Abstracts

On the motivic cohomology of schemes

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(joint work with M. Morrow)

Let k be a field and X a smooth k-scheme. The work of various authors [3, 5, 7, 10, 11, 12] constructs an Atiyah-Hirzebruch style spectral sequence (also commonly known as the *motivic spectral sequence*)

$$\mathrm{H}^{i-j}_{\mathrm{mot}}(X; \mathbb{Z}(-j)) \Rightarrow \mathrm{K}_{-i-j}(X),$$

which diffracts the algebraic K-theory of X into its motivic cohomology. We explain an extension of this spectral sequence to the case when X is a qcqs scheme over $\mathbb{F}_p$, using recent advances in the subject. Here, we discuss the story in characteristic $p > 0$; the characteristic zero story will appear in Morrow's article in this volume.

Theorem 1. *Let* $\mathrm{Sch}_{\mathbb{F}_p}$ *denote the category of qcqs* $\mathbb{F}_p$*-schemes. There exists, for each* $j \geq 0$*, presheaves*

$$\mathbb{Z}(j)^{\mathrm{mot}} : \mathrm{Sch}^{\mathrm{op}}_{\mathbb{F}_p} \to \mathcal{D}(\mathbb{Z}),$$

and a filtration on (Thomason-Trobaugh) K*-theory*

$$\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{K} \to \mathrm{K} : \mathrm{Sch}^{\mathrm{op}}_{\mathbb{F}_p} \to \mathcal{DF}(\mathbb{S})_+{}^{1},$$

which enjoy the following properties:

(1) *the filtration is multplicative and exhaustive. It is complete whenever* X *has finite valuative dimension.*
(2) *Each* $\mathrm{Fil}^{\geq j}_{\mathrm{mot}}\mathrm{K}$ *and* $\mathbb{Z}(j)^{\mathrm{mot}}$ *is a finitary, Nisnevich sheaf.*
(3) *There is a canonical identification of graded pieces*

$$\mathrm{gr}^j_{\mathrm{mot}}\mathrm{K} \simeq \mathbb{Z}(j)^{\mathrm{mot}}[2j];$$

(4) *if* ℓ *is coprime to* p *then we have a canonical equivalence:*

$$\mathbb{Z}(j)^{\mathrm{mot}}/\ell \simeq L_{\mathrm{cdh}}\tau^{\leq j} R\Gamma_{\acute{e}t}(-; \mu_\ell^{\otimes j});$$

(5) *at the prime* p*, we have a cartesian square*

$$\begin{array}{ccc} \mathbb{Z}(j)^{\mathrm{mot}}/p & \longrightarrow & \mathbb{Z}/p(j)^{\mathrm{syn}} \\ \downarrow & & \downarrow \\ R\Gamma_{\mathrm{cdh}}(-; \Omega^j_{\log})[-j] & \longrightarrow & R\Gamma_{\mathrm{eh}}(-; \Omega^j_{\log})[-j]. \end{array}$$

[1] This denotes the ∞-category of filtered spectra equipped with an augmentation.

(6) *For any $r \geq 1$, there is a first chern class*

$$c_1(\mathcal{O}(1)) \in H^2(\mathbb{Z}(1)^{\mathrm{mot}}(\mathbb{P}^r_X));$$

such that the map

$$\mathbb{Z}(j)^{\mathrm{mot}}(X) \oplus \cdots \oplus \mathbb{Z}(j-r)^{\mathrm{mot}}(X)[-2r] \xrightarrow{\pi^* \oplus \cdots \oplus \pi^*(-) \cup c_1(\mathcal{O}(1))^r} \mathbb{Z}(j)^{\mathrm{mot}}(\mathbb{P}^r_X)$$

is an equivalence, i.e., it satisfies the $\mathbb{P}^r$-bundle formula.

(7) *If X is an (essentially-)smooth $\mathbb{F}_p$-scheme, then we have a canonical identification:*

$$\mathbb{Z}(j)^{\mathrm{mot}}(X) \simeq z^j(X, \bullet)[-2j],$$

where $z^j(X, \bullet)$ is Bloch's cycle complex.

(8) *We have a canonical equivalence:*

$$L_{\mathrm{cdh}}\mathbb{Z}(j)^{\mathrm{mot}} \simeq L_{\mathbb{A}^1}\mathbb{Z}(j)^{\mathrm{mot}}.$$

The presheaf $\mathbb{Z}(j)^{\mathrm{mot}}$ is constructed by modifying a cdh-local version of the theory (which is related to "conventional motivic homotopy theory") with *syntomic cohomology*, built from prismatic cohomology [2]. The former is a presheaf of complexes, $\mathbb{Z}(j)^{\mathrm{cdh}}$, constructed by cdh-sheafifying the left Kan extension of Voevodsky's complexes from smooth k-schemes to all qcqs k-schemes; the details of this construction will appear in joint work with Tom Bachmann and Matthew Morrow [1]. An important property of this construction is its value after p-completion:

$$\mathbb{Z}_p(j)^{\mathrm{cdh}}(X) \simeq R\Gamma_{\mathrm{cdh}}(X; W\Omega^j_{\log})[-j],$$

a cdh-extension of the Geisser-Levine theorem [6]. On the other hand, we have the p-adic syntomic complexes $\mathbb{Z}_p(j)^{\mathrm{syn}}$ whose cdh-sheafification we can compute as:

$$L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}(X) \simeq R\Gamma_{\mathrm{eh}}(X; W\Omega^j_{\log})[-j];$$

whence we have a map $\mathbb{Z}_p(j)^{\mathrm{cdh}} \to L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}(X)$. The presheaf $\mathbb{Z}(j)^{\mathrm{mot}}$ is then defined as the following pullback:

$$\begin{array}{ccc} \mathbb{Z}(j)^{\mathrm{mot}} & \longrightarrow & \mathbb{Z}_p(j)^{\mathrm{syn}} \\ \downarrow & & \downarrow \\ \mathbb{Z}(j)^{\mathrm{cdh}} & \longrightarrow & L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}. \end{array}$$

Similar arguments also produce a map $\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{KH} \to L_{\mathrm{cdh}}\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{TC}$, where $\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{TC}$ is the motivic filtration coming from [2]. The motivic filtration on K-theory is then defined by an analogous pullback diagram. The construction of our motivic filtration is a filtered refinement of the following cartesian square:

$$\begin{array}{ccc} \mathrm{K} & \xrightarrow{\mathrm{tr}} & \mathrm{TC} \\ \downarrow & & \downarrow \\ \mathrm{KH} & \longrightarrow & L_{\mathrm{cdh}}\mathrm{TC}. \end{array} \tag{1}$$

This existence of this cartesian square follows from the fact 1) that fiber of the cyclotomic trace satisfies cdh-descent [9] and that 2) KH identifies with $L_{\mathrm{cdh}}K$ [8].

The key to comparing our construction with Voevodsky's original complexes is based on the following result, joint with Bachmann and Morrow:

Theorem 2. *Let X be a qcqs $\mathbb{F}_p$-scheme, then*

(1) $\mathbb{Z}(j)^{\mathrm{cdh}}(\mathbb{A}^1_X) \simeq \mathbb{Z}(j)^{\mathrm{cdh}}(X)$ *and,*
(2) *if X is essentially smooth over $\mathbb{F}_p$, then $\mathbb{Z}(j)^{\mathrm{cdh}}(X)$ recovers Voevodsky's motivic cohomology.*
(3) $L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}$ *satisfies the $\mathbb{P}^1$-bundle formula.*

These are motivic refinements of facts about localizing invariants: 1) that $L_{\mathrm{cdh}}\mathrm{K}$ is $\mathbb{A}^1$-invariant, 2) that $\mathrm{K} \simeq \mathrm{KH}$ on regular noetherian schemes and 3) $L_{\mathrm{cdh}}\mathrm{TC}$ satisfies the $\mathbb{P}^1$-bundle formula (which is a consequence of the bicartesian square (1)).

Theorem 2.(1) implies Theorem 2.(2) using the formalism of Gersten resolutions [4]. Theorem 2.2 then, in turn, produces a non-obvious map from $\mathbb{Z}(j)^{\mathrm{mot}}|_{\mathrm{EssSm}_{\mathbb{F}_p}}$ to Voevodsky's motivic cohomology which we proved to be a retraction. It then suffices to prove that a summand of $\mathbb{Z}(j)^{\mathrm{mot}}|_{\mathrm{EssSm}_{\mathbb{F}_p}}$ is zero. We reduce to checking this on all characteristic $p > 0$ fields after verifying that $\mathbb{Z}(j)^{\mathrm{mot}}|_{\mathrm{EssSm}_{\mathbb{F}_p}}$ satisfies a form of Gersten injectivity:

Theorem 3. *Let A be a regular local $\mathbb{F}_p$-algebra with fraction field F, then for all $i, j \geq 0$ the map*

$$\mathrm{H}^i(\mathbb{Z}(j)^{\mathrm{mot}}(A)) \to \mathrm{H}^i(\mathbb{Z}(j)^{\mathrm{mot}}(F)),$$

is injective.

This last result follows the observation of [4] that a $\mathbb{P}^1$-bundle formula can be used in lieu of $\mathbb{A}^1$-invariance to verify Gersten-type statements.

References

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