Minicourse on motivic cohomology of schemes

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Abstract. This represents my (surely futile) effort to keep good notes for my talks at the summer school in algebraic K-theory, held at IHES in 2023. I gave three lectures over zoom. The first was on classical motivic cohomology from a "handcrafted perspective." The second was on general methods to extend cohomology theories to outside of smooth settings, motivated by the proof of Weibel's conjecture in characteristic zero. The third was on motivic cohomology on equicharacteristic schemes, as developed jointly with Matthew Morrow.

I have tried to keep the tone of these notes "conversational" and the choice of material is somewhat idiosyncratic. Nonetheless, I hope the reader can benefit from some of the ideas and references therein. It is not an easy subject to get into, starting with the intricate web of references to navigate and these notes are written partly to alleviate the beginning learner's pain.

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CHAPTER 1

Introduction

Let us explain what these lectures are all about. This summer school is about algebraic K-theory and many of the talks are going to emphasize on the higher K-groups. In a sense, these lectures go a different direction: we want to understand the lower K-groups. So let us start with K_0. If X is a smooth quasiprojective scheme over a field k, then a key invariant of X is $\mathrm{K}_0(\mathrm{X})$, the Grothendieck group of vector bundles on X. While simple to define, $\mathrm{K}_0(\mathrm{X})$ is a rather sophisticated invariant: it contains the data of algebraic cycles on X.

Remark 0.0.1. To begin with, we note that there is map of (of abelian groups) $\det : \mathrm{K}_0(\mathrm{X}) \to \mathrm{Pic}(\mathrm{X})$ such that the composite $\mathrm{Pic}(\mathrm{X}) \to \mathrm{K}_0(\mathrm{X}) \to \mathrm{Pic}(\mathrm{X})$ is the identity (though the first inclusion is only an inclusion of multiplicative monoids). This says that Pic(X) injects into $\mathrm{K}_0(\mathrm{X})$, a rather nontrivial statement. When X is a curve, we can split Pic(X) off as an abelian group.

To see, this we note that when X is regular, $\mathrm{K}_0(\mathrm{X})$ is canonically isomorphic to $\mathrm{G}_0(\mathrm{X})$, the Grothendieck group of coherent sheaves on X. Therefore, we may speak of the dimension of supports of coherent sheaves.

Having this, we can define

$$\mathrm{F}^j\mathrm{K}_0(\mathrm{X}) = \mathrm{F}^j\mathrm{G}_0(\mathrm{X})$$

as the subgroup generated by the subset $\{[\mathcal{F}] : \mathrm{codim}_{\mathrm{X}}\mathrm{supp}\,(\mathcal{F}) \geqslant j\}$. We then obtain a decreasing filtration

$$\cdots \subset \mathrm{F}^{j+1}\mathrm{K}_0(\mathrm{X}) \subset \mathrm{F}^j\mathrm{K}_0(\mathrm{X}) \subset \cdots \mathrm{F}^0\mathrm{K}_0(\mathrm{X}) = \mathrm{K}_0(\mathrm{X}).$$

The graded ring of this filtration admits a map

$$\bigoplus_{j=0}^{\dim(\mathrm{X})} \mathrm{CH}^j(\mathrm{X}) \to \bigoplus_{j=0}^{\dim(\mathrm{X})} \mathrm{F}^j\mathrm{K}_0(\mathrm{X})/\mathrm{F}^{j+1}\mathrm{K}_0(\mathrm{X}) \tag{0.0.2}$$

given by associating to the class of $\mathrm{Z} \hookrightarrow \mathrm{X}$, the class of the structure sheaf[1] $[\mathcal{O}_{\mathrm{Z}}]$ which is a coherent $\mathcal{O}_{\mathrm{X}}$-module supported in Z. This map is surjective.

Lemma 0.0.3. *The degree i kernel of the above map is killed by $(i-1)!$. In particular the above map is an isogeny of graded groups and is isomorphism after tensoring with $\mathbb{Q}$.*

Proof. For a proof, see [**Stacks**, Tag 0FEV] or [**Ful98**, Chapters 15 and 18]. □

Higher K-groups measure the failure of K_0 to glue so that if $\mathrm{X} = \mathrm{U} \cup \mathrm{V}$ is an open cover, we get a long exact sequence

$$\cdots \to \mathrm{K}_j(\mathrm{U})\oplus\mathrm{K}_j(\mathrm{V}) \to \mathrm{K}_j(\mathrm{U}\cap\mathrm{V}) \to \mathrm{K}_{j-1}(\mathrm{X}) \to \cdots \mathrm{K}_0(\mathrm{X}) \to \mathrm{K}_0(\mathrm{U})\oplus\mathrm{K}_0(\mathrm{V}) \to \mathrm{K}_0(\mathrm{U}\cap\mathrm{V}) \to 0;$$

[1] In $\mathrm{K}_0(\mathrm{X})$, the structure sheaf is the alternating sum given by $\sum(-1)^i[\mathrm{P}_i]$ where $\mathrm{P}_\bullet \to \mathcal{O}_{\mathrm{Z}}$ is a projective resolution of $\mathcal{O}_{\mathrm{Z}}$.

which is (one of the) standard motivation for the higher K-groups. Since we are among mature audience, it is better to say that we have a cartesian square of spectra

$$\begin{tikzcd} \mathrm{K}(\mathrm{X}) \arrow[r] \arrow[d] & \mathrm{K}(\mathrm{U}) \arrow[d] \\ \mathrm{K}(\mathrm{V}) \arrow[r] & \mathrm{K}(\mathrm{U}\cap\mathrm{V}). \end{tikzcd}$$

In this light, there should be an extension of the isomorphism of Lemma 0.0.3 to higher K-groups. Even, better:

THEOREM 0.0.4. *Let* X *be a smooth, k-scheme. There exists a functorial decreasing motivic filtration*

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(\mathrm{X}) \to \mathrm{X}$$

which is multiplicative[2]*, exhaustive and complete*[3]*. The graded pieces are (shifts) of motivic cohomology:*

$$\mathrm{gr}^{\star}_{\mathrm{mot}}\mathrm{K}(\mathrm{X}) \simeq \mathbb{Z}(\star)^{\mathrm{mot}}[2\star](\mathrm{X}).$$

It has the property that there is a canonical isomorphism of graded rings:

$$\mathrm{H}^{2\star}(\mathbb{Z}(\star)^{\mathrm{mot}}(\mathrm{X})) \cong \mathrm{CH}^{\star}(\mathrm{X})$$

and satisfies the following local-to-global principle refining the one for K*-theory: if* $\mathrm{X} = \mathrm{U}\cup\mathrm{V}$ *is an open cover then*

$$\cdots \to \mathrm{H}^{i}_{\mathrm{mot}}(\mathrm{U};\mathbb{Z}(j)) \oplus \mathrm{H}^{i}_{\mathrm{mot}}(\mathrm{V};\mathbb{Z}(j)) \to \mathrm{H}^{i}_{\mathrm{mot}}(\mathrm{V}\cap\mathrm{U};\mathbb{Z}(j)) \to \mathrm{H}^{i+1}_{\mathrm{mot}}(\mathrm{X};\mathbb{Z}(j)) \to \cdots ;$$

and the spectral sequence degenerates to give the isomorphism of Lemma 0.0.3.

In particular, we get a long exact sequence which refines the familiar one for Chow groups:

$$\cdots \mathrm{H}^{2j-1}_{\mathrm{mot}}(\mathrm{V}\cap\mathrm{U};\mathbb{Z}(j)) \to \mathrm{CH}^{j}(\mathrm{X}) \to \mathrm{CH}^{j}(\mathrm{U}) \oplus \mathrm{CH}^{j}(\mathrm{V}) \to \mathrm{CH}^{j}(\mathrm{V}\cap\mathrm{U}) \to 0;$$

Indeed, what we are looking for are the motivic cohomology groups $\mathrm{H}^{i}(\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X})) =: \mathrm{H}^{i}_{\mathrm{mot}}(\mathrm{X};\mathbb{Z}(j))$ are called **motivic cohomology** and offers a theory of "higher algebraic cycles." Indeed, making Chow groups the cohomology of a complex is exactly the right thing to do to get long exact sequences such as the one above.

REMARK 0.0.5. For $j = 1$, so that $\mathrm{CH}^1(\mathrm{X}) \cong \mathrm{Pic}(\mathrm{X})$, this long exact sequence is well-known:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}(\mathrm{X})^{\times} \arrow[r] & \mathcal{O}(\mathrm{U})^{\times} \oplus \mathcal{O}(\mathrm{V})^{\times} \arrow[r] & \mathcal{O}(\mathrm{U}\cap\mathrm{V})^{\times} \arrow[dlll] \\ \mathrm{CH}^1(\mathrm{X}) \arrow[r] & \mathrm{CH}^1(\mathrm{U}) \oplus \mathrm{CH}^1(\mathrm{V}) \arrow[r] & \mathrm{CH}^1(\mathrm{U}\cap\mathrm{V}) \arrow[r] & 0 \end{tikzcd}$$

[2]In these notes multiplicative always means in the homotopy coherent sense, i.e., $\mathbb{E}_{\infty}$.

[3]In our convention a filtration is always decreasing unless otherwise stated:

$$\cdots \mathrm{Fil}^{j+1}\mathrm{E} \to \mathrm{Fil}^{j}\mathrm{E} \to \mathrm{Fil}^{j-1}\mathrm{E} \to \cdots \mathrm{E}.$$

Being **exhaustive** means that the map

$$\operatorname*{colim}_{j\to-\infty} \mathrm{Fil}^{j}\mathrm{E} \to \mathrm{E}$$

is an equivalence and being **complete** means that

$$\lim_{j\to+\infty} \mathrm{Fil}^{j}\mathrm{E} \simeq 0.$$

Remark 0.0.6. The spectral sequence (without any vanishing assumptions!) that we get from Theorem 0.0.4 displays as

$$\mathrm{H}^{i-j}(-j) \Rightarrow \mathrm{K}_{-i-j};$$

$$\begin{tikzcd}
\mathrm{H}^{-1}(0) & \mathrm{H}^{0}(0) \arrow[drr] & \mathrm{H}^{1}(0) & \mathrm{H}^{2}(0) & \mathrm{H}^{3}(0) \\
\mathrm{H}^{0}(1) & \mathrm{H}^{1}(1) \arrow[drr] & \mathrm{H}^{2}(1) & \mathrm{H}^{3}(1) & \mathrm{H}^{4}(1) \\
\mathrm{H}^{1}(2) & \mathrm{H}^{2}(2) \arrow[drr] & \mathrm{H}^{3}(2) & \mathrm{H}^{4}(2) & \mathrm{H}^{5}(2) \\
\mathrm{H}^{2}(3) & \mathrm{H}^{3}(3) \arrow[drr] & \mathrm{H}^{4}(3) & \mathrm{H}^{5}(3) & \mathrm{H}^{6}(4) \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{tikzcd} \tag{0.0.7}$$

Here comes a K-theoretic vanishing result which forces the vanishing of certain groups:

$$\mathrm{K}_{<0}(\mathrm{X}) = 0 \qquad \mathrm{X} \text{ is regular}.$$

The easiest (and, possibly the only way) to ensure that this happens is that $\mathrm{H}^i(j) = 0$ whenever $i > 2j$; these are exactly the groups that will contribute to negative K-theory. We will add this to the design requirement for motivic cohomology.

The overarching goal of this lecture is to explain the extension of Theorem 0.0.4 to schemes which are not necessarily smooth, or even finite type. Indeed, since the K-groups are defined for arbitrary schemes (not even ones which are equicharacteristic), there should be a corresponding general theory of motivic cohomology for these schemes.

For now, motivic cohomology is a quite abstract and all one knows is that there should be a relationship with both algebraic K-theory (vector bundles) and Chow groups (algebraic cycles). By now, there are plenty of good references for the theory of motivic cohomology. In the first lecture we give a "handcrafted viewpoint" which will tie in with our construction of motivic cohomology of singular schemes. What we mean by "handcrafted" is that we will specify a complex (more precisely, a presheaf of graded derived rings[4]) for each $\mathrm{X} \in \mathrm{Sm}_k$:

$$\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X}) \qquad j \geqslant 0$$

via the following cartesian square

$$\begin{tikzcd}
\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X}) \arrow[r, dashed] \arrow[d, dashed] & \prod_p \mathbb{Z}_p(j)^{\mathrm{mot}}(\mathrm{X}) \arrow[d] \\
\mathbb{Q}(j)^{\mathrm{mot}}(\mathrm{X}) \arrow[r] & \left(\prod_p \mathbb{Z}_p(j)^{\mathrm{mot}}(\mathrm{X})\right)_{\mathbb{Q}} .
\end{tikzcd}$$

In the top right and bottom right corners, the product is taken across all primes. The construction when the prime is exactly the characteristic of k is different from when the other primes. Granting this, the construction is quite uniform: for each integer $\nu \geqslant 1$, we have $\mathbb{Z}/p^{\nu}(j)^{\mathrm{mot}}(\mathrm{X})$ and maps

$$\cdots \mathbb{Z}/p^{\nu+1}(j)^{\mathrm{mot}}(\mathrm{X}) \to \mathbb{Z}/p^{\nu}(j)^{\mathrm{mot}}(\mathrm{X}) \to \cdots \mathbb{Z}/p(j)^{\mathrm{mot}}(\mathrm{X}).$$

and we take the limit

$$\mathbb{Z}_p(j)^{\mathrm{mot}}(\mathrm{X}) := \lim \mathbb{Z}/p^{\nu}(j)^{\mathrm{mot}}(\mathrm{X}).$$

[4]

We want to give a rather motivated introduction to motivic cohomology via various older results and constructions. The treatment will be somewhat ahistorical but hopefully it will give some idea of what the objects are.

CHAPTER 2

Lecture 1: on classical motivic cohomology

1. Prehistory: Bloch-Ogus-Gabber theory

Let us attempt to follow the next scholium, which might seem either bold or unmotivated at first glance:

SCHOLIUM 1.0.1. *Doing a certain truncation to an "étale motivic cohomology" theory produces algebraic cycles.*

I have not said what "étale motivic cohomology" is. Soon it will just be étale cohomology in what follows. Indeed, étale cohomology is the one cohomology theory that we have built in vast generality, so one might just guess that étale cohomology could provide a good theory of motivic cohomology. Let us push this line of thinking as far as possible.

The set-up is as follows:

($\star$) let k be a field and m an integer invertible in k and all X that will appear are smooth k-schemes.

We have the complex of étale cochains[1] $\mathrm{R\Gamma}_{\text{ét}}(\mathrm{X};\mu_m^{\otimes j})$, computing the étale cohomology of X in weight j:

$$\mathrm{H}^i(\mathrm{R\Gamma}_{\text{ét}}(\mathrm{X};\mu_m^{\otimes j})) = \mathrm{H}^i_{\text{ét}}(\mathrm{X};\mu_m^{\otimes j}).$$

In fact, the reader unfamiliar with étale cohomlogy can consider instead $k=\mathbb{C}$, and $\mathrm{C}^*_{\mathrm{B}}(\mathrm{X}^{\mathrm{an}};\mathbf{Z}/m)$ the complex of singular cochains on the analytic space of X. One way that we can understand étale cohomology is to use the change-of-site spectral sequence. We have a morphism of sites $\pi:\mathrm{X}_{\text{ét}}\to\mathrm{X}_{\mathrm{Zar}}$ where $\mathrm{X}_{\mathrm{Zar}}$ has underlying category the category of opens of X. The sheaf $\mu_m^{\otimes j}$ is a discrete sheaf of abelian groups on the étale site. But its (derived) pushforward to the Zariski site gives the following equivalence

$$\mathrm{R}\pi_*\mu_m^{\otimes j}(\mathrm{U}) \simeq \mathrm{R\Gamma}_{\text{ét}}(\mathrm{U};\mu_m^{\otimes j}).$$

The amplitude of $\mathrm{R\Gamma}_{\text{ét}}(\mathrm{U};\mu_m^{\otimes j})$ is given as follows: if X is *affine* and finite type over k then it is bounded above by the quantity

$$\mathrm{cd}_m(k)+\dim(\mathrm{X}),$$

where $\mathrm{cd}_m(k)$ is the m-cohomological dimension of the Galois cohomology of k (this is a form of the Andreotti-Frankel theorem in étale cohomology; see, for example, [**Stacks**, Tag 0F0W]). Of course if k is algebraically closed, then the quantity $\mathrm{cd}_m(k)$ is zero.

The formalism of change-of-site spectral sequences gives us a spectral sequence

$$\mathrm{H}^p_{\mathrm{Zar}}(\mathrm{X};\mathcal{H}^q(\mu_m^{\otimes j})) \Rightarrow \mathrm{H}^p_{\text{ét}}(\mathrm{X};\mu_m^{\otimes q});$$

here, $\mathcal{H}^q(\mu_m^{\otimes j})$ is th Zariski sheafification of the discrete presheaf

$$\mathrm{U}\mapsto \mathrm{H}^q_{\text{ét}}(\mathrm{U};\mu_m^{\otimes j}).$$

Given the bound, the spectral sequence looks as follows (setting $\mathrm{H}^p(q)=\mathrm{H}^p_{\mathrm{Zar}}(\mathrm{X};\mathcal{H}^q(\mu_m^{\otimes j}))$):

[1] The lift of this object to an explicit cochain does not matter for what follows.

(1.0.2)

$$\begin{tikzcd}
0 & \mathrm{H}^0(\mathrm{cd}(k)+\dim(\mathrm{X})) \arrow[drr] & \mathrm{H}^1(0) & \cdots & \mathrm{H}^{\dim(\mathrm{X})}(\mathrm{cd}(k)+\dim(\mathrm{X})) \\
0 & \vdots & \vdots & \vdots & \vdots \\
0 & \mathrm{H}^0(2) \arrow[drr] & \mathrm{H}^1(2) & \cdots & \mathrm{H}^{\dim(\mathrm{X})}(2) \\
0 & \mathrm{H}^0(1) \arrow[drr] & \mathrm{H}^1(1) & \cdots & \mathrm{H}^{\dim(\mathrm{X})}(1) \\
0 & \mathrm{H}^0(0) \arrow[drr] & \mathrm{H}^1(0) & \cdots & \mathrm{H}^{\dim(\mathrm{X})}(0) \\
0 & 0 & 0 & 0 &
\end{tikzcd}$$

It will turn out that the invariants $\mathrm{H}^i(j)$ is closer to what we want, i.e., close to the theory of algebraic cycles. Indeed, one of the themes of this series of lectures is to extract information about the E_2 page of a spectral sequence from the E_∞ page.

EXAMPLE 1.0.3. Let k be algebraically closed and assume that X is smooth *and projective.* Then the spectral sequence above tells us that

$$\mathrm{H}^{\dim(\mathrm{X})}(\dim(\mathrm{X})) \cong \mathrm{H}^{2d}_{\text{ét}}(\mathrm{X};\mu_m^{\otimes j}) \cong \mathbb{Z}/m;$$

where we have used algebraic-closedness to trivialize the Tate twist. This result gives us a computation of the Zariski cohomology group $\mathrm{H}^{\dim(\mathrm{X})}(\dim(\mathrm{X}))$ which is typically harder to access than étale cohomology. Here we see the first incarnation of a theme: what we are interested in knowing is ultimately the groups in the E_2 page of a spectral sequence and what we know is the E_∞-page.

EXAMPLE 1.0.4. Remaining algebraically closed, the invariants $\mathrm{H}^p_{\mathrm{Zar}}(\mathrm{X};\mathcal{H}^q(\mu_m^{\otimes j})) = \mathrm{H}^p_{\mathrm{Zar}}(\mathrm{X};\mathcal{H}^q(\mathbb{Z}/m))$ are quite subtle. For example, when $p=0$ and q is arbitrary, we obtain the theory of **unramified cohomology**, advocated by Colliot-Thélenè as birational invariants of smooth, projective varieties. A result of Colliot-Thélenè and Voisin also states that $\mathrm{H}^i(\mathcal{H}^{\dim(\mathrm{X})})$ for all i are birational invariants as well. The last two results depend on the Bloch-Kato conjecture/Rost-Voevodsky theorem.

This spectral sequence is very active, so we do not really like it. In the 1980's, Bloch and Ogus [**BO74**] were able to eliminate many of these groups:

THEOREM 1.0.5 (Bloch-Ogus). *If* X *is a smooth scheme over a field k, and m is invertible in k. Then*

$$\mathrm{H}^p(q) = 0 \qquad p > q.$$

The key point of this result is to construct a certain "skeletal resolution" (what should be called the Bloch-Ogus-Gersten complex) of the sheaves $\mathcal{H}^q_{\text{ét}}(\mu_m^{\otimes j})$; it takes the form

$$\mathrm{G}^\star(\mathrm{U};\mathcal{H}^q_{\text{ét}}(\mu_m^{\otimes j})) = \mathrm{H}^q_{\text{ét}}(k(\mathrm{X});\mu_m^{\otimes j}) \to \bigoplus_{x\in\mathrm{U}^{(1)}} \mathrm{H}^{q-1}_{\text{ét}}(k(\mathrm{X});\mu_m^{\otimes j-1})\cdots \to \bigoplus_{x\in\mathrm{U}^{(c)}} \mathrm{H}^{q-c}_{\text{ét}}(k(\mathrm{X});\mu_m^{\otimes j-c})\cdots.$$

This is a presheaf on $\mathrm{X}_{\mathrm{Zar}}$ as $\mathrm{U}\subset\mathrm{X}$ varies.

Theorem 1.0.6 (Quillen, Bloch-Ogus, Gabber, Colliot-Thélène-Hoobler-Kahn). [**Qui10, BO74, CTHK97**] *The canonical map*

$$\mathcal{H}^q(\mu_m^{\otimes j}) \to \mathrm{G}^\star(-;\mathcal{H}^q_{\text{ét}}(\mu_m^{\otimes j}))$$

is a flasque resolution.

In particular, for each q, the length of the resolution cannot exceed more than q itself and hence the vanishing is proved. Therefore, the spectral sequence (1.0.2) becomes sparser:

(1.0.7)

$$\begin{tikzcd}
0 & \mathrm{H}^0(\mathrm{cd}(k)+\dim(\mathrm{X})) \arrow[drr] & \mathrm{H}^1(0) & & \cdots & \mathrm{H}^{\dim(\mathrm{X})}(\mathrm{cd}(k)+\dim(\mathrm{X})) \\
0 & \vdots & \vdots & \vdots & \vdots & \\
0 & \mathrm{H}^0(2) \arrow[drr] & \mathrm{H}^1(2) & \mathrm{H}^2(2) & \cdots & 0 \\
0 & \mathrm{H}^0(1) \arrow[drr] & \mathrm{H}^1(1) & 0 & \cdots & 0 \\
0 & \mathrm{H}^0(0) \arrow[drr] & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & &
\end{tikzcd}$$

Remark 1.0.8. The sparsity of (1.0.2) has several geometric sequences; let us explore the first nontrivial information we get. Assume that k is algebraically closed and assume that X is a smooth surface. The spectral sequence then looks like:

(1.0.9)
$$\begin{tikzcd}
0 & \mathrm{H}^0(2) \arrow[drr] & \mathrm{H}^1(2) & \mathrm{H}^2(2) & 0 \\
0 & \mathrm{H}^0(1) \arrow[drr] & \mathrm{H}^1(1) & 0 & \\
0 & \mathrm{H}^0(0) & 0 & 0 &
\end{tikzcd}$$

From this, the first new thing we learn is an isomorphism

$$\mathrm{H}^1_{\mathrm{Zar}}(\mathrm{X};\mathcal{H}^2(\mathbb{Z}/m)) \cong \mathrm{H}^3_{\text{ét}}(\mathrm{X};\mathbb{Z}/m).$$

In particular, for any prime ℓ invertible in k we get an isomorphism

$$\mathrm{H}^1_{\mathrm{Zar}}(\mathrm{X};\mathcal{H}^2(\mathbb{Q}_\ell/\mathbb{Z}_\ell)) \cong \mathrm{H}^3_{\text{ét}}(\mathrm{X};\mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Now, let us assume that X is furthermore *projective*, then we have Poincaré duality in étale cohomology (re-inserting the Tate twists) which says that

$$\mathrm{H}^3_{\text{ét}}(\mathrm{X};\mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \cong \mathrm{Hom}(\mathrm{H}^{3-2=1}_{\text{ét}}(\mathrm{X};\mathbb{Z}_\ell(1)),\mu_{\ell^\infty}) \cong \mathrm{Hom}(\mathrm{T}_\ell(\mathrm{Pic}(\mathrm{X})(k)),\mu_{\ell^\infty});$$

where $(\mathrm{Pic}(\mathrm{X})(k)$ is the abelian group of k-points of the Picard variety of X, representing degree zero line bundles on X. There is another name for this, $\mathrm{Alb}(\mathrm{X})(k)_{\ell-\mathrm{tors}}$, the k-points of the

Albanese variety of X. From this discussion we obtain an isomorphism

$$\mathrm{H}^1_{\mathrm{Zar}}(\mathrm{X};\mathcal{H}^2(\mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \cong \mathrm{Alb}(\mathrm{X})(k)_{\ell^\infty-\mathrm{tors}}.$$

This is basically the key input to Bloch (ℓ invertible in k) and Milne's proof (at the characteristic) of Roitman's torsion theorem which asserts that there is a canonical isomorphism

$$\mathrm{CH}_0(\mathrm{X})^0_{\mathrm{tors}} \to \mathrm{Alb}(\mathrm{X})(k)_{\mathrm{tors}}.$$

where Alb(X) is the Albanese variety of X, the initial abelian variety under X, up to base points; the map above is by universal properties. This theorem is valid for any smooth projective variety over an algebraically closed field. Indeed, Bloch and Milne's key observation is the existence of the following factorization:

$$\begin{tikzcd}
\mathrm{H}^1_{\mathrm{Zar}}(\mathrm{X};\mathcal{H}^2(\mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \arrow[r, dashed] \arrow[dr, "\cong"'] & \mathrm{CH}^2(\mathrm{X})_{\ell^\infty-\mathrm{tors}} \arrow[d] \\
 & \mathrm{Alb}(\mathrm{X})(k)_{\ell^\infty-\mathrm{tors}};
\end{tikzcd}$$

where the top-right map is surjective. This is enough to prove that the comparison map is surjective for X a surface, which is the key nontrivial step in the proof of Roitman's theorem[2].

Another consequence is:

Corollary 1.0.10. *Let* X *be a smooth k-scheme and $j \geqslant 0$. There is a canonical isomorphism*

$$\mathrm{H}^j_{\mathrm{Zar}}(\mathrm{X};\mathcal{H}^j(\mu_m^{\otimes j})) \cong \mathrm{CH}^j(\mathrm{X}) \otimes \mathbb{Z}/m.$$

Proof. The Bloch-Ogus-Gersten complex computing $\mathrm{H}^j_{\mathrm{Zar}}$ is given by

$$\bigoplus_{x \in \mathrm{X}^{(j-1)}} \mathrm{H}^1(\kappa(x);\mu_m^{\otimes 1}) \to \bigoplus_{y \in \mathrm{X}^{(j}} \mathrm{H}^0(\kappa(y);\mathbf{Z}/m) \to 0.$$

Hilbert theorem 90, gives an isomorphism $\mathrm{H}^1(\kappa(x);\mu_m^{\otimes 1}) \cong \kappa(x)^\times/\left(\kappa(x)^\times\right)^m$, and one can identify the above complex with the mod-m reduction of the complex

$$\bigoplus_{x \in \mathrm{X}^{(j-1)}} \kappa(x)^\times \xrightarrow{\mathrm{v}} \bigoplus_{y \in \mathrm{X}^{(j}} \mathbb{Z} \to 0,$$

where the differential map is given by the "discrete valuation." This latter complex exactly computes $\mathrm{CH}^j(\mathrm{X})$. We thus conclude the result. □

2. Motivic cohomology away from the prime

Corollary 1.0.10 is very nice because it expresses algebraic cycles mod-m cohomologically. Recall that our goal is to extend Chow groups "to the left" so as to satisfy a Zariski local-to-global principle refining the one for algebraic K-theory. We note that étale cohomology itself will not do this job since the terms computing Chow groups do not necessarily survive the spectral sequence (2.0.1) because there are differentials going into the relevant groups. In fact, there is a cycle class map $\mathrm{CH}^j(\mathrm{X}) \otimes \mathbb{Z}/m \to \mathrm{H}^{2j}_{\text{ét}}(\mathrm{X};\mu_m^{\otimes j})$, constructed as an edge map in the spectral sequence, which is usually neither surjective nor injective.

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[2]To finish off, we need two more statements:

(1) that we have an isomrophism of sheaves $\mathcal{H}^2(\mathbb{Z}/m) \cong \mathcal{K}_2/m$;
(2) $\mathrm{H}^2_{\mathrm{Zar}}(\mathrm{X};\mathcal{K}_2) \cong \mathrm{CH}^2(\mathrm{X})$.

Having this, we then have a surjection $\mathrm{H}^1_{\mathrm{Zar}}(\mathrm{X};\mathcal{H}^2(\mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \to \mathrm{H}^2_{\mathrm{Zar}}(\mathrm{X};\mathcal{K}_2)_{\ell^\infty-\mathrm{tors}} \to 0$ just from the Bockstein sequence. The second claim can be attributed to Qullen [**Qui10**], but for a surface it is much older [**Blo74**]. The first claim is a special case of Merkurjev-Suslin theorem [**Mer06**] but is actually easier to prove [**Blo10**, Theorem 5.7].

Here is a candidate which, I think, is basically forced upon us. Let us take the (Zariski-local) truncation, for X a smooth scheme over a perfect field k:

$$\mathbb{Z}/m(j)^{\mathrm{mot}}|_{\mathrm{X}} := \tau^{\leqslant j}\mathrm{R}\pi_*\mu_m^{\otimes j};$$

then the spectral sequence takes the form

$$\begin{tikzcd}
0 & 0 & 0 & & \cdots & 0 \\
0 & \mathrm{H}^0(j) \arrow[drr] & \mathrm{H}^1(j) & & \cdots & \mathrm{H}^j(j) \\
0 & \vdots & \vdots & \vdots & \vdots & \\
0 & \mathrm{H}^0(2) \arrow[drr] & \mathrm{H}^1(2) & \mathrm{H}^2(2) & \cdots & 0 \\
0 & \mathrm{H}^0(1) \arrow[drr] & \mathrm{H}^1(1) & 0 & \cdots & 0 \\
0 & \mathrm{H}^0(0) \arrow[drr] & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & &
\end{tikzcd} \tag{2.0.1}$$

In particular:

Proposition 2.0.2. *We have an isomorphism*

$$\mathrm{CH}^j(\mathrm{X}) \otimes \mathbb{Z}/m \simeq \mathrm{H}^{2j}(\tau^{\leqslant j}\mathrm{R}\pi_*\mu_m^{\otimes j}(\mathrm{X})),$$

Proof. In the spectral sequence of (2.0.1), we get an isomorphism

$$\mathrm{H}^{2j}(\tau^{\leqslant j}\mathrm{R}\pi_*\mu_m^{\otimes j}(\mathrm{X})) \cong \mathrm{H}^j(j)$$

since the last group is stable in the spectral sequence. The result then follows from (1.0.10). □

This is great! We get some kind of cohomology theory. Furthermore, the top cohomology of $\tau^{\leqslant j}\mathrm{R}\pi_*\mu_m^{\otimes j}$ is exactly $2j$; this is *irrespective* of the dimension of X. Though we also get the following vanishing:

Proposition 2.0.3. *For any $j \geqslant 0$, we have that*

$$\mathrm{H}^i(\tau^{\leqslant j}\mathrm{R}\pi_*\mu_m^{\otimes j}(\mathrm{X})) = 0 \qquad i > j + \dim(\mathrm{X}).$$

Remark 2.0.4. The local-to-global principle is basically automatic given that we interpret $\tau^{\leqslant j}$ as a truncation of Zariski sheaves. Indeed, the truncation has Zariski descent which is equivalent to having Zariski excision by a result of Voevodsky's; see [**AHW16**, Theorem 3.2.5] for a reference. However, because we have performed this truncation, $\mathbb{Z}/m(j)^{\mathrm{mot}}$ is *no longer* an étale sheaf.

Example 2.0.5. Weight one motivic cohomology can easily be read off from this construction:

$$\mathrm{H}^{\star}(\mathrm{X}; \mathbb{Z}/m(1)) = \begin{cases} \mu_m(\mathcal{O}(\mathrm{X})) & \star = 0 \\ \mathrm{H}^1_{\text{ét}}(\mathrm{X}; \mu_m) & \star = 1 \\ \mathrm{Pic}(\mathrm{X})/m\mathrm{Pic}(\mathrm{X}) & \star = 2 \\ 0 & \text{else.} \end{cases}$$

More precisely, we have an equivalence[3] of Zariski sheaves:

$$\mathbb{Z}/m(1)^{\mathrm{mot}}_{\mathrm{X}} \simeq \mathbb{G}_m/m\mathbb{G}_m[-1].$$

This is one of the desiderata of a theory of motivic cohomology: that its weight one motivic cohomology computes the Zariski cohomology of $\mathbb{G}_m$, up to a shift.

3. Motivic cohomology at the prime

Taking the above as inspiration, let us discuss motivic cohomology with mod-p coefficients for smooth k-schemes where k is a perfect field of characteristic $p > 0$. The guiding principle is that the theory should not look that much different. It is well known that étale cohomology with p-torsion coefficients at the prime p is poorly behaved. In the Arbeitsgemeinschaft attached to this summer school, the syntomic cohomology of schemes is discussed and is a good replacement for étale cohomology at the characteristic; it should be regarded as the theory of "étale motivic cohomology" at the prime.

Let X be a $\mathbb{F}_p$-scheme. We have a canonical map of abelian presheaves on $\mathrm{X}_{\text{ét}}$:

$$\mathrm{dlog} : \mathbb{G}_m^{\otimes j} \to \Omega^j_{\mathrm{X}} \qquad f_1 \otimes \cdots \otimes f_j \mapsto \tfrac{df_1}{f_1} \wedge \cdots \tfrac{df_j}{f_j}. \tag{3.0.1}$$

Let τ be a Grothendieck topology, then we set the τ-sheaf for $j \geqslant 1$

$$\Omega^j_{\log,\mathrm{X},\tau} := \tau\text{-sheafification of the image of (3.0.1)};$$

if no τ is specified we will put

$$\Omega^j_{\log,\mathrm{X}} := \Omega^j_{\log,\mathrm{X},\text{ét}}.$$

Example 3.0.2. For any regular local $\mathbb{F}_p$-algebra R, the map

$$(\mathrm{R}^\times)/(\mathrm{R}^\times)^p \xrightarrow{\mathrm{dlog}} \Omega^1_{\mathrm{A},\log,\text{ét}}$$

is an isomorphism. In particular, we have an isomorphism of étale sheaves:

$$\mathbb{G}_m/p \xrightarrow{\mathrm{dlog}} \Omega^1_{\mathrm{X},\log,\text{ét}},$$

whenever X is smooth.

Remark 3.0.3 (Milne duality). One of the justification for setting $\Omega^j_{\mathrm{X},\log,\text{ét}}$ to be the theory of étale motivic cohomology is that it enjoys a form of Poincaré duality. In fact, we saw from the discussion in Remark 1.0.8 that this is one of the key ingredients to proving Roitman's theorem. If X is a smooth, projective variety over an algebraically closed field of charateristic $p > 0$, then there is a perfect pairing

$$\mathrm{H}^i_{\text{ét}}(\mathrm{X};\Omega^j_{\log,\mathrm{X}}) \otimes_{\mathbb{F}_p} \mathrm{H}^{d-i}_{\text{ét}}(\mathrm{X};\Omega^{d-j}_{\log,\mathrm{X}}) \xrightarrow{\cup} \mathrm{H}^d_{\text{ét}}(\mathrm{X};\Omega^d_{\log,\mathrm{X}}) \cong \mathbb{Z}/p.$$

which gives a duality theorem for étale cohomology with coefficients in $\Omega^j_{\log}$. If k is a finite field, the duality result asserts a perfect pairing of the form

$$\mathrm{H}^i_{\text{ét}}(\mathrm{X};\Omega^j_{\log,\mathrm{X}}) \otimes_{\mathbb{F}_p} \mathrm{H}^{d+1-i}_{\text{ét}}(\mathrm{X};\Omega^{d-j}_{\log,\mathrm{X}}) \xrightarrow{\cup} \mathrm{H}^{d+1}_{\text{ét}}(\mathrm{X};\Omega^d_{\log,\mathrm{X}}) \cong \mathbb{Z}/p;$$

reflecting the cohomological dimension of finite fields; see [**Mil76**, Theorem 1.9 and 2.4] for details or [**Bha**, Section 4.5] for a modern reference.

---

[3]Contained within this isomorphism is the fact that $\mathrm{H}^{\geqslant 2}_{\mathrm{Zar}}(\mathrm{X};\mathbb{G}_m) = 0$. In fact this holds for any noetherian, integral scheme which is locally factorial. On the latter, we have an exact sequence of Zariski sheaves (an analog of the Gersten complex):

$$0 \to \mathbb{G}_m \to k(\mathrm{X})^\times \to \bigoplus_{x \in \mathrm{X}^{(1)}} (i_x)_*\mathbb{Z} \to 0.$$

We conclude by nothing that Zariski cohomology in constant sheaves vanishes on an irreducible scheme.

REMARK 3.0.4 (Relationship with flat cohomology). For X smooth over a perfect field κ, there is an equivalence (see [**Ill79**, Theorem 3.23.2] or [**Mor19**, Corollary 3.2] for a more general and more modern reference):

$$\mathrm{R}\Gamma_{\text{ét}}(\mathrm{X}; \Omega^{j}_{\log,\mathrm{X},\text{ét}}[-1]) \simeq \mathrm{R}\Gamma_{\mathrm{fppf}}(\mathrm{X}; \mu_{p,\mathrm{X}}).$$

so that

$$\mathrm{H}^{*-1}_{\text{ét}}(\mathrm{X}; \Omega^{j}_{\log,\text{ét}}) \cong \mathrm{H}^{*}_{\mathrm{fppf}}(\mathrm{X}; \mu_{p}).$$

REMARK 3.0.5 (Relationship with algebraic cycles). An analog of the Gersten complex in this setting is the presheaf

$$\mathrm{U} \mapsto \mathrm{G}^{\star}(\mathrm{U}; \Omega^{j}_{\log,\mathrm{X}}) := \Omega^{j}_{\log,k(\mathrm{X})} \to \bigoplus_{x \in \mathrm{U}^{(1)}} \Omega^{j-1}_{\log,\kappa(x)} \to \cdots \bigoplus_{x \in \mathrm{U}^{(p)}} \Omega^{j-p}_{\log,\kappa(x)} \cdots.$$

Gros and Suwa [**GS88**] proves that the map

$$\pi_{*}\Omega^{j}_{\log,\mathrm{X}} \to \mathrm{G}^{\star}(\mathrm{U}; \Omega^{j}_{\log,\mathrm{X}})$$

is a flasque resolution, from which one can deduce a canonical isomorphism

$$\mathrm{CH}^{j}(\mathrm{X}) \otimes \mathbb{Z}/p \simeq \mathrm{H}^{j}_{\mathrm{Zar}}(\mathrm{X}; \Omega^{j}_{\log}).$$

Example 3.0.2 suggests a renormalization for étale motivic cohomology mod-p. It should be defined as the shifted étale sheaf:

$$\Omega^{j}_{\mathrm{X},\log,\text{ét}}[-j].$$

This is also motivated by the computation of syntomic cohomology over fields in characteristic $p > 0$:

$$\mathrm{R}\Gamma_{\text{ét}}(\mathrm{X}; \Omega^{j}_{\log,\text{ét}}[-j]) \simeq \mathbb{Z}/p(j)^{\mathrm{syn}}(\mathrm{X})$$

whenever X is smooth over a perfect field. Just like the case for $\mu_m^{\otimes j}$, it is also a discrete étale sheaf and the pushforward to the Zariski site is computed as

$$\mathrm{R}\pi_{*}\Omega^{j}_{\mathrm{X},\log,\text{ét}}[-j](\mathrm{U}) \simeq \mathrm{R}\Gamma_{\text{ét}}(\mathrm{U}; \Omega^{j}_{\mathrm{X},\log,\text{ét}})[-j].$$

We thus set, for X a smooth scheme over a perfect field k:

$$\tau^{\leqslant j}\mathrm{R}\pi_{*}\Omega^{j}_{\log}[-j] =: \mathbb{Z}/p(j)^{\mathrm{mot}}_{\mathrm{X}};$$

We then have the following result:

PROPOSITION 3.0.6. *We have an isomorphism*

$$\mathrm{H}^{i+j}_{\mathrm{mot}}(\mathrm{X}; \mathbb{Z}/p(j)) \cong \mathrm{H}^{i}_{\mathrm{Zar}}(\mathrm{X}; \pi_{*}\Omega^{j}_{\log}).$$

Furthermore

$$\Omega^{j}_{\log,\mathrm{X},\mathrm{Zar}} \cong \pi_{*}\Omega^{j}_{\log,\mathrm{X},\text{ét}}$$

for X *a smooth scheme over a perfect field.*

From proposition 3.0.6 we also get the exact same vanishing range as the theory away from the characteristics.

PROOF. The first claim follows by definition. The second claim is actually not quite obvious: one has to prove that for any regular local $\mathbb{F}_p$ A, the map

$$\Omega^{j}_{\log,\mathrm{A},\mathrm{Zar}} \to \pi_{*}\Omega^{j}_{\log,\mathrm{X},\text{ét}}$$

is surjective. The results of Gros and Suwa [**GS88**] and Kerz [**Ker09**] (the latter affording a Gersten exactness result for Minor K-theory) reduces this to following surjectivity statement: there is canonical surjection

$$\mathrm{K}^{\mathrm{M}}(\mathrm{F})/p \to \Omega^{j}_{\log,\mathrm{F},\text{ét}} \to 0;$$

which follows from a theorem of Bloch-Kato-Gabber; see [**Mor19**, Theorem 1.2, Corollary 4.2] for details. $\square$

To build a p-adic theory, we note that there exists extension of $\Omega^j_{\log,\mathrm{X}}$ to a $\mathbb{Z}/p^r$-linear theory for $r \geqslant 1$. Here are the key points:

(1) We have the sheaves $\mathrm{W}_r\Omega^j_{\mathrm{X}}$ which are the individual terms of the de Rham-Witt complexes of Illusie;
(2) we have dlog maps
$$\mathrm{dlog} : \mathbb{G}_m^{\otimes j} \to \mathrm{W}_r\Omega^j_{\mathrm{X}} \qquad f_1 \otimes \cdots \otimes f_j \mapsto \tfrac{d[f_1}{]}[f_1] \wedge \cdots \tfrac{d[f_j]}{[f_j]};$$
out of which we can define $\mathrm{W}_r\Omega^j_{\mathrm{X},\log,\tau}$ as above;
(3) there are natural transition maps
$$\cdots \mathrm{W}_{r+1}\Omega^j_{\log,\mathrm{X},\tau} \to \mathrm{W}_r\Omega^j_{\log,\mathrm{X},\tau} \cdots ;$$
(4) and motivic cohomology can be defined as
$$\tau^{\leqslant j}\mathrm{R}\pi_*\mathrm{W}_r\Omega^j_{\log}[-j] =: \mathbb{Z}/p^r(j)^{\mathrm{mot}}_{\mathrm{X}}.$$

Remark 3.0.7. Another description of the logarithmic Hodge-Witt sheaves uses the Cartier isomorphism; recall that there are maps (of étale sheaves)
$$\mathrm{W}_r\Omega^j_{\mathrm{X}} \xrightarrow{\mathrm{C}^{-1}} \mathrm{W}_r\Omega^j_{\mathrm{X}}/d\mathrm{W}_r\Omega^{j-1}_{\mathrm{X}};$$
and $\mathrm{W}_r\Omega^j_{\log,\mathrm{X}}$ fits as the kernel
$$0 \to \mathrm{W}_r\Omega^j_{\log,\mathrm{X}} \to \mathrm{W}_r\Omega^j_{\mathrm{X}} \xrightarrow{\pi-\mathrm{C}^{-1}} \mathrm{W}_r\Omega^j_{\mathrm{X}}/d\mathrm{W}_r\Omega^{j-1}_{\mathrm{X}};$$
where π is the obvious projection; see [**Mor19**, Corollary 4.1].

4. Rational motivic cohomology

Rational motivic cohomology can be defined as the eigenspectra of a certain filtration on K-theory; we sketch this theory and defer details to [**Rio10**] and [**CD19**, Chapter 14] and the upcoming [**EM23**]. This filtration is somewhat easier to construct in characteristic $p > 0$:

Construction 4.0.1. Let k be a perfect field of characteristic $p > 0$. The Frobenius defines natural a transformation on Sm_k:
$$\mathrm{Fr}^* : \mathrm{K} \to \mathrm{K}.$$
Rationalizing this action, we get that $\mathrm{K}_{\mathbb{Q}}$ splits into a direct sum of eigenspectra
$$\mathbb{Q}(j)^{\mathrm{mot}}[2j] := \mathrm{K}_{\mathbb{Q}}^{\mathrm{Fr}^*-p^j} \qquad j \geqslant 0.$$
The filtration is the "canonical" one given by
$$\mathrm{Fil}^{\geqslant j}\mathrm{K}_{\mathbb{Q}} := \bigoplus_{i \leqslant j} \mathbb{Q}(i)^{\mathrm{mot}}[2i].$$

In characteristic zero, we use the Adams operations in lieu of the Frobenius. We follow the construction in [**BH20**] within the world of motivic stable homotopy theory, buta use the universal properties described by Iwasa in his lectures in this series [**AI22**] instead of $\mathbb{A}^1$-invariance. For what follows we write $\mathcal{P}\mathrm{ic}$ for the Picard stack of line bundles, restricted to the category of smooth k-schemes. It can also be described as the classifying stack $\mathrm{B}\mathbb{G}_m = \mathrm{B}_{\mathrm{fppf}}\mathbb{G}_m$.

Construction 4.0.2. Let $n \geqslant 2$. We work with presheaves on Sm_k. First, consider the map of group schemes:
$$\mathbb{G}_m \to \mathbb{G}_m \qquad z \mapsto z^n.$$
This induces a map of commutative monoids:
$$(-)^n : \mathcal{P}\mathrm{ic} \to \mathcal{P}\mathrm{ic}.$$

We take Σ^∞_+, the suspension spectrum to get to presheaves of spectra on Sm_k; this yields a morphims of $\mathbb{E}_\infty$-rings:

$$\psi^n : \Sigma^\infty_+ \mathcal{P}\mathrm{ic} \to \Sigma^\infty_+ \mathcal{P}\mathrm{ic}.$$

We have the Bott element, classified by a map

$$\beta := 1 - [\mathcal{O}(1)] : \Sigma^\infty_+ \mathbb{P}^1 \to \Sigma^\infty_+ \mathcal{P}\mathrm{ic} \to \mathrm{K};$$

and the main theorem of [**AI22**] furnishes an equivalence (of $\Sigma^\infty_+ \mathcal{P}\mathrm{ic}$-modules)

$$\Sigma^\infty_+ \mathcal{P}\mathrm{ic}[\beta^{-1}] \simeq \mathrm{K}.$$

Writing $\mathrm{K}^{[n]}$ as K-theory with the $\Sigma^\infty_+ \mathcal{P}\mathrm{ic}$-module structure given by restriction along ψ^n, we obtain an $\Sigma^\infty_+ \mathcal{P}\mathrm{ic}$-module map

$$\Sigma^\infty_+ \mathcal{P}\mathrm{ic} \xrightarrow{\psi^n} \Sigma^\infty_+ \mathcal{P}\mathrm{ic} \to \mathrm{K}^{[n]};$$

inverting β along the above map gives

$$\mathrm{K} \to \mathrm{K}^{[n]}[\tfrac{1}{n\beta}] \simeq \mathrm{K}[\tfrac{1}{n}]$$

since $\psi^n(\beta) = n\beta$ by [**BH20**, Lemma 3.1] and thus we get a map

$$\psi^n : \mathrm{K}[\tfrac{1}{n}] \to \mathrm{K}[\tfrac{1}{n}]$$

As in Construction 4.0.1, we set

$$\mathbb{Q}(j)^{\mathrm{mot}}[2j] := \mathrm{K}_{\mathbb{Q}}^{\psi^j - p^j} \qquad j \geqslant 0.$$

4.1. The Beilinson motivic cohomology spectrum. Being the graded pieces of a split filtration on K-theory, $\mathbb{Q}(j)^{\mathrm{mot}}$ enjoys all the properties K-theory does on smooth schemes. In particular, it automatically enjoys:

(Zariski descent) $\mathbb{Q}(j)^{\mathrm{mot}}$ is a Zariski sheaf;

($\mathbb{A}^1$-invariance) $\mathbb{Q}(j)^{\mathrm{mot}}(\mathrm{X}) \xrightarrow{\simeq} \mathbb{Q}(j)^{\mathrm{mot}}(\mathrm{X} \times \mathbb{A}^1)$;

($\mathbb{P}^1$-bundle formula) there is a canonical first chern class map $c_1 : \mathcal{P}\mathrm{ic} \to \mathbb{Q}(1)[2]$ which induces an equivalence for $j \geqslant 1$:

$$\mathbb{Q}(j)^{\mathrm{mot}}(\mathrm{X}) \oplus \mathbb{Q}(j-1)[-2](\mathrm{X}) \xrightarrow{\pi^* \oplus c_1(\mathcal{O}(-1))\pi^*} \mathbb{Q}(j)^{\mathrm{mot}}(\mathrm{X} \times \mathbb{P}^1).$$

These three properties means that $\mathbb{Q}(j)^{\mathrm{mot}}$ assembles into a motivic spectrum in the sense of Morel-Voevodsky; but the first and the last property already shows that $\mathbb{Q}(j)^{\mathrm{mot}}$ defines a motivic spectrum in the sense of [**AI22**]. In any case, this resulting motivic spectrum is usually denoted by HB for **Beilinson motivic cohomology**; it enjoys a certain universal property formulated in [**CD19**, Corollary 14.2.16]. This lets us map out of HB, and hence out of $\mathbb{Q}(j)^{\mathrm{mot}}$ provided that our target is satisfies suitable properties. Our goal is to map into $\left(\prod_p \mathbb{Z}_p(j)^{\mathrm{mot}}\right)_{\mathbb{Q}}$, but we need it to first have the expected properties of motivic cohomology.

4.2. $\mathbb{Z}_p(j)$ as a motivic spectrum. In order to build a map out of HB, we need to show that $(\prod \mathbb{Z}_p(j))_{\mathbb{Q}}$ is assembles into a motivic spectrum; in turn we need to show that for each p, $\mathbb{Z}_p(j)$ is a motivic spectrum. In particular we discuss how one can verify:

(Zariski descent) $\mathbb{Z}_p(j)^{\mathrm{mot}}$ is a Zariski sheaf;

($\mathbb{A}^1$-invariance) $\mathbb{Z}_p(j)^{\mathrm{mot}}(\mathrm{X}) \xrightarrow{\simeq} \mathbb{Z}_p(j)^{\mathrm{mot}}(\mathrm{X} \times \mathbb{A}^1)$;

($\mathbb{P}^1$-bundle formula) there is a canonical first chern class map $c_1 : \mathcal{P}\mathrm{ic} \to \mathbb{Z}_p(1)[2]$ which induces an equivalence for $j \geqslant 1$:

$$\mathbb{Z}_p(j)^{\mathrm{mot}}(\mathrm{X}) \oplus \mathbb{Z}_p(j-1)[-2](\mathrm{X}) \xrightarrow{\pi^* \oplus c_1(\mathcal{O}(-1))\pi^*} \mathbb{Z}_p(j)^{\mathrm{mot}}(\mathrm{X} \times \mathbb{P}^1).$$

The first property has already been discussed. We note that the difficulty in establishing the two properties above lies in the fact that we have sheafified the truncation. So even though étale cohomology has these properties it is not *a priori* clear that they are preserved under sheafification.

One way to proceed is to simply appeal to the Rost-Voevodsky theorem which relates $\mathbb{Z}_p(j)^{\mathrm{mot}}$ to other models of motivic cohomology (see references) where these properties are more evident.

5. Properties of motivic cohomology

In this section, we discuss and summarize some basic properties of motivic cohomology. Throughout we have a perfect field k and that motivic cohomology is defined for schemes which are smooth over such fields.

5.0.1. *Low weights.* As we have seen throughout the text, motivic cohomology at low weights can be described:

EXAMPLE 5.0.2 (Weight zero). Motivic cohomology in weight zero always corresponds to the constant sheaf

$$\mathbb{Z} \simeq \mathbb{Z}(0)^{\mathrm{mot}};$$

and the edge map $\mathrm{K}(\mathrm{X}) \to \mathrm{H}^0_{\mathrm{Zar}}(\mathrm{X}; \mathbb{Z})$ corresponds to the rank homomorphism.

EXAMPLE 5.0.3 (Weight one). In weight one, motivic cohomology is a shift of the $\mathbb{G}_m$:

$$\mathbb{Z}(1)^{\mathrm{mot}}[1] \simeq \mathbb{G}_m;$$

in particular we have a morphism (thinking of the target as a presheaf of spaces under Dold-Kan):

$$\mathcal{P}\mathrm{ic} \to \mathbb{G}_m[1] \simeq \mathbb{Z}(1)^{\mathrm{mot}}[2],$$

corresponding to a theory of the first chern class. We also have an edge map

$$\mathrm{K}_0(\mathrm{X}) \to \mathrm{H}^2_{\mathrm{mot}}(\mathrm{X}; \mathbb{Z}(1))$$

corresponding to the determinant. In the motivic spectral sequence, the first possible live differential is the map

$$d_2 : \mathrm{H}^1_{\mathrm{mot}}(\mathrm{X}; \mathbb{Z}(1)) \to \mathrm{H}^4_{\mathrm{mot}}(\mathrm{X}; \mathbb{Z}(2)).$$

This map is, in fact, zero: since this is a functorial map in X and weight one motivic cohomology is representable by a shift of $\mathbb{G}_m$, it is determined by a class in $\mathrm{H}^4_{\mathrm{mot}}(\mathbb{G}_m; \mathbb{Z}(2)) \cong \mathrm{CH}^2(\mathbb{G}_m) = 0$. Therefore, we have an edge map:

$$\mathrm{K}_1(\mathrm{X}) \to \mathrm{H}^1_{\mathrm{mot}}(\mathrm{X}; \mathbb{Z}(1))$$

which is induced by a space-level determinant map

$$\Omega^{\infty}\mathrm{K} \to \mathcal{P}\mathrm{ic}.$$

5.0.4. *$\mathbb{P}^1$-bundle formula and $\mathbb{A}^1$-invariance.* We have discussed extensively the following:

THEOREM 5.0.5. *For any smooth k-scheme* X*, we have the following*

($\mathbb{A}^1$*-invariance*) *canonical equivalences* $\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X}) \xrightarrow{\simeq} \mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X} \times \mathbb{A}^1)$*;*

($\mathbb{P}^1$*-bundle formula*) *there is a canonical first chern class map* $c_1 : \mathcal{P}\mathrm{ic} \to \mathbb{Z}_p(1)[2]$ *which induces an equivalence for* $j \geqslant 1$*:*

$$\mathbb{Z}_p(j)^{\mathrm{mot}}(\mathrm{X}) \oplus \mathbb{Z}_p(j-1)[-2](\mathrm{X}) \xrightarrow{\pi^* \oplus c_1(\mathcal{O}(-1))\pi^*} \mathbb{Z}_p(j)^{\mathrm{mot}}(\mathrm{X} \times \mathbb{P}^1).$$

5.0.6. *Vanishing range.* By construction, we are assured of the following result, out of which we derive all the vanishing statements:

THEOREM 5.0.7. *On* $\mathrm{X}_{\mathrm{Zar}}$*,* $\mathbb{Z}(j)^{\mathrm{mot}}|_{\mathrm{X}_{\mathrm{Zar}}}$ *is locally concentrated in (cohomological) degrees* $\leqslant j$*. In other words, in the standard t-structure on* $\mathrm{Shv}(\mathrm{X}_{\mathrm{Zar}}, \mathrm{D}(\mathbb{Z}))$*, we have*

$$\mathbb{Z}(j)^{\mathrm{mot}}|_{\mathrm{X}_{\mathrm{Zar}}} \in \mathrm{Shv}(\mathrm{X}_{\mathrm{Zar}}, \mathrm{D}(\mathbb{Z}))^{\leqslant j}.$$

PROPOSITION 5.0.8. *If* X *is (essentially) smooth over* k *and* d *is the (essential) Krull dimension of* X*, then:*

(1)

5.0.9. *Relationship with Milnor* K*-theory.* Thanks to Theorem 5.0.7 there is a map from $\mathbf{Z}(j)^{\mathrm{mot}}$ to its highest cohomology sheaf

$$\mathbb{Z}(j)^{\mathrm{mot}} \to \mathcal{H}^j(\mathbb{Z}(j)^{\mathrm{mot}})[-j];$$

the identity of the latter is given by Milnor K-theory thanks to the work of Suslin-Nesterenko [], Totaro [] and Kerz.

5.0.10. *Relationship with algebraic cycles.* As already discussed we have:

THEOREM 5.0.11. *For* X *a smooth scheme over a perfect field* k*, there is a canonical isomorphism of graded rings*

$$\mathrm{CH}^{\star}(\mathrm{X}) \cong \mathrm{H}^{2\star}_{\mathrm{mot}}(\mathrm{X}; \mathbb{Z}(\star)).$$

There is another isomorphism which is quite useful but stated less often. Recall that Bloch's formula gives $\mathrm{CH}^j(\mathrm{X}) \cong \mathrm{H}^j_{\mathrm{Zar}}(\mathrm{X}; \mathcal{K}^{\mathrm{M}}_j)$. There is in fact an isomorphism

PROPOSITION 5.0.12. *With the same hypotheses as Theorem 5.0.11, there is a canonical isomorphism:*

$$\mathrm{H}^{j-1}_{\mathrm{Zar}}(\mathrm{X}; \mathcal{K}^{\mathrm{M}}_j) \cong \mathrm{H}^{2j-1}_{\mathrm{mot}}(\mathrm{X}; \mathbb{Z}(j)).$$

PROOF. We have the descent spectral sequence for a fixed $j \geqslant 0$ (which already arises from our discussion on Bloch-Ogus theories!):

$$\mathrm{H}^p_{\mathrm{Zar}}(\mathrm{X}; \mathcal{H}^q(\mathbb{Z}(j)) \Rightarrow \mathrm{H}^{p+q}(\mathrm{X}; \mathbb{Z}(j))$$

which displays as (setting $\mathrm{H}^p(q) = \mathrm{H}^p_{\mathrm{Zar}}(\mathrm{X}; \mathcal{H}^q(\mathbb{Z}(j))))$.

$$
\begin{tikzcd}
0 & 0 & 0 & & \cdots & 0 \\
0 & \mathrm{H}^0(j) \arrow[drr] & \mathrm{H}^1(j) & & \mathrm{H}^{j-1}(j) & \mathrm{H}^j(j) \\
0 & \mathrm{H}^0(j-1) & \mathrm{H}^1(j) & \cdots & \mathrm{H}^{j-1}(j-1) & 0 \\
0 & \vdots & \vdots & \vdots & \vdots & \\
0 & \mathrm{H}^0(2) \arrow[drr] & \mathrm{H}^1(2) & \mathrm{H}^2(2) & \cdots & 0 \\
0 & \mathrm{H}^0(1) \arrow[drr] & \mathrm{H}^1(1) & 0 & \cdots & 0 \\
0 & \mathrm{H}^0(0) \arrow[drr] & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & &
\end{tikzcd}
\tag{5.0.13}
$$

$\square$

5.0.14. *Relationship with algebraic* K*-theory.*

CHAPTER 3

Lecture 2: the cdh topology and applications

So far, we have discussed motivic cohomology of smooth k-schemes and now we wish to create some kind of extension to singular schemes. It is not a bad idea to see what has been done for other cohomology theories.

1. The Deligne-du Bois complex

Let k be a field of characteristic zero and $\mathrm{X} \in \mathrm{Sch}_k^{\mathrm{ft}}$ and is integral. Then we have the **Deligne-du Bois** complex constructed as follows: pick a hypercover $\mathrm{X}_\bullet \to \mathrm{X}$ where:

(1) the maps $\mathrm{X}_n \to \mathrm{X}$ are proper surjections,
(2) each term X_n is smooth over k (this uses resolution of singularities).

Then set

$$\underline{\Omega}^*_{\mathrm{X}/k} := \lim_{[n]\in\Delta} \mathrm{R}f_*\Omega^*_{\mathrm{X}_n/k}.$$

This object promotes to the (decreasingly-)filtered complex via the **du Bois filtration**:

$$\mathrm{Fil}^j_{\mathrm{dB}}\underline{\Omega}^*_{\mathrm{X}/k} := \lim_{[n]\in\Delta} \mathrm{R}f_*\Omega^{\geqslant j}_{\mathrm{X}_n/k};$$

$$\mathrm{Fil}^\star_{\mathrm{dB}}\underline{\Omega}^*_{\mathrm{X}/k} \to \underline{\Omega}^*_{\mathrm{X}/k}.$$

The graded pieces are denoted by

$$\mathrm{gr}^j_{\mathrm{dB}}\underline{\Omega}^*_{\mathrm{X}/k} \simeq \lim_{[n]\in\Delta} \mathrm{R}f_*\Omega^j_{\mathrm{X}_n/k}[-j];$$

and we set

$$\underline{\Omega}^j_{\mathrm{X}/k} := \mathrm{gr}^j_{\mathrm{dB}}\underline{\Omega}^*_{\mathrm{X}/k}[j].$$

If $k = \mathbb{C}$, Deligne's theory of cohomological descent proves that, in fact, we have a quasi-isomorphism in $\mathrm{D}(\mathrm{X};\mathbb{C})$:

$$\underline{\Omega}^*_{\mathrm{X}/\mathbb{C}} \simeq \mathbb{C}_{\mathrm{X}}.$$

However, it is not usually the case that

$$\mathrm{R}\Gamma_{\mathrm{Zar}}(\mathrm{X};\Omega^j_{\mathrm{X}/k}) \simeq \mathrm{R}\Gamma(\mathrm{X};\underline{\Omega}^j_{\mathrm{X}/k}).$$

Remark 1.0.1. There is a morphism $\Omega^*_{\mathrm{X}/k} \to \underline{\Omega}^*_{\mathrm{X}/k}$ which, in fact, admits a splitting in the derived category of sheaves of k-modules [**Bha12a**, Proposition 5.2]. In general, the Deligne-du Bois complex and the de Rham complex is not a quasi-isomorphism. The "correct" thing to compare against, however, will turn out to be the derived de Rham complex which we will explain later.

Example 1.0.2. Over any field, one can prove that $\mathrm{H}^0(\mathrm{X};\underline{\Omega}^0) = \mathcal{O}(\mathrm{X}^{\mathrm{sn}})$ where X^{sn} is the seminormalization of X (see [**HK18**, Proposition 6.2] for a general proof of [**Sch09**, Lemma 5.6] for a proof in characteristic zero and [**Voe96**, Section 3.2] for a general statement). Of course, if X is not seminormal, we will not get that this isomorphic to $\mathrm{H}^0(\mathrm{X};\mathcal{O})$ in general. In any case, $\underline{\Omega}^*_{\mathrm{X}/\mathbb{C}}$ gives rise to Hodge theory of proper but not-necessarily-smooth varieties over $\mathbb{C}$. For example, Deligne proved that the spectral sequence from the du Bois filtration collapses. One important consequence of this is a surjection of the edge map

$$\mathrm{H}^i(\mathrm{X};\mathbb{C}) \to \mathrm{H}^i(\mathrm{X};\underline{\Omega}^0_{\mathrm{X}}),$$

which is important for the study of singularities in birational geometry. When the target is isomorphic to $\mathrm{H}^i(\mathrm{X};\mathcal{O}_\mathrm{X})$, one can extend Kodaira vanishing to these possibly singular varieties. Since types of singularities should be a local condition, we formulate this condition on the level of sheaves: we say that X has **du Bois singularities** if the map $\mathcal{O}_\mathrm{X} \to \underline{\Omega}^0$ is a quasi-isomorphism.

The construction of the Deligne-du Bois complex is not satisfactory for a couple of reasons:

(1) the construction involves the choice of a proper hypercover of X and it is not clearly independent of this choice;
(2) relatedly we need to have resolution of singularities which will not hold over arbitrary characteristics.

In a bit, we will explain how to reformulate the Deligne-du Bois complex and the du Bois filtration.

This reformulation is useful in proving the following "innoncuous theorem":

Theorem 1.0.3. *Let $d = \dim(\mathrm{X})$. Then the map*

$$\mathrm{H}^d_{\mathrm{Zar}}(\mathrm{X};\Omega^j_{\mathrm{X}/k}) \to \mathrm{H}^d(\mathrm{X};\underline{\Omega}^j_{\mathrm{X}/k})$$

is a surjection. In particular for $d \geqslant 1$ and X affine, then $\mathrm{H}^d(\mathrm{X};\underline{\Omega}^j_{\mathrm{X}/k}) = 0$.

In this generality, this is a result of Cortiñas, Haesemeyer, Schlichting and Weibel [**CnHSW08**]. We will soon see its significance in the study of the K-theory of singularities.

2. Weibel's conjecture

By K(X) we mean the non-connective K-theory of X and by $\mathrm{K}_{\geqslant 0}(\mathrm{X})$ we mean the connective part. The negative K-groups are the negative homotopy groups of K but they are actually quite easy to define and depends only on K_0.

Definition 2.0.1. If $\mathrm{F} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Ab}$ is a functor, then we write

$$\mathrm{F}_{-1}(\mathrm{X}) := \mathrm{coker}(\mathrm{F}(\mathrm{X}\times\mathbb{A}^1_+) \oplus \mathrm{F}(\mathrm{X}\times\mathbb{A}^1_-) \to \mathrm{F}(\mathrm{X}\times\mathbb{G}_m)).$$

Iterating this construction, we get the presheaf F_{-n} for $n \geqslant 1$.

The negative K-theory groups identify as the n-fold contraction of K_0; hence it was defined (by Bass) before K-theory became a spectrum. In particular, we get Bass' fundamental sequence

$$0 \to \mathrm{K}_n(\mathrm{X}) \to \mathrm{K}_n(\mathrm{X}\times\mathbb{A}^1) \oplus \mathrm{K}_n(\mathrm{X}\times\mathbb{A}^1) \to \mathrm{K}_n(\mathrm{X}\times\mathbb{G}_m) \to \mathrm{K}_{n-1}(\mathrm{X}) \to 0; \tag{2.0.2}$$

where the last map admits a natural splitting. This sequence says that the higher K-groups, in fact, *refines* the lower ones.

Remark 2.0.3. Using the formula above, Negative K-theory has a concrete interpretation: for $n > 0$, $\mathrm{K}_{-n}(\mathrm{X})$ is a quotient of $\mathrm{K}_0(\mathrm{X}\times\mathbb{G}_m^{\times n})$. Any element in the image of $\mathrm{K}_0(\mathrm{X}\times\mathbb{A}^n) \to \mathrm{K}_0(\mathrm{X}\times\mathbb{G}_m^{\times n})$ vanishes. Roughly, it is a measure of to what extend we can extend (virtual) vector bundles from $\mathrm{X}\times\mathbb{G}_m^{\times n}$ to $\mathrm{X}\times\mathbb{A}^n$.

Example 2.0.4. Let C be the (affine) nodal curve over k. Then we have a cdh square of the form:

$$\begin{tikzcd} \mathrm{Spec}\,k \sqcup \mathrm{Spec}\,k \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d, "q"] \\ \mathrm{Spec}\,k \arrow[r, "j"] & \mathrm{C}. \end{tikzcd}$$

We have an exact sequence (which *does not* extend to the left):

$$\big(\mathrm{K}_0(k) \cong \mathrm{K}_0(\mathbb{A}^1)\big) \oplus \mathrm{K}_0(k) \to \mathrm{K}_0(\mathrm{Spec}\,k)^{\oplus 2} \to \mathrm{K}_{-1}(\mathrm{C}) \to 0;$$

which reads as

$$\mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z} \to \mathrm{K}_{-1}(\mathrm{C}) \to 0.$$

But now, the matrix is given by

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

hence is only rank 1 and $\mathrm{K}_{-1}(\mathrm{C}) = \mathbb{Z}$.

The following conjecture (now a theorem) was, for a long time, one of the guiding conjectures about the K-theory of non-smooth schemes.

THEOREM 2.0.5 (Kerz, Strunk, Tamme; Weibel's conjecture). *If* X *is a noetherian scheme, then*

$$\mathrm{K}_{<-\dim(\mathrm{X})}(\mathrm{X}) = 0.$$

We note that, in characteristic zero, Theorem 2.0.5 was proved by Cortiñas, Haesemyer, Schlichting and Weibel. The goal for the rest of this lecture is to give a fan fiction account of their proof and how to arrive at the Theorem 3.0.1 from this point of view (which was the key statement for them as well).

3. The cdh topology

In any account of the proof of Theorem 2.0.5, one introduces the cdh topology. For simplicity, we will work with the categoy $\mathrm{Sch}_k^{\mathrm{ft}}$ of finite type k-schemes, though the cdh topology is well behaved in broader contexts as well.

DEFINITION 3.0.1. Let k be a field.

(1) A **Nisnevich square** is a cartesian square in $\mathrm{Sch}_k^{\mathrm{ft}}$:

$$\begin{tikzcd} \mathrm{V} \arrow[r] \arrow[d] & \mathrm{X}' \arrow[d, "q"] \\ \mathrm{U} \arrow[r, "j"] & \mathrm{X}, \end{tikzcd} \tag{3.0.2}$$

where j is an open immersion and q is an étale morphism inducing an isomorphism $q^{-1}(\mathrm{X} \smallsetminus \mathrm{U})_{\mathrm{red}} \to (\mathrm{X} \smallsetminus \mathrm{U})_{\mathrm{red}}$.

(2) An **abstract blowup square** is a cartesian square in $\mathrm{Sch}_k^{\mathrm{ft}}$:

$$\begin{tikzcd} \mathrm{Y}' \arrow[r] \arrow[d] & \mathrm{X}' \arrow[d, "p"] \\ \mathrm{Y} \arrow[r, "i"] & \mathrm{X}, \end{tikzcd} \tag{3.0.3}$$

where p is a proper morphism and i is a closed immersion such that the induced map $p : \mathrm{X}' \smallsetminus \mathrm{Y} \to \mathrm{X} \smallsetminus \mathrm{Y}$ is an isomorphism.

(3) Let $\mathcal{C}$ be a stable ∞-category. A presheaf

$$\left(\mathrm{Sch}_k^{\mathrm{ft}}\right)^{\mathrm{op}} \to \mathcal{C},$$

is cdh-**excisive** if it converts both Nisnevich squares and abstract blowup squares to pullbacks in $\mathcal{C}$.

Nisnevich squares are more familiar to a wider audience so let us explain how abstract blowup squares work.

EXAMPLE 3.0.4. If $Z \hookrightarrow X$ is a closed immersion of codimension $\geqslant 2$, then the blouwp along Z produces an actual blowup square which is, in particular, an abstract blowup square:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathrm{Bl}_Z X \arrow[d, "p"] \\ Z \arrow[r, "i"] & X. \end{tikzcd}$$

EXAMPLE 3.0.5. Let $A \hookrightarrow B$ be a finite extension of finite type k-algebras. We have the the conductor ideal $\mathrm{Ann}_A(B) = \{a \in A : aB \subset A\}$. The cartesian square in schemes

$$\begin{tikzcd} \operatorname{Spec} B/B\mathrm{Ann}_A(B) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d, "p"] \\ \operatorname{Spec} A/\mathrm{Ann}_A(B) \arrow[r, "i"] & \operatorname{Spec} A, \end{tikzcd}$$

is an abstract blowup square which is not an actual blowup.

EXAMPLE 3.0.6. Here is a degenerate-looking example but turns out to be quite important. Let $X_{\mathrm{red}} \hookrightarrow X$ be the closed immersion given by the reduced locus of X; then

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \emptyset \arrow[d, "p"] \\ X_{\mathrm{red}} \arrow[r, "i"] & X, \end{tikzcd}$$

is an abstract blowup square. In particular, this proves any cdh-excisive presheaf is nilinvariant.

The formalism of cd-structures of Voevodsky [**Voe10**] proves that we can interpret cdh-excision sheaf-theoretically. The cdh topology is the one generated by the pretopology where covers are $\{U \to X, X' \to X\}$ extracted from Nisnevich squares and $\{Y \to X, X' \to X\}$ extracted from abstract blowup squares. The upshot of the sheaf-theoretic interpretation of the Nisnevich topology is that we can speak of stalks which are henselian valuation rings. Also important is that we have the cdh-sheafification functor:

$$\mathrm{PSh}\left(\mathrm{Sch}_k^{\mathrm{ft}}\right) \xrightarrow{L_{\mathrm{cdh}}} \mathrm{Shv}_{\mathrm{cdh}}\left(\mathrm{Sch}_k^{\mathrm{ft}}\right)$$

left adjoint to the inclusion of cdh sheaves into presheaves.

3.1. Revisiting the Deligne-du Bois complex. This is an extended example in a new language. Let us revisit the Deligne-du Bois complex using the language of cdh topology. To the author's knowledge, this point of view was advocated by Huber and her collaborators; see the series [**Hub16, HKK17, HK18**].

CONSTRUCTION 3.1.1 (Derived de Rham complex). The derived de Rham complex of Bhatt and Illusie (see [**Bha12a, Bha12b**] or [**SZ18**, Section 2] and [**Mat22**, Section 7] for expositions) is the left Kan extension of the functor sending $R \mapsto \Omega^*_{R/k}$ polynomial k-algebras to all k-algebras; it is then extended to all schemes via right Kan extension; we write this as

$$X \mapsto L\Omega_{X/k}.$$

There is a **Hodge completed version**, $L\widehat{\Omega}_{X/k}$ where one forces to Hodge filtration to be complete; on a k-algebra R, it is computed via the formula

$$L\widehat{\Omega}_{R/k} \simeq \lim_j \operatorname*{colim}_{\Delta^{\mathrm{op}}} \Omega^{\leqslant j}_{P_\bullet/k}$$

where $\mathrm{P}_\bullet \to \mathrm{R}$ is a simplicial resolution of R by polynomial algebras. All in all, we get a filtered object

$$\mathrm{Fil}^\star_{\mathrm{Hdg}}\mathrm{L}\widehat{\Omega}_{\mathrm{X}/k} \to \mathrm{L}\widehat{\Omega}_{\mathrm{X}/k};$$

whose graded pieces are given by the cotangent complex and its (derived) wedge powers:

$$\mathrm{gr}^\star_{\mathrm{Hdg}}\mathrm{L}\widehat{\Omega}_{\mathrm{X}/k} \simeq \mathbb{L}^j_{\mathrm{X}/k}[-j].$$

The above construction works in any characteristic. If k is a field of characteristic zero, then Bhatt [**?**, Proposition 5.2] proves that there is a canonical equivalence

$$\mathrm{L}\widehat{\Omega}_{\mathrm{X}/k} \xrightarrow{\simeq} \underline{\Omega}^*_{\mathrm{X}/k}$$

for any $\mathrm{X} \in \mathrm{Sch}^{\mathrm{ft}}_k$. The key point is that, all along, the Deligne-du Bois complex is an explicit model for cdh-sheafification of the de Rham complex and that $\mathrm{L}\widehat{\Omega}_{\mathrm{X}/k}$ is cdh-excisive (we refer to an appendix for an account of this). While it is true that the Hodge filtration maps to the du Bois filtration [**?**, Proposition 5.2]

$$\mathrm{Fil}^\star_{\mathrm{Hdg}}\mathrm{L}\widehat{\Omega}_{\mathrm{X}/k} \to \mathrm{Fil}^\star_{\mathrm{dB}}\underline{\Omega}^*_{\mathrm{X}/k};$$

we can already see on graded pieces that the map is not an equivalence: it induces a map

$$\mathbb{L}^j_{\mathrm{X}/k} \to \Omega^j_{\mathrm{X}/k}[0] \to \mathrm{L}_{\mathrm{cdh}}\Omega^j_{\mathrm{X}/k}[0].$$

This map is in fact the cdh-sheafification of the wedge powers of the cotangent complex, a fact which is true in any characteristics. The content here is that $\mathbb{L}^j_{(-)/k}$ is cdh-locally discrete.

Lemma 3.1.2. *Let* $\mathrm{X} \in \mathrm{Sch}^{\mathrm{ft}}_k$*, where k is any field. Then there is a canonical equivalence:*

$$\mathrm{L}_{\mathrm{cdh}}\mathbb{L}^j_{\mathrm{X}/k} \simeq \mathrm{L}_{\mathrm{cdh}}\Omega^j_{\mathrm{X}/k}.$$

Proof. There is a comparison map $\mathbb{L}^j_{(-)/k} \to \mathcal{H}_0(\mathbb{L}^j_{(-)/k}) \simeq \Omega^j_{(-)/k}$; to prove the result it suffices to check that the induced map on stalks is an equivalence. The following statement suffices:

($\star$) each valuation ring containing k is a colimit of its filtered subalgebras.

Indeed, the formation of the L^j's and Ω^j's preserve filtered colimits and, on smooth k-algebras, $\mathrm{L}^j_{\mathrm{A}/k} \simeq \Omega^j_{\mathrm{A}/k}[0]$. In turn, the statement ($\star$) follows easily from resolution of singularities. Without this, Gabber and Romero were able to prove the equivalence $\mathrm{L}^j_{\mathrm{V}/k} \simeq \Omega^j_{\mathrm{V}/k}[0]$ in [**GR03**, Theorem 6.5.8]. $\square$

Lemma 3.1.2 illustrates an important principle in trying to study motivic cohomology of singular schemes *without* resolution of singularities: local statements of valuation rings usually suffice for our purposes. The next theorem is similar.

4. Weibel's conjecture in characteristic zero

We begin by working over any field k (though most of the initial discussion works in arbitrary generality). To begin the proof of Theorem 2.0.5 we consider the canonical sheafification map $\mathrm{K} \to \mathrm{L}_{\mathrm{cdh}}\mathrm{K}$. We can ask for an approximation of Weibel's conjecture.

Theorem 4.0.1. *Let* X *be a noetherian scheme of dimension d, then* $\pi_{<-d}\mathrm{L}_{\mathrm{cdh}}\mathrm{K}(\mathrm{X}) = 0$. *Furthermore,*

$$\pi_d\mathrm{L}_{\mathrm{cdh}}\mathrm{K}(\mathrm{X}) \cong \mathrm{H}^d_{\mathrm{cdh}}(\mathrm{X};\mathbb{Z}).$$

Proof. We have a spectral sequence

$$\mathrm{H}^p_{\mathrm{cdh}}(\mathrm{X};\mathcal{K}_q) \Rightarrow \pi_{q-p}\mathrm{L}_{\mathrm{cdh}}\mathrm{K}(\mathrm{X});$$

though it is not clear at all this spectral sequence converges. It is theorem, due to [], that it actually does and the proof is interesting though it will lead us further afield. Nonetheless, we see that it suffices to prove the local version of this statement:

($\star$) for any henselian valuation ring V, we have that $\pi_{<0}\mathrm{K}(\mathrm{V}) = 0$.

This follows from a theorem of Osofsky (from the 70's) [**?**] or a more recent one by Datta [**Dat17**, Theorem 5.1]. We discuss the later statement since it's much cleaner: for any valuation ring (possibly not containing a field) of finite rank (so finite Krull dimension), the global dimension is $\leqslant 2$. This is a remarkable statement since, for noetherian rings, global dimension is $\leqslant \dim(\mathrm{R})$ so there is some hidden "regularity" property for valuation rings which seem unexpected. In any case, for any stably coherent ring of finite global dimension it is well-known that negative K-theory vanishes. □

From this, it suffices to compare K and $\mathrm{L_{cdh}K}$. Indeed, we will try to prove the following more refined statement: for any d-dimensional X, the fiber of $\mathrm{K(X)} \to \mathrm{L_{cdh}K(X)}$ is supported in $\pi_{\geqslant -d-1}$. This will also prove the isomorphisms

$$\mathrm{K}_{-d}(\mathrm{X}) \cong \mathrm{L_{cdh}K}_{-d}(\mathrm{X}) \cong \mathrm{H}^d_{\mathrm{cdh}}(\mathrm{X};\mathbb{Z}), \tag{4.0.2}$$

which is part of the package of Weibel's conjecture[1].

To study this fiber, we will use **trace methods**. In particular, trace methods we will reduce Weibel's conjecture to Theorem 3.0.1 in characteristic zero and also tell us something interesting in characteristic $p > 0$.

4.1. A motivic proof of (4.0.1). In other lectures in this summer school, we learned about **topological cyclic homology** and other theories. In particular there is a cyclotomic trace map

$$\mathrm{tr} : \mathrm{K} \to \mathrm{TC}.$$

In characteristic zero, TC is just $\mathrm{HC}^-(-/\mathbb{Q})$. The key input is the following very deep result:

THEOREM 4.1.1 (Cortiñas, Land–Tamme). *Let* X *be a qcqs scheme. The the square*

$$\begin{array}{ccc} \mathrm{K(X)} & \longrightarrow & \mathrm{TC(X)} \\ \downarrow & & \downarrow \\ \mathrm{L_{cdh}K(X)} & \longrightarrow & \mathrm{L_{cdh}TC(X)}^2. \end{array} \tag{4.1.2}$$

is cartesian. If X *is a* $\mathbb{Q}$*-scheme, then we have a cartesian square*

$$\begin{array}{ccc} \mathrm{K(X)} & \longrightarrow & \mathrm{HC}^-(\mathrm{X}/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \mathrm{L_{cdh}K(X)} & \longrightarrow & \mathrm{L_{cdh}HC}^-(\mathrm{X}/\mathbb{Q}). \end{array} \tag{4.1.3}$$

The proof of this result is outside the scope of these lectures, but might be discussed in Tamme's lecture. In any case, we conclude that in characteristic zero the fiber of $\mathrm{K(X)} \to \mathrm{L_{cdh}K(X)}$ is equivalent to the fiber of $\mathrm{HC}^-(\mathrm{X}/\mathbb{Q}) \to \mathrm{L_{cdh}HC}^-(\mathrm{X}/\mathbb{Q})$. We will access this fiber by relating it to geometry using the Hochschild-Kostant-Rosenberg filtration on HC^-:

THEOREM 4.1.4. [**Lod92, Wei97, TV15, Ant19, Rak20, MRT22**] *Let k be a discrete commutative rings. For any qcqs k-scheme* X*, there exists a functorial, complete, multiplicative filtration* $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(\mathrm{X}/k)$ *on* $\mathrm{HC}^-(\mathrm{X}/k)$ *whose graded pieces for $j \in \mathbb{Z}$ are given by*

$$\mathrm{gr}^j_{\mathrm{HKR}}\mathrm{HC}^-(\mathrm{X}/k) \simeq \mathrm{R\Gamma}(\mathrm{X}, \mathrm{L}\widehat{\Omega}^{\geqslant j}_{-/k})[2j].$$

Moreover:

[1]In fact, if we know the following additional fact: $\mathrm{L_{cdh}K} \simeq \mathrm{KH}$ where KH is $\mathbb{A}^1$-invariant K-theory then the isomorphism (4.0.2) implies Weibel vanishing. Recall that a scheme X is said to be K_n-regular if for all $r \geqslant 0$, $\mathrm{K}_n(\mathrm{X}) \cong \mathrm{K}_n(\mathrm{X} \times \mathbb{A}^r)$. Then (4.0.2) and the fact that $\mathrm{L_{cdh}K} \simeq \mathrm{KH}$ is $\mathbb{A}^1$-invariant tells us that X is K_{-d}-regular. Now, X being K_{-d} regular implies that it is $\mathrm{K}_{\leqslant -d}$-regular by [**Wei13**, Theorem V.8.6] and so $\mathrm{K}_{\leqslant -d}(\mathrm{X}) \cong \mathrm{KH}_{\leqslant -d}(\mathrm{X})$ which is again $\pi_{\leqslant -d}\mathrm{L_{cdh}K(X)}$ and thus zero by Theorem 4.0.1.

(1) If X *is quasisyntomic over* k,[3] *then this filtration is exhaustive.*
(2) If k *is a* **Q**-*algebra, then the filtration admits a natural splitting:*

$$\mathrm{HC}^-(\mathrm{X}/k) \simeq \prod_{j\in\mathbb{Z}} \mathrm{gr}^j_{\mathrm{HKR}}\mathrm{HC}^-(\mathrm{X}/k).$$

REMARK 4.1.5.

From Theorem 4.1.4, we equip $\mathrm{L_{cdh}HC}^-(-/\mathbb{Q})$ with $\mathrm{L_{cdh}}$ of the HKR filtration. Its graded pieces are thus given by cdh-global sections

$$\mathrm{gr}^j_{\mathrm{HKR}}\mathrm{L_{cdh}HC}^-(\mathrm{X}/k) \simeq \mathrm{R\Gamma_{cdh}}(\mathrm{X}, \mathrm{L}\widehat{\Omega}^{\geqslant j}_{-/k})[2j].$$

Now, in order to access the fiber, F, we are reduced to understanding the fiber sequence of spectra

$$\mathrm{F(X)} \to \mathrm{HC}^-(\mathrm{X}/\mathbb{Q}) \to \mathrm{L_{cdh}HC}^-(\mathrm{X}/\mathbb{Q}).$$

This fiber sequence is refined by filtrations whose graded pieces are ($\mathrm{W}(j)$ is defined as the fiber and stands for "Weibel"):

$$\mathrm{W}(j)(\mathrm{X})[2j] \to \mathrm{R\Gamma}(\mathrm{X}, \widehat{\mathrm{L}\Omega}^{\geqslant j}_{-/k})[2j] \to \mathrm{R\Gamma_{cdh}}(\mathrm{X}, \widehat{\mathrm{L}\Omega}^{\geqslant j}_{-/k})[2j].$$

We claim the following result:

LEMMA 4.1.6. *For any scheme* $\mathrm{X} \in \mathrm{Sch}^{\mathrm{ft}}_k$, *we have the following:*

$$\mathrm{W}(j)(\mathrm{X}) = \begin{cases} 0 & j \leqslant 0 \\ \mathrm{Fib}\left(\mathrm{R\Gamma_{Zar}}(\mathrm{X}; \mathrm{L}\Omega^{\leqslant j-1}) \to \mathrm{R\Gamma_{Zar}}(\mathrm{X}; \mathrm{L}\Omega^{\leqslant j-1})\right)[-1]^4 & j \geqslant 1. \end{cases}$$

Furthermore,
(1) $\mathrm{H}^{>j+d+1}(\mathrm{W}(j)(\mathrm{X})) = 0$
(2) $\mathrm{H}^{j+d+1}(\mathrm{W}(j)(\mathrm{X})) \cong \mathrm{coker}(\mathrm{H}^d_{\mathrm{Zar}}(\mathrm{X}; \Omega^{j-1}_{-/k}) \to \mathrm{H}^d_{\mathrm{Zar}}(\mathrm{X}; \Omega^{j-1}_{-/k}))$.

PROOF. When $j \leqslant 0$, $\mathrm{L}\Omega^{\geqslant j}_{\mathrm{X}/k} \simeq \mathrm{L}\Omega_{\mathrm{X}/k}$ and thus satisfy cdh descent in characteristic zero, so that $\mathrm{W}(j) = 0$ in this range. To prove the second claim, we use that derived de Rham complex satisfy cdh-descent and the fiber sequence

$$\mathrm{L}\Omega^{\geqslant j}_{\mathrm{X}/k} \to \mathrm{L}\Omega_{\mathrm{X}/k} \to \mathrm{L}\Omega^{\leqslant j-1}_{\mathrm{X}/k}.$$

Now, the vanishing range of (1) follows from the fact that Zariski and cdh cohomological dimension of X are both bounded above by the Krull dimension. For the top cohomology, we have an isomorphism

$$\mathrm{H}^{j+d+1}(\mathrm{W}(j)(\mathrm{X})) \cong \mathrm{coker}\left(\mathrm{H}^d_{\mathrm{Zar}}(\mathrm{X}, \mathrm{L}\Omega^{\leqslant j-1}_{\mathrm{X}/k}) \to \mathrm{H}^d_{\mathrm{cdh}}(\mathrm{X}, \mathrm{L}\Omega^{\leqslant j-1}_{\mathrm{X}/k})\right).$$

To massage it to the terms displayed in (2), first use the spectral sequence from the Hodge (for $\tau = \mathrm{Zar}$) and the du Bois (for $\tau = \mathrm{cdh}$) filtrations:

$$\mathrm{E}_1^{ij} = \mathrm{H}^j_\tau(\mathrm{X}; \mathbb{L}^i_{\mathrm{X}/\mathbb{Q}}) \Rightarrow \mathrm{H}^{i+j}_\tau(\mathrm{X}; \mathrm{L}\Omega^{\leqslant d}_{\mathrm{X}/\mathbb{Q}}); \tag{4.1.7}$$

for $\tau = \mathrm{Zar}$ and cdh to conclude that

$$\mathrm{H}^d_\tau(\mathrm{X}, \mathrm{L}\Omega^{\leqslant j-1}_{\mathrm{X}/k}) \cong \mathrm{H}^d_\tau(\mathrm{X}; \mathbb{L}^{j-1}_{\mathrm{X}}),$$

and observe that $\mathrm{H}^d_\tau(\mathrm{X}; \mathbb{L}^{j-1}_{\mathrm{X}}) \cong \mathrm{H}^d_\tau(\mathrm{X}; \Omega^{j-1}_{\mathrm{X}})$ where, for $\tau = \mathrm{cdh}$ this follow from Lemma 3.1.2 and for $\tau = \mathrm{Zar}$, this follows from $\mathcal{H}^0(\mathbb{L}^{j-1}_{\mathrm{X}/k}) \cong \Omega^{j-1}_{\mathrm{X}/k}$ and the descent spectral sequence.

$\square$

[3]i.e., for each affine open $\mathrm{Spec\,A} \subseteq \mathrm{X}$, the cotangent complex $\mathbb{L}_{\mathrm{A}/k} \in \mathrm{D(A)}$ has Tor amplitude in $[-1, 0]$.

Now we have the spectral sequence filtering F:

$$\mathrm{H}^{i-j}(\mathrm{W}(\mathrm{X})(-j)) \Rightarrow \mathrm{F}_{-i-j}(\mathrm{X});$$

and it displays as (say $d = 2$)

$$\begin{tikzcd}
0 & 0 \arrow[drr] & 0 & 0 & 0 & \\
0 & 0 \arrow[drr] & 0 & 0 & 0 & 0 \\
\mathrm{H}^0(1) & \mathrm{H}^1(1) \arrow[drr] & \mathrm{H}^2(1) & \mathrm{H}^3(1) & \mathrm{H}^4(1) & 0 \\
\mathrm{H}^1(2) & \mathrm{H}^2(2) \arrow[drr] & \mathrm{H}^3(2) & \mathrm{H}^4(2) & \mathrm{H}^5(2) & 0 \\
\mathrm{H}^2(3) & \mathrm{H}^3(3) \arrow[drr] & \mathrm{H}^4(3) & \mathrm{H}^5(3) & \mathrm{H}^6(4) & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{4.1.8}$$

At this point note that:

(1) the vanishing is already proved!

(2) we need the term $\mathrm{H}^{j+d+1}(1) \cong \operatorname{coker}\left(\mathrm{H}^d_{\mathrm{Zar}}(\mathrm{X}, \mathrm{L}\Omega^{\leqslant j-1}_{\mathrm{X}/k}) \to \mathrm{H}^d_{\mathrm{cdh}}(\mathrm{X}, \mathrm{L}\Omega^{\leqslant j-1}_{\mathrm{X}/k})\right)$ to vanish to obtain the more refined isomorphism $\mathrm{K}_{-d}(\mathrm{X}) \cong (\mathrm{L}_{\mathrm{cdh}}\mathrm{K})_{-d}(\mathrm{X})$; this is precisely the "innocuous theorem."

CHAPTER 4

Lecture 3: motivic cohomology of schemes

Last time, we left off with the reduction of Weibel's conjecture in characteristic zero to a statement about surjectivity between Zariski cohomology onto du Bois cohomology, interpreted as cdh cohomology. This reduction relies on the observation that the map $\mathrm{HC}^-(\mathrm{X}/\mathbb{Q}) \to \mathrm{L}_{\mathrm{cdh}}\mathrm{HC}^-(\mathrm{X}/\mathbb{Q})$ is a morphism of filtered spectra and hence we can endow the fiber with a filtration whose graded pieces we can understand using geometry. The goal of this lecture is to expand on this idea to give a motivic filtration on K-theory itself and explain some applications.

At the most basic level, and relating back to our initial motivation concerning algebraic cycles, what we have tried to do is to offer a construction of the functor to graded rings:

$$\mathrm{X} \mapsto \mathrm{H}^{2\star}(\mathrm{X}; \mathbb{Z}(\star));$$

which, restricted to schemes which are smooth over fields, is the usual Chow ring of algebraic cycles on X. We will see later that we can describe some parts of our theory in terms of previously considered cycle theories, but these groups still remain quite mysterious from a geometric viewpoint. The point, in line with our philosophy of describing E_2-pages using E_∞-pages, is that whatever cycle theory one constructs must reflect certain properties of algebraic K-theory, an invariant that is universally defined for all schemes.

Such an extension was asked for by Srinivas in his ICM address [**Sri10**]. We also remark that the usual theory of cycles/higher Chow groups is not naturally a cohomology theory; its natural functoriality is *covariant* along proper morphisms and hence is a Borel-Moore homology.

1. Constructing the motivic filtration

The starting point is Theorem 4.1.1. To build a filtration on K-theory, we build filtrations on TC and $\mathrm{L}_{\mathrm{cdh}}\mathrm{K}$ and prove a compatibility result about refining the cdh-sheafified trace map to a filtered map.

THEOREM 1.0.1 (E.-Morrow). *Let $\mathbb{F} = \mathbb{F}_p$ (resp. $\mathbb{Q}$). As presheaves on $\mathrm{Sm}_{\mathbb{F}}$ the maps*

$$\mathrm{K} \to \mathrm{TC} \qquad \textit{(resp. } \mathrm{K} \to \mathrm{HC}^-(-/\mathbb{Q})\textit{)}$$

refines uniquely to a filtered morphism

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K} \to \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC} \qquad \textit{(resp. } \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K} \to \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{HC}^-(-/\mathbb{Q})\textit{).}$$

PROOF IDEAS. Let us illustrate why one might believe this in a couple of instances. We recall that the Nesterenko-Suslin isomorphism gives an identification of graded rings

$$\mathrm{H}^\star(\mathrm{L}; \mathbf{Z}(\star)^{\mathrm{mot}}) \cong \mathrm{K}^{\mathrm{M}}_\star(\mathrm{L}),$$

for any field L. This can be promoted (via the Gersten resolution on motivic cohomology and on Milnor K-theory by Kerz) to an isomorphism of Zariski sheaves

$$\mathcal{H}^\star(\mathbf{Z}(\star)) \cong \mathcal{K}^{\mathrm{M}}_\star.$$

This isomorphism describes top degree motivic cohomology (Zariski-locally) and lets us build maps out of motivic cohomology.

(Char 0) We recall that there is a map regulator map:

$$\mathbb{Z}(j)^{\mathrm{mot}} \to \mathcal{H}^j(\mathbb{Z}(j)^{\mathrm{mot}})[-j] \xrightarrow{\mathrm{dlog}} \Omega^j_{-/\mathbb{Q}}[-j];$$

$$[f_1, \cdots f_j] \mapsto \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_j}{f_j};$$

which defines a map of Zariski sheaves

$$\mathbb{Z}(j)^{\mathrm{mot}} \to \Omega^j[-j] \to \Omega^{\geqslant j}_{-/\mathbb{Q}}.$$

Note that in the t-structure for Zariski sheaves, $\mathbb{Z}(j)^{\mathrm{mot}}$ is $-j$-connective while $\Omega^{\geqslant j}_{-/\mathbb{Q}}$ is $-j$-truncated. Enhancing this observation to the filtered level gives us the desired filtered refinement.

(Char p) We focus on mod-p^r filtrations. The theorem of Geisser-Levine in particular implies that the motivic filtration on K/p^r for smooth $\mathbb{F}_p$-schemes is just the Zariski Postnikov filtration. Indeed, they proved that $\mathbb{Z}/p^r(j)^{\mathrm{mot}}$ is Zariski-locally discrete and concentrated in cohomological degree j, whence it identifies with $\mathrm{W}_r\Omega^j_{\mathrm{X,log,Zar}}$. This is the same story for TC/p^r and the BMS filtration: this time it identifies with the étale Postnikov filtration. Therefore, the motivic to BMS filtered map just witnesses étale sheafification.

□

The reason why Theorem 1.0.1 is useful, even if it is only stated for smooth schemes is because of the following result of Bhatt-Lurie [**EHK$^+$20**, Proposition A.0.1]:

THEOREM 1.0.2 (Bhatt-Lurie). *Connective* K*-theory is left Kan extended from smooth k-schemes, i.e., the map*

$$\mathrm{L}^{\mathrm{Sm}}\mathrm{K}|_{\mathrm{Sm\,CAlg}_k} \to \mathrm{K}_{\geqslant 0}$$

is an equivalence.

PROOF. Let us given Hoyois' proof as in [**EHK$^+$20**, Remark A.0.9]. The goal is to write the K-theory presheaf on CAlg_k as a colimit of smooth k-algebras. Since group completion preserves colimits, it suffices to prove that the stack of bundles of rank at most $\leqslant n$ $\mathrm{Vect}_{\leqslant n}$, restricted to CAlg_k, is a colimit of smooth k-algebras. We will write it as a geometric realization of essentially smooth k-algebras. We have a effective epimorphism[1]

$$\mathrm{Gr}_{\leqslant n} \to \mathrm{Vect}_{\leqslant n};$$

where the target is the ind-smooth ind-scheme of Grassmanians classifying rank at most n-bundles. This is not affine, but we can pick a Jouanalou device, $\mathrm{J}_n \to \mathrm{Gr}_n$ where J_n is affine and the map is an ind-vector bundle torsor and is thus an effective epimorphism. Altogether, we form the Čech nerve $\mathrm{J}_n^{\times_{\mathrm{Vect}_{\leqslant n}}\bullet}$ whose colimit is $\mathrm{Vect}_{\leqslant n}$ and the terms are affine because $\mathrm{Vect}_{\leqslant n}$ is smooth with affine diagonal. □

REMARK 1.0.3. Mathew has given an axiomatization of this result, recorded in [**EHK$^+$20**, Proposition A.0.1, A.0.4]. Actually Fulton has recorded a similar-looking statement on the level of K_0 in [**Ful75**]. Basically his result says that any K_0-class on a quasiprojective scheme over a field is pulled back from K_0 of a smooth scheme, basically constructed from the Grassmanian. This is one of the key inputs in his extension of the theory of algebraic cycles to singular schemes.

Let us now gather the ingredients together to construct the motivic filtration:

CONSTRUCTION 1.0.4. Over $\mathbb{F}_p$, let us take the left Kan extension and cdh-sheafification of the filtered map obtained in Theorem 1.0.1, which results in a map

$$\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{L}_{\mathrm{cdh}}\mathrm{K}(\mathrm{X}) \to \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{L}_{\mathrm{cdh}}\mathrm{TC}(\mathrm{X})$$

[1]We mean here an epimorphism on all A-points; of course a typical Artin stack will have a smooth epi from a scheme but this notion is usually demanded only locally.

We then construct the motivic filtration as a pullback

$$\begin{tikzcd} \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(\mathrm{X}) \arrow[r, dotted] \arrow[d, dotted] & \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(\mathrm{X}) \arrow[d] \\ \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{L}_{\mathrm{cdh}}\mathrm{K}(\mathrm{X}) \arrow[r] & \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{L}_{\mathrm{cdh}}\mathrm{TC}(\mathrm{X}). \end{tikzcd} \tag{1.0.5}$$

Over $\mathbb{Q}$, the same procedure leads to the construction of the motivic filtration on K-theory as a pullback:

$$\begin{tikzcd} \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(\mathrm{X}) \arrow[r, dotted] \arrow[d, dotted] & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(\mathrm{X}/\mathbb{Q}) \arrow[d] \\ \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{L}_{\mathrm{cdh}}\mathrm{K}(\mathrm{X}) \arrow[r] & \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{L}_{\mathrm{cdh}}\mathrm{HC}^-(\mathrm{X}/\mathbb{Q}). \end{tikzcd} \tag{1.0.6}$$

Perhaps one of the key points of this construction is its simplicity: it seems inevitable from Theorem 1.0.7. Having said that, the following theorem summarizes the properties of motivic cohomology which do not, at all, come for free. Most of the results are joint work with Morrow and some relies on another work with Bachmann and Morrow:

Theorem 1.0.7 (E.-Morrow, Bachmann-E.-Morrow). *Let $\mathbb{F}$ be a prime field and X a quasicompact, quasiseparated k-scheme. Then there exists functorial complexes*

$$\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X}) \in \mathbf{D}(\mathbb{Z}) \qquad j \geqslant 0$$

such that

(Descent) *The functor*

$$\mathrm{X} \mapsto \mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X}),$$

defines a Nisnevich sheaf and converts cofiltered limits of schemes to filtered colimits (in other words, they are finitary).

(Atiyah-Hirzebruch SS) *There is a functorial, exhaustive, multiplicative, $\mathbb{N}$-indexed filtration*

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(\mathrm{X}) \to \mathrm{K}(\mathrm{X}),$$

resulting in a spectral sequence:

$$\mathrm{E}_2^{i,j} = \mathrm{H}^{i-j}(\mathbb{Z}(-j)^{\mathrm{mot}}(\mathrm{X})) \Rightarrow \mathrm{K}_{-i-j}(\mathrm{X}), \tag{1.0.8}$$

which is convergent whenever X has finite valuative dimension. It degenerates rationally. From now on write

$$\mathrm{H}^i_{\mathrm{mot}}(\mathrm{X};\mathbb{Z}(j)) := \mathrm{H}^i(\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X})).$$

(Low weights) *When $j = 0$, we have that*

$$\mathbb{Z}(0)^{\mathrm{mot}} \simeq \mathrm{L}_{\mathrm{cdh}}\mathbb{Z}.$$

When $j = 1$ we have a natural map

$$\mathrm{R}\Gamma_{\mathrm{Nis}}(-;\mathbb{G}_m)[-1] \to \mathbb{Z}(1)^{\mathrm{mot}}$$

which is a $\tau^{\leqslant 3}$-equivalence, and therefore isomorphisms:

$$\mathrm{H}^j_{\mathrm{mot}}(\mathrm{X};\mathbb{Z}(1)) = \begin{cases} 0 & j = 0 \\ \mathcal{O}(\mathrm{X})^\times & j = 1 \\ \mathrm{Pic}(\mathrm{X}) & j = 2 \\ \mathrm{H}^3_{\mathrm{Nis}}(\mathrm{X};\mathbb{G}_m) & j = 3. \end{cases}$$

(Projective bundle formula) *The isomorphism $\mathrm{H}^2_{\mathrm{mot}}(\mathrm{X};\mathbb{Z}(1)) \cong \mathrm{Pic}(\mathrm{X})$ induces natural classes $c_1(\mathcal{O}(-1)) \in \mathrm{H}^2_{\mathrm{mot}}(\mathbb{P}^r_{\mathrm{X}};\mathbb{Z}(1))$ for all $r \geqslant 1$ which induces a natural isomorphism for all $j \geqslant 1$:*

$$\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X}) \oplus \cdots \oplus \mathbb{Z}(j-r)^{\mathrm{mot}}(\mathrm{X})[-2r] \xrightarrow{\pi^*\oplus\cdots\oplus\pi^*(-)\cup c_1(\mathcal{O}(-1))^r} \mathbb{Z}(j)^{\mathrm{mot}}(\mathbb{P}^r_{\mathrm{X}})$$

(Classical comparison) *Let* X *be a regular* $\mathbb{F}$*-scheme. Then the canonical map*

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(\mathrm{X}) \to \mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{L}_{\mathrm{cdh}}\mathrm{K}(\mathrm{X})$$

is an equivalence. Furthermore, if X *is smooth over a field, then we have further identification with the classical motivic filtration. In particular* $\mathbb{Z}(j)^{\mathrm{mot}}$ *agrees with previous definitions of motivic cohomology for schemes which are smooth over a field.*

(Singular Nesterenko-Suslin) *For any local* k*-algebra* A*, we have a canonical isomorphism*

$$\widehat{\mathrm{K}}^{\mathrm{M}}_{j}(\mathrm{A}) \cong \mathrm{H}^{j}_{\mathrm{mot}}(\operatorname{Spec}\mathrm{A}; \mathbb{Z}(j)).$$

(Connective comparison) *If* A *is a local* $\mathbb{F}$*-algebra, then*

$$\mathrm{L}^{\mathrm{sm}}\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{A}) \simeq \tau^{\leqslant j}\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{A}).$$

($\mathbb{A}^1$-comparison) *The canonical map* $\mathbb{Z}(j)^{\mathrm{mot}} \to \mathrm{L}_{\mathrm{cdh}}\mathbb{Z}(j)^{\mathrm{mot}}$ *induces equivalences:*

$$\mathrm{L}_{\mathbb{A}^1}\mathbb{Z}(j)^{\mathrm{mot}} \simeq \mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{cdh}}\mathbb{Z}(j)^{\mathrm{mot}} \simeq \mathrm{L}_{\mathrm{cdh}}\mathbb{Z}(j)^{\mathrm{mot}}.$$

In particular, the cdh*-sheafification of* $\mathbb{Z}(j)^{\mathrm{mot}}$ *is* $\mathbb{A}^1$*-invariant.*

(étale comparison) *If* $m \geqslant 2$ *is invertible in* $\mathbb{F}$*, then there is a natural isomorphism*

$$\mathrm{H}^{i}_{\mathrm{mot}}(\mathrm{X}; \mathbb{Z}/m(j)) \cong \mathrm{H}^{i}_{\mathrm{\acute{e}t}}(\mathrm{X}; \mathbb{Z}/m(j)) \qquad i \leqslant j.$$

(p-adic comparison) *If* $\mathbb{F} = \mathbb{F}_p$*, then for any* $r \geqslant 1$ *there is a cartesian square*

(1.0.9)
$$\begin{tikzcd} \mathbb{Z}/p^r(j)^{\mathrm{mot}}(\mathrm{X}) \arrow[r] \arrow[d] & \mathbb{Z}/p^r(j)^{\mathrm{syn}}(\mathrm{X}) \arrow[d] \\ \mathrm{R}\Gamma_{\mathrm{cdh}}(\mathrm{X}; \mathrm{W}_r\Omega^{j}_{\mathrm{log}})[-j] \arrow[r] & \mathrm{R}\Gamma_{\mathrm{eh}}(\mathrm{X}; \mathrm{W}_r\Omega^{j}_{\mathrm{log}})[-j]. \end{tikzcd}$$

(Hodge comparison) *If* $\mathbb{F} = \mathbb{Q}$ *is characteristic zero, then we have a cartesian square*

(1.0.10)
$$\begin{tikzcd} \mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma_{\mathrm{Zar}}(\mathrm{X}, \widehat{\mathrm{L}\Omega^{\geqslant j}_{(-)/k}}) \arrow[d] \\ \mathrm{L}_{\mathrm{cdh}}z(\mathbb{A}^j, 0)(\Delta^{\bullet} \times -)[-2j](\mathrm{X}) \arrow[r] & \mathrm{R}\Gamma_{\mathrm{cdh}}(\mathrm{X}, \widehat{\mathrm{L}\Omega^{\geqslant j}_{(-)/k}}); \end{tikzcd}$$

where the bottom left corner is a cdh *version of motivic cohomology modeled by Friedlander-Voevodsky in terms of algebraic cycles* [**FV00b**].

(Cycles comparison) *We have*

$$\mathrm{H}^{2j}_{\mathrm{mot}}(\mathrm{X}; \mathbb{Z}(j)) = \begin{cases} \mathrm{CH}^{j}(\mathrm{X}) & \text{if } \mathrm{X} \text{ is smooth} \\ \mathrm{Pic}(\mathrm{X}) & j = 1 \\ \mathrm{CH}^{\mathrm{LW}}_{0}(\mathrm{X}) & \mathrm{X} \text{ is reduced, noetherian surface} \end{cases}$$

where $\mathrm{CH}^{\mathrm{LW}}_{0}(\mathrm{X})$ *is Levine-Weibel's zero cycles group* [**LW85**].

(Soulé-Weibel vanishing) *If* X *is noetherian and finite dimensional, then*

$$\mathrm{H}^{i}_{\mathrm{mot}}(\mathrm{X}; \mathbb{Z}(j)) = 0 \qquad i > j + \dim(\mathrm{X}).$$

(Blowup descent) *Let* X *be noetherian and suppose that* $\mathrm{Z} \hookrightarrow \mathrm{X}$ *is closed immersion. Then we have a pro-cartesian square:*

$$\begin{tikzcd} \mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X}) \arrow[r] \arrow[d] & \{\mathbb{Z}(j)^{\mathrm{mot}}(r\mathrm{Z})\} \arrow[d] \\ \mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{Bl}_{\mathrm{Z}}(\mathrm{X})) \arrow[r] & \{\mathbb{Z}(j)^{\mathrm{mot}}(r\mathrm{E})\}, \end{tikzcd}$$

where E *is the exceptional divisor of the blowup* $\mathrm{Bl}_{\mathrm{Z}}(\mathrm{X}) \to \mathrm{X}$.

1.1. Several remarks on the theorem.

REMARK 1.1.1 (What are the new ingredients?). It is quite reasonable to ask what makes Theorem 1.0.7 possible; rather, what makes it only provable in recent years. I offer several answers:

(1) most obviously, Theorem 4.1.1 has really unveiled what K-theory is made of; this pullback square is the conceptual lynchpin of Theorem 1.0.7.
(2) The motivic filtration on TC is a key input and is a product of [**BMS19**], while the theory of derived de Rham complex and its relationship to Hochschild type invariants have been clarified enormously in [**Bha12a, Ant19, Rak20, MRT22**].
(3) We do note, however, that prismatic cohomology could be avoided in constructing the BMS-filtration on TC/p^r, just using 1) the Geisser-Hesselholt theorem that TC/p^r is étale K-theory [**GH99**], 2) the Geisser-Levine theorem identifying the graded pieces of the motivic filtration on K/p^r [**GL00**] and 3) left Kan extension methods.
(4) There is a key technical ingredient of the above result and it's a purely site-theoretic statement:
$$\mathrm{L}_{\mathrm{cdh}}\mathrm{L}_{\text{ét}} \simeq \mathrm{L}_{\text{éh}}$$

REMARK 1.1.2 (Comparison to $\mathbb{A}^1$-invariant motivic homotopy theory). In the joint work with Bachmann, we examined the cdh-local motivic cohomology:
$$\mathbb{Z}(j)^{\mathrm{cdh}} := \mathrm{L}_{\mathrm{cdh}}\mathrm{L}^{\mathrm{Sm}}\mathbb{Z}(j)^{\mathrm{mot}}.$$
One of the main things that we proved is that $\mathbb{Z}(j)^{\mathrm{cdh}}$ is $\mathbb{A}^1$-invariant. This is surprising and refines the next surprising result (at least in the equicharacteristic situations), Theorem 1.1.3. We further prove that this compares well with Spitzweck's extension of motivic cohomology to non-smooth schemes [**Spi18**]. We can also compute cdh-motivic cohomology away from the prime as
$$\mathbb{Z}/m(j)^{\mathrm{cdh}} \simeq \mathrm{L}_{\mathrm{cdh}}\tau^{\leqslant j}\mathrm{R}\Gamma_{\text{ét}}(-,\mu_m^{\otimes j})(\mathrm{X});$$
and a cdh-sheafiifed Geisser-Levine theorem
$$\mathbb{Z}/p^r(j)^{\mathrm{cdh}} \simeq \mathrm{R}\Gamma_{\mathrm{cdh}}(-;\mathrm{W}_r\Omega^j_{\log})[-j] \qquad r \geqslant 1.$$

THEOREM 1.1.3. *The* cdh-*sheaf* $\mathrm{L}_{\mathrm{cdh}}\mathrm{K}$ *is* $\mathbb{A}^1$*-invariant. Furthermore, we have a canonical equivalence* $\mathrm{L}_{\mathrm{cdh}}\mathrm{K} \simeq \mathrm{KH}$.

PROOF. The key points are that KH is cdh sheaf and that K-theory is $\mathbb{A}^1$-invariant on valuation rings. The former has, by now, several proofs: the original is due to Cisinski [**Cis13**] though Haesemeyer isolated such a statement in characteristic zero as being key in the story in his thesis [**Hae04**]; Cisinski observed that the formalism of stable motivic homotopy theory and Ayoub's proper base change theorem [**Ayo08a, Ayo08b**] proves cdh descent for any invariant which are "stable under pullback" in **SH**. The second approach is Land-Tamme's theory of truncating invariants [**LT19**] gives a noncommutative way to access such a statement. Lastly, a simple proof of this result in characteristic p was observed by Kelly and Morrow in [**KM21**]. The fact that K-theory is $\mathbb{A}^1$-invariant on valuation rings can again be deduced from [**Dat17**, Theorem 5.1] as in [**KM21**, Theorem 3.3]. □

REMARK 1.1.4 (Motivic-étale comparison). Say m is invertible in k. The étale comparison statement of Theorem 1.0.7 in fact comes from the assertion that we have an equivalence:
$$\mathbb{Z}/m(j)^{\mathrm{mot}}(\mathrm{X}) \xrightarrow{\simeq} \mathrm{L}_{\mathrm{cdh}}\tau^{\leqslant j}\mathrm{R}\Gamma_{\text{ét}}(-,\mu_m^{\otimes j})(\mathrm{X}) \qquad j \geqslant 1,$$
plus Deligne's cohomological descent theorem which asserts that $\mathrm{R}\Gamma_{\text{ét}}(-,\mu_m^{\otimes j})$ satisfies cdh descent. This refines the equivalence (first proved by Weibel) that $\mathrm{K}[\frac{1}{p}] \simeq \mathrm{KH}[\frac{1}{p}]$.

REMARK 1.1.5 (Motivic-syntomic comparison). Suppose that p is the characteristic of k. The cofiber of the map $\mathbb{Z}/p(j)^{\mathrm{mot}} \to \mathbb{Z}/p(j)^{\mathrm{syn}}$ is a very reasonable cohomology theory; we have a fiber sequence:
$$\mathbb{Z}/p(j)^{\mathrm{mot}} \to \mathbb{Z}/p(j)^{\mathrm{syn}} \to \mathrm{R}\Gamma_{\mathrm{cdh}}(-;\widetilde{\nu}(j))[-j-1].$$

The (étale) sheaves $\widetilde{\nu}(j)$ has several incarnations. On an $\mathbb{F}_p$-algebra A we have

(1) via the (higher) Artin-Schreier sequences we have:

$$\widetilde{\nu}(j)(\mathrm{A}) \cong \mathrm{H}^1_{\text{ét}}(\mathrm{A}, \Omega^j_{\log});$$

(2) it is computed as the top degree mod-p syntomic cohomology [**AMMN22**, Corollary 5.43]:

$$\widetilde{\nu}(j)(\mathrm{A}) \cong \mathrm{H}^{j+1}(\mathbb{Z}/p(j)^{\mathrm{syn}}(\mathrm{A}));$$

(3) if A is furthermore local, it is exactly a homotopy group of connective infinitesimal K-theory [**CMM21**, Theorem 4.29, 6.11]:

$$\widetilde{\nu}(j)(\mathrm{A}) \cong \pi_{j-1}\mathrm{cofib}(\mathrm{K}_{\geqslant 0}(\mathrm{A})/p \to \mathrm{TC}(\mathrm{A})/p).$$

Remark 1.1.6 (Motivic Dundas-Goodwillie-McCarthy theorem). Let $\mathrm{A} \to \mathrm{B}$ a nilpotent extension of $\mathbb{F}$-algebras; this means that the map is surjective and the ideal kerne I is nilpotent. The relative motivic cohomology:

$$\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{A}, \mathrm{I}) := \mathrm{Fib}\left(\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{A}) \to \mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{B})\right);$$

identifies with familiar looking terms:

$$\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{A}, \mathrm{I}) \simeq \mathrm{Fib}\left(\mathbb{Z}(j)^{\mathrm{syn}}(\mathrm{A}) \to \mathbb{Z}(j)^{\mathrm{syn}}(\mathrm{B})\right) \qquad \mathbb{F} = \mathbb{F}_p;$$

and

$$\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{A}, \mathrm{I}) \simeq \mathrm{Fib}\left(\mathrm{L}\widehat{\Omega}^{\leqslant j-1}_{\mathrm{A}/\mathbb{Q}} \to \mathrm{L}\widehat{\Omega}^{\leqslant j-1}_{\mathrm{B}/\mathbb{Q}}\right)[-1] \qquad \mathbb{F} = \mathbb{Q}$$

Remark 1.1.7 (Algebraic cycles and left Kan extension). Taking the left Kan extension of $\mathbb{Z}(j)^{\mathrm{mot}}$, we get a theory $\mathbb{Z}(j)^{\mathrm{lke}}$ which, on affines, are the graded pieces of a filtration on connective K-theory. This theory has the advantage of being explicitly connected to algebraic cycles. First, pick $\mathrm{P}_\bullet \to \mathrm{A}$ where each term P_m is an ind-smooth $\mathbb{F}$-algebra and each face map $\mathrm{P}_{m+1} \to \mathrm{P}_m$ is a henselian surjection. The $\mathbb{Z}(j)^{\mathrm{lke}}(\mathrm{A})$ is given by the $-2j$-shift of the total bicomplex:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\cdots \arrow[r] & z^j(\mathrm{P}_2, 2) \arrow[r] \arrow[d] & z^j(\mathrm{P}_2, 1) \arrow[r] \arrow[d] & z^j(\mathrm{P}_2, 0) \arrow[d] \\
\cdots \arrow[r] & z^j(\mathrm{P}_1, 2) \arrow[r] \arrow[d] & z^j(\mathrm{P}_1, 1) \arrow[r] \arrow[d] & z^j(\mathrm{P}_1, 0) \arrow[d] \\
\cdots \arrow[r] & z^j(\mathrm{P}_0, 2) \arrow[r] & z^j(\mathrm{P}_0, 1) \arrow[r] & z^j(\mathrm{P}_0, 0)
\end{tikzcd}$$

There is always a comparison map

$$\mathbb{Z}(j)^{\mathrm{lke}} \to \mathbb{Z}(j)^{\mathrm{mot}};$$

which is, in general, witnesses a $\tau^{\leqslant j}$-truncation. In particular, it is an equivalence when evaluated on A, a local ring of dimension zero. From this we get cycle-theoretic descriptions of motivic cohomology of things like the fattened points.

2. A computational sampler

We now offer two computational samplers, none of which is particularly new. We hope that the motivic approach are conceptual, clarifying alternatives to the original.

2.1. Levine-Weibel Chow groups. In [**LW85**], a cohomological theory of zero cycles was proposed by Levine and Weibel;

$$X \mapsto \mathrm{CH}_0^{\mathrm{LW}}(X);$$

of course X should have some limitations: at least it should be reduced and maybe it is quasiprojective. The best results should also only hold whenever X is normal so that it is regular in codimension one.

Among others[2], the desired properties include:

(1) there should be a cycle class map $\mathrm{CH}_0^{\mathrm{LW}} \to \mathrm{K}_0(X)$; in particular the easiest way to guarantee this is to only use the *smooth points* of X to generate the theory[3];
(2) in fact the image of the cycle class map should be $\mathrm{F}^{\dim(X)}\mathrm{K}_0(X)$, the bottom-layer of the codimension filtration on K_0 (or, the part of K_0 generated by structure sheaves of smooth points);
(3) in fact $\mathrm{CH}_0^{\mathrm{LW}}$ should be the home for the "top chern class" of a vector bundle which controls splitting problems for vector bundles of affine schemes [**Mur94**];
(4) for (reasonable) curves, it should recover the Picard group;
(5) for projective schemes over an algebraically closed field k it should be the k-points of an abelian variety, extending Roitman's theorem to a more general context.

Levine and Weibel basically proved (1) and (3) and work of other authors have made much progress to other aspects of this theory. The state of the art, comparing all other approaches, is the work of Binda and Krishna [**BK22**]. Our goal is to recast this theory of zero cycles using the newly minted motivic cohomology. We will not attempt to be comprehensive, or even detailed, but give the reader a sense of how to use this theory of motivic cohomology and relate it to invariants which have been studied before. We will focus on the case of a surface, but the reader can easily extrapolate generalizations of the results that will follow.

For the next two sections, we will refer to the diagram which is an abstract blowup square

(2.1.1)
$$\begin{tikzcd} E \arrow[r] \arrow[d] & X' \arrow[d, "p"] \\ Y \arrow[r, "i"] & X, \end{tikzcd}$$

where X is a surface[4] over k, X′ is smooth, Y is zero dimensional (since we have assumed that X is normal, its singularities are zero dimensional) and E is a curve.

2.2. Krishna-Srinivas on zero cycles on normal surfaces. Let k be a field, and X a normal surface, then we have the following formula which describes $\mathrm{CH}_0^{\mathrm{LW}}(X)$ in terms of a resolution of X.

THEOREM 2.2.1. *As in (2.1.1), we have an exact sequence for $r \gg 0$:*

$$\mathrm{H}^1_{\mathrm{Zar}}(X'; \mathcal{K}_2) \to \mathrm{H}^1_{\mathrm{Zar}}(rE; \mathcal{K}_2) \to \mathrm{CH}_0^{\mathrm{LW}}(X) \to \mathrm{CH}_0(X') \to 0.$$

REMARK 2.2.2. From this result, we deduce that $\mathrm{CH}_0^{\mathrm{LW}}(X) \cong \mathrm{CH}_0(X')$ whenever $\mathrm{H}^1_{\mathrm{Nis}}(X'; \mathcal{K}_2) \to \mathrm{H}^1_{\mathrm{Nis}}(rE; \mathcal{K}_2)$ is surjective for $r \gg 0$.

Krishna and Srinivas [**KS02**, Theorem 1.3] proves that this happens whenever:

(1) $\mathrm{H}^2(X; \mathcal{O}_X) \cong \mathrm{H}^2(X'; \mathcal{O}_{X'})$ in characteristic zero;
(2) $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(E_{\mathrm{red}})$ is surjective.

[2] What comes is the author's own idiosyncratic list of demands

[3] This renders the usual Fulton-style definition of zero cycles as not the right thing to look at in this context.

[4] By a surface we mean a reduced, connected 2-dimensional finite type scheme over k.

Proof. Pro-cdh descent gives a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & \{\mathrm{H}^3_{\mathrm{mot}}(r\mathrm{Y};\mathbb{Z}(2))\}_r \oplus \mathrm{H}^3_{\mathrm{mot}}(\mathrm{X}';\mathbb{Z}(2)) \arrow[r] & \{\mathrm{H}^3_{\mathrm{mot}}(r\mathrm{E};\mathbb{Z}(2))\} \arrow[dll] \\ \mathrm{H}^4_{\mathrm{mot}}(\mathrm{X};\mathbb{Z}(2)) \arrow[r] & \mathrm{H}^4_{\mathrm{mot}}(\mathrm{X}';\mathbb{Z}(2)) \oplus \{\mathrm{H}^4_{\mathrm{mot}}(r\mathrm{Y};\mathbb{Z}(2))\}_r \arrow[r] & 0 \end{tikzcd}$$

But now:

(1) $\mathrm{H}^4_{\mathrm{mot}}(r\mathrm{Y};\mathbb{Z}(2))$ and $\mathrm{H}^3_{\mathrm{mot}}(r\mathrm{Y};\mathbb{Z}(2))$ vanishes by Soulé-Weibel vanishing applied to the zero-dimensional Y;
(2) the descent spectral sequence, Soulé-Weibel vanishing applied to the curve $r\mathrm{E}$ and singular Nesterenko-Suslin identifies $\mathrm{H}^3_{\mathrm{mot}}(r\mathrm{E};\mathbb{Z}(2)) \cong \mathrm{H}^1_{\mathrm{Nis}}(r\mathrm{E};\widehat{\mathcal{K}}^{\mathrm{M}}_2) \cong \mathrm{H}^1_{\mathrm{Nis}}(r\mathrm{E};\mathcal{K}_2)$;
(3) cycles comparison tells us that $\mathrm{H}^4_{\mathrm{mot}}(\mathrm{X};\mathbb{Z}(2))$ is indeed the Levine-Weibel zero cycle group.

We thus get an exact sequence (noting the classical identification $\mathrm{H}^3_{\mathrm{mot}}(\mathrm{X}';\mathbb{Z}(2)) \cong \mathrm{H}^1_{\mathrm{Zar}}(\mathrm{X};\mathcal{K}_2)$:

$$\mathrm{H}^1_{\mathrm{Nis}}(\mathrm{X}';\mathcal{K}_2) \to \{\mathrm{H}^1_{\mathrm{Nis}}(r\mathrm{E};\mathcal{K}_2)\} \to \mathrm{CH}^{\mathrm{LW}}_0(\mathrm{X}) \to \mathrm{CH}_0(\mathrm{X}') \to 0.$$

But now, we know that the pro-abelian group is isomorphic to a constant group. Since the transition maps are surjective (it is the top cohomology and the map of sheaves are surjective), they must be eventually isomorphisms. ☐

2.3. Weibel on K-theory of normal surfaces. In a different, but related direction Weibel studied the negative K-groups of singular surfaces. The phenomena that he focused on is the failure of $\mathbb{A}^1$-invariance (more generally of $\mathbb{A}^r$-invariance) in the setting of negative K-groups. If $\mathcal{F}$ is a presheaf of spectra, defined on schemes, then we write $\mathrm{N}\mathcal{F}$ to be the fiber of the map $\mathcal{F}(\mathrm{X}\times\mathbb{A}^1) \xrightarrow{0^*} \mathcal{F}(\mathrm{X})$. Using the pullback map, we get a decomposition

$$\mathrm{N}\mathcal{F}(\mathrm{X}) \oplus \mathcal{F}(\mathrm{X}) \cong \mathcal{F}(\mathrm{X}\times\mathbb{A}^1).$$

Let us recall several facts on the NK-groups. First, we recall that the notion of K_n**-regularity**: a scheme X is K_n-regular if for all $r \geqslant 0$, $\mathrm{K}_n(\mathrm{X}) \xrightarrow{\cong} \mathrm{K}_n(\mathrm{X}\times\mathbb{A}^r)$. It is a fact that if X is K_n-regular, then it is K_{n-1}-regular[5] [**Wei13**, Theorem V.8.6]. However, the following two statements are false in general:

(1) if X is K_n-regular, then it is K_{n+1}-regular;
(2) if $\mathrm{NK}_n(\mathrm{X}) = 0$, then X is K_n-regular.

Indeed the paper [**CnHWW10**] is dedicated to giving counterexamples to the second statement, which was a question of Bass'. This phenomenon is counter to what happens to Picard groups [**Swa80**, Theorem 1].

Theorem 2.3.1. [**Wei01**, Theorem 5.9] *Let* X *be a normal surface over a perfect field. For any resolution* $\mathrm{X}' \to \mathrm{X}$ *such that the exceptional locus* E *has smooth components and normal crossings, then the following are equivalent:*

(1) $\mathrm{NK}_{-1}(\mathrm{X}) = 0$*;*
(2) for any curve on X', Y' *such that* $\mathrm{Y}'_{\mathrm{red}} = \mathrm{E}$*, we have that* $\mathrm{Pic}(\mathrm{Y}'_{\mathrm{red}}) = \mathrm{Pic}(\mathrm{E})$.

Both statements are also furthermore equivalent to X *being* K_{-1}*-regular.*

Proof. We claim that, in fact:

$$\mathrm{NK}_{-1}(\mathrm{X}) \cong \{\mathrm{NPic}(r\mathrm{E})\}. \tag{2.3.2}$$

[5] Let us a sketch a proof of this; it uses Vorst' localization theorem. First, we can assume X is affine and also prove a sharper claim: if $\mathrm{K}_n(\mathrm{R}) \cong \mathrm{K}_n(\mathrm{R}[s,t])$ then $\mathrm{NK}_{n-1}(\mathrm{R}) = 0$. Indeed, we substitute R for $\mathrm{R}[t]$ and use induction to prove the general claim. Now, by hypothesis, $\mathrm{NK}_n(\mathrm{R}[s]) = 0$; applying Vorst's localization theorem [**Wei13**, Lemma V.8.5] we get that $\mathrm{NK}_n(\mathrm{R}[s,s^{-1}]) \cong \mathrm{NK}_n(\mathrm{R}[s])_{[s]} \cong 0$. But, by Bass, $\mathrm{NK}_{n-1}(\mathrm{R})$ is a summand of $\mathrm{NK}_n(\mathrm{R}[s,s^{-1}])$ whence the result.

for any such Y'. First, note that since X is a surface, motivic Soulé-Weibel vanishing and the low-weights identification of motivic cohomology gives us an exact sequence

$$0 \to H^3_{mot}(X; \mathbb{Z}(1)) \to K_{-1}(X) \to H^1_{cdh}(X; \mathbb{Z}) \to 0;$$

since the application of "N" is exact on cohomology theories and cdh-cohomology with $\mathbb{Z}$-coefficients is $\mathbb{A}^1$-invariant, we have that

$$NH^3_{mot}(X; \mathbb{Z}(1)) \cong NK_{-1}(X).$$

Now, apply pro-cdh descent to obtain an exact sequence:

$$\begin{tikzcd} \cdots \arrow[r] & \{NH^2_{mot}(rY; \mathbb{Z}(1))\}_r \oplus NH^2_{mot}(X'; \mathbb{Z}(1)) \arrow[r] & \{NH^2_{mot}(rE; \mathbb{Z}(1))\} \arrow[dll] \\ NH^3_{mot}(X; \mathbb{Z}(1)) \arrow[r] & NH^3_{mot}(X'; \mathbb{Z}(1)) \oplus \{NH^3_{mot}(rY; \mathbb{Z}(1))\}_r \arrow[r] & 0 \end{tikzcd}$$

Here:

(1) the terms involving smooth schemes disappear since motivic cohomology is $\mathbb{A}^1$-invariant on them;
(2) $NH^*(-; \mathbb{Z}(1))$ will spread to at most $1 + (1 + \dim(X))$ since it is a summand of $H^*(- \times \mathbb{A}^1; \mathbb{Z}(1))$. In particular, $NH^3_{mot}(rY; \mathbb{Z}(1)) = 0$ for all r.
(3) $NH^2_{mot}(rY; \mathbb{Z}(1)) = 0$: $H^2_{mot}(-; \mathbb{Z}(1))$ always computes the Picard group and Swan's criterion [**Swa80**] says that we have $\mathbb{A}^1$-invariance if the reduced locus of rY is seminormal; the latter is a product of fields.

Therefore we conclude:

$$\{NPic(rE)\} \cong \{NH^2_{mot}(rE; \mathbb{Z}(1))\} \cong NH^3_{mot}(X; \mathbb{Z}(1)) \cong NK_{-1}(X).$$

To finish off, we use Swan's criterion again. Since any reduced curve with smooth components and normal crossings are seminormal, we see that [**Swa80**, Theorem 1] says that $\{NPic(rE)\}$ is pro-zero if and only if $Pic(rE) \cong Pic(E)$ as desired.

But now, [**Swa80**, Theorem 1] also asserts that NPic implies Pic-regularity (the converse is trivial). Therefore, we conclude that $NK_{-1}(X)$ implies that X is K_{-1}-regular, again from (2.3.2). □

3. Motivic Soulé-Weibel vanishing

We have seen that the motivic filtration offers a slick way to prove Weibel vanishing results, at least in characteristic zero. In fact, I have claimed more vanishing than is required for just Weibel vanishing in K-theory as in the "innocuous theorem". I now want to explain how to prove "innocuous theorem" and its variants in characteristic $p > 0$ which is a direct consequence of the motivic Soulé-Weibel vanishing.

THEOREM 3.0.1. *Let* X *be a noetherian* $\mathbb{F}$*-scheme of dimension* d*. The following maps are surjective:*

(1) for $j \geqslant 1$ *and* $\mathbb{F} = \mathbb{Q}$*,*

$$H^d_{Zar}(X; \Omega^{j-1}_{-/\mathbb{Q}}) \to H^d_{cdh}(X; \Omega^{j-1}_{-/\mathbb{Q}});$$

(2) for $j \geqslant 0$

$$H^d_{Zar}(X; \widetilde{\nu}(j)) \to H^d_{cdh}(X; \widetilde{\nu}(j));$$
$$H^{d-1}_{Zar}(X; \widetilde{\nu}(j)) \to H^{d-1}_{cdh}(X; \widetilde{\nu}(j));$$
$$H^d_{Zar}(X; \widehat{K}^M_j) \to H^d_{cdh}(X; \widehat{K}^M_j).$$

For the rest of these lectures, I want to explain the next result which is implies the Motivic Soulé-Weibel vanishing. Let us set

$$W(j) := \mathrm{Fib}\left(\mathbb{Z}(j)^{mot} \to \mathbb{Z}(j)^{cdh}\right).$$

THEOREM 3.0.2 (Motivic Soulé-Weibel vanishing). *For any noetherian equicharacteristic scheme* X*, we have*

$$\mathrm{H}^i_{\mathrm{mot}}(\mathrm{W}(j)(\mathrm{X})) = 0 \qquad i > j + \dim(\mathrm{X}).$$

This is a refinement of Weibel vanishing for equicharacteristic schemes. In particular, it results in the Kerz-Strunk-Tamme vanishing theorem for equicharacteristic schemes, see Remark 3.0.4.

First, we point out an axiomatic version of the Weibel vanishing theorem:

PROPOSITION 3.0.3. *If k is a ring, and*

$$\mathrm{W} : \mathrm{Sch}_k^{\mathrm{op}} \to \mathbf{D}(\mathbb{Z})$$

is a finitary Nisnevich sheaf such that:

(1) $\mathrm{L_{cdh}W} \simeq 0$*;*
(2) W *satisfies pro-cdh descent on noetherian schemes;*
(3) for any nilpotent extension of rings $\mathrm{A} \to \mathrm{B}$*, the fiber of* $\mathrm{W(A)} \to \mathrm{W(B)}$ *is connective.*

Then for any noetherian scheme X*,* W(X) *is* $-\dim(\mathrm{X})$*-connective.*

PROOF SKETCH. We break this proof down into steps:

(Step 1) by descent and the assumption on the fiber of nilpotent extensions, we see that the fiber of $\mathrm{W(X)} \to \mathrm{W(X_{red})}$ is $-\dim(\mathrm{X})$-connective so we might as well assume that X is reduced. We then proceed by induction on d. When $d = 0$, since X is reduced, it is a disjoint union of spectra of fields. These are cdh-points and so $\mathrm{W(X)} \simeq \mathrm{L_{cdh}W(X)} \simeq 0$ by assumption (1).

(Step 2) Next, by a combination of inductive hypothesis and the following version of "Grothendieck's lemma:"

($\star$) If $\mathcal{F}$ is a Nisnevich sheaf of abelian groups on a noetherian scheme X such that for all $x \in \mathrm{X}$ satisfying $\dim \overline{\{x\}} > d$ we have that $\mathcal{F}_x = 0$, then $\mathrm{H}^i_{\mathrm{Nis}}(\mathrm{X}; \mathcal{F}) = 0$ for $i > d$.

we conclude that for $n > d$ we have that

$$\mathrm{H}^n(\mathrm{W(X)}) \cong \mathrm{H}^0_{\mathrm{Nis}}(\mathrm{X}; \mathcal{H}^n(\mathrm{W}));$$

whence on the category of noetherian k-schemes of dimension $\leqslant d$, $\mathrm{H}^n(\mathrm{W}(-))$ is a Nisnevich sheaf. By now suffices to kill $\mathrm{H}^n(\mathrm{W}(-))$ after a Nisnevich cover; in fact we will use this to kill elements of $\mathrm{H}^n(\mathrm{W}(-))$ after a modification.

(Step 3) By assumption 1, we have that $\mathrm{H}^n(\mathrm{W}(-))$ is zero cdh-locally, whence we can find a cdh cover that kills any class $\alpha \in \mathrm{H}^n(\mathrm{W(X)})$. By a structural result about the cdh topology, we can (basically) find a cdh cover of the form $\mathrm{X}_2 \xrightarrow{q} \mathrm{X}_1 \xrightarrow{p} \mathrm{X}$ where p is a modification and q is a Nisnevich cover such that α dies on X_2[6]. Since $\mathrm{H}^n(\mathrm{W}(-))$ is, in particular, Nisnvevich separated α dies upon pullback to q.

(Step 4) At this point, we see that any class $\alpha \in \mathrm{H}^n(\mathrm{W(X)})$ can be killed my an appropriate modification. But, by pro-cdh descent, we see that $\mathrm{H}^n(\mathrm{W(X)}) \cong \mathrm{H}^n(\mathrm{W(X')})$ for any modification X' since infinitesimal thickenings of both the center and the exceptional loci are all of dimension $< d$ and thus zero by the inductive hypothesis.

∎

REMARK 3.0.4. The above axiomatics, at least a version for spectra which works verbatim, can be used to draw a geodesic from pro-cdh descent to Weibel vanishing by applying it to $\mathrm{W} = \mathrm{Fib}(\mathrm{K} \to \mathrm{L_{cdh}K})$.

We now prove:

THEOREM 3.0.5. *The presheaves* W(j) *satisfies the conditions of Proposition 3.0.3.*

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PROOF. The most interesting thing to prove is pro-cdh descent for $\mathrm{W}(j)$. This amounts to proving pro-cdh descent for $\mathbb{Z}(j)^{\mathrm{syn}}$ in characteristic $p>0$ and for $\mathrm{L}\widehat{\Omega}^{\geqslant j}_{-/\mathbb{Q}}$ in characteristic zero. We indicate the proof for the former. Here is a way to control the mod-p version of syntomic cohomology in coherent terms. We have an increasing, finite filtration

$$\mathrm{Fil}_0\mathbb{F}_p(j)\to\mathrm{Fil}_1\mathbb{F}_p(j)\to\cdots\mathrm{Fil}_{2j+1}\mathbb{F}_p(j)\to\mathrm{Fil}_{2j+2}\mathbb{F}_p(j)$$

whose graded pieces are, in increasing order:

$$\mathbb{L}^{j}_{-/\mathbb{F}_p}[-j-1],\mathbb{L}^{j-1}_{-/\mathbb{F}_p}[-j]\cdots\mathbb{L}^{0}_{-/\mathbb{F}_p}[-1],\mathbb{L}^{0}_{-/\mathbb{F}_p}[0],\mathbb{L}^{1}_{-/\mathbb{F}_p}[-1]\cdots\mathbb{L}^{j}_{-/\mathbb{F}_p}[-j].$$

Therefore, we conclude that $\mathbb{F}_p(j)^{\mathrm{syn}}$ enjoys pro-cdh descent. In order to conclude the result for $\mathbb{Z}_p(j)^{\mathrm{syn}}$ we need to verify that, in fact, the cohomology groups of the total fiber

$$\mathrm{A}_r:=\mathrm{H}^n(\mathbb{Z}_p^{\mathrm{syn}}(\mathrm{X},\mathrm{X}',r\mathrm{Y}))$$

are all bounded p-power torsion. □

PROOF OF THEOREM 3.0.1. Unpacking the definition of $\mathrm{W}(j)$ in characteristic zero and characteristic $p>0$ yields the following:

($\mathbb{F}=\mathbb{Q}$) This is already done in Lemma 4.1.6.

($\mathbb{F}=\mathbb{F}_p$) Taking $\mathrm{W}(j)/p$, we get that $\mathrm{W}(j)/p=\mathrm{Fib}(\mathbb{F}_p(j)^{\mathrm{syn}}\to\mathrm{L}_{\mathrm{cdh}}\mathbb{F}_p(j)^{\mathrm{syn}})$. On the other hand, the syntomic comparison of Remark 1.1.5, coupled with the sequence

$$\mathrm{L}_{\mathrm{Nis}}\tau^{\leqslant j}\mathbb{F}_p(j)^{\mathrm{syn}}\to\mathbb{F}_p(j)^{\mathrm{syn}}\to\mathrm{R}\Gamma_{\mathrm{Nis}}(-;\widetilde{\nu}(j))[-j-1]$$

gives an exact sequence

$$\mathrm{fib}(\mathrm{L}_{\mathrm{Nis}}\tau^{\leqslant j}\mathbb{F}_p(j)^{\mathrm{syn}}\to\mathrm{L}_{\mathrm{cdh}}\tau^{\leqslant j}\mathbb{F}_p(j)^{\mathrm{syn}}\to\mathrm{W}(j)/p\to\mathrm{fib}(\mathrm{R}\Gamma_{\mathrm{Nis}}(-;\widetilde{\nu}(j))[-j-1]\to\mathrm{R}\Gamma_{\mathrm{cdh}}(-;\widetilde{\nu}(j))[-j-1]).$$

The reader is then encouraged to unpack what this means which results in the theorem after knowing that $\mathrm{H}^j(\mathbb{F}_p^{\mathrm{syn}}(j)(-))\cong\widehat{\mathrm{K}}^{\mathrm{M}}_j$ by [?].

□

APPENDIX A

The motivic spectral sequence revisited

We will retell the story of the motivic spectral sequence via the following theorem:

THEOREM 0.0.1. *Let* X *be a smooth, k-scheme. There exists a functorial, decreasing motivic filtration*

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(\mathrm{X}) \to \mathrm{X}$$

which is multiplicative, exhaustive and complete. Such that (a shift) of the graded pieces are canonically equivalent with the motivic cohomology constructed in lecture 1:

$$\mathrm{gr}^{\star}_{\mathrm{mot}}\mathrm{K}(\mathrm{X}) \simeq \mathbb{Z}(\star)^{\mathrm{mot}}[2\star](\mathrm{X}).$$

We note that the proof of this theorem will still be nontrivial: it relies heavily on the Bloch-Kato conjectures and motivic infinite loop space theory. If there is any value to this proof, is that it clarifies the role of the prickly moving lemmas which has plagued the subject for a while. We hope that the reader will appreciate this perspective.

The main difference of our exposition from the standard ones is that we follow Voevodsky in setting the filtration up using motivic homotopy theory where the formal properties can be easily established. Only later on are the graded pieces identified explicitly.

PROOF. We will break this proof down into steps.

(Step 1) We work in $\mathbf{SH}(\mathbb{F})$, the ∞-category of motivic spectra over a prime field $\mathbb{F} = \mathbb{F}_p$ or $\mathbb{Q}$; the general case follows by taking pullbacks. We have the motivic spectrum representing K-theory KGL and we have the slice filtration

$$f^{\star}_{\mathrm{slice}}\mathrm{KGL} \to \mathrm{KGL}.$$

For any (essentially-)smooth $\mathbb{F}$ scheme X the filtration is defined as

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(\mathrm{X}) := \underline{\mathrm{map}}(\Sigma^{\infty}_{\mathrm{T}}\mathrm{X}_{+}, f^{\star}_{\mathrm{slice}}\mathrm{KGL})$$

where $\underline{\mathrm{map}}$ is the mapping spectrum in $\mathbf{SH}(\mathbb{F})$. By formal reasons, this filtration is exhaustive. Furthermore, the $\mathbb{E}_{\infty}$-ring structure on KGL refiles to a filtered $\mathbb{E}_{\infty}$-structure on the slice filtration by [], hence $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(\mathrm{X}) \to \mathrm{K}(\mathrm{X})$ is multiplicative. The graded pieces of this filtration are exactly the slices $s^{\star}_{\mathrm{slice}}\mathrm{KGL}$.

(Step 2) KGL is Bott periodic by a certain map $\Sigma^{2,1}\mathrm{KGL} \xrightarrow{\simeq} \mathrm{KGL}$ which formally implies an equivalence

$$\Sigma^{2j,j}s^0\mathrm{KGL} \simeq s^j\mathrm{KGL}.$$

Our goal is thus to identify the motivic spectrum $s^0\mathrm{KGL}$ with the motivic spectra built from lecture 1 in such a way that

$$\underline{\mathrm{map}}(\Sigma^{\infty}_{\mathrm{T}}\mathrm{X}_{+}, \mathrm{gr}^{j}_{\mathrm{slice}}\mathrm{KGL}) \simeq \mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X})[2j].$$

(Step 3) We first consider the motivic spectrum $\mathrm{H}\mathbb{Z}^{\mathrm{Hoy}}$, built from motivic infinite loop spaces by Hoyois. This is obtained by taking the framed suspension spectrum of $\mathbb{Z}$ where the degree map equips $\mathbb{Z}$ with the structure of framed transfers [**Hoy18**, Section 4]. We claim that there is a morphism in $\mathbf{SH}(\mathbb{F})$:

$$f^0\mathrm{KGL} \to \mathrm{H}\mathbb{Z}^{\mathrm{Hoy}}$$

which is an equivalence after taking s^0. We will be done if the fiber of the map $f^0\mathrm{KGL} \to \mathrm{H}\mathbb{Z}^{\mathrm{Hoy}}$ turns out to be 1-effective. $\mathrm{H}\mathbb{Z}^{\mathrm{Hoy}}$ is manifestly very effective, and $f^0\mathrm{KGL}$ is very effective as well (the very effective cover coincides with the effective cover in this case) []; we want to apply the criterion of [**BE19**, Proposition 5.4] (this only relies on motivic infinite loop spaces) to the fiber of this map. In other words, we want to prove that the Σ^∞ of the fiber of $\Omega^\infty f^0\mathrm{KGL} \to \Omega^\infty\mathrm{H}\mathbb{Z}^{\mathrm{Hoy}}$ is 1-effective. The latter map coincides with the map $\mathrm{Gr} \times \mathbb{Z} \to \mathbb{Z}$ as framed motivic spaces []. But now Gr is rational, so we are done.

(Step 4) We now have to relate $\mathrm{H}\mathbb{Z}^{\mathrm{Hoy}}$ to construction in lecture 1. In order to do so we assemble $\mathbb{Z}(j)^{\mathrm{mot}}$ into a $\mathbb{A}^1$-invariant $\mathbb{P}^1$-spectrum which we call $\mathrm{H}\mathbb{Z}'$ and prove that there is a canonical equivalence

$$\mathrm{H}\mathbb{Z}^{\mathrm{Hoy}} \simeq \mathrm{H}\mathbb{Z}'.$$

Here are the list of claims:

(a) each $\mathbb{Z}(j)^{\mathrm{mot}}$ is an $\mathbb{A}$-invariant, Nisnevich sheaf;
(b) $\mathbb{Z}(\star)^{\mathrm{mot}}$ has the $\mathbb{P}^1$-bundle formula;
(c) the resulting spectrum $\mathrm{H}\mathbb{Z}'$ is very effective.

Having all of this, we repeat the proof of [**Hoy18**, Theorem 19] where only need to know that $\mathrm{H}\mathbb{Z}'$ satisfies (1) $\Omega^\infty\mathrm{H}\mathbb{Z}' = \mathbb{Z}$ and (2) $\mathrm{H}\mathbb{Z}'$ is very effective.

(Step 5)

$\square$

APPENDIX B

Annotated references

1. References for lecture 1

Here we collect references which are useful for more details on the material presented in the first lecture.

1.0.1. *Chow groups,* K_0*.*

(1) The standard reference for this classical material is the book [**Ful98**].
(2) Nowadays, the stacks project is developing material on this stuff, [**Stacks**, Tag 02P3].

1.0.2. *Bloch-Ogus theories and algebraic cycles.*

(1) The original reference [**BO74**] is still excellent.
(2) Nowadays the argument has been axiomatized in [**CTHK97**] which is a standard reference for Gersten resolutions.
(3) Wome of the key geometric maneuvers in Bloch-Ogus theory, Bloch's higher chow groups and so on are explained very well in [**Kai21**, Sections 3 and 4].
(4) The proof of Roitman's theorem can be found in [**Blo10**] and [**Mil82**] and follow roughly the same strategy.

1.0.3. *Logarithmic Hodge-Witt sheaves and Milnor* K*-theory.*

(1) a modern account of these sheaves is in the paper [**Mor19**]; it also contains references to original sources.
(2) a textbook account of the Bloch-Kato-Gabber theorem, comparing Milnor K-theory mod-p and the logarithmic Hodge-Witt sheaves is in [**GS17**].

1.0.4. *Motivic cohomology.* There are many different models for motivic cohomology of smooth k-schemes. We will not attempt to give a full list of references, but only an idiosyncratic one which will hopefully be useful for the reader.

(1) The theory of motivic cohomology via Bloch's higher Chow groups was first constructed in [**Blo86a**]; a textbook reference for this material with complete proofs can be found in [**Lev98**, Chapter II].
(2) The theory of motivic cohomology via finite correspondences has an original textbook reference [**FV00a**] and an later one [**MVW06**].
(3) Motivic cohomology can also be defined via the formalism of motivic homotopy theory (see more for below) as the zeroth-slice of the slice filtration; the original reference is [**Voe02a, Voe02b**].
(4) There is also a model for motivic cohomology, by Voevodsky, via symmetric powers which works in characteristic zero [**Voe04**]; this is also explained in [**ABH23**, Section 5].
(5) The construction of motivic cohomology in these notes are inspired by Spitzweck's construction [**Spi18**] which works over arbitrary bases.
(6) Another construction that works over any base is Hoyois' in [**Hoy18**] and uses motivic infinite loop space theory.

1.0.5. *"Classical" motivic filtrations.* Motivic cohomology are the graded pieces of a motivic filtration on algebraic K-theory; the latter is more refined that the individual motivic complexes.

(1) Bloch first proposed his cycle complexes as a candidate in [**Blo86b**] and the construction of its relationship with algebraic K-theory was sketched in a preprint with

Lichtenbaum. Later, Friedlander and Suslin [**FS02**] globalized the Bloch-Lichtenbaum construction to smooth schemes over a field.

(2) Levine revisited Bloch's complexes [**Lev94**] and gave a different method for globalizing the Bloch-Lichtenbaum spectral sequence [**Lev01**]. The first complete account, to our knowledge, of the motivic spectral sequence is Levine's machinery of homotopy coniveau tower [**Lev06, Lev08**].

(3) Voevodsky proposed a reconstruction of the motivic spectral sequence using his newly-minted theory of motivic homotopy theory [**Voe02b**] using the slice filtration [**Voe02a**]. He broke down the construction of the motivic spectral sequence into a series of conjectures internal to stable motivic homotopy theory. The required conjectures were solved by Levine in [**Lev08**]; see also [**BE19**] for simplifications of parts of the proof.

2. References for lecture 2

Here we collect references which are useful for more details on the material presented in the first lecture.

2.0.1. *Hodge theory of singular schemes.*

(1) Du Bois' original paper is [**DB81**]. The key properties of the Deligne du Bois complex is [**DB81**, Théorème 4.5]. Of particular note is the degeneration of the spectral sequence arising from what we call the du Bois filtration in these notes.

(2) Of course, the gold standard is still the original series by Deligne [**Del71a, Del71b, Del74**].

(3) A standard reference (though quite old-fashioned) for Hodge theory and related matters is [**PS08**].

(4) derived de Rham cohomology in characteristic zero is extensively discussed in [**Bha12a**].

(5)

2.0.2. K-*theory of singular schemes in characteristic zero.* In the 2000's, Cortiñas, Weibel and Haesemeyer initiated a study of K-theory of

(1) the paper which suggests Weibel's conjecture [**Wei01**] is still a gem of examples and computations; it is the prelude to the very successful and influential program to study K-theory of singular schemes using cdh methods.

(2) That the fiber of the Goodwillie chern character in characteristic zero is a cdh sheaf was proved by Cortiñas in [**Cn98**].

(3) Weibel's conjecture in characteristic zero was proved in [**CnHSW08**].

(4) Similar ideas were employed to tackle Vorst's conjecture in characteristic zero [**CnHW08**].

(5) Another application of these ideas were to give counterexamples to Bass' conjecture in characteristic zero [**CnHWW10**].

3. References for lecture 3

3.0.1. *Motivic and* cdh *ideas.*

APPENDIX C

Valuation rings

One of the key points of the extension of motivic cohomology to singular schemes is that a careful study of valuation rings can replace resolution of singularities. Let us recall some basics about the theory of valuation rings.

Definition 0.0.1. A **valuation ring** is a domain such that its collection of ideals is totally ordered by inclusion.

The following are equivalent definitions of a valuation ring:

Proposition 0.0.2. *The following are equivalent for a domain* R *with field of fraction* F*:*

(1) it is a valuation ring;
(2) the collection of principal ideals is totally ordered by inclusion;
(3) every finitely generated prime ideal is principal;
(4) each element $f \in \mathrm{F}$ *is either in* R *or* $f^{-1} \in \mathrm{R}$.
(5) there is a valuation[1] ν *on* F *such that* R *is the associated valuation ring;*

Proof. We prove the claims in turn:

(1)⇒(2) This is clear.

(2)⇒(3) Let I be generated by at two elements $\mathrm{I} = (f, g)$. Then, by (2), either $(f) \subset (g)$ or $(g) \subset (f)$. Say we are in the first case, then $g|f$ and thus $\mathrm{I} = (g)$ and thus I is principal.

(3)⇒(4) Let $x \in \mathrm{K}$ so that $x = \frac{f}{g}$ where $f, g \in \mathrm{R}$. By assumption, the ideal (f, g) is principal so that either $f|g$ or $g|f$ and thus either $x \in \mathrm{R}$ or $x^{-1} \in \mathrm{R}$.

(4)⇒(5) We set $\Gamma := \mathrm{F}^\times/\mathrm{R}^\times$ and $\nu : \mathrm{F}^\times \to \Gamma$ is the canonical map, evidently a group homomorphims for the multiplicative structures. The group Γ is given a totally ordered structure where $\nu([f]) \geqslant \nu([g])$ if and only if $\frac{f}{g} \in \mathrm{R} \smallsetminus \{0\}$. One can then check that, from the property in (4), that this defines a total order on $\mathrm{F}^\times$ such that the nonarchimedean inequality $\nu(f + g) \geqslant \min(\nu(f), \nu(g))$ is satisfied.

(5)⇒(1) If $x \in \mathrm{F}$ then it has some valuation $\nu(x) \in \Gamma$. If $\nu(x) \geqslant 0$, then it must be in R. Otherwise $\nu(x) < 0$ so that $-\nu(x) = \nu(x^{-1}) > 0$ and $x^{-1} \in \mathrm{R}$.

□

Some terminology that the reader should keep in mind after the equivalences of Proposition 0.0.2:

(1) the **value group** of R is $\Gamma := \mathrm{K}^\times/\mathrm{R}^\times$;
(2) in any of the equivalent cases above, we say that R is a **valuation ring on** F. Of course F can admit many valuation rings on it.

The value group is an interesting invariant of valuation rings and measures its complexity. We discuss this notion in light of some examples:

Example 0.0.3. The following are familiar, valuation rings which are also *discrete* (its value group is a subgroup of $\mathbb{Z}$):

[1] Recall that a valuation on K is a group homomorphism $\mathrm{K}^\times \to \Gamma$ where Γ is a totally ordered group such that $\nu(f+g) \geqslant \min(\nu(f), \nu(g))$ whenever f, g are elements such that $f+g \neq 0$. The subset $\mathrm{R} := \{f : \nu(f) \geqslant 0\}$ is called the associated valuation ring; it is indeed a ring. This ring is a local ring with maximal ideal $\mathfrak{m} = \{f : \nu(f) \neq 0\}$. The value group is defined to be the image $\nu(\mathrm{K}^\times)$,

(1) If $\mathrm{R} = \mathrm{F}$, then R is itself a valuation ring; its value group is the trivial group.
(2) the ring $\mathbb{Z}_p$ is the most typical example of a *mixed characteristic* valuation ring; more generally if k is perfect then $\mathrm{W}(k)$ is a mixed characteristic discrete valuation ring.
(3) the power series ring $\mathrm{K}[[t]]$, here K is a field, is a typical example of an *equicharacteristic* valuation ring.
(4) We can also construct valuation rings by defining the valuation. Let K be a field; for each irreducible polynomial $f \in \mathrm{K}[t]$, there is a valuation on $\mathrm{K}(t)$ (where t is an indeterminate) given by $\nu_f(g) = a$ where $g = f^a g'$ where the numerator and denominator of g' are coprime to f. This defines a valuation on $\mathrm{K}(t)$ whose valuation ring is the localization of $\mathrm{K}[t]$ at the prime ideal (f).

These are all examples of valuation rings of *rank one* which are also discrete and the only examples of noetherian valuation rings (other than fields); see Lemma 0.0.6. There is a notion of a **rank** of a valuation ring: it is simply the Krull dimension of a valuation ring but can also be characterized in terms of Γ. All discrete valuation rings are rank $\leqslant 1$ but we also have rank $= 1$ valuation rings which are *not* discrete:

(1) the algebraic closure of $\mathbb{Q}_p$, denoted by $\overline{\mathbb{Q}}_p$ admits a valuation ring with value group $\mathbb{Q}$; its value group is $\mathbb{Q}$;
(2) an even larger valuation ring of rank 1 is the p-adic complex numbers which is the (topological) completion of $\overline{\mathbb{Q}}_p$.

Example 0.0.4. The following is an "stupid construction" of a valuation ring with any value group Γ. Let K be a field, consider the group ring $k[\Gamma]$ so that elements are *finitely supported* formal sums $\sum_{\gamma\in\Gamma} x_\gamma \gamma$ where $x_\gamma \in k$. Define the valuation

$$\nu : \mathrm{R} \smallsetminus \{0\} \to \Gamma \qquad \nu(\sum_{\gamma\in\Gamma} x_\gamma\gamma) = \min\{\gamma : x_\gamma \neq 0\},$$

which is well defined by the finitely supported condition. We extend ν to a valuation on the fraction field of R by

$$\nu(\tfrac{x}{y}) = \nu(x) - \nu(y).$$

Example 0.0.5.

Lemma 0.0.6. *A valuation ring is noetherian if and only if it is a discrete valuation ring or a field.*

Proof. See [**Stacks**, Tag 00I8]. ∎

0.1. Accessing a valuation ring via its value group.

Lemma 0.1.1 (Rank of valuations). *Let* V *be a valuation ring with value group* Γ. *There is an inclusion-reversing bijection*

Remark 0.1.2. One might think that any domain R whose $\operatorname{Spec} \mathrm{R}$ is totally ordered is a valuation ring. This is not the case: consider the completion of the ring $k[x, y]/(x^2 - y^3)$ at the ideal $(0, 0)$. This is a 1-dimensional local domain and hence its prime spectrum consists of the generic point and the special point. However, this ring is *not* integrally closed (Lemma 0.1.4) and hence fails to be a valuation ring. In fact, the germ of functions (at the singular point) of a singular curve always provides such an example.

Remark 0.1.3. The spectrum of a valuation is a totally ordered topological space. Since (0) is a prime ideal, it is the generic point of the prime spectrum. However, it is not the case that a domain whose spectrum is totally ordered is a valuation ring.

Lemma 0.1.4. *The following are properties of valuation rings:*

(1) they are local rings;
(2) they are integrally closed.

In other words, R *is a local, normal domain.*

PROOF. Let R be a valuation ring with fraction field F.

(1) Since the prime ideals of a valuation ring is totally ordered, there is a maximal ideal.
(2) Suppose that $\alpha \in \mathrm{F}$ satisfies an polynomial equation

$$\alpha^n + c_{n-1}\alpha^{n-1} + \cdots c_0 = 0 \qquad c_j \in \mathrm{R}.$$

By the valuation property, either $\alpha \in \mathrm{R}$ or $\alpha^{-1} \in \mathrm{R}$. In the first case we are done, otherwise we can write

$$\alpha = -c_{n-1} + c_{n-2}\alpha^{-1} + \cdots c_0\alpha^{-n+1};$$

which shows that α is actually in R.

□

In fact, Lemma 0.1.4.(2) has a sharper version:

THEOREM 0.1.5 (Krull). *Let* E *be a field extension of* F *and* R *a valuation ring of* F*. Then the integral closure of* R *in* E *is computed as the intersection:*

$$\widetilde{\mathrm{R}} = \bigcap_{\mathrm{V}\ \mathit{val.\ ring\ of}\ \mathrm{E}, \mathrm{V}\cap\mathrm{E}=\mathrm{R}} \mathrm{V}.$$

PROOF. By Lemma 0.1.4.(2), we see that V is integrally closed in E. Since $\widetilde{\mathrm{R}}$ is characterized as the minimal subring of E which is integrally closed and contains R, we see that $\widetilde{\mathrm{R}} \subset \mathrm{V}$ for any V in the indexing set of the intersection. This means that $\mathrm{R} \subset \bigcap_{\mathrm{V\ val.\ ring\ of\ E,V\cap E=R}} \mathrm{V}$.
It suffices to prove the reverse inclusion.

□

LEMMA 0.1.6. *Let* $\mathrm{R} \to \mathrm{S}$ *be a ring map. Then for any* $s \in \mathrm{S}$ *either* s *is integral over* R *or* $\mathrm{R}[s^{-1}]/(s^{-1})^2$ *is nonzero.*

0.2. Constructing new valuation rings from old ones. Before we discuss how to construct new valuation rings from old ones, we let us prove a rather nontrivial characterization of valuation rings. Recall that if A, B are local rings, then we say that B **dominates** A if $\mathrm{A} \subset \mathrm{B}$ and $\mathrm{A} \cap \mathfrak{m}_\mathrm{B} = \mathfrak{m}_\mathrm{A}$.

PROPOSITION 0.2.1. *Let* F *be a field and* $\mathrm{R} \subset \mathrm{F}$ *is a subring. Then* R *is a valuation ring with fraction field* F *if and only if it is maximal with respect to the domination relation among subrings contained in* F*.*

PROOF. See [**Stacks**, Tag 090Q]. □

This result implies that there are plenty of valuation rings around. The proof is left to the reader as a consequence of Proposition 0.2.1 and Zorn's lemma

COROLLARY 0.2.2. *Let* F *be a field and* $\mathrm{A} \subset \mathrm{F}$ *be local subring. Then there exists a valuation ring of* F *dominating* A*.*

0.2.3. *Extending valuations.* It is useful to speak of extension of valuation rings in the language of valued fields. Suppose that $i : \mathrm{E} \hookrightarrow \mathrm{F}$ is a field extension and assume that v are w are valuations on E and F respectively with valuation rings $\mathcal{O}_\mathrm{E}$ and $\mathcal{O}_\mathrm{F}$ respectively. Then i is an **extension of valued fields** if the valuation on w restricts on E to v. If this was the case, it is easy to see that these equalities hold:

(1) $\mathcal{O}_\mathrm{F} \cap \mathrm{E} = \mathcal{O}_\mathrm{E}$
(2) $\mathcal{O}_\mathrm{F}^\times \cap \mathrm{E} = \mathcal{O}_\mathrm{E}^\times$
(3) $\mathfrak{m}_{\mathcal{O}_\mathrm{F}} \cap \mathrm{E} = \mathfrak{m}_{\mathcal{O}_\mathrm{E}}$

[2] We consider $\mathrm{R}[s^{-1}]$ to be the R-subalgebra of $\mathrm{S}[s^{-1}]$.

$\mathcal{O}_F \cap E = \mathcal{O}_E$ and that $\mathfrak{m}_{\mathcal{O}_F} \cap E = \mathfrak{m}_{\mathcal{O}_E}$, i.e., $\mathcal{O}_F$ dominates $\mathcal{O}_E$. We have the following result concerning the existence of extensions of valuations:

THEOREM 0.2.4. *Let* $E \hookrightarrow F$ *be a field extension, and* v *a valuation on* E*. Then there exists a valuation* w *on* F *which extends* v*.*

EXAMPLE 0.2.5. Let F be a field and v a valuation on it with value group Γ. Then there exists a a valuation on $\overline{F}$ which extends v and whose value group is $\Gamma_{\mathbb{Q}}$. todo

0.2.6. *Algebraic extensions.* Theorem 0.2.4 typically gives us no control over how extension of valuation rings look like. The following result is more satisfactory:

THEOREM 0.2.7. *Let* E *be an algebraic extension of* F*. Let* R *be a valuation ring of* F *with valuation* ν*. Then:*

(1) *the integral closure* $\widetilde{R}$ *of* R *in* E *is the intersection of all valuation rings of* E *containing* R*;*
(2) *the ring* R *is a Prüfer domain: its localizations at a prime ideal,* $R_{\mathfrak{p}}$*, is a valuation ring of* E*;*
(3) *there is bijection between* $\operatorname{MaxSpec}$ *of* $\widetilde{R}$ *and valuations on* E *extending* ν*.*

Here's a very pleasant corollary concerning henselian valuation rings, one of the main characters of our story. It proves a certain rigidity property of algebraic extensions of henselian valuations.

COROLLARY 0.2.8. *Let* R *be a henselian valuation ring on* F *with valuation* v*. Then for any algebraic extension* E *of* F*, there exists a* unique *extension of the valuation* v *to* E*.*

PROOF. After Theorem 0.2.7.(3), it suffices to prove that $\operatorname{MaxSpec} \widetilde{R}$ is a singleton, i.e., $\widetilde{R}$ is local. This is Lemma 0.2.9.

□

LEMMA 0.2.9. *Let* R *be a hensel local subring of a field* F*. Then its integral closure in* F *is also local.*

PROOF. Let us use the characterization of hensel local rings as those local rings such that for any finite morphism $R \to S$, S is a finite product of local rings. Now, $\widetilde{R}$ is a filtered colimit of finite morphisms over F. Hence, by the aforementioned characterization, $\widetilde{R}$ must be a colimit of finite products of local rings. But now, we note that each of these finite products is contained in F, hence must be a domain and hence each product can only have a most one component. This implies that each finite extension of R must be a local ring. Since being local is stable under filtered colimits, we conclude the result. □

0.2.10. *Henselization.*

0.3. Valuative dimension. If A is an integral domain with fraction field K, its valuative dimension $\dim_v(A)$ is the supremum of n over all chains

$$A \subset R_0 \subset R_1 \subset \cdots \subset R_n \subset K$$

where each R_i is a valuation ring and $R_i \neq R_{i+1}$. Equivalently, $\dim_v(A)$ is the supremum of the ranks of the valuation rings R such that $A \subset R \subset K$.

If A is any commutative ring, Jaffard defines

$$\dim_v(A) = \sup_{\mathfrak{p} \in \operatorname{Spec} A} \dim_v(A/\mathfrak{p}).$$

This recovers the previous definition when A is integral, by [**Jaf60**, p. 55, Lemme 1].

0.4. The theorem of Datta. We include a sketch of

THEOREM 0.4.1 (Datta). *If* V *is a valuation ring of finite rank, then the global dimension of* V *is* $\leqslant 2$.

REMARK 0.4.2. Let us recall that the global dimension of a ring can be defined [**Stacks**, Tag 065T] as

$$\mathrm{gldim}(\mathrm{R}) := \sup\{\mathrm{pd}_{\mathrm{R}}(\mathrm{R}/\mathrm{I}) : \mathrm{I} \text{ is an ideal of } \mathrm{R}\}.$$

A theorem of Serre says that if R is a regular (which means noetherian) local ring if and only if its global dimension is finite, in which case the global and Krull dimension coincide [**Stacks**, Tag 00OC]. The above theorem shows a very strange phenomena whereby the global dimension of V is finite and smaller than the Krull dimension of V.

0.5. The Gabber-Romero theorem.

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