}(-, D'gy)) \\
&\simeq \lim_{(x \to y) \in \mathrm{TwAr}(\mathcal{C}^{\mathrm{op}})} \mathrm{map}_{\mathcal{C}'}(fDx, D'gy) \\
&\simeq \mathrm{Nat}(fD, D'g^{\mathrm{op}})
\end{aligned}$$

as required. $\square$

This allows us to frame the following important definitions.

Definition 5.6 (Duality preservation)**.** Given a hermitian functor $(f, \eta) : (\mathcal{C}, \Qoppa) \to (\mathcal{C}', \Qoppa')$ I.1.1.6 says $(f \times f)^* B_{\Qoppa'} \simeq B_{f^*\Qoppa'}$, so we get transformation

$$\beta_\eta : B_\Qoppa \Rightarrow (f \times f)^* B_{\Qoppa'}$$

We then denote by

$$\tau_\eta : f D_\Qoppa \Rightarrow D_{\Qoppa'} f^{\mathrm{op}}$$

the natural transformation corresponding to the data B_η by (5.5) and the equivalence $(f \times f)^* B_{\Qoppa'} \simeq B_{f^*\Qoppa'}$. We say that a hermitian functor is *duality-preserving* if this τ_η is an equivalence.

Remark 5.7. Note that all these non-degeneracy conditions depend only on the (symmetric) bilinear part of a quadratic functor.

Definition 5.8 (I.1.2.8)**.** A hermitian structure $\Qoppa$ is called *Poincare* if $B_\Qoppa$ is perfect. We let $\mathrm{Cat}^p_\infty \subseteq \mathrm{Cat}^h_\infty$ denote the non-full subcategory spanned by Poincare categories and duality-preserving functors. Let $\mathrm{Fun}^p(\mathcal{C}) \subseteq \mathrm{Fun}^q(\mathcal{C})$ denote associated non-full subcategory.

Remark 5.9 (Structure vs property)**.** Recall that this means that a Hermitian structure is Poincare if it is perfect non-degenerate. These last two adjectives are properties and so Hermitian is structure but Poincare is property.

Observation 5.10. The constructions $\Qoppa^q_B$ and $\Qoppa^s_B$ of (1.10) given a symmetric bilinear B are Poincare iff B is perfect.

Construction 5.11 (Perfect duality)**.** Recall that we have a C_2-action on $\mathrm{Cat}^{\mathrm{ex}}_\infty$ given by taking opposites, and the (homotopy) fixed points of this are precisely small stables $\mathcal{C}$ together with an equivalence $\mathcal{C}^{\mathrm{op}} \simeq \mathcal{C}$ and higher coherences. In particular, by the duality functor construction we get a forgetful functor

$$\mathrm{Cat}^p_\infty \to (\mathrm{Cat}^{\mathrm{ex}}_\infty)^{\mathrm{op}} \quad :: \quad (\mathcal{C}, \Qoppa) \mapsto (\mathcal{C}, \mathcal{D}_\Qoppa)$$

Example 5.12. Here's the archetypal example to keep in mind. Let R be a commutative ring and $\mathcal{C} = \mathcal{D}^{\mathrm{perf}}(R)$. Then we have a natural symmetric bilinear functor $B_R : \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ given by

$$B_R(X, Y) := \mathrm{Hom}_R(X \otimes_R Y, R)$$

and we define B_{-R} to be the one where the symmetry equivalence $B_{-R}(X \otimes_R Y, R) \xrightarrow{\simeq} B_{-R}(Y \otimes_R X, R)$ is given by minus the one of B_R. While $(B_R^\Delta)_{hC_2}$ and $(B_R^\Delta)^{hC_2}$ give the classical quadratic and symmetric forms, $(B_{-R}^\Delta)_{hC_2}$ and $(B_{-R}^\Delta)^{hC_2}$ give the anti-quadratic and anti-symmetric forms.

Definition 5.13 (Shifts of quadratics)**.** For $\Qoppa : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ a quadratic and $n \in \mathbb{Z}$ we denote by

$$\Qoppa^{[n]}(x) := \Sigma^n \Qoppa(x)$$

and call it the *n-fold shift*.

Observation 5.14. Note that we have following easy identifications

(1) $B_{\Qoppa^{[n]}} \simeq \Sigma^n B_\Qoppa$

(2) $L_{\Qoppa^{[n]}} \simeq \Sigma^n L_\Qoppa$

(3) $D_{\Qoppa^{[n]}} \simeq \Sigma^n D_\Qoppa$

In particular, $\Qoppa$ is non-degenerate or perfect iff $\Qoppa^{[n]}$ is, for all n. Hence a Hermitian category $(\mathcal{C}, \Qoppa)$ is Poincare iff $(\mathcal{C}, \Qoppa^{[n]})$ is for all n.

Warning 5.15 (Paragraph after I.1.2.18)**.** In general it's not true that the pullback of a perfect quadratic functor is perfect, and so we can't pullback Poincare structures willy-nilly. Consider the example of $\mathrm{Mod}(H\mathbb{Z})^{\mathrm{perf}} \xrightarrow{f:=H\mathbb{Q}\otimes -} \mathrm{Mod}(H\mathbb{Q})^{\mathrm{perf}}$. The pullback $f^*\Qoppa^{s}_{\mathbb{Q}}$ is not even non-degenerate. To see this, recall from (1.10) that $B_{\Qoppa^s_{\mathbb{Q}}}(x,y) \simeq \mathrm{map}_{\mathbb{Q}}(x \otimes y, \mathbb{Q})$. Hence we get for $x, y \in \mathrm{Mod}(H\mathbb{Z})^{\mathrm{perf}}$

$$f^* B_{\mathbb{Q}}(x,y) := \mathrm{map}_{\mathbb{Q}}(\mathbb{Q} \otimes x \otimes y, \mathbb{Q}) \simeq \mathrm{map}_{\mathbb{Z}}(x \otimes y, \mathbb{Q}) \simeq \mathrm{map}_{\mathbb{Z}}(x, \mathbb{Q} \otimes D_{\mathbb{Z}}(y))$$

where $\mathbb{Q} \otimes D_{\mathbb{Z}}(y) \in \mathrm{Mod}(H\mathbb{Z})$ is no longer a compact object, so $f^* B_{\mathbb{Q}}(-, y)$ cannot be representable.

Definition 5.16. Let $(\mathcal{C}, \Qoppa)$ be a hermitian category and $x \in \mathcal{C}$.

(1) A *hermitian form on* x is defined to be a point $q \in \Omega^\infty \Qoppa(x)$. We can then get the *category of hermitian objects in* $(\mathcal{C}, \Qoppa)$ $\mathrm{He}(\mathcal{C}, \Qoppa)$ to be the unstraightening of $\Omega^\infty \Qoppa : \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$. We define $\mathrm{Fm}(\mathcal{C}, \Qoppa) := \mathrm{He}(\mathcal{C}, \Qoppa)^{\simeq}$ the anima of hermitian objects.

(2) If $\Qoppa$ was non-degenerate then we can consider

$$\Omega^\infty \Qoppa(x) \to \Omega^\infty B_{\Qoppa}(x,x) = \mathrm{Map}_{\mathcal{C}}(x, D_{\Qoppa} x)$$

coming from (1.8). In this case, a Hermitian object (x, q) determines $q_\# : x \to D_{\Qoppa} x$. We say that a Hermitian form is *Poincare* if $q_\#$ is an equivalence. Let $\mathrm{Pn}(\mathcal{C}, \Qoppa) \subseteq \mathrm{Fm}(\mathcal{C}, \Qoppa)$ denote the full subgroupoid of Poincare objects. In fact, it's easy to see that we can promote this to a functor $\mathrm{Pn} : \mathrm{Cat}^p_\infty \to \mathrm{An}$. We should view these as hermitian forms with a unimodularity condition.

Warning 5.17. $\mathrm{Cat}^p_\infty \subseteq \mathrm{Cat}^h_\infty$ is not full since we need that the hermitian functors preserve dualities.