NOTES ON THE HERMITIAN Q-CONSTRUCTION AND THE GROTHENDIECK–WITT SPECTRUM

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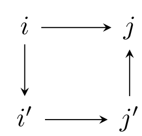
Let $(\mathcal{C}, \Qoppa)$ be a Poincaré ∞-category. Recall that we defined the *Grothendieck–Witt group* $\mathrm{GW}_0(\mathcal{C}, \Qoppa)$ to be the quotient of the commutative monoid $\pi_0(\mathrm{Pn}(\mathcal{C}, \Qoppa))$ of Poincaré objects by the relation $(x, q) \sim (\mathrm{hyp}(w))$ for all (x, q) that admit a Lagrangian $w \longrightarrow x$. Our goal in these notes is to define space and spectrum level enhancements of the Grothendieck–Witt group. Let's first recall how this is done for the K-group of a stable ∞-category by means of the *Q-construction*.

0.1. **Definition.** Let $\mathcal{C}$ be a stable ∞-category. The *Q-construction* is the ∞-category $\mathrm{Span}(\mathcal{C})$ of spans in $\mathcal{C}$.[1]

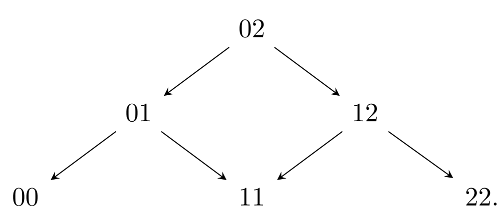
We may then define the K-theory space $K(\mathcal{C})$ to be $\Omega|\mathrm{Span}(\mathcal{C})|$, the loop space on the classifying space of the Q-construction. Stated more precisely, our first goal in these notes is to define a hermitian enhancement of the ∞-category $\mathrm{Span}(\mathcal{C})$, where we equip objects and spans with certain Poincaré structure.

1. The hermitian Q-construction

Recall the *twisted arrow ∞-category* $\mathrm{TwAr}(K)$ has objects arrows $[i \longrightarrow j]$ in K and morphisms



so that morphisms are covariant in the source and contravariant in the target. For example, a picture of the twisted arrow category $\mathrm{TwAr}(\Delta^2)$ is given by



1.1. **Definition.** Let $\mathcal{C}$ be a stable ∞-category. We define $\mathrm{Q}_n(\mathcal{C})$ to be the stable ∞-category given by the full subcategory of $\mathrm{Fun}(\mathrm{TwAr}(\Delta^n), \mathcal{C})$ on those functors that send all squares to cartesian squares.

[1] In the setting of stable ∞-categories, no conditions on the spans are imposed. This stands in contrast to Quillen's original Q-construction which was defined in the context of exact categories.

1.2. **Remark.** If $n = 0$, then $\mathrm{Q}_0(\mathcal{C}) = \mathcal{C}$. If $n = 1$, then $\mathrm{Q}_1(\mathcal{C})$ is the ∞-category whose objects are spans in $\mathcal{C}$ (which should not be confused with $\mathrm{Span}(\mathcal{C})$ itself!). If $n = 2$, then $\mathrm{Q}_2(\mathcal{C})$ is the ∞-category of compositions of spans in $\mathcal{C}$, etc.

Now, it is more or less clear that $\mathrm{Q}_\bullet(\mathcal{C})$ defines a Segal object in ∞-categories, and $\iota\mathrm{Q}_\bullet(\mathcal{C})$ defines a complete Segal space. The ∞-category associated to this complete Segal space is then $\mathrm{Span}(\mathcal{C})$, i.e., it is the *Q-construction* on $\mathcal{C}$.

1.3. **Definition.** Let $(\mathcal{C}, \Qoppa)$ be a hermitian ∞-category. We define $\mathrm{Q}_n(\mathcal{C}, \Qoppa)$ to be the hermitian ∞-category with underlying stable ∞-category $\mathrm{Q}_n(\mathcal{C})$, and with quadratic functor $\Qoppa_n : \mathrm{Q}_n(\mathcal{C})^{\mathrm{op}} \longrightarrow \mathbf{Sp}$ defined by

$$\Qoppa_n(c_{\bullet\bullet}) = \Qoppa(c_{00}) \times_{\Qoppa(c_{01})} \dots \times_{\Qoppa(c_{(n-1)n})} \Qoppa(c_{nn}).$$

1.4. **Example.** Let us unpack the example of $\mathrm{Q}_1(\mathcal{C}, \Qoppa)$. In this case, if $(\mathcal{C}, \Qoppa)$ is Poincaré with duality D then so is $\mathrm{Q}_1(\mathcal{C}, \Qoppa)$ with duality

$$D_1 : (x \longleftarrow y \longrightarrow z) \mapsto (Dx \longleftarrow Dx \times_{Dy} Dz \longrightarrow Dz).$$

A Poincaré object in $\mathrm{Q}_1(\mathcal{C}, \Qoppa)$ is then given by a hermitian object

$$[x \xleftarrow{\alpha} w \xrightarrow{\beta} x',\ \overline{q} = (q, q', \eta : \alpha^* q \simeq \beta^* q') \in \Omega^\infty(\Qoppa(x) \times_{\Qoppa(w)} \Qoppa(x'))]$$

such that the map $\overline{q}_\sharp$ given by

$$\begin{array}{ccccc} x & \longleftarrow & w & \longrightarrow & x' \\ \downarrow{\scriptstyle q_\sharp} & & \downarrow{\scriptstyle (\overline{q}_\sharp)_{01}} & & \downarrow{\scriptstyle q'_\sharp} \\ Dx & \longleftarrow & Dx \times_{Dw} Dx' & \longrightarrow & Dx' \end{array}$$

is levelwise an equivalence (where $(\overline{q}_\sharp)_{01}$ is defined by the image of η under $\Omega^\infty \Qoppa(w) \longrightarrow \Omega^\infty B(w, w) = \mathrm{Map}(w, Dw)$). In other words, (x, q) and (x', q') are Poincaré objects of $(\mathcal{C}, \Qoppa)$ and the span is a *cobordism* thereof.

Note that if $x' = 0$, then such a hermitian object in $\mathrm{Q}_1(\mathcal{C}, \Qoppa)$ is precisely the data of an *isotropic object* over (x, q), and the condition for the hermitian object to be Poincaré corresponds to the condition that the isotropic object be a *Lagrangian*, i.e., that

$$w \longrightarrow x \simeq Dx \longrightarrow Dw$$

is an exact sequence (with nullhomotopy furnished by η).

Let us also explain the geometric intuition behind this notion. Suppose $(\mathcal{C}, \Qoppa) = (D^p(\mathbb{Z}), \Qoppa^{s[-n]}_{\mathbb{Z}})$. Then examples of Poincaré objects are furnished by $(C^*(M), q^{[M]})$ for closed oriented n-manifolds M, where $q^{[M]}$ is defined by cup-product and pairing against the fundamental class. A cobordism from $(C^*(M), q^{[M]})$ to $(C^*(N), q^{[N]})$ can then be defined by an oriented cobordism W from M to N, where the fundamental class $[W]$ determines a path from $q^{[M]}$ to $q^{[N]}$. Here, we invoke Lefschetz duality to see that the Poincaré condition holds.

The phenomenon that $\mathrm{Q}_1(\mathcal{C}, \Qoppa)$ is Poincaré if $(\mathcal{C}, \Qoppa)$ is extends more generally to all n:

1.5. **Lemma.** *Suppose $(\mathcal{C}, \Qoppa)$ is Poincaré. Then $\mathrm{Q}_n(\mathcal{C}, \Qoppa)$ is Poincaré.*

Proof. Let $\mathcal{J}_n \subset \mathrm{TwAr}(\Delta^n)$ be the subposet on ii and $i(i+1)$. Note that the restriction $\mathrm{Q}_n(\mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{J}_n, \mathcal{C})$ is an equivalence with inverse given by right Kan extension. We may then identify $(\mathrm{Q}_n(\mathcal{C}), \Qoppa_n)$ with the cotensor $(\mathcal{C}, \Qoppa)^{\mathcal{J}_n}$ in hermitian ∞-categories, and this is Poincaré since $\mathcal{J}_n$ is the category of simplices of the simplicial complex given by sticking n closed intervals end to end.

Note also that the duality D_n on $(\mathcal{C}, \Qoppa)^{\mathcal{J}_n}$ is given by

$$\begin{aligned} &[c_{00} \longleftarrow c_{01} \longrightarrow \dots \longleftarrow c_{(n-1)n} \longrightarrow c_{nn}] \mapsto \\ &\qquad [Dc_{00} \longleftarrow Dc_{00} \times_{Dc_{01}} Dc_{11} \longrightarrow \dots \longleftarrow Dc_{(n-1)(n-1)} \times_{Dc_{(n-1)n}} Dc_{nn} \longrightarrow Dc_{nn}] \end{aligned}$$

□

It should now be clear that Poincaré objects of $\mathrm{Q}_n(C, \Qoppa)$ are given by n-fold compositions of cobordisms. We next consider the functoriality in $[n] \in \Delta$. Suppose $(\mathcal{C}, \Qoppa)$ is Poincaré.

1.6. **Lemma.** *For any map $f : [n] \longrightarrow [m]$ in Δ, the hermitian functor $f^* : \mathrm{Q}_m(\mathcal{C}, \Qoppa) \longrightarrow \mathrm{Q}_n(\mathcal{C}, \Qoppa)$ is Poincaré.*

Proof. We first remark that the hermitian structure $\eta : \mathfrak{Q}_m \Rightarrow f^*\mathfrak{Q}_n$ is given by the obvious map

$$\mathfrak{Q}(x_{00}) \times ... \times \mathfrak{Q}(x_{mm}) \longrightarrow \mathfrak{Q}(x_{f(0)f(0)}) \times ... \times \mathfrak{Q}(x_{f(n)f(n)})$$

natural in $x_{\bullet\bullet} \in \mathrm{Q}_m(\mathcal{C})$. Now by the formula for the duality, it is clear (at least intuitively) that the duality commutes with f^* if f is a degeneracy or an outer face map. The case of an inner face map is slightly more complicated since f^* will involve terms right Kan extended from $\mathcal{I}_m$. For example, if $f = d_1 : [1] \subset [2]$, we observe that for $x_{\bullet\bullet} \in \mathrm{Q}_2(\mathcal{C})$, since

$$D(x_{00}) \times_{D(x_{02})} D(x_{22}) \simeq D(x_{11})$$

we get that

$$\begin{aligned} D(x_{00}) \times_{D(x_{02})} D(x_{22}) &\simeq D(x_{00}) \times_{D(x_{01})} D(x_{01}) \times_{D(x_{02})} D(x_{12}) \times_{D(x_{12})} D(x_{22}) \\ &\simeq D(x_{00}) \times_{D(x_{01})} D(x_{11}) \times_{D(x_{12})} D(x_{22}) \end{aligned}$$

which is the main step in showing that $D_1 f^* \simeq f^* D_2$, where both composites evaluate on $x_{\bullet\bullet}$ as

$$D(x_{00}) \longleftarrow D(x_{00}) \times_{D(x_{02})} D(x_{22}) \longrightarrow D(x_{22}).$$

□

1.7. **Lemma.** *Let $f : [n] \longrightarrow [m]$ be an injection in Δ. Then $f^* : \mathrm{Q}_n(\mathcal{C}, \mathfrak{Q}) \longrightarrow \mathrm{Q}_m(\mathcal{C}, \mathfrak{Q})$ is a split Poincaré-Verdier projection.*

Proof. We spell out a few cases in low degrees. First consider $f = d_0 : [0] \subset [1]$ so that $f^* : \mathrm{Q}_1(\mathcal{C}) \longrightarrow \mathcal{C}$ sends $[x \longleftarrow w \longrightarrow x']$ to x'. Then f^* admits a left adjoint $L : c \mapsto [0 = 0 \longrightarrow c]$, and $(L^*\mathfrak{Q}_1)(c) \simeq \mathfrak{Q}(c)$.

Next consider $f = d_1 : [1] \subset [2]$. Then f^* admits a left adjoint

$$L : [x \longleftarrow w \longrightarrow x'] \mapsto [x \longleftarrow w = w = w \longrightarrow x']$$

and

$$(L^*\mathfrak{Q}_2)(x \longleftarrow w \longrightarrow x') \simeq \mathfrak{Q}(x) \times_{\mathfrak{Q}(w)} \mathfrak{Q}(x') \simeq \mathfrak{Q}_1(x \longleftarrow w \longrightarrow x').$$

Finally consider $f = d_0 : [1] \subset [2]$. Then f^* admits a left adjoint

$$L : [x \longleftarrow w \longrightarrow x'] \mapsto [0 = 0 \longrightarrow x \longleftarrow w \longrightarrow x']$$

and again it is clear that $L^*\mathfrak{Q}_2 \simeq \mathfrak{Q}_1$. □

1.8. **Lemma.** *$\mathrm{Q}_\bullet(C, \mathfrak{Q})$ is a complete Segal object of $\mathbf{Cat}^p_\infty$.*

Proof. This follows by the known result for the underlying stable ∞-categories and the explicit formula for the pullback of Poincaré ∞-categories (which is computed as a pullback of hermitian ∞-categories): recall that given a span

$$(\mathcal{C}, \mathfrak{Q}) \xrightarrow{(f,\eta)} (\mathcal{C}'', \mathfrak{Q}'') \xleftarrow{(g,\xi)} (\mathcal{C}', \mathfrak{Q}')$$

the quadratic functor $\widetilde{\mathfrak{Q}}$ on $\mathcal{C} \times_{\mathcal{C}''} \mathcal{C}'$ is given by

$$(c, c', f(c) \simeq g(c')) \mapsto \lim \left(\mathfrak{Q}(c) \xrightarrow{\eta_c} (\mathfrak{Q}'' f)(c) \simeq (\mathfrak{Q}'' g)(c') \xleftarrow{\xi_{c'}} \mathfrak{Q}'(c') \right).$$

For example, consider the claim that the commutative square

$$\begin{array}{ccc} \mathrm{Q}_2(\mathcal{C}, \mathfrak{Q}) & \longrightarrow & \mathrm{Q}_{[1,2]}(\mathcal{C}, \mathfrak{Q}) \\ \downarrow & & \downarrow \\ \mathrm{Q}_{[0,1]}(\mathcal{C}, \mathfrak{Q}) & \longrightarrow & \mathrm{Q}_{[1]}(\mathcal{C}, \mathfrak{Q}) \end{array}$$

is a pullback square of Poincaré ∞-categories. This translates to the assertion that

$$\mathfrak{Q}_2(c_{\bullet\bullet}) \simeq \lim \left(\mathfrak{Q}_1(c_0 \leftarrow c_{01} \to c_1) \longrightarrow \mathfrak{Q}(c_1) \longleftarrow \mathfrak{Q}_1(c_1 \leftarrow c_{12} \to c_2) \right),$$

which is clear. □

Recall that a square in $\mathbf{Cat}_\infty^p$ is said to be *split Poincaré-Verdier* if its vertical morphisms are split Poincaré-Verdier. We then say that a functor $\mathcal{F} : \mathbf{Cat}_\infty^p \longrightarrow \mathbf{Spc}$ is *additive* if it is reduced and sends split Poincaré-Verdier squares to cartesian squares of spaces. (We note in passing that a simple argument shows that after the reduced assumption the latter condition is equivalent to sending split Poincaré-Verdier sequences to fiber sequences).

1.9. **Example.** Let $\mathcal{F} = \mathrm{Pn}$ be the functor that sends $(\mathcal{C}, \Omega)$ to its space of Poincaré objects. Then since Pn is corepresented by $(\mathbf{Sp}^\omega, \Omega^u)$, Pn is additive. Similarly, the functor $\mathcal{F} = \mathrm{Cr}$ that sends $(\mathcal{C}, \Omega)$ to $\iota\mathcal{C}$ is additive, being corepresented by $\mathrm{Hyp}(\mathbf{Sp}^\omega)$.

Combining Lemma 1.8 and Lemma 1.7, we see that for any additive functor $\mathcal{F} : \mathbf{Cat}_\infty^p \longrightarrow \mathbf{Spc}$, the composite $\mathcal{F}\mathrm{Q}_\bullet(\mathcal{C}, \Omega) : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Spc}$ is a Segal space. Moreover, if $\mathcal{F}$ preserves pullbacks then $\mathcal{F}\mathrm{Q}_\bullet(\mathcal{C}, \Omega)$ is a complete Segal space. In particular, we see that $\mathrm{Pn}\mathrm{Q}_\bullet(\mathcal{C}, \Omega)$ is a complete Segal space.

1.10. **Definition.** We define the *$\mathcal{F}$-cobordism* ∞-category $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Omega)$ to be the ∞-category associated to the Segal space $\mathcal{F}\mathrm{Q}(\mathcal{C}, \Omega^{[1]})$. If $\mathcal{F} = \mathrm{Pn}$, we write $\mathrm{Cob}(\mathcal{C}, \Omega) = \mathrm{Cob}^{\mathrm{Pn}}(\mathcal{C}, \Omega)$.

To belabor the obvious, $\mathrm{Cob}(\mathcal{C}, \Omega)$ is the ∞-category whose objects are given by Poincaré objects in $(\mathcal{C}, \Omega^{[1]})$ and whose morphisms are given by cobordisms thereof.

1.11. **Remark.** We note that $\mathrm{Map}_{\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Omega)}(0, 0) \simeq \mathcal{F}(\mathcal{C}, \Omega)$. This makes $\Omega|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Omega)|$ a good candidate for the *group completion* of $\mathcal{F}$, as we will see below.

1.12. **Example.** We run through a few different examples of cobordism categories.

1. If $\mathcal{F} = \mathrm{Cr}$, it is clear that $\mathrm{Cob}^{\mathrm{Cr}}(\mathcal{C}, \Omega) = \mathrm{Span}(\mathcal{C})$.
2. For any additive functor $\mathcal{F}$, we have $\mathrm{Cob}^{\mathcal{F}}(\mathrm{Hyp}(\mathcal{C})) \simeq \mathrm{Span}^{\mathcal{F}\circ\mathrm{Hyp}}(\mathcal{C})$, where $\mathrm{Span}^{\mathcal{G}}(-)$ is defined for any additive functor $\mathcal{G} : \mathbf{Cat}_\infty^{\mathrm{ex}} \longrightarrow \mathbf{Spc}$ by analogy with Definition 1.10. Indeed, one observes that $Q_n\mathrm{Hyp}(\mathcal{C}) \simeq \mathrm{Hyp}Q_n(\mathcal{C})$ by juggling the universal properties, so $\mathcal{F}Q_\bullet\mathrm{Hyp}(\mathcal{C}) \simeq \mathcal{F}\mathrm{Hyp}Q_\bullet(\mathcal{C})$. In particular, for $\mathcal{F} = \mathrm{Pn}$, since $\mathrm{PnHyp} = \mathrm{Cr}$ we have $\mathrm{Cob}(\mathrm{Hyp}(\mathcal{C})) \simeq \mathrm{Span}(\mathcal{C})$.
3. We claim that $\mathrm{Cob}(\mathcal{C}, \Omega^s) \simeq \mathrm{Span}(\mathcal{C})^{hC_2}$, where the C_2-action on $\mathrm{Span}(\mathcal{C}) \simeq |\iota Q_\bullet\mathcal{C}|$ is induced by the levelwise action of D_Ω on $\iota Q_\bullet\mathcal{C}$. This follows from two assertions: (i) the Poincaré structures on $Q_n(\mathcal{C}, \Omega^s)$ are all symmetric; (ii) the space of Poincaré objects in $(\mathcal{D}, \Phi^s)$ is given by $\iota\mathcal{D}^{hC_2}$. (We also use that limits of ∞-categories are computed levelwise in terms of complete Segal spaces since the inclusion $\mathbf{Cat}_\infty \subset s\mathbf{Spc}$ is right adjoint.)
4. A basic feature of $\mathrm{Span}(\mathcal{C})$ is that $\mathrm{Span}(\mathcal{C}) \simeq \mathrm{Span}(\mathcal{C})^{\mathrm{op}}$. The same holds true for the cobordism category $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Omega)$ and for the same reason. Indeed, one has $Q_\bullet(\mathcal{C}, \Omega) \simeq Q_{\mathrm{rev}(\bullet)}(\mathcal{C}, \Omega)$ (the reversed simplicial object) in view of the natural isomorphism $\mathrm{TwAr}(\Delta^n) \cong \mathrm{TwAr}((\Delta^n)^{\mathrm{op}})$.

We are now ready to define the *Grothendieck–Witt space* of a Poincaré ∞-category.

1.13. **Definition.** Given a Poincaré ∞-category $(\mathcal{C}, \Omega)$, we define its *Grothendieck–Witt space* to be

$$\mathrm{GW}(\mathcal{C}, \Omega) := \Omega|\mathrm{Cob}(\mathcal{C}, \Omega)|,$$

where the basepoint is given by the zero Poincaré object.

1.14. **Remark.** Note that $\mathrm{Q}_\bullet$ preserves products. Since $\mathcal{F}$ is additive and geometric realization commutes with products, it follows that $|\mathrm{Cob}^{\mathcal{F}}(-)|$ preserves products. Now since $\mathbf{Cat}_\infty^p$ is semiadditive, $|\mathrm{Cob}^{\mathcal{F}}(-)|$ canonically lifts to Mon_{E_∞}, the ∞-category of E_∞-monoids in spaces. Therefore, GW canonically lifts to Gp_{E_∞}, the ∞-category of grouplike E_∞-monoids in spaces, which identifies with connective spectra. We warn the reader that this spectrum should *not* be confused with the Grothendieck–Witt spectrum, which is a canonical delooping of GW to a generally non-connective spectrum that we will define below.

Actually, we can compute $\pi_0|\mathrm{Cob}^{\mathcal{F}}(-)|$ to be a group so that $|\mathrm{Cob}^{\mathcal{F}}(-)|$ is also valued in grouplike E_∞ monoids:

1.15. **Lemma.** *The Poincaré endofunctor* $(\mathrm{id}_{\mathcal{C}}, -\mathrm{id}_{\Omega})$ *of* $(\mathcal{C}, \Omega)$ *induces the inversion on* $\pi_0|\mathrm{Cob}^{\mathcal{F}}(-)|$.

Moreover, we can explicitly compute $\pi_0|\mathrm{Cob}^{\mathcal{F}}(-)|$:

1.16. **Proposition.** *We have a pushout square of commutative monoids*

$$\begin{array}{ccc} \pi_0(\mathcal{F}(\mathrm{Met}(\mathcal{C},\text{Ϙ}^{[1]}))) & \xrightarrow{\mathrm{met}} & \pi_0\mathcal{F}(\mathcal{C},\text{Ϙ}^{[1]}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \pi_0\left|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\text{Ϙ})\right| \end{array}$$

where the righthand vertical functor is induced by the inclusion of the degree 0 *simplices.*

Proof. As with any Segal space, we have that $\pi_0|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\text{Ϙ})|$ may be computed as the coequalizer of the face maps $d_0, d_1 : \pi_0\mathcal{F}\mathrm{Q}_1(\mathcal{C},\text{Ϙ}^{[1]}) \longrightarrow \pi_0\mathcal{F}(\mathcal{C},\text{Ϙ}^{[1]})$. Now consider the split Poincaré–Verdier sequence

$$\mathrm{Met}(\mathcal{C},\text{Ϙ}^{[1]}) \longrightarrow \mathrm{Q}_1(\mathcal{C},\text{Ϙ}^{[1]}) \longrightarrow d_1(\mathcal{C},\text{Ϙ}^{[1]}).$$

Noting that the other face map yields the Poincaré functor met, after applying $\mathcal{F}$ we get the commutative square in question as well as a surjective map $f : \mathrm{coker(met)} \longrightarrow \pi_0|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\text{Ϙ})|$. Now, the commutative monoid coker(met) is in fact a group by the same argument that we used to show the L groups are groups. It remains to see that f is injective to conclude. So suppose $x \in \pi_0\mathcal{F}(\mathcal{C},\text{Ϙ}^{[1]})$ is an element that maps to 0 in $\pi_0|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\text{Ϙ})|$. We then have $w \in \pi_0\mathcal{F}\mathrm{Q}_1(\mathcal{C},\text{Ϙ}^{[1]})$ such that $d_0(w) = x$ and $d_1(w) = 0$. Again, by applying the additive functor $\mathcal{F}$ to the above split Poincaré–Verdier sequence we get a lift of w to $\pi_0\mathcal{F}(\mathrm{Met}(\mathcal{C},\text{Ϙ}^{[1]}))$, so $x = 0$ in coker(met). □

1.17. **Example.** If $\mathcal{F} = \mathrm{Pn}$, then Proposition 1.16 shows that $\pi_0|\mathrm{Cob}(\mathcal{C},\text{Ϙ})| \cong L_{-1}(\mathcal{C},\text{Ϙ})$.

1.18. **Corollary.** $|\mathrm{Cob}^{\mathcal{F}}(\mathrm{Met}(\mathcal{C},\text{Ϙ}))|$ *and* $|\mathrm{Cob}^{\mathcal{F}}(\mathrm{Hyp}\mathcal{C})|$ *are both connected spaces.*

Proof. This follows from Proposition 1.16 once we note that met : $\mathrm{Met}(\mathrm{Met}(\mathcal{C},\text{Ϙ})) \longrightarrow \mathrm{Met}(\mathcal{C},\text{Ϙ})$ and met : $\mathrm{Met}(\mathrm{Hyp}\mathcal{C}) \longrightarrow \mathrm{Hyp}(\mathcal{C})$ are both split. □

2. Additivity

Let $\mathcal{F} : \mathbf{Cat}^p_\infty \longrightarrow \mathbf{Spc}$ be an additive functor.

2.1. **Theorem** (Additivity theorem)**.** *The functor* $|\mathrm{Cob}^{\mathcal{F}}(-)| : \mathbf{Cat}^p_\infty \longrightarrow \mathbf{Spc}$ *is additive.*

The additivity theorem can be deduced from the fibration theorem.

2.2. **Theorem** (Fibration theorem)**.** *Suppose* $(p,\theta) : (\mathcal{D},\Phi) \longrightarrow (\mathcal{E},\Psi)$ *is a split Poincaré–Verdier projection. Then*

$$(p,\theta)_* : \mathrm{Cob}^{\mathcal{F}}(D,\Phi) \longrightarrow \mathrm{Cob}^{\mathcal{F}}(\mathcal{E},\Psi)$$

is a bicartesian fibration.

Proof of Theorem 2.1. Given a bicartesian fibration $f : \mathcal{X} \longrightarrow \mathcal{Y}$, note that all the fibers $\mathcal{X}_y$ are weak homotopy equivalent to each other. Indeed, for all $\alpha : y \longrightarrow y'$, we have an adjunction

$$\alpha_! : \mathcal{X}_y \rightleftarrows \mathcal{X}_{y'} : \alpha^*,$$

and adjoint functors induce weak homotopy equivalences. Moreover, for any $y \in \mathcal{Y}$, we get that

$$\begin{array}{ccc} |\mathcal{X}_y| & \longrightarrow & |\mathcal{X}| \\ \downarrow & & \downarrow \\ \{y\} & \longrightarrow & |\mathcal{Y}| \end{array}$$

is a pullback square of spaces. It then follows that for any split Poincaré–Verdier sequence

$$(\mathcal{C},\text{Ϙ}) \longrightarrow (\mathcal{D},\Phi) \longrightarrow (\mathcal{E},\Psi),$$

if we can show that

$$\begin{array}{ccc} \mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\text{Ϙ}) & \longrightarrow & \mathrm{Cob}^{\mathcal{F}}(\mathcal{D},\Phi) \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathrm{Cob}^{\mathcal{F}}(\mathcal{E},\Psi) \end{array}$$

is a pullback square of ∞-categories, then

$$\begin{array}{ccc} \left|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Omega)\right| & \longrightarrow & \left|\mathrm{Cob}^{\mathcal{F}}(\mathcal{D},\Phi)\right| \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \left|\mathrm{Cob}^{\mathcal{F}}(\mathcal{E},\Psi)\right| \end{array}$$

is a pullback square of spaces by Theorem 2.2. We now note that for a *finite* poset P such that the cotensor $(-)^P$ preserves Poincaré ∞-categories, $(-)^P : \mathbf{Cat}^p_\infty \longrightarrow \mathbf{Cat}^p_\infty$ is both a left and right adjoint and hence preserves split Poincaré–Verdier sequences. Thus, $\mathrm{Q}_\bullet$ and hence $\mathcal{F}\mathrm{Q}_\bullet$ are additive functors, so we have a pullback square of Segal spaces, and an easy argument shows that this remains a pullback square upon completion. □

In the remainder of this section we will partially sketch the proof of Theorem 2.2 when $\mathcal{F} = \mathrm{Pn}$. As a warmup, we first explain how to prove the fibration theorem for K-theory. Let $p : \mathcal{D} \longrightarrow \mathcal{E}$ be a split Verdier projection. Then we claim that

(i) p is a bicartesian fibration.
(ii) $p_* : \mathrm{Span}(\mathcal{D}) \longrightarrow \mathrm{Span}(\mathcal{E})$ is a bicartesian fibration.

For (i), let $g \dashv p \dashv h$ be adjoints. Then for $x \in \mathcal{D}$ and an edge $\overline{\alpha} : px \longrightarrow \overline{y} \in \mathcal{E}$, we claim that a p-cocartesian lift of $\overline{\alpha}$ is furnished by taking the map α in the pushout square

$$\begin{array}{ccc} gpx & \xrightarrow{g\overline{\alpha}} & g\overline{y} \\ \downarrow{\scriptstyle \epsilon_x} & & \downarrow \\ x & \xrightarrow{\alpha} & y. \end{array}$$

Indeed, this following directly from the mapping space criterion to be a p-cocartesian edge. Dually, given $x \in \mathcal{D}$ and an edge $\overline{\beta} : \overline{y} \longrightarrow px \in \mathcal{E}$, a p-cartesian lift of $\overline{\beta}$ is given by the map β in the pullback square

$$\begin{array}{ccc} y & \xrightarrow{\beta} & x \\ \downarrow & & \downarrow{\scriptstyle \eta} \\ h\overline{y} & \xrightarrow{h\overline{\beta}} & hpx. \end{array}$$

For (ii), the idea is that given $x \in \mathrm{Span}(\mathcal{D})$ and a span $px \xleftarrow{\overline{\alpha}} \overline{w} \xrightarrow{\overline{\beta}} \overline{y}$, a p_*-cocartesian lift is given by $x \xleftarrow{\alpha} \overline{\alpha}^* x = w \xrightarrow{\beta} \overline{\beta}_! \overline{\alpha}^* x = y$ (here $\overline{\alpha}^*$ and $\overline{\beta}_!$ denote the cartesian and cocartesian transition functors encoded by p), and dually for p_*-cartesian lifts.

2.3. **Remark.** This approach to the additivity theorem in K-theory via the fibration theorem for span categories is due to Steimle. In addition, step (ii) (that is, the demonstration that for a bicartesian fibration, the passage to span categories remains a bicartesian fibration) is Barwick's 'unfurling' construction.

We now explain how to bring Poincaré structures into the game. Suppose given a Poincaré object (x, q) in $(\mathcal{D}, \Phi)$ and a cobordism

$$(px, \theta q) \xleftarrow{\overline{\alpha}} (\overline{w}, \overline{\eta}) \xrightarrow{\overline{\beta}} (\overline{y}, \overline{r})$$

(so $\overline{\eta} : \overline{\alpha}^* \overline{q} \simeq \overline{\beta}^* \overline{r}$ and the induced map $\overline{w} \longrightarrow D\overline{x} \times_{D\overline{w}} D\overline{y}$ is an equivalence). We then want to endow the p_*-cocartesian lift $x \xleftarrow{\alpha} w \xrightarrow{\beta} y$ with cobordism structure; this will be our $(p, \theta)_*$-cocartesian lift of the cobordism. First, for the hermitian structure on y, we use the following claim:

2.4. **Lemma.** *For any p-cocartesian edge $\beta : w \longrightarrow y$, we have that $\Phi(y) \xrightarrow{\simeq} \Phi(w) \times_{\Psi(pw)} \Psi(py)$ is an equivalence.*

Given this, we then let $r := (\alpha^* q, \overline{r}, \overline{\eta} : \overline{\alpha^* q} = \overline{\alpha}^* \overline{q} \simeq \overline{r}) \in \Omega^\infty \Phi(y)$ be the hermitian form on y. Also, we let $\eta : \alpha^* q \simeq \beta^* r$ be any lift of $\overline{\eta}$.

We then verify the Poincaré condition on $(x, q) \xleftarrow{\alpha} (w, \eta) \xrightarrow{\beta} (y, r)$. For this, we use the following claim:

2.5. **Lemma.** *The duality D_1 on $Q_1(D, \Phi)$ preserves p_*-cocartesian spans.*

Given this, we have that in the diagram

$$\begin{array}{ccccc} x & \xleftarrow{\alpha} & w & \xrightarrow{\beta} & y \\ \downarrow{\scriptstyle q_\sharp} & & \downarrow{\scriptstyle \eta_\sharp} & & \downarrow{\scriptstyle r_\sharp} \\ Dx & \xleftarrow{\alpha'} & Dx \times_{Dw} Dy & \xrightarrow{\beta'} & Dy \end{array}$$

the morphism α' is p-cartesian and the morphism β' is p-cocartesian. Now since $q_\sharp$ is an equivalence by assumption, it follows from the uniqueness of p-cartesian and p-cocartesian lifts that $\eta_\sharp$ and $r_\sharp$ are equivalences. We deduce that the span is in fact a cobordism.

We omit the proof that this cobordism is actually a $(p, \theta)_*$-cocartesian lift. Once one shows this, the construction of $(p, \theta)^*$-cartesian lifts is entirely dual.

Proof of Lemma 2.4. Since β is p-cocartesian, we have that $y \simeq w \cup_{\epsilon, gpw, gp\beta} gpy$, i.e., $gpw \simeq w \times_{\beta, y, \epsilon} gpy$. Since (p, θ) is a split Poincaré–Verdier projection, $\Phi g \simeq \Psi$. Thus, it suffices to see that Φ of the square in question is cartesian. This holds if and only if $B_\Phi(\mathrm{cof}(gp\beta), \mathrm{cof}(\epsilon)) = 0$. But we have that

$$\begin{aligned} B_\Phi(\mathrm{cof}(gp\beta), \mathrm{cof}(\epsilon)) &\simeq \hom_{\mathcal{D}}(\mathrm{cof}(gp\beta), D_\Phi \mathrm{cof}(\epsilon)) \\ &\simeq \hom_{\mathcal{E}}(\mathrm{cof}(p\beta), pD_\Phi \mathrm{cof}(\epsilon)) \\ &\simeq \hom_{\mathcal{E}}(\mathrm{cof}(p\beta), D_\Psi p \mathrm{cof}(\epsilon)) \simeq 0 \end{aligned}$$

since $p(\epsilon) \simeq \mathrm{id}$. □

Proof of Lemma 2.5. We claim that

1. D of a p-cocartesian edge is p-cartesian, and vice-versa. Indeed, if $g \dashv p \dashv h$, we see that $Dh \simeq gD$. Therefore, D transforms a cartesian square

$$\begin{array}{ccc} gpw & \longrightarrow & gpx \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} \\ w & \longrightarrow & x \end{array}$$

into a cartesian square

$$\begin{array}{ccc} Dx & \longrightarrow & Dw \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\ hpDx & \longrightarrow & hpDw, \end{array}$$

and vice-versa.

2. A pullback of a p-cartesian edge is p-cartesian, and similarly a pullback of a p-cocartesian edge is p-cocartesian. This follows immediately from our description of such edges.

Now suppose given a span $[x \xleftarrow{\alpha} w \xrightarrow{\beta} y]$ with α p-cartesian and β p-cocartesian. By the above two facts, we see that in the cartesian square

$$\begin{array}{ccc} Dx \times_{Dw} Dy & \longrightarrow & Dy \\ \downarrow & & \downarrow \\ Dx & \longrightarrow & Dw \end{array}$$

the horizontal edges are p-cocartesian and the vertical edges are p-cartesian. The conclusion follows. □

3. Consequences of additivity

Consider the 'diamond' diagram of Poincaré ∞-categories

$$\begin{array}{ccccc} & & (\mathcal{C},\Qoppa) & & \\ & & \big\downarrow{\scriptstyle s} & \searrow^{=} & \\ \operatorname{Met}(C,\Qoppa) & \hookrightarrow & \mathrm{Q}_1(\mathcal{C},\Qoppa) & \xrightarrow{d_1} & (\mathcal{C},\Qoppa) \\ & \searrow_{\operatorname{lag}} & \big\downarrow{\scriptstyle p^{\mathrm{hyp}}} & & \\ & & \operatorname{Hyp}(\mathcal{C}) & & \end{array}$$

Here, p^{hyp} is induced by the functor $p : \mathrm{Q}_1(\mathcal{C}) \longrightarrow \mathcal{C}$ that sends a span $x \longleftarrow \alpha w \xrightarrow{\beta} y$ to $\operatorname{fib}(\alpha)$, so as a functor p^{hyp} sends the span to $(\operatorname{fib}(\alpha), D\operatorname{cof}(\beta))$, and $\operatorname{lag}(x \xrightarrow{\beta} y) = (w, D\operatorname{cof}(\beta)$.

One can check that both the horizontal and vertical sequences are split Poincaré–Verdier sequences. By a formal argument, it follows that for any *grouplike* additive functor $\mathcal{F} : \mathbf{Cat}^p_\infty \longrightarrow \mathbf{Spc}$ (such as $\mathrm{GW} = \Omega|\mathrm{Cob}|$ or $|\mathrm{Cob}|$, but excluding Pn), we have splittings

$$\mathcal{F}(\mathrm{Q}_1(\mathcal{C},\Qoppa)) \simeq \mathcal{F}(\mathcal{C},\Qoppa) \times \mathcal{F}(\operatorname{Met}(\mathcal{C},\Qoppa)) \simeq \mathcal{F}(\mathcal{C},\Qoppa) \times \mathcal{F}(\operatorname{Hyp}\mathcal{C})$$

and an equivalence

$$\operatorname{lag} : \mathcal{F}(\operatorname{Met}(\mathcal{C},\Qoppa)) \xrightarrow{\simeq} \mathcal{F}(\operatorname{Hyp}\mathcal{C})$$

with inverse induced by $\operatorname{can} : \operatorname{Hyp}(\mathcal{C}) \longrightarrow \operatorname{Met}(\mathcal{C},\Qoppa)$, $(x,y) \mapsto (x \longrightarrow x \oplus Dy)$, since $\operatorname{lag} \circ \operatorname{can} = \mathrm{id}$.

3.1. **Example.** For $\mathcal{F} = \mathrm{GW}$, we get that $\mathrm{GW}(\operatorname{Met}(\mathcal{C},\Qoppa)) \simeq \Omega|\operatorname{Span}(\mathcal{C})| \simeq K(\mathcal{C})$.

Now consider the metabolic (split Poincaré–Verdier) sequence

$$(\mathcal{C},\Qoppa^{[-1]}) \longrightarrow \operatorname{Met}(\mathcal{C},\Qoppa) \xrightarrow{\operatorname{met}} (\mathcal{C},\Qoppa),$$

where the inclusion is given by $x \mapsto [x \longrightarrow 0]$ and the projection met is the target functor. Applying GW and using Example 3.1, we get the *Bott-Genauer* sequence

$$\mathrm{GW}(\mathcal{C},\Qoppa^{[-1]}) \xrightarrow{\operatorname{fgt}} K(\mathcal{C}) \xrightarrow{\operatorname{hyp}} \mathrm{GW}(\mathcal{C},\Qoppa).$$

We similarly get a fiber sequence involving $\Omega|\mathrm{Cob}^{\mathcal{F}}(-)|$ for every additive $\mathcal{F}$. Moreover, we can identify the boundary map as follows:

3.2. **Lemma.** *For any additive functor $\mathcal{F}$, the diagram*

$$\begin{array}{ccc} & \mathcal{F}(\mathcal{C},\Qoppa) & \\ \swarrow & & \searrow \\ \Omega\left|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Qoppa)\right| & \xrightarrow{\qquad\partial\qquad} & \left|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Qoppa)\right| \end{array}$$

commutes. Here, the left diagonal arrow is the inclusion as endomorphisms of 0*, and the right diagonal arrow is the inclusion as degree* 0 *simplices.*

Our next goal is to identify π_0 of GW as the previously defined Grothendieck–Witt group. Actually, we will state the result for an arbitrary additive functor $\mathcal{F}$. To this end, consider the commutative diagram

$$\begin{array}{ccccc} \pi_0\mathcal{F}\operatorname{Hyp}\mathcal{C} & \xleftarrow{\quad\operatorname{lag}\quad} & \pi_0\mathcal{F}\operatorname{Met}(\mathcal{C},\Qoppa) & \xrightarrow{\quad\operatorname{met}\quad} & \pi_0\mathcal{F}(\mathcal{C},\Qoppa) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_1\left|\mathrm{Cob}^{\mathcal{F}}(\operatorname{Hyp}\mathcal{C})\right| & \underset{\cong}{\xleftarrow{\operatorname{lag}}} & \pi_1\left|\mathrm{Cob}^{\mathcal{F}}(\operatorname{Met}(\mathcal{C},\Qoppa))\right| & \xrightarrow{\operatorname{met}} & \pi_1\left|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Qoppa)\right| \end{array}$$

where the lower functor lag is an isomorphism as above. This yields a commutative square

$$\begin{array}{ccc}
\pi_0\mathcal{F}\mathrm{Met}(\mathcal{C},\Qoppa) & \xrightarrow{\mathrm{met}} & \pi_0\mathcal{F}(\mathcal{C},\Qoppa) \\
\big\downarrow{\scriptstyle \mathrm{lag}} & & \big\downarrow \\
\pi_0\mathcal{F}\mathrm{Hyp}\mathcal{C} & \longrightarrow & \pi_1\left|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Qoppa)\right|
\end{array}$$

3.3. **Theorem.** *This is a pushout square of commutative monoids. In particular,* $\pi_1|\mathrm{Cob}(\mathcal{C},\Qoppa)|$ *identifies with the Grothendieck–Witt group.*

Proof. We sketch the proof when $\mathcal{F} = \mathrm{Pn}$. Note first that for $(\mathcal{C},\Qoppa) = \mathrm{Hyp}(\mathcal{D})$, this square is

$$\begin{array}{ccc}
\pi_0\iota\,\mathrm{Ar}(\mathcal{D}) & \xrightarrow{t} & \pi_0\iota\mathcal{D} \\
\big\downarrow{\scriptstyle (s,\mathrm{cof})} & & \big\downarrow \\
\pi_0(\iota\mathcal{D})^{\times 2} & \longrightarrow & \pi_0K(\mathcal{D})
\end{array}$$

which is a pushout square by the known computation for π_0 of the K-theory space. Write GW_0 for the pushout of the square in question and $\gamma : \mathrm{GW}_0 \longrightarrow \pi_1\,|\mathrm{Cob}|$ for the comparison map that we want to show is an isomorphism. In view of the metabolic sequence, we then have the commutative ladder

$$\begin{array}{ccccccccc}
\mathrm{GW}_0(\mathcal{C},\Qoppa^{[-1]}) & \longrightarrow & \mathrm{GW}_0(\mathrm{Hyp}\mathcal{C}) & \longrightarrow & \mathrm{GW}_0(\mathcal{C},\Qoppa) & \longrightarrow & L_0(\mathcal{C},\Qoppa) & \longrightarrow & 0 \\
\big\downarrow{\scriptstyle \gamma^{[-1]}} & & \big\downarrow{\scriptstyle \cong} & & \big\downarrow{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle \cong} & & \\
\pi_1\left|\mathrm{Cob}(\mathcal{C},\Qoppa^{[-1]})\right| & \longrightarrow & \pi_1\,|\mathrm{Cob}(\mathrm{Met}(\mathcal{C},\Qoppa))| & \longrightarrow & \pi_1\,|\mathrm{Cob}(\mathcal{C},\Qoppa)| & \longrightarrow & \pi_0\left|\mathrm{Cob}(\mathcal{C},\Qoppa^{[-1]})\right| & \longrightarrow & 0
\end{array}$$

where the upper horizontal maps are given by the composites

$$\mathrm{GW}_0(\mathcal{C},\Qoppa^{[-1]}) \longrightarrow \mathrm{GW}_0(\mathrm{Met}(\mathcal{C},\Qoppa)) \xrightarrow{\mathrm{lag}} \mathrm{GW}_0(\mathrm{Hyp}\mathcal{C})$$

and

$$\mathrm{hyp} : \mathrm{GW}_0(\mathrm{Hyp}\mathcal{C}) \xrightarrow{\mathrm{can}} \mathrm{GW}_0(\mathrm{Met}(\mathcal{C},\Qoppa)) \xrightarrow{\mathrm{met}} \mathrm{GW}_0(\mathcal{C},\Qoppa),$$

while the middle vertical map is

$$\mathrm{GW}_0(\mathrm{Hyp}\mathcal{C}) \xrightarrow{\mathrm{can}} \mathrm{GW}_0(\mathrm{Met}(\mathcal{C},\Qoppa)) \xrightarrow{\gamma^{\mathrm{met}}} \pi_1\,|\mathrm{Cob}(\mathrm{Met}(\mathcal{C},\Qoppa))|\,,$$

which is an isomorphism as we have explained. Here, we have invoked Lemma 3.2 to see that the righthand square commutes.

By the additivity theorem, we have that the bottom sequence is exact, using that $|\mathrm{Cob}(\mathrm{Met}(\mathcal{C},\Qoppa))|$ is connected by Corollary 1.18. Moreover, by Proposition 1.16 we have that $L_0(\mathcal{C},\Qoppa) \cong \pi_0|\mathrm{Cob}(\mathcal{C},\Qoppa^{[-1]})|$. Now, it is not difficult to check exactness of the upper sequence. We then invoke the 4-lemma to first establish that γ, and hence $\gamma^{[-1]}$, is surjective. Another application of the 4-lemma then shows that γ is also injective. ☐

We next explain how to deduce Karoubi's fundamental theorem from the Bott-Genauer sequence.

3.4. **Definition.** Let $(\mathcal{C},\Qoppa)$ be a Poincaré ∞-category. We let

$$\mathcal{U}(\mathcal{C},\Qoppa) := \mathrm{fib}(K(\mathcal{C}) \xrightarrow{\mathrm{hyp}} \mathrm{GW}(\mathcal{C},\Qoppa)), \quad \mathcal{V}(\mathcal{C},\Qoppa) := \mathrm{fib}(\mathrm{GW}(\mathcal{C},\Qoppa) \xrightarrow{\mathrm{fgt}} K(\mathcal{C})).$$

The Bott-Genauer sequence now immediately implies the following:

3.5. **Theorem** (Karoubi fundamental theorem)**.** *We have natural equivalences*

$$\mathcal{V}(\mathcal{C},\Qoppa) \simeq \Omega\mathcal{U}(\mathcal{C},\Qoppa^{[2]}) \simeq \Omega\mathrm{GW}(\mathcal{C},\Qoppa^{[1]}).$$

3.6. **Example.** Together with Karoubi periodicity, we deduce that for a ring R and invertible module with involution M, we have equivalences

$$\mathcal{V}^q(R,M) \simeq \Omega\mathcal{U}^q(R,-M), \quad \mathcal{V}^s(R,M) \simeq \Omega\mathcal{U}^s(R,-M)$$

for the homotopy quadratic and homotopy symmetric Poincaré structures, and

$$\mathcal{V}^{gs}(R,M) \simeq \Omega\mathcal{U}^{ge}(R,-M), \quad \mathcal{V}^{ge}(R,M) \simeq \Omega\mathcal{U}^{gq}(R,-M)$$

for the genuine symmetric, genuine even, and genuine quadratic Poincaré structures.

3.7. **Remark.** In relating Karoubi's fundamental theorem in this setting to the classical one, we implicitly use the theorem of Hebestreit–Steimle that the Grothendieck–Witt space agrees with the classical version defined via group completion (for the *genuine* Poincaré structures). Given this, Example 3.6 proves a conjecture of Giffen and Karoubi concerning an extension of the classical Karoubi fundamental theorem to the situation where 2 is not invertible in R.

4. Group completion

Let $\mathcal{F} : \mathbf{Cat}^p_\infty \longrightarrow \mathbf{Spc}$ be an additive functor. As we have already used, we have that $\mathrm{Map}_{\mathrm{Cob}^{\mathcal{F}}}(0,0) \simeq \mathcal{F}$, so we have a cartesian square of ∞-categories

$$\begin{array}{ccc} \mathcal{F}(\mathcal{C},\Qoppa) & \longrightarrow & \mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Qoppa)_{0/} \\ \big\downarrow & & \big\downarrow \\ 0 & \longrightarrow & \mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Qoppa). \end{array}$$

4.1. **Theorem** ("Baby" group completion theorem)**.** *Suppose $\mathcal{F}$ is in addition grouplike. Then*

$$\mathcal{F}(\mathcal{C},\Qoppa) \xrightarrow{\simeq} \Omega|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Qoppa)|.$$

Since $\Omega|\mathrm{Cob}^{\mathcal{F}}|$ is always grouplike and additive, we deduce:

4.2. **Corollary.** $\Omega|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Qoppa)|$ *admits a canonical delooping.*

4.3. **Example.** We define the *Grothendieck–Witt spectrum* $\mathbb{GW}$ to be the Ω-spectrum

$$\mathbb{GW}(\mathcal{C},\Qoppa) := (\mathrm{GW}(\mathcal{C},\Qoppa), \mathrm{GW}^{(1)}(\mathcal{C},\Qoppa), \mathrm{GW}^{(2)}(\mathcal{C},\Qoppa), ...)$$

where $\mathrm{GW}^{(n)}$ denotes the n-fold delooping of GW supplied by Theorem 4.1. We note that $\mathbb{GW}$ is in general a *non-connective* spectrum. We will see that the negative L-groups contribute to the negative homotopy groups of $\mathbb{GW}$.

For the proof of Theorem 4.1, it is convenient to introduce a simplicial model of the slice category $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Qoppa)_{0/}$.

4.4. **Definition.** Let S be a simplicial object. The *décalage* of S is given by

$$\mathrm{dec}(S)_n := S_{1+n}.$$

We then define the simplicial object $\mathrm{Null}_\bullet(\mathcal{C},\Qoppa)$ via the pullback

$$\begin{array}{ccc} \mathrm{Null}_\bullet(\mathcal{C},\Qoppa) & \longrightarrow & \mathrm{dec}(\mathrm{Q}_\bullet(\mathcal{C},\Qoppa)) \\ \big\downarrow & & \big\downarrow {\scriptstyle (i_0)^*} \\ 0 & \longrightarrow & \mathrm{const}(\mathcal{C},\Qoppa) \end{array}$$

where $(i_0)_n : [0] \subset [n+1]$ is the inclusion of 0. For example, we have that $\mathrm{Null}_0(\mathcal{C},\Qoppa) = \mathrm{Met}(\mathcal{C},\Qoppa)$ as the kernel of the split Poincaré–Verdier projection $d_1 : \mathrm{Q}_1(\mathcal{C},\Qoppa) \longrightarrow (\mathcal{C},\Qoppa)$. In general, this pullback square is objectwise a split Poincaré–Verdier square.

4.5. **Definition.** We define $\pi : \mathrm{Null}_\bullet(\mathcal{C},\Qoppa) \longrightarrow \mathrm{Q}_\bullet(\mathcal{C},\Qoppa)$ by the map

$$\mathrm{Null}_n(\mathcal{C},\Qoppa) \subset \mathrm{Q}_{1+n}(\mathcal{C},\Qoppa) \xrightarrow{d_0} \mathrm{Q}_n(\mathcal{C},\Qoppa),$$

which is natural in $[n] \in \Delta$. At the level of objects, this forgets the leftmost leg $[0 \longleftarrow x_{01}]$ of the zigzag $x_{\bullet\bullet}$.

We also define $i : \mathrm{const}(\mathcal{C},\Qoppa^{[-1]} \longrightarrow \mathrm{Null}_\bullet(\mathcal{C},\Qoppa)$ by the map

$$i_n : (\mathcal{C},\Qoppa^{[-1]}) \longrightarrow \mathrm{Null}_n(\mathcal{C},\Qoppa)$$

that sends x to the zigzag

$$\begin{array}{ccccccc} & & x & & \dots & & 0 \\ & \swarrow & & \searrow & & & & \searrow \\ 0 & & & & 0 & & \dots & & 0 \end{array}$$

We note that it's important to shift Ω in order to get a split Poincaré–Verdier inclusion.

We thus get a commutative square

$$\begin{array}{ccc} \mathrm{const}(\mathcal{C}, \Omega^{[-1]}) & \xrightarrow{i} & \mathrm{Null}_\bullet(\mathcal{C}, \Omega) \\ \downarrow & & \downarrow \pi \\ 0 & \longrightarrow & Q_\bullet(\mathcal{C}, \Omega) \end{array}$$

that one can show is a split Poincaré–Verdier sequence in each degree. Therefore, we get a pullback square of Segal spaces

$$\begin{array}{ccc} \mathcal{F}(\mathcal{C}, \Omega) & \xrightarrow{i} & \mathcal{F}\mathrm{Null}_\bullet(\mathcal{C}, \Omega^{[1]}) \\ \downarrow & & \downarrow \pi \\ 0 & \longrightarrow & \mathcal{F}Q_\bullet(\mathcal{C}, \Omega^{[1]}) \end{array}$$

and it's not difficult to see that this becomes a pullback square of ∞-categories upon completion. In fact, by general nonsense about the décalage construction we see that this models the pullback square of ∞-categories considered at the beginning of this section.

Our goal is then to show that upon geometric realization, we obtain a pullback square of spaces (assuming $\mathcal{F}$ is also grouplike). To this end, we have the following criterion of Rezk:

4.6. **Lemma** (Rezk's equifibration lemma). *Consider a pullback square*

$$\begin{array}{ccc} X_\bullet & \longrightarrow & Y_\bullet \\ \downarrow & & \downarrow \tau \\ Z_\bullet & \longrightarrow & W_\bullet \end{array}$$

*of functors $I \longrightarrow$ **Spc**. Suppose that τ is equifibered in the sense that for every morphism $i \longrightarrow j$ in I, we have a pullback square of spaces*

$$\begin{array}{ccc} Y(i) & \xrightarrow{\tau_i} & W(i) \\ \downarrow & & \downarrow \\ Y(j) & \xrightarrow{\tau_j} & W(j). \end{array}$$

Then upon taking the colimit over I, our square becomes a pullback square of spaces.

Proof. This is a simple exercise with the descent criterion for colimits in an ∞-topos, considered for the ∞-topos of spaces. □

Theorem 4.1 now follows directly from Lemma 4.6 after showing that $\pi : \mathcal{F}\mathrm{Null}_\bullet(\mathcal{C}, \Omega) \longrightarrow \mathcal{F}\mathrm{Q}_\bullet(\mathcal{C}, \Omega)$ is equifibered. We note that for this, it suffices after the Segal condition to check the low degree cases. For instance, we need to show that the square

$$\begin{array}{ccc} \mathcal{F}\mathrm{Null}_2(\mathcal{C}, \Omega^{[1]}) & \xrightarrow{\pi} & \mathcal{F}\mathrm{Q}_2(\mathcal{C}, \Omega^{[1]}) \\ \downarrow d_i & & \downarrow d_i \\ \mathcal{F}\mathrm{Null}_1(\mathcal{C}, \Omega^{[1]}) & \xrightarrow{\pi} & \mathcal{F}\mathrm{Q}_1(\mathcal{C}, \Omega^{[1]}) \end{array}$$

is cartesian for $i = 0, 1, 2$. When $i = 1, 2$, this square prior to applying $\mathcal{F}$ is split Poincaré–Verdier, whereas when $i = 0$, taking vertical fibers over 0 yields the map $\mathrm{can} : \mathcal{F}(\mathrm{Hyp}\mathcal{C}) \longrightarrow \mathcal{F}(\mathrm{Met}(\mathcal{C}, \Omega))$, which we saw is an equivalence when $\mathcal{F}$ is in addition grouplike.

We end by recording the actual group completion theorem, whose proof requires substantial new ideas that we will not discuss. This is a hermitian enhancement of the "Q = Σ" theorem in K-theory.

4.7. **Theorem** (Group completion)**.** *Let* $\mathcal{F} : \mathbf{Cat}_\infty^p \longrightarrow \mathbf{Spc}$ *be any additive functor. Then the square*

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & * \\ \big\downarrow & & \big\downarrow \\ * & \longrightarrow & \left|\mathrm{Cob}^{\mathcal{F}}(-)\right| \end{array}$$

exhibits $|\mathrm{Cob}^{\mathcal{F}}(-)|$ *as the suspension of* $\mathcal{F}$ *in* $\mathrm{Fun}^{\mathrm{add}}(\mathbf{Cat}_\infty^p, \mathbf{Spc})$*, where the superscript denotes that we take the full subcategory of additive functors. Moreover,* $\mathcal{F} \longrightarrow \Omega\Sigma\mathcal{F}$ *is an equivalence when* $\mathcal{F}$ *is also grouplike.*

Given Theorem 4.7, it follows formally that $\mathcal{F} \longrightarrow \Omega\Sigma\mathcal{F}$ computes the group completion in $\mathrm{Fun}^{\mathrm{add}}(\mathbf{Cat}_\infty^p, \mathbf{Spc})$. We also deduce a theorem establishing an analogous universal property for the canonical deloopings as a formal consequence.

4.8. **Example.** Let $\mathcal{F} = \mathrm{Pn}$. Then Theorem 4.7 establishes the *universality* of (unstable) Grothendieck-Witt theory (as a functor under Pn).

Let $\mathcal{F} = \mathrm{Cr}$. Then Theorem 4.7 yields a strengthening of the universality theorem for K-theory as proven by Blumberg, Gepner, and Tabuada.