THURSDAY SEMINAR: HERMITIAN K-THEORY OF ∞-CATEGORIES

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Talk 1 **Hermite and Poincaré.** Define the notion of a quadratic functor [CDH+20b, 1.1.4], its associated cross-effects and polarization, and discuss examples of them by relating them to classical hermitian forms on vector spaces [CDH+20b, 1.1.16-17] and perform the computation in [CDH+20b, 1.1.21]. Time permitting, explain Goodwillie calculus and the notion of n-excisive functors to contextualize the discussion. Define the notions of **Hermitian** and **Poincaré** ∞-categories; define also the associated categories with duality. Explain the notion of recollement and use this language to prove the classification theorem for Hermitian and Poincaré structures [CDH+20b, 1.3.12-13].

Talk 2 **Examplefest, Poincaré objects.** Recall the usual theory of various hermitian forms on vector spaces (if this is not done in the previous talk). Discuss in detail the examples [CDH+20b, 1.2.12-1.2.15,1.2.18]. Discuss the notion of a Poincaré object as an object in $\mathcal{C}$ with a certain hermitian form on it subject to a unimodularity/nondegeneracy condition [CDH+20b, Definition 2.1.3], discuss the important example arising from a compact orientable manifold [CDH+20b, 2.1.8]. Recall the classical notion of a hyperbolic form on a vector space. Motivate and define hyperbolic Poincaré structures [CDH+20b, 2.2.1] and its categorification [CDH+20b, 2.2.2]; skip the C_2-refinement but discuss the notion of isotropic, Lagrangian and metabolic objects [CDH+20b, Definition 2.3.1] and give lots of examples [CDH+20b, 2.3.2-2.3.4].

Talk 3 **L_0 and GW_0.** Introduce the Witt ring of symmetric bilinear forms on a commutative ring R modulo "splits forms" [HM73, 1.7]; introduce its additive and multiplicative structure. Prove that, over fields, a Witt class is uniquely determined by its anisotropic component [HM73, III.1.7]. Do a parallel discussion of Grothendieck-Witt theory and discuss that Witt theory is obtained from Grothendieck-Witt theory by modding out the ideal generated by the hyperbolic form. Perhaps mention the relationship with arithmetic geometry via Milnor's conjecture. Give examples of some known Witt rings of fields following [HM73, III]. Use the above to motivate the definition of the L-groups [CDH+20b, 2.3.11] and the Grothendieck-Witt groups [CDH+20b, 2.4.1]. Explain thoroughly the picture in [CDH+20b, 2.3.8] and the square (58). Prove the algebraic Thom isomorphism [CDH+20b, 2.3.20].

Talk 4 **Poincaré structures Examplefest (Ben Antieau).** To motivate this, talk about work of Kato and Sah on the difference between quadratic Witt theory and symmetric Witt theory in characteristic two [Sah72, Kat82]. Then, discuss examples of Poincaré structures. First, define modules with involutions and the associated symmetric and quadratic hermitian structures [CDH+20b, 3.1.1,3.1.5]. Discuss modules with genuine involution [CDH+20b, 3.2.2-3.2.6]. One of the main points of the present work is the tower [CDH+20b, 3.2.7] interpolating between symmetric and quadratic Poincaré structures; highlight this. Prove periodicity results for these structures [CDH+20b, 3.4.2, 3.4.10,3.4.11]. Describe the universal Poincaré ∞-category [CDH+20b, 4.1.3]. The rest of the talk is up to the speaker to play around with all the examples in [CDH+20b, 4]. For geometric topologists, perhaps they should look at [CDH+20b, 4.4] and discuss "visible Poincaré structures" while algebraists might explain [CDH+20b, 2].

Talk 5 **PV sequences.** We now turn to [CDH+20c]. I advise the reader to look back to [CDH+20b, 6] as necessary for the formalism of the ∞-category of Poincaré ∞-categories and use them as necessary. Define Poincaré-Verdier (PV) sequences [CDH+20c, 1.1.1], squares [CDH+20b, 1.1.5], additive and localizing functors [CDH+20b, 1.5.4] and characterize inclusions and projections [CDH+20b, 1.1.5]. Discuss two important examples: the metabolic fiber sequence [CDH+20b, 1.2.5] and localization of rings [CDH+20b, 1.4.8]. Time permitting, give more examples [CDH+20b, 1.4]. It might be a good idea to motivate these via their analogs in algebraic K-theory/stable ∞-categories.

Talk 6 **Hermitian Q, Additivity and GW (Jay Shah)** Recall that Q-construction for algebraic K-theory as motivation; state that we are working towards constructing Hermitian/GW-theory in the same fashion. Proceed to define the Hermitian Q-construction [CDH+20c, 2.1]; concentrate on explaining the following aspects: (1) that it is a complete Segal object, (2) it preserves Poincaré structures; draw the simplices(!) and explain why it deserves to be called "cobordism" category. (3) its face maps are PV-projections. Extract the cobordism category as in [CDH+20c, 2.3], explain all the examples as in [CDH+20c, 2.3.3]. At this moment, we can define the GW-space: take the first equivalence of [CDH+20c, 4.1.4] as definition. Now state that the goal is to deloop into the Grothendieck-Witt spectrum: the key to this is the additivity theorem [CDH+20c, 2.5.1]. Mention the K-theoretic version of this result, noting that it is a special case [CDH+20c, 2.7] but do not prove the result. Use it to prove delooping and group completion theorems for the Grothendieck-Witt space [CDH+20c, 3.3.4, 3.3.6]. Define the Grothendieck-Witt spectrum [CDH+20c, 3.4.3, 4.2] and reconcile the definition of the Grothendieck-Witt space from the previous talk with the one coming from group completion. Prove (at this point quite easily) the Bott-Genauer sequence and Karoubi's fundamental theorem [CDH+20c, 4.2].

Talk 7 **L-theory and the fundamental square** As motivation, recall that fiber square expressing genuine fixed C_2-fixed points in terms of geometric, Tate and homotopy fixed points. Then state the main result [CDH+20c, 3.6.7] and the associated recollement [CDH+20b, 3.6.8]; explain the punchline that: Hermitian K-theory = K-theory + L-theory glued along something. In particular, if we pretend to know the K-theory of something, then what is left is to know L-theory. Of course this entails defining what a bordism invariant functor is [CDH+20c, 3.5]. Define the bordification functor via the ρ-construction [CDH+20c, 3.6.12] starting with these "ad's" [CDH+20c, 3.6.10] and define the L-theory spectrum [CDH+20c, 4.4]. Prove [CDH+20c, 4.4.2], Lurie's localization theorem for L-theory which expresses higher L-groups as L_0 of something else; emphasize that this adds to the computability of L-theory. Mention the universal property of L-theory [CDH+20c, 4.4.12]. At the end give a "high-level" overview of how one can produce the Real algebraic K-theory C_2-spectrum. Define genuine C_2-equivariant refinements of quadratic functors [CDH+20b, 7.4.17] and hyperbolic Mackey functors [CDH+20b, 7.4.18]. use this to define the real algebraic K-theory spectrum [CDH+20c, 4.5.2] and prove genuine Karoubi periodicity [CDH+20c, 3.7.7] and discuss its consequences.

Talk 8 **Surgery and the L-theory of fields** Begin by explaining geometric surgery [CDH+20c, 2.4.2] and explain how surgery is a way to produce cobordisms. Mention that surgery, in the context of L-theory, is a way to find good representative of abstract elements; this will be the content of this talk and the next. Discuss the surgery equivalence [CDH+20c, 2.4.3], mention [CDH+20c, 2.4.5] but not necessarily in any detail. Now we use the technology we have developed to do something quite concrete: go through Lectures 11-13 in Lurie's notes to compute the L-theory of fields. One might need the exposition of surgery and Lagrangian surgery from paper 3 [CDH+20a, 1.1.13-15]

Talk 9 **More surgery** This is the start of the third paper [CDH+20a]. The goal are two identifications:

(a) the identification of genuine symmetric L-groups with the L-groups of short complexes [CDH+20a, 1.2.18]; and
(b) the L-groups of regular coherent rings in a range [CDH+20a, 1.3.7].

Roughly proceed as follows: define m-quadratic Poincaré structures [CDH+20a, 1.1.2] and L-groups with connectivity conditions [CDH+20a, 1.2.1] then identify the L-groups of an m-quadratic Poincaré structure with quadratic L-groups in a range [CDH+20a, 1.2.3, 1.2.8]. Gather the results to get [CDH+20a, 1.2.18]. Next, define r-symmetric Poincaré structures and do symmetric surgery [CDH+20a, 1.3.1, 1.3.4] and identify the symmetric L-groups for a duality compatible with a t-structure [CDH+20a, 1.3.3] and then do the computation in [CDH+20a, 1.3.7]. There is a lot of material here so pick and choose whichever highlights the idea that L-theory is computable via surgery techniques!

Talk 10 **The L-theory of $\mathbb{Z}$** We can now attempt to compute the Grothendieck-Witt/L-theory of Dedekind domains. Discuss the homotopy limit problem [CDH+20a, 3.1.1,3.1.6]. Then sketch the proof of [CDH+20a, 3.1.6] as follows: 1) prove dévissage [CDH+20a, 2.1.8] and deduce the localization sequence [CDH+20a, 2.1.9], identify the boundary map [CDH+20a, 2.2.1]. Though not strictly necessary prove rigidity for quadratic L-groups [CDH+20a, 2.1.13]. Discuss computational consequences about the Grothendieck-Witt groups of the integers [CDH+20a, 3.2].

Talk 11 **Quadratic-symmetric duality** The spectra $\mathrm{L}^s(\mathbb{Z})$ and $\mathrm{L}^q(\mathbb{Z})$ are Anderson dual as $\mathrm{L}^s(\mathbb{Z})$-modules. This is a theorem of Hebestreit, Land and Nikolaus [HLN21]. Explain this paper.

Talk 12 **Quadratic trace methods (Jay Shah)** Discuss the quadratic enrichment of the Dundas-Goodwillie-McCarthy theorem due to Harpaz-Nikolaus-Shah.

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