Motivic cohomology of equicharacteristic schemes

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Abstract

We construct a theory of motivic cohomology for quasi-compact, quasi-separated schemes of equal characteristic, which is related to non-connective algebraic K-theory via an Atiyah–Hirzebruch spectral sequence, and to étale cohomology in the range predicted by Beilinson and Lichtenbaum. On smooth varieties over a field our theory recovers classical motivic cohomology, defined for example via Bloch's cycle complex. Our construction uses trace methods and (topological) cyclic homology.

As predicted by the behaviour of algebraic K-theory, the motivic cohomology is in general sensitive to singularities, including non-reduced structure, and is not $\mathbb{A}^1$-invariant. It nevertheless has good geometric properties, satisfying for example the projective bundle formula and pro cdh descent.

Further properties of the theory include a Nesterenko–Suslin comparison isomorphism to Milnor K-theory, and a vanishing range which simultaneously refines Weibel's conjecture about negative K-theory and a vanishing result of Soulé for the Adams eigenspaces of higher algebraic K-groups. We also explore the relation of the theory to algebraic cycles, showing in particular that the Levine–Weibel Chow group of zero cycles on a surface arises as a motivic cohomology group.

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1 INTRODUCTION

The vision of motivic cohomology is due to Beilinson and Lichtenbaum [13, 14, 82]. For a reasonable class of schemes X they predicted the existence of natural complexes of abelian groups $\mathbb{Z}(j)^{\mathrm{mot}}(X)$, for $j \geq 0$, satisfying various relations to algebraic K-theory and étale cohomology. Perhaps the most important of these relations is a desired *Atiyah–Hirzebruch spectral sequence*

$$E_2^{i,j} = H^{i-j}_{\mathrm{mot}}(X, \mathbb{Z}(-j)) \implies \mathrm{K}_{-i-j}(X), \tag{1.1}$$

relating the *motivic cohomology groups* $H^i_{\mathrm{mot}}(X, \mathbb{Z}(j)) := H^i(\mathbb{Z}(j)^{\mathrm{mot}}(X))$ to the algebraic K-groups of X. They asked that this spectral sequence would degenerate rationally and identify the rationalised motivic cohomology $H^i_{\mathrm{mot}}(X, \mathbb{Z}(j)) \otimes_{\mathbb{Z}} \mathbb{Q}$ with the Adams eigenspace $\mathrm{K}_{i-2j}(X)^{(j)}_{\mathbb{Q}}$. Meanwhile, motivic cohomology with finite coefficients $H^i_{\mathrm{mot}}(X, \mathbb{Z}/\ell(j)) := H^i(\mathbb{Z}(j)^{\mathrm{mot}}(X)/\ell)$ was expected to coincide with étale cohomology $H^i_{\mathrm{\acute{e}t}}(X, \mu_\ell^{\otimes j})$ in the range $i \leq j$, whenever $\ell > 0$ is invertible on X. Note that any such theory of motivic cohomology must necessarily fail to be $\mathbb{A}^1$-invariant for sufficiently singular X, i.e., the maps $H^i_{\mathrm{mot}}(\mathbb{A}^1_X, \mathbb{Z}(j)) \to H^i_{\mathrm{mot}}(X, \mathbb{Z}(j))$ are not in general isomorphisms, since algebraic K-theory also fails to be $\mathbb{A}^1$-invariant on general schemes.

In this article, which builds on our joint work with T. Bachmann about cdh-local motivic cohomology [7], we construct such motivic complexes $\mathbb{Z}(j)^{\mathrm{mot}}(X)$ whenever X is a quasi-compact, quasi-separated scheme of equal characteristic.[1] For the rest of the introduction let $\mathbb{F}$ be a prime field, i.e., $\mathbb{Q}$ or $\mathbb{F}_p$ for some prime number p.

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[1] By *equal characteristic* we mean that the structure map $X \to \operatorname{Spec} \mathbb{Z}$ factors through $\operatorname{Spec} \mathbb{Q}$ or $\operatorname{Spec} \mathbb{F}_p$ for some prime number p. The main definition of this paper, namely gluing filtrations on $\mathrm{KH}(X)$ and $\mathrm{TC}(X)$ in order to define motivic cohomology of X, can be adapted to work without the equicharacteristic assumption, i.e., on any qcqs scheme, but the resulting theory is incomplete in mixed characteristic. See forthcoming work of Bouis, whose earlier work on the syntomic cohomology of valuation rings [29] means that the theory currently works best in certain highly ramified situations, such as for schemes defined over a perfectoid valuation ring.

Theorem 1.1. *There exist finitary Nisnevich sheaves*

$$\mathbb{Z}(j)^{\mathrm{mot}} : \mathrm{Sch}_{\mathbb{F}}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{D}(\mathbb{Z})$$

for $j \geq 0$, such that the following properties hold for any qcqs $\mathbb{F}$-scheme X:

1. *There exists a functorial, multiplicative, $\mathbb{N}$-indexed filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ on the non-connective algebraic K-theory $K(X)$, such that the graded pieces are naturally given by*

$$\mathrm{gr}^j_{\mathrm{mot}}\mathrm{K}(X) \simeq \mathbb{Z}(j)(X)^{\mathrm{mot}}[2j]$$

for $j \geq 0$. In particular, writing $H^i_{\mathrm{mot}}(X,\mathbb{Z}(j)) := H^i(\mathbb{Z}(j)^{\mathrm{mot}}(X))$ for the corresponding motivic cohomology groups*, there exists an Atiyah-Hirzebruch spectral sequence*

$$E_2^{ij} = H^{i-j}_{\mathrm{mot}}(X,\mathbb{Z}(-j)) \implies \mathrm{K}_{-i-j}(X).$$

If X has finite valuative dimension, then the filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ is complete and the Atiyah–Hirzebruch spectral sequence is convergent.

2. *Rational structure: the Atiyah–Hirzebruch spectral sequence degenerates rationally and there are natural isomorphisms*

$$H^i_{\mathrm{mot}}(X,\mathbb{Z}(j)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{K}_{2j-i}(X)^{(j)}_{\mathbb{Q}}$$

for all $i \in \mathbb{Z}$ and $j \geq 0$, where the right side refers to Adams eigenspaces of rationalised K-theory.

3. *Relation to étale cohomology: for any integer $\ell > 0$ invertible in $\mathbb{F}$, there are natural equivalences*

$$\tau^{\leq j}(\mathbb{Z}(j)^{\mathrm{mot}}(X)/\ell) \simeq \tau^{\leq j} R\Gamma_{\text{ét}}(X, \mu_\ell^{\otimes j})$$

for $j \geq 0$.

4. *Relation to syntomic cohomology: if $\mathbb{F} = \mathbb{F}_p$ then for any $r > 0$ there are natural equivalences*

$$\tau^{\leq j}(\mathbb{Z}(j)^{\mathrm{mot}}(X)/p^r) \simeq \tau^{\leq j}(\mathbb{Z}_p(j)^{\mathrm{syn}}(X)/p^r)$$

for $j \geq 0$, where $\mathbb{Z}_p(j)^{\mathrm{syn}}(X)$ denotes the weight-j syntomic *cohomology of X in the sense of [5, 20].*

5. *Weight zero: there is a natural equivalence*

$$\mathbb{Z}(0)^{\mathrm{mot}}(X) \simeq R\Gamma_{\mathrm{cdh}}(X,\mathbb{Z})$$

where the right side denotes cdh cohomology with coefficients in the constant sheaf $\mathbb{Z}$.

6. *Weight one: there is a natural map*

$$R\Gamma_{\mathrm{Nis}}(X,\mathbb{G}_m)[-1] \longrightarrow \mathbb{Z}(1)^{\mathrm{mot}}(X),$$

which is an equivalence in degrees ≤ 3.

7. *Projective bundle formula: For any $j, r \geq 0$, the powers of the first Chern class of the tautological bundle $c_1(\mathcal{O}(1)) \in \mathrm{Pic}(\mathbb{P}^r_X) \cong H^2_{\mathrm{mot}}(\mathbb{P}^r_X,\mathbb{Z}(1))$ induce a natural equivalence*

$$\bigoplus_{i=0}^{r} \mathbb{Z}(j-i)^{\mathrm{mot}}(X)[-2i] \xrightarrow{\sim} \mathbb{Z}(j)^{\mathrm{mot}}(\mathbb{P}^r_X).$$

8. *Blow-up formula: Given any regular closed immersion $Y \to X$ (i.e., X admits an open affine cover such that, on each such affine, Y is defined by a regular sequence), then $\mathbb{Z}(j)^{\mathrm{mot}}$ carries the cartesian square of schemes*

$$\begin{tikzcd} Y \times_X \mathrm{Bl}_Y(X) \arrow[r] \arrow[d] & \mathrm{Bl}_Y(X) \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

to a cartesian square in $\mathrm{D}(\mathbb{Z})$.

9. *Finally, suppose X is a smooth scheme over a field. Then there are equivalences*
$$\mathbb{Z}(j)^{\mathrm{mot}}(X) \simeq z^j(X,\bullet)[-2j]$$
for $j \geq 0$, where $z^j(X,\bullet)$ is Bloch's cycle complex of X. Moreover the filtration $\mathrm{Fil}^\star_{\mathrm{mot}}$ on $K(X)$ is naturally equivalent, as multiplicative filtered spectra, with the filtration coming from Voevodsky's slice filtration as in [112].

The original approach to motivic cohomology was that of Bloch [24], in terms of his cycle complexes $z^j(X,\bullet)$ for algebraic varieties X. Ignoring certain technicalities (such as functoriality, multiplicative structure, quasi-projectivity hypotheses,...), the work of Bloch, Bloch–Lichtenbaum [26], Friedlander–Suslin [44], and Levine [78, 79] show that the complexes $z^j(X,\bullet)[-2j]$ satisfy a variant of the conjectural framework of Beilinson and Lichtenbaum; the crucial difference is that $z^j(-,\bullet)$ is covariant in the algebraic variety and the Atiyah–Hirzebruch spectral sequence converges not to the K-theory of X but rather to the G-theory. (In terms of Voevodsky's approach [88, 111] via triangulated categories of motives, Bloch's cycle complex appears as Borel–Moore homology.) However, restricting attention to smooth algebraic varieties X, the motivic complexes
$$\mathbb{Z}(j)^{\mathrm{cla}}(X) := z^j(X,\bullet)[-2j]$$
do have all desired properties (and the technicalities can be overcome using motivic stable homotopy theory and the slice filtration); we will call this theory the *classical motivic cohomology* of the smooth algebraic variety X. Theorem 1.1(9) states that the new theory of this paper reduces to the classical theory in the smooth case.

Remark 1.2. Although the focus of this article is to extend motivic cohomology beyond smooth algebraic varieties, our results have applications to the smooth case. For example, we will see in Corollary 6.4 that Theorem 1.1(9) implies that the canonical map $R\Gamma_{\mathrm{\acute{e}t}}(X,\Omega^j_{\log}) \to R\Gamma_{\mathrm{\acute{e}h}}(X,\Omega^j_{\log})$ is an equivalence for any smooth variety X over a field of characteristic p. The analogous equivalence between the Nisnevich and cdh cohomologies is contained in the joint work with Bachmann [7]. Such results, which are required for example in Geisser's theory of arithmetic cohomology [48], seem to have been previously out of reach without assuming resolution of singularities.

1.1 Relation to $\mathbb{A}^1$-invariant motivic cohomology

This article depends on our joint work with Bachmann [7], in which we revisit the theory of $\mathbb{A}^1$-invariant motivic cohomology. Although much of that project works for arbitrary qcqs schemes, we restrict our summary here to the simpler equicharacteristic context. See §3.3 for further details.

For each $j \geq 0$ let
$$\mathbb{Z}(j)^{\mathrm{cdh}} : \mathrm{Sch}^{\mathrm{qcqs,op}}_{\mathbb{F}} \longrightarrow \mathrm{D}(\mathbb{Z})$$
be the cdh sheafification of the left Kan extension of classical motivic cohomology $\mathbb{Z}(j)^{\mathrm{cla}}$ from smooth $\mathbb{F}$-schemes to qcqs $\mathbb{F}$-schemes; we call it *cdh-local motivic cohomology* and note that it provides a cdh analogue of Bloch's cycle complex. It was already studied by Friedlander, Suslin, and Voevodsky [115] in the case of singular algebraic varieties assuming resolution of singularities.

In [7] we establish various properties of this cdh-local motivic cohomology, without any assumption on resolution of singularities, proving in particular that it is $\mathbb{A}^1$-invariant: namely, for any qcqs equicharacteristic scheme X, the canonical maps $\mathbb{Z}(j)^{\mathrm{cdh}}(\mathbb{A}^1_X) \to \mathbb{Z}(j)^{\mathrm{cdh}}(X)$ are equivalences. Moreover, the cohomology groups $H^i_{\mathrm{cdh}}(X,\mathbb{Z}(j)) := H^i(\mathbb{Z}(j)^{\mathrm{cdh}}(X))$ fit into an Atiyah–Hirzebruch spectral sequence converging to the KH-groups of X, and with finite coefficients away from the characteristic they are related to étale cohomology. See Theorem 3.5 for more precise statements. In short, the complexes $\mathbb{Z}(j)^{\mathrm{cdh}}(X)$, for $j \geq 0$, satisfy a variant of Beilinson and Lichtenbaum's vision, except that they are $\mathbb{A}^1$-invariant and related not to $\mathrm{K}(X)$ but rather to $\mathrm{KH}(X)$. Although it is not required for the present article, we also prove in [7] that this cdh-local motivic cohomology coincides with the motivic cohomology represented by the zeroth slice of the unit $\mathbf{1}_X$, or equivalently by the motivic Eilenberg–Maclane spectrum $\mathrm{H}\mathbb{Z}_X$, in Morel–Voevodsky's stable homotopy category $\mathcal{SH}(X)$; see Remark 3.6 for details. In other words, we have no doubt that $\mathbb{Z}(j)^{\mathrm{cdh}}$, for $j \geq 0$, provide the "correct" theory of $\mathbb{A}^1$-invariant motivic cohomology.

The theory of this paper is designed so that, for any qcqs equicharacteristic scheme X, the canonical map $\mathrm{K}(X) \to \mathrm{KH}(X)$ is compatible with the motivic filtrations on each side, thereby inducing comparison maps
$$\mathbb{Z}(j)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(j)^{\mathrm{cdh}}(X) \tag{1.2}$$

from the new non-$\mathbb{A}^1$-invariant motivic cohomology to the $\mathbb{A}^1$-invariant, cdh-local theory. This comparison map has the following properties, thereby refining to the level of motivic cohomology known comparisons between K-theory and KH-theory:

Theorem 1.3 (See Thms. 6.1, 4.10, and 4.24)**.** *Let $j \geq 0$ and let $\mathbb{F}$ be a prime field.*

1. *The map (1.2) identifies $\mathbb{Z}(j)^{\mathrm{cdh}}$ with the $\mathbb{A}^1$-localisation and the cdh-sheafification of $\mathbb{Z}(j)^{\mathrm{mot}}$. That is, on the category of qcqs $\mathbb{F}$-schemes, there are natural equivalences of $\mathrm{D}(\mathbb{Z})$-valued presheaves:*
$$L_{\mathbb{A}^1}\mathbb{Z}(j)^{\mathrm{mot}} \simeq \mathbb{Z}(j)^{\mathrm{cdh}} \simeq L_{\mathrm{cdh}}\mathbb{Z}(j)^{\mathrm{mot}}.$$
2. *For any qcqs $\mathbb{F}$-scheme and integer $\ell > 0$ invertible in $\mathbb{F}$, the map (1.2) is an equivalence mod ℓ:*
$$\mathbb{Z}(j)^{\mathrm{mot}}(X)/\ell \xrightarrow{\sim} \mathbb{Z}(j)^{\mathrm{cdh}}(X)/\ell.$$
3. *For any qcqs $\mathbb{F}_p$-scheme X, the map (1.2) is an equivalence after inverting p:*
$$\mathbb{Z}(j)^{\mathrm{mot}}[\tfrac{1}{p}] \xrightarrow{\sim} \mathbb{Z}(j)^{\mathrm{cdh}}[\tfrac{1}{p}].$$
4. *For any regular Noetherian $\mathbb{F}$-scheme X, the map (1.2) is an equivalence.*

Part (1) of the theorem refines the fact that KH-theory is both the $\mathbb{A}^1$-localisation of K-theory (by definition) and its cdh-sheafification (as we will recall at the start of §1.2). Parts (2) and (3) refine in equicharacteristic results of Weibel [119] that $\mathrm{K}(A)/\ell \xrightarrow{\sim} \mathrm{KH}(A)/\ell$ (resp. $\mathrm{K}(A)[\frac{1}{p}] \xrightarrow{\sim} \mathrm{KH}(A)[\frac{1}{p}]$) for rings A in which ℓ is invertible (resp. in which $p = 0$). Finally, part (4) refines in equicharacteristic the equivalence between K-theory and KH-theory for regular Noetherian rings.

An input to establishing part (3) of the previous theorem, which is essential to controlling our motivic cohomology in characteristic p, is to show that rationalised syntomic cohomology $\mathbb{Q}_p(j)^{\mathrm{syn}} := \mathbb{Z}_p(j)^{\mathrm{syn}}[\frac{1}{p}]$ is a cdh sheaf on qcqs $\mathbb{F}_p$-schemes (see Corollary 4.20). Perhaps this can be proved directly, but our approach is rather to reduce it to the aforementioned fact that $\mathrm{K}[\frac{1}{p}] = \mathrm{KH}[\frac{1}{p}]$ on such schemes; the reduction argument passes through the cartesian square (1.3) below and so ultimately depends on trace methods. This extraction of information about cohomology theories from localising invariants is a theme which runs throughout this paper and [7]; we will return to it in Remark 1.7 when discussing the projective bundle formula.

1.2 The construction of $\mathbb{Z}(j)^{\mathrm{mot}}$ via trace methods

Our construction of $\mathbb{Z}(j)^{\mathrm{mot}}$ is inspired by trace methods in algebraic K-theory. For any qcqs scheme X let $\mathrm{TC}(X)$ denote its topological cyclic homology, and $\mathrm{K}^{\mathrm{inf}}(X)$ the fibre of the trace map $\mathrm{K}(X) \to \mathrm{TC}(X)$. The presheaf $\mathrm{K}^{\mathrm{inf}}$ is nil-invariant by the Dundas–Goodwillie–McCarthy theorem [37], and even a cdh sheaf by Kerz–Strunk–Tamme [70] and Land–Tamme [76]. Coupled with the surprising fact that Weibel's KH-theory is equivalent to the cdh-sheafification of K-theory (first proved by Haesemeyer in characteristic zero [55] and Kerz–Strunk–Tamme [70] in general), we arrive at a cartesian square for any qcqs scheme

$$\begin{tikzcd}
\mathrm{K}(X) \arrow[r] \arrow[d] & \mathrm{TC}(X) \arrow[d] \\
\mathrm{KH}(X) \arrow[r] & L_{\mathrm{cdh}}\mathrm{TC}(X),
\end{tikzcd} \tag{1.3}$$

where the bottom map is the cdh-sheafified trace map. We define the motivic filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}$ on $\mathrm{K}(X)$ by glueing existing filtrations on $\mathrm{KH}(X)$, $\mathrm{TC}(X)$, and $L_{\mathrm{cdh}}\mathrm{TC}(X)$:

1. For any qcqs $\mathbb{F}_p$-scheme X, Bhatt, Scholze, and the second author [20] have defined a filtration on $\mathrm{TC}(X)$ whose graded pieces are $\mathbb{Z}_p(j)^{\mathrm{syn}}(X)[2j]$ for $j \geq 0$; here $\mathbb{Z}_p(j)^{\mathrm{syn}}(X)$ is the *syntomic cohomology* of X, which modulo p is a derived version of the étale cohomology of Illusie–Milne's sheaves $\Omega^j_{X,\log}$. Cdh sheafifying this filtration over qcqs $\mathbb{F}_p$-schemes induces a filtration on $L_{\mathrm{cdh}}\mathrm{TC}(X)$.
2. For any qcqs $\mathbb{Q}$-scheme X, its topological cyclic homology $\mathrm{TC}(X)$ identifies with its negative cyclic homology $\mathrm{HC}^-(X/\mathbb{Q})$. Antieau [4] has defined a filtration on $\mathrm{HC}^-(X/\mathbb{Q})$, extending previous work of Loday [83] and Weibel [116], whose graded pieces are $R\Gamma(X, \widehat{L\Omega}^{\geq j}_{-/\mathbb{Q}})[2j]$ for $j \in \mathbb{Z}$. Here $\widehat{L\Omega}_{-/\mathbb{Q}}$ is the Hodge-completed *derived de Rham complex* equipped with its Hodge filtration, as studied notably by Illusie [60, 61] and Bhatt [15]. As in characteristic p, cdh sheafifying then induces a compatible filtration on $L_{\mathrm{cdh}}\mathrm{TC}(X) = L_{\mathrm{cdh}}\mathrm{HC}^-(X/\mathbb{Q})$.

3. For any qcqs equicharacteristic scheme X, our joint work with Bachmann [7], briefly discussed above in §1.1, defines a motivic filtration on $\mathrm{KH}(X)$ by left Kan extending from the smooth context and then cdh sheafifiying. The graded pieces are $\mathbb{Z}(j)^{\mathrm{cdh}}(X)[2j]$ for $j \geq 0$, and it turns out that the filtration agrees with Voevodsky's slice filtration coming from motivic stable homotopy theory.

The following compatibility of these filtrations is not difficult to prove but is fundamental to our construction; we refer to Corollary 4.8 and Proposition 4.22 for more precise statements:

Proposition 1.4. *For any qcqs equicharacteristic scheme X, the cdh-sheafified trace map $\mathrm{KH}(X) \to \mathrm{TC}(X)$ respects the filtrations on each side.*

We consequently define our motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}$ on $\mathrm{K}(X)$ by glueing the existing filtrations on the three other corners of the square (1.3). Passing to graded pieces yields the following description of our motivic cohomology:

Theorem 1.5 (See Thms. 4.10 and 4.24)**.** *For $j \geq 0$ and any qcqs scheme X over $\mathbb{Q}$ (resp. $\mathbb{F}_p$), there is a natural cartesian square in $D(\mathbb{Z})$*

$$\begin{tikzcd}
\mathbb{Z}(j)^{\mathrm{mot}}(X) \arrow[r] \arrow[d] & R\Gamma(X, \widehat{L\Omega}^{\geq j}_{-/\mathbb{Q}}) \arrow[d] & \textit{resp.} & \mathbb{Z}(j)^{\mathrm{mot}}(X) \arrow[r] \arrow[d] & \mathbb{Z}_p(j)^{\mathrm{syn}}(X) \arrow[d] \\
\mathbb{Z}(j)^{\mathrm{cdh}}(X) \arrow[r] & R\Gamma_{\mathrm{cdh}}(X, \widehat{L\Omega}^{\geq j}_{-/\mathbb{Q}}). & & \mathbb{Z}(j)^{\mathrm{cdh}}(X) \arrow[r] & L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}(X).
\end{tikzcd} \tag{1.4}$$

The cartesian squares (1.4) encapsulate the central idea of our construction of motivic cohomology. They say that the motivic complex $\mathbb{Z}(j)^{\mathrm{mot}}(X)$ is a modification of the cdh-local, $\mathbb{A}^1$-invariant theory $\mathbb{Z}(j)^{\mathrm{cdh}}(X)$ (which is governed by algebraic cycles) by derived de Rham/syntomic cohomology. In particular, in characteristic zero the left square of (1.4) yields a fibre sequence

$$\mathbb{Z}(j)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(j)^{\mathrm{cdh}}(X) \longrightarrow \mathrm{cofib}\left(R\Gamma(X, L\Omega^{<j}_{-/\mathbb{Q}}) \to R\Gamma_{\mathrm{cdh}}(X, \Omega^{<j}_{-/\mathbb{Q}})\right)[-1];$$

this plays the role of the weight-j motivic component of the well-known fibre sequence

$$\mathrm{K}(X) \longrightarrow \mathrm{KH}(X) \longrightarrow \mathrm{cofib}\big(\mathrm{HC}(X) \to L_{\mathrm{cdh}}\mathrm{HC}(X)\big)[1]$$

arising from (1.3), which was used throughout Cortiñas–Haesemeyer(–Schlichting)–Weibel's work [34, 35] on the K-theory of singular varieties in characteristic zero. The present paper may in fact be roughly understood as a refinement of their work from the level of K-theory to that of motivic cohomology, as well as providing an extension to finite characteristic.

The squares (1.4) also provide a refinement of the trace map and its main property to the level of motivic cohomology:

Corollary 1.6. *On the category of qcqs schemes over $\mathbb{Q}$ (resp. $\mathbb{F}_p$), there exists for each $j \geq 0$ a "weight-j motivic trace map" (namely the top horizontal arrow in (1.4))*

$$\mathbb{Z}(j)^{\mathrm{mot}} \longrightarrow R\Gamma(-, \widehat{L\Omega}^{\geq j}_{-/\mathbb{Q}}), \qquad \textit{resp. } \mathbb{Z}(j)^{\mathrm{mot}} \longrightarrow \mathbb{Z}_p(j)^{\mathrm{syn}},$$

whose fibre is a cdh sheaf.

Remark 1.7 (Projective bundle formula)**.** As already stated in Theorem 1.1(8), our motivic cohomology satisfies the projective bundle formula. For Grothendieck this was one of the most fundamental desired properties of any cohomology theory, and it means that the motivic cohomology assembles into a motivic spectrum in the sense of Annalla–Iwasa [3]. But it is also an essential input into proving the comparison theorems with classical motivic cohomology (Theorem 1.1(9)) and with the $\mathbb{A}^1$-invariant theory in the regular case (Theorem 1.3(4)), as we use Gabber's technique [46, 53] axiomatised by Colliot-Thélène–Hoobler–Kahn [33].

Remarkably, our proof of the projective bundle formula depends on the theory of localising invariants (at least in characteristic p – in characteristic zero it is sufficient to use strong resolution of singularities). Indeed, exploiting the fact that cdh-local motivic cohomology and syntomic cohomology are known to have this property (by [7] and [18] respectively; the proof in [7] also uses localising invariants), the problem reduces via the right square in (1.4) to showing that cdh-sheafified syntomic cohomology satisfies the projective bundle formula. We prove this in Theorem 5.14, using the fact that the square (1.3) is cartesian and so allows us to upgrade $L_{\mathrm{cdh}}\mathrm{TC}$ to a localising invariant.

1.3 Relation to Milnor K-theory and lisse motivic cohomology

When F is a field, a theorem of Nesterenko–Suslin [94], later reproved by Totaro [109], produces a natural isomorphism between the classical motivic cohomology $H^j_{\mathrm{mot}}(F,\mathbb{Z}(j))$ with the Milnor K-group $\mathrm{K}^M_j(F)$. On the one hand, this gives a "cohomological interpretation" of the Milnor K-groups defined via generators and relations. On the other hand it shows that the graded ring $\bigoplus_{j\geq 0} H^j_{\mathrm{mot}}(F,\mathbb{Z}(j))$ is generated by elements in degree 1 and with relations in degree 2.

In his thesis [67], Kerz extended the Nesterenko–Suslin isomorphism to the generality of regular local rings containing an infinite field, thereby settling a conjecture of Beilinson. He later eliminated the hypothesis that the field be infinite, using the improved Milnor K-theory $\widehat{\mathrm{K}}^M_j$ which he and Gabber had introduced [68].

Our motivic cohomology satisfies the Nesterenko–Suslin isomorphism for arbitrary local rings containing a field, without any regularity hypotheses:

Theorem 1.8 (Nesterenko–Suslin isomorphism; see Thm. 7.12)**.** *Let A be a local ring containing a field. Then the isomorphism $A^\times \cong H^1_{\mathrm{mot}}(A,\mathbb{Z}(1))$ of Theorem 1.1(6) induces, by multiplicativity, isomorphisms*

$$\widehat{\mathrm{K}}^M_j(A) \xrightarrow{\simeq} H^j_{\mathrm{mot}}(A,\mathbb{Z}(j)) \tag{1.5}$$

for all $j \geq 1$.

The proof of Theorem 1.8 is intertwined with a comparison theorem relating our motivic cohomology to a more naive version of the theory obtained by simply left Kan extending classical motivic cohomology. More precisely, for any $\mathbb{F}$-algebra A we define $\mathbb{Z}(j)^{\mathrm{lse}}(A) \in \mathrm{D}(\mathbb{Z})$ to be the left Kan extension, from smooth $\mathbb{F}$-algebras, of weight-j classical motivic cohomology. More explicitly, there exists a simplicial resolution $P_\bullet \xrightarrow{\sim} A$ whose terms are ind-smooth $\mathbb{F}$-algebras and whose face maps are henselian surjections; then $\mathbb{Z}(j)^{\mathrm{lse}}(A)$ is given by the totalisation of the simplicial complex $\mathbb{Z}(j)^{\mathrm{cla}}(P_\bullet)$. We call $\mathbb{Z}(j)^{\mathrm{lse}}(A)$ the weight-j, *lisse motivic cohomology* of A to emphasise the fact that it controlled by smooth algebras. The complexes $\mathbb{Z}(j)^{\mathrm{lse}}(A)[2j]$, for $j \geq 0$, appear as the graded pieces of a motivic filtration on the connective algebraic K-theory $K^{\mathrm{cn}}(A)$; see §3.2 for more details. For any $\mathbb{F}$-algebra A there is a natural comparison map

$$\mathbb{Z}(j)^{\mathrm{lse}}(A) \longrightarrow \mathbb{Z}(j)^{\mathrm{mot}}(A), \tag{1.6}$$

and we prove the following, which in degree j is the Nesterenko–Suslin isomorphism:

Theorem 1.9 (see Thm. 7.7)**.** *For any local $\mathbb{F}$-algebra A and $j \geq 0$, the map (1.6) induces an equivalence*

$$\mathbb{Z}(j)^{\mathrm{lse}}(A) \xrightarrow{\sim} \tau^{\leq j}\mathbb{Z}(j)^{\mathrm{mot}}(A).$$

The theorem states that, in degrees less than or equal to the weight, our motivic cohomology is Zariski locally controlled by classical motivic cohomology; in particular, in this range it is closely related to algebraic cycles. This is the next topic we discuss.

1.4 Relations to algebraic cycles

One of the key features of the classical motivic cohomology of smooth algebraic varieties X is its description in terms of algebraic cycles, via Bloch's cycle complex; this yields in particular isomorphisms

$$H^{2j}_{\mathrm{mot}}(X,\mathbb{Z}(j)) \cong \mathrm{CH}^j(X) \tag{1.7}$$

for each $j \geq 0$.

In the case of a singular algebraic variety X, various definitions of Chow groups have been proposed. A first possibility is Fulton's [45], but his theory is a Borel–Moore homology theory related more to $G(X)$ than $\mathrm{K}(X)$. Another is Baum–Fulton–Macpherson's [11] theory of cohomological Chow groups, essentially obtained by left Kan extending CH^j from smooth algebraic varieties to arbitrary varieties; it is thus related, at least superficially, to the lisse motivic cohomology $\mathbb{Z}(j)^{\mathrm{lse}}$ introduced above in §1.3. Levine [80] refined Baum–Fulton–Macpherson's idea by (roughly speaking) restricting the class of smooth varieties appearing in the left Kan extension procedure to better control the algebraic cycles. The case of zero cycles is particularly well-developed, and the *Levine–Weibel* Chow group $\mathrm{CH}^{\mathrm{LW}}_0(\mathrm{X})$ of zero cycles [81] on a singular variety X has found concrete applications towards K-theoretic problems such as the splitting of vector bundles on affine varieties [75, 93]. For a modern text on Levine–Weibel's group, we refer the reader to work of Binda–Krishna [21, 22].

In §9 we study the relationship between our motivic cohomology and algebraic cycles. We are not sure what to expect in general, but we can show in the case of surfaces that our theory captures the Levine–Weibel group of zero cycles:

Theorem 1.10 (See Thm. 9.5)**.** *Let X be a reduced, equi-dimensional, quasi-projective surface over an infinite field k; then there is a natural isomorphism*

$$H^4_{\mathrm{mot}}(X, \mathbb{Z}(2)) \cong \mathrm{CH}_0^{\mathrm{LW}}(X). \tag{1.8}$$

Whereas one often views (1.7) as a description of motivic cohomology in terms of algebraic cycles, we suggest adopting the alternative point of view on (1.8), namely it provides a new description of zero cycles on singular surfaces. Indeed, bearing in mind the main idea presented after Theorem 1.5, it says that the Levine–Weibel group of zero cycles of a surface is somehow built from cdh-local zero cycles and derived de Rham/syntomic cohomology.

Example 1.11 (cdh-local zero cycles on surfaces)**.** Let X be as in Theorem 1.10. The proof of the Soulé–Weibel vanishing Theorem 1.13 below implies in addition that the canonical map

$$\mathrm{CH}_0^{\mathrm{LW}}(X) = H^4_{\mathrm{mot}}(X, \mathbb{Z}(j)) \longrightarrow H^4_{\mathrm{cdh}}(X, \mathbb{Z}(2)) = H^2_{\mathrm{cdh}}(X, \widehat{K}_2^M)$$

is surjective, where the right equality follows from a cdh-local version of the Nesterenko–Suslin isomorphism. In other words, any "cdh-local zero cycle" comes from an honest zero cycle. Our results also imply that this map is an isomorphism modulo any integer invertible in the base field k, and an isomorphism after inverting p if k has characteristic $p > 0$. This may well be known to experts: such comparisons have certainly been established previously for *projective* varieties in arbitrary dimensions [22, Thms. 1.6 & 1.7].

Another context in which algebraic cycles appear in motivic cohomology is the theory of Chow groups with modulus, building on Bloch–Esnault's earlier notion of additive Chow groups [25]. The set-up of the theory varies, but suppose for simplicity that X is a smooth algebraic variety equipped with an effective divisor D such that D_{red} is a simple normal crossing divisor. The theory defines various "Chow groups on X with modulus D", which it is hoped will ultimately correspond to a piece of the motivic cohomology of X relative to D; these Chow groups with modulus should in particular be related to the K-theory of X relative to D, but at present the evidence of this relation is limited. A common theme in the subject [25, 102, 103], already present in the original work of Bloch–Esnault, is that Chow groups with modulus, although they are defined purely in terms of algebraic cycles, often contain groups of differential forms, Witt vectors, or more generally de Rham–Witt groups; that is, the theory offers a cycle-theoretic description of the latter objects. Our theory provides a systematic framework to obtain similar descriptions, exemplified as follows, for which the reader should recall that lisse motivic cohomology is described by algebraic cycles:

Example 1.12 (See Ex. 9.3)**.** Let k be a perfect field of characteristic $p > 0$, and $j, e \geq 0$. Then the lisse motivic cohomology of $k[x]/x^e$ relative to its residue field, i.e., the fibre of $\mathbb{Z}(j)^{\mathrm{lse}}(k[x]/x^e) \to \mathbb{Z}(j)^{\mathrm{lse}}(k)$, is naturally equivalent to $(\mathbb{W}_{ej}(k)/V^e\mathbb{W}_j(k))[-1]$.

1.5 Negative K-groups and Soulé–Weibel vanishing

A major stimulus in the development of the algebraic K-theory of singular schemes has been the problem of understanding their negative K-groups, in which the central conjecture for many years was Weibel's vanishing conjecture: for a Noetherian scheme X of finite dimension, he predicted that the negative K-groups $K_{-n}(X)$ would vanish for $n > \dim X$. Following numerous special cases (see the start of Section 8 for references), the conjecture was proved in general by Kerz–Strunk–Tamme [70]. Meanwhile, concerning the positive K-groups of a Noetherian ring A, Soulé [104, Corol. 1] had proved much earlier the vanishing of the Adams eigenspaces $\mathrm{K}_n(A)^{(j)}_{\mathbb{Q}}$ whenever $n > 0$ and $j > n + \dim A$. The following integral motivic vanishing theorem strenghtens and unifies these two results in the equicharacteristic case:

Theorem 1.13 (Motivic Soulé–Weibel vanishing; see Thm. 8.1)**.** *Let $j \geq 0$ and let X be a Noetherian equicharacteristic scheme of finite dimension. Then $H^i_{\mathrm{mot}}(X, \mathbb{Z}(j)) = 0$ for all $i > j + \dim X$.*

Kerz–Strunk–Tamme's proof of Weibel vanishing depended on first establishing that K-theory satisfied pro cdh descent on Noetherian schemes, again following various special cases which had been proved earlier. It seems in fact that pro cdh descent, which is an analogue of the formal functions theorem from coherent cohomology, is one of the most fundamental properties of algebraic K-theory. In any case, as well as its appearance in the proof of Weibel vanishing, it has applications to the study of algebraic cycles on singular varieties [74, 75, 89]. We prove that our motivic cohomology also has this property:

Theorem 1.14 (Pro cdh descent for motivic cohomology; see Thm. 8.2)**.** *On the category of Noetherian equicharacteristic schemes, the presheaf $\mathbb{Z}(j)^{\mathrm{mot}}$ satisfies pro cdh descent for each $j \geq 0$. That is, given any abstract blowup square of Noetherian equicharacteristic schemes*

$$\begin{tikzcd}
Y' \arrow[r] \arrow[d] & X' \arrow[d] \\
Y \arrow[r] & X
\end{tikzcd}$$

the associated square of pro complexes

$$\begin{tikzcd}
\mathbb{Z}(j)^{\mathrm{mot}}(X) \arrow[r] \arrow[d] & \mathbb{Z}(j)^{\mathrm{mot}}(X') \arrow[d] \\
\{\mathbb{Z}(j)^{\mathrm{mot}}(rY)\}_r \arrow[r] & \{\mathbb{Z}(j)^{\mathrm{mot}}(rY')\}_r
\end{tikzcd}$$

is cartesian.

1.6 Other recent approaches to motivic cohomology

1.6.1 Kelly–Saito's pro-cdh-local motivic cohomology

Kelly and Saito [66] have recently defined a Grothendieck topology, called the *pro-cdh topology*, on qcqs schemes with the following property: a presheaf F on qcqs schemes, valued in Sp or $\mathrm{D}(\mathbb{Z})$, is a pro-cdh sheaf if and only if it is both a Nisnevich sheaf and, for every abstract blow-up square of qcqs schemes denoted as in Theorem 1.14, the associated square

$$\begin{tikzcd}
F(X) \arrow[r] \arrow[d] & \lim_r F(rY) \arrow[d] \\
F(X') \arrow[r] & \lim_r F(rY')
\end{tikzcd}$$

is cartesian. They define *pro-cdh-local motivic cohomology*

$$\mathbb{Z}(j)^{\mathrm{pcdh}} : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{D}(\mathbb{Z})$$

to be the pro-cdh sheafification of the left Kan extension of classical motivic cohomology from smooth $\mathbb{Z}$-schemes to all qcqs schemes. That is, the definition mimics that of $\mathbb{Z}(j)^{\mathrm{cdh}}$, but replacing the cdh topology by their coarser pro-cdh topology.

For any Noetherian scheme X, its pro-cdh local motivic cohomology fits into an Atiyah–Hirzebruch spectral sequence converging to $\mathrm{K}(X)$; to prove this one uses that, on Noetherian schemes, K-theory is the pro-cdh sheafification of connective K-theory.

By combining some of Kelly–Saito's main theorems about their topology (in particular the fact that it has enough points, and the description of the points) with some of our own (including Theorems 1.1(9), 1.9, and 1.14), one obtains the following comparison:

Theorem 1.15 (See [66])**.** *For any Noetherian equicharacteristic scheme X and $j \geq 0$, there is a natural equivalence*

$$\mathbb{Z}(j)^{\mathrm{pcdh}}(X) \xrightarrow{\sim} \mathbb{Z}(j)^{\mathrm{mot}}(X). \tag{1.9}$$

Thus, on Noetherian equicharacteristic schemes, Kelly–Saito's approach offers an alternative definition of the same motivic cohomology of this paper; their definition is not restricted to equal characteristic and does not require trace methods. On the other hand we are not aware at present whether their approach can be used to establish, for example, the projective bundle formula, the Nesterenko–Suslin isomorphism, the comparisons to $\mathbb{A}^1$-invariant motivic cohomology, or the relation to zero cycles on surfaces. In the generality of non-Noetherian schemes, the two theories differ and pro-cdh-local motivic cohomology is not finitary. The two sides of (1.9) thus seem to have quite different flavours; we hope that the comparison between then will serve as a powerful tool in future work (for example, the pro-cdh approach should ultimately lead to a more conceptual proof of the Soulé–Weibel vanishing bound).

1.6.2 Annala–Hoyois–Iwasa's non-$\mathbb{A}^1$-invariant motivic homotopy theory

Annala, Hoyois, and Iwasa [1] are currently developing a theory of non-$\mathbb{A}^1$-invariant motivic homotopy theory, building on earlier work of Annala–Iwasa [2, 3]. A theory of motivic cohomology in their framework is provided by forcing the left Kan extension of classical motivic cohomology, from smooth schemes to qcqs schemes, to satisfy the projective bundle bundle and Nisnevich descent. We all hope it coincides in the equicharacteristic case with the motivic cohomology constructed in the present paper.

1.6.3 Park's yeni higher Chow groups

For any algebraic variety X, Park [97] has defined complexes of cycles $\mathbf{z}^j(X,\bullet)$, for $j \geq 0$, which coincide with Bloch's cycle complexes $z^j(X,\bullet)$ when X is smooth. His complexes are Zariski locally supported in negative cohomological degrees and therefore their cohomology groups cannot fit into an Atiyah–Hirzebruch spectral sequence converging to the K-groups of X. From his construction (by locally embedding X into a smooth variety and looking at certain cycles on the formal completion of the embedding), it seems plausible that $\mathbf{z}^j(X,\bullet)[-2j]$ is Zariski locally an explicit approximation of the lisse motivic cohomology $\mathbb{Z}(j)^{\mathrm{lse}}(X)$.

1.7 Outline of the paper

We briefly summarize the contents of the paper. After reviewing conventions regarding sheaves and filtrations in §2 we recall previously known constructions of motivic cohomology in §3. Of note is the lisse version of motivic cohomology reviewed in §3.2 which is produced simply by left Kan extending and the cdh-local version of motivic cohomology which is jointly produced with Bachmann, recalled in §3.3.

Our construction of motivic cohomology is explained in §4. Specifically, the characteristic zero version is given in Definition 4.9 and the characteristic $p > 0$ version is given in Definition 4.23. We briefly discuss an extension of the theory to derived schemes in §3.3. In §5 we prove the projective bundle formula and the blowup formula for motivic cohomology. This is the technical heart of the paper. In particular, we prove a $\mathbb{P}^1$-bundle formula for cdh-sheafified syntomic cohomology in §5.2, adopting techniques that we developed in the joint paper with Bachmann in [7]. The projective bundle formula is key in comparing our construction to previous constructions of motivic cohomology in the smooth setting. This is discussed, more generally, under comparison with $\mathbb{A}^1$-invariant versions of motivic cohomology in §6.

The last part of the paper is dedicated to deeper properties of motivic cohomology. In §7 we describe a portion of our motivic cohomology using lisse motivic cohomology. Using this, in §7.2, we prove the singular Nesterenko–Suslin isomorphism. In §8 we prove the motivic Soulé–Weibel vanishing. The key ingredient is a pro cdh descent result which we establish in §8.1. We then finish off the paper by examining how our theory relates to algebraic cycles in §9.

This paper has two appendices. In Appendix A, we prove a technical result establishing, under certain hypotheses, that the cdh-sheafification of an étale sheaf is an éh-sheaf. In Appendix B, we discuss rational motivic cohomology and prove a spectrum-level, multiplicative refinement of a theorem of Cortiñas, Haesemeyer and Weibel on the compatibility between Adams operations on rationalized K-theory and negative cyclic homology. This appendix is important in controlling the rational parts of our theory.

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2 Some notation and conventions

In this section, we collect notation and conventions which we will use throughout the paper. We freely use the language of ∞-categories as developed in [85, 87]

2.1 Sheaves

Let $(\mathcal{C}, \tau)$ be an ∞-site and $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ a presheaf, where $\mathcal{D}$ is a stable, presentable ∞-category with a t-structure. The examples of $\mathcal{D}$ that will appear in this paper are mainly the category Sp of spectra and the derived category D(A) where A is a discrete coefficient ring, each equipped with the standard t-structure; we will also see filtered variants of these categories. One says that F is *discrete* if it factors through the heart $\mathcal{D}^{\heartsuit} \subseteq \mathcal{D}$; in the previous example Sp (resp. D(R)), a discrete presheaf means a discrete presheaf of abelian groups (resp. of R-modules). We will use the following terminology and notation:

1. We write L_τ to be the endofunctor $L_\tau : \mathrm{PShv}(\mathcal{C}) \to \mathrm{PShv}(\mathcal{C})$ reflecting onto the subcategory of τ-sheaves $\mathrm{Shv}_\tau(\mathcal{C})$; this functor is referred to as *sheafification.*
2. If F is a discrete presheaf, then we write $R\Gamma_\tau(-, F)$ then for the τ-cohomology of F; in other words we have an equivalence of τ-sheaves
$$L_\tau F \simeq R\Gamma_\tau(-, F).$$
3. Given another topology τ' which is finer than τ then there is an adjunction
$$\epsilon^* : \mathrm{Shv}_\tau \rightleftarrows \mathrm{Shv}_{\tau'} : \epsilon_*.$$
whose unit gives rise to a canonical map in $L_\tau F \to \epsilon_* L_{\tau'} F$ in $\mathrm{Shv}_\tau(\mathcal{C})$. (We note that in the case $\mathcal{D} = \mathrm{D(R)}$, then ϵ_* is often denoted in the literature as $R\epsilon_*$.) Often we regard $L_\tau F$ and $L_{\tau'} F$ as presheaves on $\mathcal{C}$ and simply write the previous map as $L_\tau F \to L_{\tau'} F$; the context should always make it clear how we are viewing the objects.

2.2 Filtrations

For $\mathcal{C}$ a stable ∞-category, the associated stable ∞-categories of *filtered objects* and *graded objects* are
$$\mathcal{C}^{\mathbb{Z}^{\mathrm{op}}} := \mathrm{Fun}((\mathbb{Z}, \geq)^{\mathrm{op}}, \mathcal{C}) \qquad \text{and} \qquad \mathcal{C}^{\mathbb{Z}^\delta} := \mathrm{Fun}(\mathbb{Z}^\delta, \mathcal{C}),$$
where $(\mathbb{Z}, \geq)$ denotes the totally ordered set of the integers and $\mathbb{Z}^\delta$ is the discrete category of the integers. Our filtrations are thus, by convention, $\mathbb{Z}$-indexed and always decreasing.[2] The functor of taking associated graded is written as usual by $\mathrm{gr}^\star : \mathcal{C}^{\mathbb{Z}^{\mathrm{op}}} \to \mathcal{C}^{\mathbb{Z}^\delta}$.

We tend to write filtered objects of $\mathcal{C}$ as $\mathrm{Fil}^\star M$, where M is an object of $\mathcal{C}$; this notation implicitly means that there is a morphism $\mathrm{Fil}^{-\infty} M := \mathrm{colim}_{j \to \infty} \mathrm{Fil}^j M \to M$ in $\mathcal{C}$. The filtration is said to be *exhaustive* when the latter morphism is an equivalence. The filtration is said to be $\mathbb{N}$-*indexed* when $\mathrm{Fil}^j M \to M$ is an equivalence for all $j \leq 0$ (or, equivalently, the filtration is exhaustive and $\mathrm{gr}^j M = 0$ for $j < 0$). The filtration is said to be *complete* if $\lim_{j \to \infty} \mathrm{Fil}^j M = 0$.

When $\mathcal{C} = \mathrm{D}(\mathbb{Z})$, Sp, etc., then our filtrations are often complete because they satisfy the stronger property of being *bounded* ("uniformly homologically bounded below" to be more precise): for us this means that there exists $d \geq 0$ such that $\mathrm{Fil}^j M$ is supported in cohomological degrees $\leq d - j$ for any $j \in \mathbb{Z}$. Then $\mathrm{gr}^j M$ is also supported in cohomological degrees $\leq d - j$, and the associated spectral sequence of the filtered complex/spectrum lies in the left half-plane $\{x \leq d\}$. Conversely, if the filtration

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[2] The exception is when we occasionally encounter finite filtrations, in which case we implicitly impose that the filtration be both exhaustive and complete, and we may allow it to be increasing if it makes the indexing easier to follow.

is already known to be complete then boundedness can be checked via the graded pieces: taking the inverse limit, $\mathrm{gr}^j M$ being supported in cohomological degrees $\leq d-j$ for all $j \in \mathbb{Z}$ implies the same about all $\mathrm{Fil}^j M$.

Assume now that $\mathcal{C}$ is presentably symmetric monoidal. Then $\mathcal{C}^{\mathbb{Z}^{\mathrm{op}}}$ and $\mathcal{C}^{\mathbb{Z}^{\delta}}$ admit canonical symmetric monoidal structures given by Day convolution, which ensures that taking associated graded promotes to a strong symmetric monoidal functor. In particular, we have the ∞-category of *filtered* $\mathbb{E}_\infty$*-algebras* $\mathrm{CAlg}(\mathcal{C}^{\mathbb{Z}^{\mathrm{op}}})$ and *graded* $\mathbb{E}_\infty$*-algebras* $\mathrm{CAlg}(\mathcal{C}^{\mathbb{Z}^{\delta}})$ such that $\mathrm{gr}^\star$ promotes to a strong symmetric monoidal functor $\mathrm{gr}^\star : \mathrm{CAlg}(\mathcal{C}^{\mathbb{Z}^{\mathrm{op}}}) \to \mathrm{CAlg}(\mathcal{C}^{\mathbb{Z}^{\delta}})$. Rather abusively, we tend to summarise this wealth of information by speaking simply of *multiplicative filtrations* or *multiplicative graded objects*. We also often consider maps between such structured objects and call them maps which are *multiplicative*.

Our main case of interest are when $\mathcal{C}$ is the ∞-category Sp of spectra or the derived ∞-category $\mathrm{D}(R)$ of modules over some discrete ring R. In these cases we write FSp and $\mathrm{DF}(R)$ for the associated categories of filtered objects.[3]

2.3 Left Kan extensions

Given a fully faithful inclusion of categories $\iota : \mathcal{C} \subseteq \mathcal{C}'$ and a functor $F : \mathcal{C} \to \mathcal{D}$ valued in a presentable ∞-category $\mathcal{D}$, we write $L_{\mathcal{C}'/\mathcal{C}}F : \mathcal{C}' \to \mathcal{D}$ for the corresponding left Kan extension. We will use the following standard facts:

1. Left Kan extension provides a left adjoint to the restriction functor $\iota^* : \mathrm{Fun}(\mathcal{C}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$
$$\iota_! : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D});$$
which is furthermore fully faithful [87, Prop. 4.3.2.17].

2. Let R be a commutative base ring. In the special case that $\mathcal{C} \subset \mathcal{C}'$ is the inclusion $\mathrm{CAlg}_R^{\mathrm{Sm}} \subset \mathrm{CAlg}_R$, and we are given a functor $\mathrm{CAlg}_R^{\mathrm{sm}} \to \mathrm{CAlg}(\mathcal{D})$, the left Kan extension $L_{\mathcal{C}'/\mathcal{C}}F : \mathcal{C}' \to \mathcal{D}$ upgrades to a functor $L_{\mathcal{C}'/\mathcal{C}}F : \mathcal{C}' \to \mathrm{CAlg}(\mathcal{D})$. The key point here is that, for an R-algebra S, the diagram $(\mathrm{CAlg}_R^{\mathrm{sm}})_{/S}$ is *sifted*, whence the colimit computing the left Kan extension in $\mathrm{CAlg}(\mathcal{D})$ is computed in $\mathcal{D}$; see [85, Corol. 3.2.3.2] fo a reference. This also works for the inclusion $\mathrm{CAlg}_R^{\mathrm{Sm}} \subset \mathrm{CAlg}_R^{\mathrm{ani}}$ of smooth R-algebra into animated R-algebras. Therefore, without further comment, we take for granted that the left Kan extensions appearing in this paper preserves multiplicative structures.

3 Recollections of other cohomologies

3.1 Classical motivic cohomology of smooth schemes

We briefly recall motivic cohomology for smooth schemes over a field k; we will often say "classical motivic cohomology" when we wish to draw a comparison with our forthcoming generalisation. For $j \geq 0$ and for any given $X \in \mathrm{Sm}_k$, its weight-j motivic complex $\mathbb{Z}(j)^{\mathrm{cla}}(X)$ is given by a shift of Bloch's cycle complex, namely
$$\mathbb{Z}(j)^{\mathrm{cla}}(X) = z^j(X, \bullet)[-2j]. \tag{3.1}$$

Bloch's cycle complex is a priori only functorial for flat morphisms between smooth k-schemes, which is insufficient for our purposes (notably for left Kan extending beyond smooth k-schemes), and its multiplicative properties are unclear (especially in mixed characteristic, although that is irrelevant for the present paper), but these problems are resolved via motivic stable homotopy theory. Indeed, Voevodsky's motivic cohomology (as constructed via the theory of finite correspondences in [115]) is representable in the stable motivic homotopy category $\mathcal{SH}(k)$ via the motivic Eilenberg–Maclane spectrum, thereby defining functorial weight j motivic cohomology as a presheaf
$$\mathbb{Z}(j)^{\mathrm{cla}} : \mathrm{Sm}_k^{\mathrm{op}} \longrightarrow \mathrm{D}(\mathbb{Z})$$
such that (3.1) holds for any fixed smooth k-scheme X. The graded object $\mathbb{Z}(\star)^{\mathrm{cla}}[2\star]$ is moreover multiplicative.

Furthermore, Voevodsky's slice filtration [113] equips K-theory with a multiplicative, complete $\mathbb{N}$-indexed filtration on smooth k-schemes, i.e.
$$\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K} : \mathrm{Sm}_k^{\mathrm{op}} \longrightarrow \mathrm{FSp} \tag{3.2}$$

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[3] DF being standard notation and SpF looking strange.

such that $\mathrm{Fil}^0_{\mathrm{cla}}K = K$, whose associated graded is $\mathbb{Z}(\star)^{\mathrm{cla}}[2\star]$. (Levine's homotopy coniveau tower [79] is another approach to defining such a filtration for any given X, but there again seem to be technicalities surrounding functoriality and multiplicativity; see however the recent paper [38]). This filtration induces the *Atiyah–Hirzebruch*, or *slice*, *spectral sequence*

$$E_2^{ij} = H^{i-j}(\mathbb{Z}(-j)^{\mathrm{cla}}(X)) \implies \mathrm{K}_{-i-j}(X)$$

functorially in $X \in \mathrm{Sm}_k$.

The above motivic filtration on $\mathrm{K}(X)$ is bounded in that $\mathrm{Fil}^j_{\mathrm{cla}}\mathrm{K}(X)$ is supported in homological degrees $\geq \dim X - j$ (in particular, $\mathbb{Z}(j)^{\mathrm{cla}}(X)$ is supported in cohomological degrees $\leq j + \dim X$). Adams operations imply that the motivic filtration splits rationally, i.e., there is a natural equivalence of filtered spectra $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)_{\mathbb{Q}} \xrightarrow{\sim} \prod_{j\geq 0}\mathbb{Q}(j)^{\mathrm{cla}}(X)[2j]$, and that the Atiyah–Hirzebruch spectral degenerates rationally; see §B.2 for details.

3.2 Lisse motivic cohomology

The simplest way to extend motivic cohomology to arbitrary algebras over fields is via left Kan extension of the classical theory:

Definition 3.1. Fix a prime field $\mathbb{F}$ (i.e., $\mathbb{F}_p$ for some prime number $p \geq 2$ or $\mathbb{Q}$) and $j \geq 0$. We define *weight-j, lisse motivic cohomology*

$$\mathbb{Z}(j)^{\mathrm{lse}} := L_{\mathrm{CAlg}_{\mathbb{F}}/\mathrm{CAlg}^{\mathrm{sm}}_{\mathbb{F}}}\mathbb{Z}(j)^{\mathrm{cla}} : \mathrm{CAlg}_{\mathbb{F}} \to \mathrm{D}(\mathbb{Z})$$

as the left Kan extension of classical motivic cohomology $\mathbb{Z}(j)^{\mathrm{cla}}$ along the inclusion $\mathrm{CAlg}^{\mathrm{sm}}_{\mathbb{F}} \subseteq \mathrm{CAlg}_{\mathbb{F}}$ of smooth $\mathbb{F}$-algebras into all $\mathbb{F}$-algebras.

We warn the reader that $\mathbb{Z}(j)^{\mathrm{lse}}$ is not in general a Zariski sheaf, already in the case $j = 1$:

Example 3.2 ($j = 1$)**.** Recall that $\mathbb{Z}(1)^{\mathrm{cla}} = R\Gamma_{\mathrm{Zar}}(-, \mathbb{G}_m)[-1]$, which is the same as $(\tau^{\leq 1}R\Gamma_{\mathrm{Zar}}(-, \mathbb{G}_m))[-1]$ on smooth $\mathbb{F}$-schemes. Since both units and Pic, as functors $\mathrm{CAlg}_{\mathbb{F}} \to \mathrm{D}(\mathbb{Z})$, are left Kan extended from smooth $\mathbb{F}$-algebras, we deduce that there is a natural equivalence $\mathbb{Z}(1)^{\mathrm{lse}}(A) \simeq (\tau^{\leq 1}R\Gamma_{\mathrm{Zar}}(A, \mathbb{G}_m))[-1]$ for any $\mathbb{F}$-algebra A. However, the truncated presheaf itself is not a Zariski sheaf and the natural map $\tau^{\leq 1}R\Gamma_{\mathrm{Zar}}(-, \mathbb{G}_m)) \to R\Gamma_{\mathrm{Zar}}(-, \mathbb{G}_m)$ witnesses the target as the Zariski sheafification.

Lisse motivic cohomology occurs as the graded pieces of a motivic filtration on *connective* K-theory K^{cn}:

Proposition 3.3. *Let A be an $\mathbb{F}$-algebra. Then there exists a natural, $\mathbb{N}$-indexed, multiplicative filtration $\mathrm{Fil}^\star_{\mathrm{lse}}\mathrm{K}^{\mathrm{cn}}(A)$ on the connective K-theory $\mathrm{K}^{\mathrm{cn}}(A)$ with graded pieces*

$$\mathrm{gr}^j_{\mathrm{lse}}\mathrm{K}^{\mathrm{cn}}(A) \simeq \mathbb{Z}(j)^{\mathrm{lse}}(A)[2j]$$

for $j \geq 0$; moreover $\mathbb{Z}(j)^{\mathrm{lse}}(A)$ is supported in cohomological degrees $\leq 2j$. If A is local then $\mathbb{Z}(j)^{\mathrm{lse}}(A)$ is supported in cohomological degrees $\leq j$ and the filtration is bounded.

Proof. The desired filtration follows by left Kan extending the classical motivic filtration (3.2), since $\mathrm{K}^{\mathrm{cn}} : \mathrm{CAlg}_{\mathbb{F}} \to \mathrm{Sp}$ is left Kan extended from smooth $\mathbb{F}$-algebras [41, Ex. A.0.6]. The bound holds in the smooth case and is preserved by left Kan extension.

If A is local then $\mathrm{Fil}^j_{\mathrm{lse}}\mathrm{K}^{\mathrm{cn}}(A)$ is supported in homological degrees $\geq j$; indeed, this connectivity bound holds Zariski locally on smooth $\mathbb{F}$-algebras by the Gersten conjecture in motivic cohomology, and is again preserved by left Kan extension. □

Remark 3.4 ($\mathbb{Z}(j)^{\mathrm{lse}}$ is a cycle complex)**.** Our interest in $\mathbb{Z}(j)^{\mathrm{lse}}$ is not just as an intermediate tool nor because it is the "easiest" extension of motivic cohomology beyond smooth schemes, but because it is defined purely in terms of algebraic cycles. This is already clear from the definition, since it is the left Kan extension of the cycle-theoretic $\mathbb{Z}(j)^{\mathrm{cla}} = z^j(-, \bullet)[-2j]$ from smooth algebras, but we spell it out more explicitly. Given a $\mathbb{F}$-algebra A, we may pick a simplicial resolution $P_\bullet \to A$ where each term P_m is an ind-smooth $\mathbb{F}$-algebra and each face map $P_{m+1} \to P_m$ is a henselian surjection. Then the formalism of left Kan extension from smooth algebras implies that there is a natural equivalence

$$\operatorname*{colim}_{m\in\Delta^{\mathrm{op}}} \mathbb{Z}(j)^{\mathrm{cla}}(P_m) \xrightarrow{\sim} \mathbb{Z}(j)^{\mathrm{lse}}(A)$$

(in this line and below we implicitly extend $\mathbb{Z}(j)^{\mathrm{cla}}$ and $z^j(-,\bullet)$ from smooth to ind-smooth $\mathbb{F}$-algebras, by taking filtered colimits). Expanding each $\mathbb{Z}(j)^{\mathrm{cla}}(P_m) = z^j(P_m,\bullet)[-2j]$ as a complex of cycles, we see that the left side of the previous line is the $[-2j]$-shift of the totalisation of the bicomplex (really bisimplicial abelian group)

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\ \cdots \arrow[r] & z^j(P_2,2) \arrow[r] \arrow[d] & z^j(P_2,1) \arrow[r] \arrow[d] & z^j(P_2,0) \arrow[d] \\ \cdots \arrow[r] & z^j(P_1,2) \arrow[r] \arrow[d] & z^j(P_1,1) \arrow[r] \arrow[d] & z^j(P_1,0) \arrow[d] \\ \cdots \arrow[r] & z^j(P_0,2) \arrow[r] & z^j(P_0,1) \arrow[r] & z^j(P_0,0) \end{tikzcd} \tag{3.3}$$

In conclusion $\mathbb{Z}(j)^{\mathrm{lse}}(A)$ admits a description in terms of various algebraic cycles on the affine schemes $\mathbb{A}^n_{P_m}$, for $n,m \geq 0$

We will discuss comparisons between $\mathbb{Z}(j)^{\mathrm{lse}}$ and our new motivic cohomology in §7.

3.3 cdh-local motivic cohomology

This paper builds on the cdh-local, $\mathbb{A}^1$-invariant version of motivic cohomology laid out in forthcoming joint work with Bachmann [7]. We apologise for the logical inconsistency of releasing the current paper first, and offer as explanation that the theory in [7] is developed for arbitrary qcqs schemes but simplifies over fields. Here we present a brief overview of the theory in that case. Let $\mathbb{F}$ be a prime field.

Recall that an *abstract blowup square* is a cartesian square of qcqs $\mathbb{F}$-schemes

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & X' \arrow[d, "p"] \\ Y \arrow[r, "i"'] & X, \end{tikzcd} \tag{3.4}$$

where i is a finitely presented closed immersion and p is a finitely presented, proper morphism inducing an isomorphism $X' \setminus Y' \xrightarrow{\simeq} X \setminus Y$. On $\mathrm{Sch}^{\mathrm{qcqs}}$ the *cdh topology* is the Grothendieck topology generated by the pretopology given by maps $\{Y \to X, X' \to X\}$ for all abstract blow up squares as above and by the Nisnevich pretopology. A result of Voevodsky [114], generalized in [40, Prop. 2.1.5] and [9, App. A] in the non-noetherian setting, states that cdh sheaves are exactly those presheaves which convert both Nisnevich squares and abstract blowup squares to cartesian squares.

The main object of study of [7] is

$$\mathbb{Z}(j)^{\mathrm{cdh}} := L_{\mathrm{cdh}} L_{\mathrm{Sch}_{\mathbb{F}}^{\mathrm{qcqs,op}}/\mathrm{Sm}_{\mathbb{F}}^{\mathrm{op}}} \mathbb{Z}(j)^{\mathrm{cla}} : \mathrm{Sch}_k^{\mathrm{qcqs,\,op}} \longrightarrow \mathrm{D}(\mathbb{Z}), \tag{3.5}$$

namely the cdh sheafification of the left Kan extension of classical motivic cohomology $\mathbb{Z}(j)^{\mathrm{cla}} : \mathrm{Sm}_{\mathbb{F}}^{\mathrm{op}} \to \mathrm{D}(\mathbb{Z})$ along the inclusion $\mathrm{Sm}_{\mathbb{F}}^{\mathrm{op}} \subseteq \mathrm{Sch}_{\mathbb{F}}^{\mathrm{qcqs,\,op}}$. In terms of universal properties, $\mathbb{Z}(j)^{\mathrm{cdh}} : \mathrm{Sch}_k^{\mathrm{qcqs,\,op}} \to \mathrm{D}(\mathbb{Z})$ is initial among cdh sheaves on $\mathrm{Sch}_{\mathbb{F}}^{\mathrm{qcqs,\,op}}$ whose restriction to smooth k-schemes is equipped with a map from $\mathbb{Z}(j)^{\mathrm{cla}}$. Assuming resolution of singularities, this construction is essentially due to Friedlander, Suslin, and Voevodsky for finite type k-schemes [115].

In [7], joint with Bachmann, we establish the following properties of this *cdh-local motivic cohomology*:

Theorem 3.5 ([7])**.** *Cdh-local motivic cohomology $\mathbb{Z}(j)^{\mathrm{cdh}} : \mathrm{Sch}_{\mathbb{F}}^{\mathrm{qcqs,op}} \to \mathrm{D}(\mathbb{Z})$, for $j \geq 0$, has the following properties for any qcqs $\mathbb{F}$-scheme X:*

1. *There exist a functorial, multiplicative, $\mathbb{N}$-indexed filtration $\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)$ on the homotopy invariant K-theory $\mathrm{KH}(X)$, such that the graded pieces are naturally given by $\mathrm{gr}^j_{\mathrm{cdh}}\mathrm{KH}(X) \simeq \mathbb{Z}(j)^{\mathrm{cdh}}[2j]$, for $j \geq 0$. In particular, writing $H^i_{\mathrm{cdh}}(X,\mathbb{Z}(j)) := H^i(\mathbb{Z}(j)^{\mathrm{cdh}}(X))$ for the corresponding* cdh-local motivic cohomology groups*, there exists an Atiyah-Hirzebruch spectral sequence*

$$\mathrm{H}^{i-j}_{\mathrm{cdh}}(X,\mathbb{Z}(-j)) \implies \mathrm{KH}_{-i-j}(X).$$

   If X has finite valuative dimension $\leq d$ then this filtration is bounded: more precisely, $\mathrm{Fil}^j_{\mathrm{cdh}}\mathrm{KH}(X)$ *is supported in cohomological degrees $\leq d-j$.*

2. *Finitariness:* $\mathbb{Z}(j)^{\mathrm{cdh}}$ *is a finitary cdh sheaf.*
3. *Relation to étale cohomology: for any integer* $\ell > 0$ *invertible in* $\mathbb{F}$*, there are natural equivalences*

   $$\mathbb{Z}(j)^{\mathrm{cdh}}/\ell \simeq L_{\mathrm{cdh}}\tau^{\leq j} R\Gamma_{\acute{e}t}(-, \mu_\ell^{\otimes j}).$$

   for $j \geq 0$.
4. *Relation to syntomic cohomology: if* $\mathbb{F} = \mathbb{F}_p$ *then for any* $r \geq 0$ *there are natural equivalences*

   $$\mathbb{Z}(j)^{\mathrm{cdh}}/p^r \simeq R\Gamma_{\mathrm{cdh}}(-, W_r\Omega^j_{\log})[-j].$$

   for $j \geq 0$.
5. $\mathbb{A}^1$*-invariance: the map*

   $$\mathbb{Z}(j)^{\mathrm{cdh}}(X) \longrightarrow \mathbb{Z}(j)^{\mathrm{cdh}}(X \times \mathbb{A}^1),$$

   induced by the projection $X \times \mathbb{A}^1 \to X$*, is an equivalence for each* $j \geq 0$.
6. *Weight one: there is a natural equivalence*

   $$\mathbb{Z}(1)^{\mathrm{cdh}}(X) \simeq R\Gamma_{\mathrm{cdh}}(X, \mathbb{G}_m)[-1].$$

7. *Projective bundle formula: the powers of the first Chern class of the tautological bundle* $c_1(\mathcal{O}(1)) \in \mathrm{Pic}(\mathbb{P}^r_X) \to H^2_{\mathrm{cdh}}(\mathbb{P}^r_X, \mathbb{Z}(1))$ *induce a natural equivalence*

   $$\bigoplus_{i=0}^{r} \mathbb{Z}(j-i)^{\mathrm{cdh}}(X)[-2i] \xrightarrow{\sim} \mathbb{Z}(j)^{\mathrm{cdh}}(\mathbb{P}^r_X), \tag{3.6}$$

8. *Comparison to classical motivic cohomology: for any field* $k \supseteq \mathbb{F}$ *and smooth* k*-scheme* X*, there are equivalences*

   $$\mathbb{Z}(j)^{\mathrm{cdh}}(X) \simeq z^j(X, \bullet)[-2j]$$

   for $j \geq 0$.

For our purposes, the main point of Theorem 3.5 is that there is a motivic filtration on KH of any qcqs equicharacteristic scheme such that its graded pieces are a version of motivic cohomology which is $\mathbb{A}^1$-invariant, satisfies cdh descent, and agrees with all known definitions of motivic cohomology on smooth k-schemes. This filtration on KH is defined by cdh sheafifying Proposition 3.3. The deepest parts of the theorem are the $\mathbb{A}^1$-invariance and projective bundle formula assertions, and the comparison to classical motivic cohomology; these are proved using similar arguments to those of §5 and §6. Details will of course appear in [7].

Remark 3.6 (Comparison to motivic homotopy theory)**.** We briefly discuss $\mathbb{A}^1$-invariant motivic homotopy theory and the slice filtration, though we stress that the results presented in this remark are not required for the current article. The summary of this remark is that cdh-local motivic cohomology is the same as the $\mathbb{A}^1$-invariant motivic cohomology coming from stable homotopy theory.

For a qcqs scheme X let $\mathcal{SH}(X)$ denote its ∞-category of motivic spectra, as introduced by Morel and Voevodsky [110]; for a modern approach see [100] [9, §4]. Examples of such motivic spectra include the unit object (or motivic sphere) $\mathbf{1}_X$, the motivic spectrum KGL_X representing homotopy invariant K-theory of smooth X-schemes, and a motivic Eilenberg–Maclane spectrum $\mathrm{H}\mathbb{Z}_X$ constructed by Spitzweck [105].

Any motivic spectrum $E \in \mathcal{SH}(X)$ may be equipped with a functorial *slice filtration*

$$\cdots \to \mathrm{f}^{j+1}E \to \mathrm{f}^j E \to \mathrm{f}^{j-1}E \to \cdots \to E$$

in $\mathcal{SH}(X)$, whose graded pieces are denoted by $s^j E := \mathrm{cofib}(\mathrm{f}^{j+1}E \to \mathrm{f}^j E)$. In [7], using previous work of Bachmann [6], we establish natural equivalences in $\mathcal{SH}(X)$

$$\mathrm{H}\mathbb{Z}_X \simeq s^0(\mathbf{1}_X) \simeq s^0(\mathrm{KGL}_X), \tag{3.7}$$

thereby settling Conjectures 1, 7, and 10 of Voevodsky [112] for arbitrary qcqs schemes.

The first equivalence of (3.7) shows that the cohomology theory represented by $\mathrm{H}\mathbb{Z}_X$, which is typically considered to be the correct theory of $\mathbb{A}^1$-invariant motivic cohomology, is the same as the cohomology theory represented by $s^0(\mathbf{1}_X)$. The second equivalence, combined with the machinery of the slice filtration, justifies this point of view since it implies that this cohomology theory is related to the KH-groups of X via an Atiyah–Hirzebruch spectral sequence.

In [7] we also prove, at least for qcqs schemes X of equal characteristic, that the $\mathbb{A}^1$-invariant motivic cohomology represented by $\mathrm{H}\mathbb{Z}_X$ coincides with the cdh-local motivic cohomology, and that the filtration on $\mathrm{KH}(X)$ coming from the slice filtration on KGL_X coincides with the filtration of Theorem 3.5.

3.4 (Topological) cyclic homology

The constructions of this paper depend on trace methods in algebraic K-theory. Recall that if E is a spectrum with S^1-action, then we can functorially associate several other spectra: its *homotopy fixed points* E^{hS^1}, its *homotopy orbits* E_{hS^1} and its *Tate fixed points*

$$E^{tS^1} := \mathrm{cofib}(\mathrm{Nm} : E_{hS^1}[1] \to E^{hS^1}).$$

The same formalism exists if we replace S^1 by any of its finite subgroups, for example if p is a prime then one has

$$E^{tC_p} := \mathrm{cofib}(\mathrm{Nm} : E_{hC_p} \to E^{hC_p}).$$

According to [95], a *cyclotomic spectrum* is a spectrum with an S^1-action E equipped with S^1-equivariant maps

$$\phi_p : E \to E^{tC_p}$$

for all primes p; here E^{tC_p} is given the residual $S^1/C_p \simeq S^1$-action. In the situation of algebraic geometry, we have the functor of topological Hochschild homology landing in the ∞-category of cyclotomic spectra:

$$\mathrm{THH} : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{CycSp} \qquad X \mapsto \mathrm{THH}(\mathrm{Perf}(X)) =: \mathrm{THH}(X).$$

Let E be a cyclotomic spectrum which is bounded below (which will always be the case in our situations of interest). Firstly, for each prime number p its *p-adic topological cyclic homology* is defined to be the p-complete spectrum

$$\mathrm{TC}(E;\mathbb{Z}_p) := \mathrm{fib}\left(\phi_p^{hS^1} - \mathrm{can} : \left(E_p^{\wedge}\right)^{hS^1} \longrightarrow \left(E^{tC_p}\right)^{hS^1} \simeq \left(E_p^{\wedge}\right)^{tS^1}\right).$$

Here we have used that E^{tC_p} is p-complete and that $E^{tC_p} \simeq (E_p^{\wedge})^{tC_p}$ by [95, Len. I.2.9], and can is the canonical map from fixed points to the Tate construction. We assemble these p-adic constructions to define the *integral topological cyclic homology* $\mathrm{TC}(E)$ of E as the pullback

$$\begin{tikzcd}
\mathrm{TC}(E) \arrow[r] \arrow[d] & (E_{\mathbb{Q}})^{hS^1} \arrow[d] \\
\prod_p \mathrm{TC}(E;\mathbb{Z}_p) \arrow[r] & \prod_p \left(E_p^{\wedge}[\tfrac{1}{p}]\right)^{hS^1}
\end{tikzcd} , \tag{3.8}$$

where the bottom map is the product over p of the compositions $\mathrm{TC}(E;\mathbb{Z}_p) \to \left(E_p^{\wedge}\right)^{hS^1} \to (E_p^{\wedge}[\frac{1}{p}]))^{hS^1}$.

By a standard abuse of notation, for a scheme X, we write $\mathrm{TC}(X)$ in place of $\mathrm{TC}(\mathrm{THH}(X))$, and similarly for the p-adic variant.

Remark 3.7. The square (3.8) imitates the original definition of integral topological cyclic homology defined by Goodwillie [37, Lem. 6.4.3.2]. Indeed, [95, Thm. II.4.11] proves that the definitions agree for bounded below cyclotomic spectra.

There is a morphism of localizing invariants (in the sense of [27]) called the *cyclotomic trace*, or just *trace map* for short

$$\mathrm{tr} : \mathrm{K} \longrightarrow \mathrm{TC}.$$

A major result about this map is the Dundas–Goodwillie–McCarthy theorem [37], stating that its fibre $\mathrm{K}^{\mathrm{inf}}$ is insensitive to nilpotent thickenings. In the language of [76], $\mathrm{K}^{\mathrm{inf}}$ is even *truncating*: for any connective $\mathbb{E}_1$-ring A, the map $\mathrm{K}^{\mathrm{inf}}(A) \to \mathrm{K}^{\mathrm{inf}}(\pi_0 A)$ is an equivalence. This property implies not only nil-invariance [76, Corol. 3.5] but even cdh descent, whence one obtains the following fundamental square:

Theorem 3.8 (Kerz–Strunk–Tamme, Land–Tamme)**.** *Let X be a qcqs scheme. The the square*

$$\begin{tikzcd} \mathrm{K}(X) \arrow[r] \arrow[d] & \mathrm{TC}(X) \arrow[d] \\ \mathrm{KH}(X) \arrow[r] & L_{\mathrm{cdh}}\mathrm{TC}(X). \end{tikzcd} \tag{3.9}$$

is cartesian.

Proof. This follows from the facts that the canonical map $L_{\mathrm{cdh}}\mathrm{K}(X) \to \mathrm{KH}(X)$ is an equivalence [70, Thm. 6.3] (see also [65, Rem. 3.4]) and that $\mathrm{K}^{\mathrm{inf}}$ satisfies cdh descent [76]. □

To be clear, the bottom horizontal arrow in the previous diagram is obtained by cdh sheafifying the trace map $\mathrm{K} \to \mathrm{TC}$. Indeed, as we commented in the proof, we have $\mathrm{KH} \simeq L_{\mathrm{cdh}}\mathrm{K}$ and therefore there is an induced *cdh-local trace map* $\mathrm{KH} \to L_{\mathrm{cdh}}\mathrm{TC}$; it will play an important role in the construction of our motivic cohomology.

4 DEFINITION OF $\mathbb{Z}(j)^{\mathrm{mot}}(X)$

In this section we introduce our theory of motivic cohomology and the motivic filtration on algebraic K-theory. We also establish a number of other fundamental properties, such as finitariness, to justify that the definition is not unreasonable.

4.1 Characteristic zero

We begin with reminders on cyclic homology. First note that for any $X \in \mathrm{Sch}_{\mathbb{Q}}^{\mathrm{qcqs}}$ we have

$$\mathrm{THH}(X) \simeq \mathrm{THH}(X) \otimes_{\mathrm{THH}(\mathbb{Q})} \mathbb{Q} \simeq \mathrm{HH}(X/\mathbb{Q})$$

where the second equivalence is formal and the first follows from the fact that $\mathrm{THH}(\mathbb{Q}) \simeq \mathbb{Q}$. Similarly, the integral topological cyclic homology $\mathrm{TC}(X)$, as defined by the pullback square (3.8), coincides with the *negative cyclic homology* of X; indeed, the latter is defined by $\mathrm{HC}^-(X/\mathbb{Q}) := (\mathrm{HH}(X/\mathbb{Q}))^{hS^1}$ and the square (3.8) collapses to an equivalence

$$\mathrm{TC}(X) \xrightarrow{\sim} \mathrm{HC}^-(X/\mathbb{Q})$$

(since the bottom terms of the square vanish as $\mathrm{THH}(X)$ has vanishing p-completion). The cyclotomic trace becomes the more classical *Goodwillie trace*

$$\mathrm{tr} : \mathrm{K}(X) \longrightarrow \mathrm{HC}^-(X/\mathbb{Q}),$$

and Theorem 3.8 is rewritten as the cartesian square

$$\begin{tikzcd} \mathrm{K}(X) \arrow[r] \arrow[d] & \mathrm{HC}^-(X/\mathbb{Q}) \arrow[d] \\ \mathrm{KH}(X) \arrow[r] & L_{\mathrm{cdh}}\mathrm{HC}^-(X/\mathbb{Q}). \end{tikzcd} \tag{4.1}$$

We remark that the bottom right term in the previous diagram is poor notation, which we will nevertheless continue to use; it should really be written $(L_{\mathrm{cdh}}\mathrm{HC}^-(-/\mathbb{Q}))(X)$.

Remark 4.1 (Replacing $\mathbb{Q}$ by a general case k)**.** More generally, let k be a discrete commutative ring. For any qcqs k-scheme X, let $\mathrm{HC}^-(X/k) := \mathrm{HH}(X/k)^{hS^1}$ denote its negative cyclic homology relative to k.

Then there is a natural map $\mathrm{TC}(X) \to \mathrm{HC}^-(X/k)$ constructed as follows. Firstly, from square (3.8) we see that $\mathrm{TC}(X)$ naturally maps to the pull back of

$$\begin{tikzcd} {} \arrow[r, dotted] \arrow[d, dotted] & (\mathrm{THH}(X)_{\mathbb{Q}})^{hS^1} \arrow[d] \\ \prod_p (\mathrm{THH}(X)_p^\wedge)^{hS^1} \arrow[r] & \prod_p \left(\mathrm{THH}(X)_p^\wedge[\tfrac{1}{p}]\right)^{hS^1}. \end{tikzcd} \tag{4.2}$$

Removing the hS^1 from the three corners, the pullback of the square is $\mathrm{THH}(X)$; passing to homotopy fixed points preserves pullback squares (and commutes with products), whence the pullback of the square (4.2) is the *negative topological cyclic homology* $\mathrm{TC}^-(X) := \mathrm{THH}(X)^{hS^1}$. This constructs a natural map $\mathrm{TC}(X) \to \mathrm{TC}^-(X)$, which may then be composed with S^1-fixed points of $\mathrm{THH}(X) \to \mathrm{HH}(X/k)$.

Composing with the cyclotomic trace thereby defines a trace map $\mathrm{K}(X) \to \mathrm{HC}^-(X/k)$ relative to k; of course it would have been sufficient to define this in the case $k = \mathbb{Z}$ and then compose with the canonical map $\mathrm{HC}^-(X/\mathbb{Z}) \to \mathrm{HC}^-(X/k)$.

To construct our motivic filtration on K-theory in characteristic zero we first recall the Hochschild–Kostant–Rosenberg filtration on negative cyclic homology, which relies on the theory of derived de Rham cohomology of Illusie [61] and Bhatt [15, 16]. Since the following two results do not require any characteristic zero hypothesis,[4] let k be a discrete commutative ring and recall, for any k-algebra R, the *Hodge-completed derived de Rham cohomology* $\widehat{L\Omega}_{R/k} \in \mathrm{D}(k)$ of R and its complete $\mathbb{N}$-indexed *Hodge filtration* $\widehat{L\Omega}^{\geq\star}_{R/k}$. For $j \geq 0$ the cofibre of the map $\widehat{L\Omega}^{\geq j}_{R/k} \to \widehat{L\Omega}_{R/k}$ is $L\Omega^{<j}_{R/k}$, which admits a finite decreasing filtration with graded pieces (in increasing order)

$$R, L_{R/k}[-1], L^2_{R/k}[-2], \dots, L^{j-1}_{R/k}[-j+1].$$

By fpqc descent of $L_{-/k}$ and its wedge powers on k-algebras [20, Thm. 3.1], right Kan extension defines a unique fpqc sheaf

$$\mathrm{Sch}_k^{\mathrm{qcqs,op}} \longrightarrow \mathrm{D}(k), \qquad X \mapsto R\Gamma(X, \widehat{L\Omega}_{-/k})$$

whose value on affines $\mathrm{Spec} R$ is $\widehat{L\Omega}_{R/k}$; similarly for $\widehat{L\Omega}^{\geq j}_{-/k}$, $L\Omega^{<j}_{-/k}$, and each wedge power of $L_{-/k}$. Alternatively, the fpqc sheaf $R\Gamma(-, \widehat{L\Omega}_{-/k})$ is equivalent to the Nisnevich sheafification of $X \mapsto \widehat{L\Omega}_{\mathcal{O}_X(X)/k}$, and similarly for the variants.

The following is the HKR filtration on negative cyclic homology:

Theorem 4.2 (HKR filtration [4, 98, 92])**.** *Let k be a discrete commutative ring. For any qcqs k-scheme X, there exists a functorial, complete, multiplicative filtration* $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X/k)$ *on* $\mathrm{HC}^-(X/k)$ *whose graded pieces for $j \in \mathbb{Z}$ are given by*

$$\mathrm{gr}^j_{\mathrm{HKR}}\mathrm{HC}^-(X/k) \simeq R\Gamma(X, \widehat{L\Omega}^{\geq j}_{-/k})[2j].$$

Furthermore, if X is quasisyntomic over k[5] *then this filtration is exhaustive.*

Remark 4.3. If k is a $\mathbb{Q}$-algebra and X is smooth over k, then this result is essentially due to Loday [83]. Dropping the hypothesis that X be smooth, but remaining in characteristic zero, the product decomposition of the previous theorem is due to Weibel, under the name of the "Hodge decomposition" and written in Adams operator type notation as "$\mathrm{HN}(X/k) \simeq \prod_j \mathrm{HN}^{(j)}(X/k)$" in [116, 35].

Remark 4.4 (Variant: cdh-local HKR filtration)**.** Cdh sheafifying the HKR filtration levelwise we see that, for any qcqs k-scheme X, there exists a functorial, multiplicative filtration

$$\mathrm{Fil}^\star_{\mathrm{HKR}} L_{\mathrm{cdh}}\mathrm{HC}^-(X/k) := L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-/k)(X),$$

on $L_{\mathrm{cdh}}\mathrm{HC}^-(X/k)$ whose graded pieces for $j \in \mathbb{Z}$ are given by

$$\mathrm{gr}^j_{\mathrm{HKR}} L_{\mathrm{cdh}}\mathrm{HC}^-(X/k) \simeq R\Gamma_{\mathrm{cdh}}(X, \widehat{L\Omega}^{\geq j}_{-/k})[2j].$$

Here we denote by

$$\mathrm{Sch}_k^{\mathrm{qcqs,op}} \to \mathrm{D}(k), \qquad X \mapsto R\Gamma_{\mathrm{cdh}}(X, \widehat{L\Omega}_{-/k})$$

the cdh sheafification of the presheaf $R\Gamma(-, \widehat{L\Omega}_{-/k})$, or equivalently the cdh sheafification of the presheaf $X \mapsto \widehat{L\Omega}_{\mathcal{O}_X(X)/k}$.

Similar notation will be used for $L\Omega^{<j}_{-/k}$ and each wedge power of $L_{-/k}$, though we stress that the canonical maps

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[4] In any case this extra degree of generality should eventually be necessary for extending the theory of this paper to mixed characteristic.

[5] i.e., for each affine open $\mathrm{Spec} A \subseteq X$, the cotangent complex $L_{A/k} \in D(A)$ has Tor amplitude in $[-1, 0]$.

The following is probably known to experts but we could not find a standalone reference in the required degree of generality:

Lemma 4.5 (Cdh descent of derived de Rham cohomology in characteristic zero)**.** *For any $\mathbb{Q}$-algebra k, the two presheaves*

$$\begin{aligned} \mathrm{Sch}_k^{\mathrm{qcqs,op}} &\longrightarrow \mathrm{D}(k) \\ X &\mapsto R\Gamma(X, \widehat{L\Omega}_{-/k}) \\ X &\mapsto \mathrm{HC}^-(X/k)/\mathrm{Fil}^0_{\mathrm{HKR}}\mathrm{HC}^-(X/k) \end{aligned}$$

satisfy cdh descent.

Proof. The cited references for Theorem 4.2 also construct an HKR filtration on periodic cyclic homology: for a qcqs k-scheme X, this is a functorial, complete, multiplicative filtration $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(X/k)$ on $\mathrm{HP}(X/k)$ whose graded pieces for $j \in \mathbb{Z}$ are given by

$$\mathrm{gr}^j_{\mathrm{HKR}}\mathrm{HP}(X/k) \simeq R\Gamma(X, \widehat{L\Omega}_{-/k})[2j].$$

The references show that the canonical map $\mathrm{HC}^-(X/k) \to \mathrm{HP}(X/k)$ respects the HKR filtrations, i.e., naturally upgrades to a filtered map, given on graded pieces by the canonical maps $R\Gamma(X, \widehat{L\Omega}^{\geq j}_{-/k}) \to R\Gamma(X, \widehat{L\Omega}_{-/k})$.

Since k is a $\mathbb{Q}$-algebra, the HKR filtration on $\mathrm{HP}(X/k)$ is naturally split [10], i.e., there is a natural equivalence $\mathrm{HP}(X/k) \simeq \prod_{n\in\mathbb{Z}} R\Gamma(X, \widehat{L\Omega}_{-/k})[2n]$ such that the HKR filtration on the left matches the product filtration $\prod_{n\leq -j}$ on the right.

The presheaf $R\Gamma(-, \widehat{L\Omega}_{-/k}) : \mathrm{Sch}_k^{\mathrm{qcqs,op}} \to \mathrm{D}(k)$ is thus a direct summand of the presheaf $\mathrm{HP}(-/k)$; but the latter is a cdh sheaf thanks to the theory of truncating invariants [34, Corol. 3.13] [76, Cor. A.6], so therefore the former is also a cdh sheaf.

The cited references for Theorem 4.2 also implicity prove that the canonical map $\mathrm{HC}^-(-/k) \to \mathrm{HP}(-/k)$ induces an equivalence $\mathrm{HC}^-(-/k)/\mathrm{Fil}^0_{\mathrm{HKR}}\mathrm{HC}^-(-/k) \xrightarrow{\simeq} \mathrm{HP}(-/k)/\mathrm{Fil}^0_{\mathrm{HKR}}\mathrm{HP}(-/k)$. By the aforementioned splitting the latter is equivalent to $\prod_{n\leq -1} R\Gamma(-, \widehat{L\Omega}_{-/k})[2n]$ and is therefore a cdh sheaf since we have shown that $R\Gamma(-, \widehat{L\Omega}_{-/k})$ is a cdh sheaf. ☐

We next prove the following compatibility, informally stating that for any smooth k-scheme X the trace map $\mathrm{K}(X) \to \mathrm{HC}^-(X/k)$ naturally carries the classical motivic filtration on the left to the HKR filtration on the right. In fact, it is rather the cdh-local analogue below (Corollary 4.8) which is crucial to our construction, but the smooth case is required for the proof of the cdh case and also to formulate the comparison map to classical motivic cohomology (Construction 4.34):

Proposition 4.6. *Let $k_0 \to k$ be a quasismooth*[6] *map of rings, where k is a field. Then the trace map $\mathrm{K} \to \mathrm{HC}^-(-/k_0)$, viewed as a map between spectra-valued presheaves on Sm_k, admits a unique, multiplicative extension to a map of filtered presheaves of spectra $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K} \to \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-/k_0)$.*

Proof. There is a t-structure on $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k; \mathrm{Sp})$, which denotes the stable ∞-category of Zariski sheaves of spectra on smooth k-schemes. This t-structure is described as follows:

- its non-negative part $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k; \mathrm{Sp})_{\geq 0}$ is given by those sheaves of spectra $\mathcal{F}$ such that the homotopy sheaves $\underline{\pi}_n\mathcal{F}$ vanish for all $n < 0$;
- its non-positive part $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k; \mathrm{Sp})_{\leq 0}$ is given by those sheaves of spectra $\mathcal{F}$ such that $\pi_n(\mathcal{F}(X))$ vanishes for all $X \in \mathrm{Sm}_k$ and all $n > 0$.

This is a specialization of a much more general result on ∞-topoi as in [86, Prop. 1.3.2.7].

Now let $j \in \mathbb{Z}$ and observe the following facts about the connectivity of the filtrations on K and HC^- with respect to the above t-structure:

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[6] i.e., the cotangent complex L_{k/k_0} is supported in degree 0 and Ω^1_{k/k_0} is a flat k-module. In this paper we only require the trivial situation that $\mathbb{Q} = k_0 = k$, but we record the more general statement for future use.

1. $\mathrm{Fil}^{\geq j}_{\mathrm{cla}}\mathrm{K}$ is j-connective. This follows from standard vanishing bounds in motivic cohomology, though a little care is required since a priori taking homotopy sheaves might not commute with viewing K-theory as a complete filtered object. Let X be a smooth k-scheme and set $d := \dim X$.

   Recall first that, for any $i \geq 0$, the cohomology sheaves $\mathcal{H}^n(\mathbb{Z}(i)^{\mathrm{cla}}) = \underline{\pi}_{-n}\mathbb{Z}(i)^{\mathrm{cla}}$ on X_{Zar} vanish for $n > i$ (by Gersten injectivity to reduce to the case of a field); therefore, the motivic complex $\mathbb{Z}(j)^{\mathrm{cla}}(X)$ of X itself vanishes in cohomological degrees $> 2j$ (by the Gersten resolution), and also in degrees $> j + d$ (for dimension reasons). Using the dimension bound we see, for any affine open $\mathrm{Spec}(R) \subseteq X$ in place of X and $i \geq j + d$, that $\mathrm{gr}^i_{\mathrm{cla}}\mathrm{K}(R) = \mathbb{Z}(i)^{\mathrm{cla}}(R)[2i]$ is supported in cohomological degrees $\leq -j$; by completeness of the motivic filtration we deduce the same for $\mathrm{Fil}^{\geq j+d}_{\mathrm{cla}}K(R)$, for all open affines $\mathrm{Spec}(R) \subseteq X$. In particular, $\mathrm{Fil}^{\geq j+d}_{\mathrm{cla}}\mathrm{K}$ is j-connective on X_{Zar}.

   But now the problem reduces, by a finite induction, to checking that $\mathrm{gr}^i_{\mathrm{cla}}\mathrm{K}$ is i-connective for each $i \geq 0$ (in fact, just for $i = j, \dots, j + d - 1$), or in other words that the Zariski cohomology sheaves $\mathcal{H}^n(\mathbb{Z}(i)^{\mathrm{cla}})$ vanish for $n > i$. But this was already explained in the previous paragraph and so completes the proof.

2. On the other hand, $\mathrm{Fil}^{<j}_{\mathrm{HKR}}\mathrm{HC}^-(-/k_0) := \mathrm{HC}^-(-/k_0)/\mathrm{Fil}^j_{\mathrm{HKR}}\mathrm{HC}^-(-/k_0)$ is $j-1$-truncated for the t-structure. Indeed, for any smooth k-algebra R and $i \in \mathbb{Z}$, the i^{th} graded piece of the HKR filtration on $\mathrm{HC}^-(R/k_0)$ is given by

   $$\mathrm{gr}^i_{\mathrm{HKR}}\mathrm{HC}^-(R/k_0) = \widehat{L\Omega}^{\geq i}_{R/k_0}[2i] \simeq \Omega^{\geq i}_{R/k_0}[2i]$$

   since the composition $k_0 \to k \to R$ is quasismooth; the graded piece therefore vanishes in cohomological degrees $< -i$. By induction it follows, for any $i < j$, that the cofibre of $\mathrm{Fil}^j_{\mathrm{HKR}}\mathrm{HC}^-(R/k_0) \to \mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{HC}^-(R/k_0)$ vanishes in cohomological degrees $\leq -j$. Finally let $i \to \infty$, recalling from Theorem 4.2(1) that the filtration is exhaustive in this case, to deduce that $\mathrm{Fil}^{<j}_{\mathrm{HKR}}\mathrm{HC}^-(R/k_0)$ vanishes in cohomological degrees $\leq -j$ (i.e., homotopical degrees $> j - 1$).

Therefore, by general results on t-structures, the mapping space

$$\mathrm{Map}_{\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k;\mathrm{Sp})}(\mathrm{Fil}^{\geq j}_{\mathrm{cla}}\mathrm{K}, \mathrm{Fil}^{<j}_{\mathrm{HKR}}\mathrm{HC}^-(-/k_0))$$

is contractible for each j. By induction, the trace map $\mathrm{K} \to \mathrm{HC}^-(-/k_0)$ therefore uniquely refines to compatible maps $\mathrm{Fil}^j_{\mathrm{cla}}\mathrm{K} \to \mathrm{Fil}^j_{\mathrm{HKR}}\mathrm{HC}^-(-/k_0)$ for all $j \geq 0$, as desired.

To ensure multiplicativity, one uses the Postnikov t-structure on Zariski sheaves of filtered spectra as introduced in [98, Cons. 3.3.6-7]. This is a t-structure which wraps together the t-structure on Zariski sheaves and the t-structure on the filtered derived category. The (co-)connective part consists of filtered Zariski sheaves $F^\star$ such that $F^j \in \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k;\mathrm{Sp})_{\geq j}$ (resp. $F^j \in \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k;\mathrm{Sp})_{\leq j}$) for all $j \in \mathbb{Z}$. Furthermore, the truncation functor $\tau^P_{\geq 0}$ admits a lax symmetric monoidal structure such that the counit map $\tau^P_{\geq 0} \to \mathrm{id}$ is a morphism of lax symmetric monoidal functors. In particular, if $F^\star$ is a filtered, multiplicative sheaf then the map $\tau^P_{\geq 0}F^\star \to F^\star$ is multiplicative.

Our proof shows that, for the Postnikov t-structure, firstly $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}$ is connective, and secondly $\mathrm{cofib}(\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-/k_0) \to \mathrm{HC}^-(-/k_0))$ is -1-truncated where the target is given the constant filtration; therefore the map $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-/k_0) \to \mathrm{HC}^-(-/k_0)$ is a $\tau^P_{\geq 0}$-equivalence. So we obtain a multiplicative map of filtered objects

$$\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K} \xleftarrow{\simeq} \tau^P_{\geq 0}\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K} \to \tau^P_{\geq 0}\mathrm{HC}^-(-/k_0) \simeq \tau^P_{\geq 0}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-/k_0) \to \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-/k_0),$$

as desired. □

Remark 4.7. In fact the t-structure argument in the above proposition proves slightly more: with the same hypotheses as in Proposition 4.6, *any morphism* $K \to \mathrm{HC}^-(-/k_0)$ promotes uniquely to a filtered map intertwining the classical motivic and HKR filtration. This will be used later to compute Adams operations on rationalized motivic cohomology.

It follows that the cdh-local trace map $\mathrm{KH}(X) \to L_{\mathrm{cdh}}\mathrm{HC}^-(X/\mathbb{Q})$, for qcqs $\mathbb{Q}$-schemes X, also naturally carries the cdh-local motivic filtration on the left to the cdh-local HKR filtration on the right:

Corollary 4.8. *The cdh-local trace map* $\mathrm{KH} \to L_{\mathrm{cdh}}\mathrm{HC}^-(-/\mathbb{Q})$*, viewed as a map between spectra-valued presheaves on* $\mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{Q}}$*, admits a unique multiplicative extension to a map of filtered presheaves*

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{KH} \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}L_{\mathrm{cdh}}\mathrm{HC}^-(-/\mathbb{Q})$$

(the filtration on the left being the cdh-local motivic filtration of Theorem 3.5(1); the filtration on the right is the cdh-local HKR filtration of Remark 4.4).

Proof. Given a smooth $\mathbb{Q}$-scheme X, we claim that the canonical maps of filtered spectra $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X) \to \mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{KH}(X)$ and $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-/\mathbb{Q}) \to \mathrm{Fil}^\star_{\mathrm{HKR}}L_{\mathrm{cdh}}\mathrm{HC}^-(-/\mathbb{Q})$ are equivalences. The first follows from the equivalence $\mathrm{K}(X) \xrightarrow{\sim} \mathrm{KH}(X)$ and the equivalences $\mathbb{Z}(j)^{\mathrm{cla}}(X) \xrightarrow{\sim} \mathbb{Z}(j)^{\mathrm{cdh}}(X)$ of Theorem 3.5(8). The second is a standard consequence of strong resolution of singularities.

Consequently, any filtered upgrade of the cdh-local trace map restricts to the unique filtered upgrade of the trace map for smooth $\mathbb{Q}$-schemes (from Proposition 4.6), and conversely the filtered upgrade of the cdh-local trace map is then necessarily given by the following composition:

$$\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH} \longrightarrow L_{\mathrm{cdh}}L_{\mathrm{Sch}_{\mathbb{Q}}^{\mathrm{qcqs,op}}/\mathrm{Sm}_{\mathbb{Q}}^{\mathrm{op}}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-/\mathbb{Q}) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}L_{\mathrm{cdh}}\mathrm{HC}^-(-/\mathbb{Q}).$$

Here the first map is given by left Kan extending the filtered trace map for smooth $\mathbb{Q}$-schemes along $\mathrm{Sm}_{\mathbb{Q}}^{\mathrm{op}} \subseteq \mathrm{Sch}_{\mathbb{Q}}^{\mathrm{qcqs,\,op}}$, then cdh sheafifying. The second map is the cdh sheafification of the canonical map $L_{\mathrm{Sch}_{\mathbb{Q}}^{\mathrm{qcqs,op}}/\mathrm{Sm}_{\mathbb{Q}}^{\mathrm{op}}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-/\mathbb{Q}) \to \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-/\mathbb{Q})$. Multiplicativity follows from the fact that both left Kan extension and sheafification are multiplicative operations. ☐

We may now construct our motivic cohomology and motivic filtration on qcqs $\mathbb{Q}$-schemes:

Definition 4.9. For a qcqs $\mathbb{Q}$-scheme X, let $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ be the filtered spectrum defined as the pullback (in filtered $\mathbb{E}_\infty$-algebras) of the diagram

$$\begin{tikzcd}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X) \arrow[r, dotted] \arrow[d, dotted] & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X/\mathbb{Q}) \arrow[d] \\
\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X) \arrow[r] & \mathrm{Fil}^\star_{\mathrm{HKR}}L_{\mathrm{cdh}}\mathrm{HC}^-(X/\mathbb{Q}).
\end{tikzcd} \tag{4.3}$$

Here the bottom horizontal arrow is the unique filtered upgrade of the cdh-local trace map $\mathrm{KH}(X) \to L_{\mathrm{cdh}}\mathrm{HC}^-(X/\mathbb{Q})$ provided by Corollary 4.8, and the right vertical arrow is the canonical map.

For $j \in \mathbb{Z}$, define the *weight-j motivic cohomology* of X to be

$$\mathbb{Z}(j)^{\mathrm{mot}}(X) := (\mathrm{gr}^j_{\mathrm{mot}}\mathrm{K}(X))[-2j],$$

which we will see in Theorem 4.10 lies in $D(\mathbb{Z})$ and vanishes for $j < 0$. The associated motivic cohomology groups, for $i \in \mathbb{Z}$, are $H^i_{\mathrm{mot}}(X, \mathbb{Z}(j)) := H^i(\mathbb{Z}(j)^{\mathrm{mot}}(X))$.

In the following theorem we collect some of the immediate, but fundamental, properties of this motivic cohomology theory for qcqs $\mathbb{Q}$-schemes:

Theorem 4.10. *Let $j \in \mathbb{Z}$. For any qcqs $\mathbb{Q}$-scheme X, the weight-j motivic cohomology has the following properties:*

1. *$\mathbb{Z}(j)^{\mathrm{mot}}(X) = 0$ for $j < 0$.*
2. *There is a natural pullback square*

$$\begin{tikzcd}
\mathbb{Z}(j)^{\mathrm{mot}}(X) \arrow[r] \arrow[d] & R\Gamma(X, \widehat{L\Omega}^{\geq j}_{-/\mathbb{Q}}) \arrow[d] \\
\mathbb{Z}(j)^{\mathrm{cdh}}(X) \arrow[r] & R\Gamma_{\mathrm{cdh}}(X, \widehat{L\Omega}^{\geq j}_{-/\mathbb{Q}}).
\end{tikzcd}$$

3. *Fundamental fibre sequence: there is a natural fibre sequence*

$$\mathbb{Z}(j)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(j)^{\mathrm{cdh}}(X) \longrightarrow \mathrm{cofib}\Big(R\Gamma(X, L\Omega^{<j}_{-/\mathbb{Q}}) \to R\Gamma_{\mathrm{cdh}}(X, \Omega^{<j}_{-/\mathbb{Q}})\Big)[-1]$$

4. *For any integer $m \geq 0$, the map $\mathbb{Z}(j)^{\mathrm{mot}}(X)/m \to \mathbb{Z}(j)^{\mathrm{cdh}}(X)/m$ is an equivalence.*
5. *The presheaf $\mathbb{Z}(j)^{\mathrm{mot}} : \mathrm{Sch}_{\mathbb{Q}}^{\mathrm{qcqs}} \to \mathrm{D}(\mathbb{Z})$ is a finitary Nisnevich sheaf.*

Proof. We obtain the pullback square of part (2), even for all $j \in \mathbb{Z}$ if we define $\mathbb{Z}(j)^{\mathrm{cdh}}(X) = 0$ for $j < 0$, by taking graded pieces in the cartesian square (4.3). As a reminder, the graded pieces of $\mathrm{KH}(X)$ are described by Theorem 3.5; those of $\mathrm{HC}^-(X/\mathbb{Q})$ by Theorem 4.2; and those of its cdh sheafification by Remark 4.4.

In particular, when $j < 0$ we have established the existence of a cartesian square

$$\begin{tikzcd}
\mathbb{Z}(j)^{\mathrm{mot}}(X) \arrow[r] \arrow[d] & R\Gamma(X, \widehat{L\Omega}_{-/\mathbb{Q}}) \arrow[d] \\
0 \arrow[r] & R\Gamma_{\mathrm{cdh}}(X, \widehat{L\Omega}_{-/\mathbb{Q}}).
\end{tikzcd}$$

But the right vertical arrow is an equivalence because Hodge-completed derived de Rham cohomology satisfies cdh descent in characteristic zero by Lemma 4.5. Therefore $\mathbb{Z}(j)^{\mathrm{mot}}(X) = 0$ for $j < 0$.

To obtain the fundamental fibre sequence, compute the cofibre of the right vertical arrow in part (2) as follows: compare the fibre sequence

$$R\Gamma(X, \widehat{L\Omega}^{\geq j}_{-/\mathbb{Q}}) \longrightarrow R\Gamma(X, \widehat{L\Omega}_{-/\mathbb{Q}}) \longrightarrow R\Gamma(X, L\Omega^{<j}_{-/\mathbb{Q}})$$

to its cdh sheafified version, and use the following two facts: firstly, cdh sheafifying the middle term of the fibre sequence does not change it, by Lemma 4.5; secondly, the canonical map $R\Gamma_{\mathrm{cdh}}(X, L\Omega^{<j}_{-/\mathbb{Q}}) \to R\Gamma_{\mathrm{cdh}}(X, \Omega^{<j}_{-/\mathbb{Q}})$ is an equivalence, either by resolution of singularities or by Gabber–Ramero's results on the cotangent complex of valuation rings [47, Thm. 6.5.12 & Corol. 6.5.21].

Part (4) follows from the pullback square of (2), since the complexes on the right side of the square are rational.

For part (5), recall that wedge powers $L^i_{-/\mathbb{Q}}$ of the cotangent complex commute with filtered colimits of rings; therefore, by Zariski descent and a finite induction, $R\Gamma(X, L\Omega^{<j}_{-/\mathbb{Q}})$ is finitary. Cdh sheafifying preserves finitariness, so $R\Gamma_{\mathrm{cdh}}(-, L\Omega^{<j}_{-/\mathbb{Q}})$ is also finitary. Finally, $\mathbb{Z}(j)^{\mathrm{cdh}}$ is finitary by Theorem 3.5(2). We now deduce finitariness of $\mathbb{Z}(j)^{\mathrm{mot}}$ from the fundamental fibre sequence. □

Example 4.11 (Weight 0)**.** The right vertical arrow in Theorem 4.10(2) is an equivalence when $j = 0$, by cdh descent of Hodge-completed derived de Rham cohomology; therefore the same is true of the left vertical arrow. That is, there is a natural equivalence

$$\mathbb{Z}(0)^{\mathrm{mot}}(X) \xrightarrow{\sim} \mathbb{Z}(0)^{\mathrm{cdh}}(X) = R\Gamma_{\mathrm{cdh}}(X, \mathbb{Z})$$

for any qcqs $\mathbb{Q}$-scheme X (the equality in the previous line easily following from the definition of $Z(0)^{\mathrm{cdh}}$ in §3.3). We will see in Example 4.26 that the same holds in finite characteristic.

Next we state some of the fundamental properties of the motivic filtration: namely, $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ is indeed a filtration on $\mathrm{K}(X)$, as suggested by the notation, and so there is the desired Atiyah–Hirzebruch spectral sequence:

Theorem 4.12. *Let X be a qcqs $\mathbb{Q}$-scheme. Then the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ is $\mathbb{N}$-indexed, multiplicative, and satisfies $\mathrm{Fil}^0_{\mathrm{mot}}\mathrm{K}(X) = \mathrm{K}(X)$. If X has finite valuative dimension, then:*

1. *the filtration is bounded and so induces a bounded multiplicative Atiyah–Hirzebruch spectral sequence*

$$E_2^{ij} = H^{i-j}_{\mathrm{mot}}(X, \mathbb{Z}(-j)) \implies \mathrm{K}_{-i-j}(X);$$

2. *the filtration is rationally split, i.e., there is a natural, multiplicative equivalence of filtered spectra*

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)_{\mathbb{Q}} \xrightarrow{\sim} \bigoplus_{j\geq 0} \mathbb{Q}(j)^{\mathrm{mot}}(X)[2j],$$

3. *and the spectral sequence degenerates rationally.*

Proof. Theorem 4.10(1) already shows that the filtered object $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ is $\mathbb{N}$-graded. By definition $\mathrm{Fil}^0_{\mathrm{mot}}\mathrm{K}(X)$ is defined via a pullback square

$$\begin{tikzcd}
\mathrm{Fil}^0_{\mathrm{mot}}\mathrm{K}(X) \arrow[r] \arrow[d] & \mathrm{Fil}^0_{\mathrm{HKR}}\mathrm{HC}^-(X/\mathbb{Q}) \arrow[d] \\
\mathrm{Fil}^0_{\mathrm{mot}}\mathrm{KH}(X) \arrow[r] & L_{\mathrm{cdh}}\mathrm{Fil}^0_{\mathrm{HKR}}\mathrm{HC}^-(X/\mathbb{Q}),
\end{tikzcd}$$

which admits a map to the pullback square (4.1). Since $\mathrm{Fil}^0_{\mathrm{mot}}\mathrm{KH}(X) \xrightarrow{\sim} \mathrm{KH}(X)$, the claim reduces to checking that the square

$$\begin{tikzcd}
\mathrm{Fil}^0_{\mathrm{HKR}}\mathrm{HC}^-(X/\mathbb{Q}) \arrow[r] \arrow[d] & \mathrm{HC}^-(X/\mathbb{Q}) \arrow[d] \\
L_{\mathrm{cdh}}\mathrm{Fil}^0_{\mathrm{HKR}}\mathrm{HC}^-(X/\mathbb{Q}) \arrow[r] & L_{\mathrm{cdh}}\mathrm{HC}^-(X/\mathbb{Q}),
\end{tikzcd}$$

is a pullback, i.e., that the cofibre $\mathrm{HC}^-(-/\mathbb{Q})/\mathrm{Fil}^0_{\mathrm{HKR}}\mathrm{HC}^-(-/\mathbb{Q})$ satisfies cdh descent on $\mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{Q}}$. This was explained in Lemma 4.5.

Next suppose that X has finite valuative dimension d. We know from Theorem 3.5(1) that $\mathrm{Fil}^j_{\mathrm{cdh}}\mathrm{KH}(X)$ is supported in cohomological degrees $\leq d-j$. Now, $\widehat{L\Omega}^{\geq j}_{-/\mathbb{Q}}$ is supported in cohomological degrees $\leq j$; by Zariski or cdh sheafifying, it follows that $\mathrm{Fil}^j_{\mathrm{HKR}}\mathrm{HC}^-(X/\mathbb{Q})$ and $\mathrm{Fil}^j_{\mathrm{HKR}}L_{\mathrm{cdh}}\mathrm{HC}^-(X/\mathbb{Q})$ are both supported in homological degrees $\geq d-j$. From the defining pullback square (4.6), we then see that $\mathrm{Fil}^j_{\mathrm{mot}}\mathrm{K}(X)$ is supported in homological degrees $\geq d+1-j$, which is good enough to prove the desired boundedness (but not the optimal bound: see §8).

The splitting result follows from the result on Adams operations in Proposition 4.13 below by a standard argument. Indeed, we start with the multiplicative maps of graded objects

$$\mathrm{gr}^\star_{\mathrm{mot}}\mathrm{K}(X) \longleftarrow \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X) \longrightarrow \mathrm{K}(X),$$

where the right-most term is regarded as a constant graded object and the middle term is the graded object $j \mapsto \mathrm{Fil}^j_{\mathrm{mot}}\mathrm{K}(X)$. Upon rationalizing and taking eigenspectra, Proposition 4.13 then produces multiplicative equivalences of graded objects

$$\mathrm{gr}^\star_{\mathrm{mot}}\mathrm{K}(X)_{\mathbb{Q}} \simeq (\mathrm{gr}^\star_{\mathrm{mot}}\mathrm{K}(X)_{\mathbb{Q}})^{\psi^\ell-\ell^\star} \longleftarrow (\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X))^{\psi^\ell-\ell^\star} \longrightarrow \mathrm{K}(X)^{\psi^\ell-\ell^\star};$$

which proves the claim. The rational degeneration follows. □

The splitting and degeneration parts of the parts of the previous theorem were consequences of a finer result, namely the existence of Adams operations acting in the desired way on motivic cohomology:

Proposition 4.13. *Let X be a qcqs $\mathbb{Q}$-scheme and $\ell \geq 2$. Then there exists a natural, multiplicative automorphism ψ^ℓ of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)_{\mathbb{Q}}$ such that, for each $j \geq 0$, the induced automorphism on $\mathrm{gr}^j_{\mathrm{mot}}\mathrm{K}(X)_{\mathbb{Q}} = \mathbb{Q}(j)^{\mathrm{mot}}(X)[2j]$ is multiplication by ℓ^j.*

Proof. Taking left Kan extension and cdh sheafification extends the compatibility of Corollary B.10 along the cdh local trace map of Corollary 4.8. Hence, the rationalized version of the diagram (4.3) can be promoted to a diagram of filtered spectra with multiplicative endomorphisms ψ^ℓ; these are in fact automorphisms since we can define $\psi^{-\ell}$ which furnishes an inverse. By construction, we then have a filtered, multiplicative automorphism ψ^ℓ on $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)_{\mathbb{Q}}$. The action on graded pieces is then determined by the action of ψ^ℓ on each individual graded pieces of the rationalized filtration which, in turn, can be checked on smooth $\mathbb{Q}$-schemes. This last assertion is Corollary B.7 and [98, Prop. 6.4.12]. □

4.2 Characteristic $p > 0$

We first give a with a quick overview of syntomic cohomology in characteristic p, in the sense of [20]. The reader should refer to [20, §8] and [5, §6.2] for more details.

For any $\mathbb{F}_p$-algebra A, let $W_r\Omega^j_{A,\log}$ denote the global sections of the subsheaf $W_r\Omega^j_{\log}$ of the de Rham–Witt sheaf $W_r\Omega^j_{\mathrm{Spec}A}$ which is generated étale locally (or, equivalently, Zariski locally [91, Corol. 4.2(i)]) by $\frac{d[f_1]}{f_1} \wedge \cdots \wedge \frac{d[f_j]}{f_j}$ for units $f_1, \ldots, f_j$. Alternatively [91, Corol. 4.2(iii)], $W_r\Omega^j_A$ is the kernel of the Artin–Schreier map

$$C^{-1} - 1 : W_r\Omega^j_A \longrightarrow W_r\Omega^j_A/dV^{r-1}\Omega^j_A. \tag{4.4}$$

Since the de Rham-Witt sheaves have no higher cohomology on affines and the Artin–Schreier map is étale locally surjective, the previous observations may alternatively be expressed as a fibre sequence

$$R\Gamma_{\text{ét}}(A, W_r\Omega^j_{\log}) \longrightarrow W_r\Omega^j_A \xrightarrow{C^{-1}-1} W_r\Omega^j_A/dV^{r-1}\Omega^j_A;$$

in particular the cohomology of $R\Gamma_{\text{ét}}(A, W_r\Omega^j_{\log})$ is concentrated in degrees zero and one.

Definition 4.14. *Mod-p^r, weight-j syntomic cohomology* of $\mathbb{F}_p$-algebras

$$\mathbb{Z}_p(j)^{\text{syn}}(-)/p^r : \text{CAlg}_{\mathbb{F}_p} \to \text{D}(\mathbb{Z})$$

is defined to be the left Kan extension of $R\Gamma_{\text{ét}}(-, W_r\Omega^j_{\log})[-j]$ along the inclusion $\text{CAlg}^{\Sigma}_{\mathbb{F}_p} \subseteq \text{CAlg}_{\mathbb{F}_p}$; here $\text{CAlg}^{\Sigma}_{\mathbb{F}_p}$ denotes the category of finitely generated polynomial $\mathbb{F}_p$-algebras. When $r = 1$ we will often write $\mathbb{F}_p(j)^{\text{syn}}$ to simplify notation.

Taking the inverse limit over r, the weight-j syntomic cohomology of an $\mathbb{F}_p$-algebra A is defined by

$$\mathbb{Z}_p(j)^{\text{syn}}(A) := \lim_r \mathbb{Z}_p(j)^{\text{syn}}(A)/p^r.$$

Remark 4.15.

1. Taking $\mathbb{Z}_p(j)^{\text{syn}}(A)$ modulo p^r does recover $\mathbb{Z}_p(j)^{\text{syn}}(A)/p^r$ as it was initially defined, thanks to Illusie's short exact sequence of étale sheaves $0 \to W_s\Omega^j_{\log} \xrightarrow{p^r} W_{r+s}\Omega^j_{\log} \to W_r\Omega^j_{\log} \to 0$ on smooth $\mathbb{F}_p$-schemes [62, §I.5.7].
2. For any $\mathbb{F}_p$-algebra A and $r \geq 1$ there is, by construction, a natural comparison map

$$\mathbb{Z}_p(j)^{\text{syn}}(A)/p^r \longrightarrow R\Gamma_{\text{ét}}(A, W_r\Omega^j_{\log})[-j].$$

It is an equivalence whenever A is regular Noetherian, or more generally Cartier smooth [65, Prop. 5.1].

Syntomic cohomology can be loosely controlled via the cotangent complex through the following lemma:

Lemma 4.16. *For any $\mathbb{F}_p$-algebra A, the complex $\mathbb{F}_p(j)^{\text{syn}}(A)$ admits a natural finite increasing filtration in $D(\mathbb{F}_p)$, of length $2(j+1)$, with graded pieces given in increasing order by*

$$L^j_{A/\mathbb{F}_p}[-j-1], L^{j-1}_{A/\mathbb{F}_p}[-j], L^{j-2}_{A/\mathbb{F}_p}[-j+1], \ldots, L^0_{A/\mathbb{F}_p}[-1],$$
$$L^0_{A/\mathbb{F}_p}[0], L^1_{A/\mathbb{F}_p}[-1], L^2_{A/\mathbb{F}_p}[-2], \ldots, L^j_{A/\mathbb{F}_p}[-j].$$

Proof. The key is to show the following claim: for R any smooth $\mathbb{F}_p$-algebra, then $\Omega^j_R/d\Omega^{j-1}_R$ admits a natural finite increasing filtration (in $\text{D}(\mathbb{F}_p)$, not as submodules) of length $2j+1$ with graded pieces in increasing order

$$\Omega^j_R, \Omega^{j-1}_R[1], \Omega^{j-2}_R[2], \ldots, \Omega^0_R[j],$$
$$\Omega^0_R[j+1], \Omega^1_R[j], \Omega^2_R[j-1], \ldots, \Omega^{j-1}_R[2].$$

The case $j = 0$ (when the bottom row of the listed graded pieces is empty) is trivial; we proceed by induction to treat the case $j > 0$, so assume that we already have the filtration on $\Omega^{j-1}_R/d\Omega^{j-2}_R$, i.e.,

$$\text{Fil}_0(\Omega^{j-1}_R/d\Omega^{j-2}_R) \to \text{Fil}_1(\Omega^{j-1}_R/d\Omega^{j-2}_R) \to \cdots \to \text{Fil}_{2j-1}(\Omega^{j-1}_R/d\Omega^{j-2}_R) = \Omega^{j-1}_R/d\Omega^{j-2}_R,$$

with the desired graded pieces. Then we define, for $i = 1, \ldots, 2j$, the filtered step $\text{Fil}_i(\Omega^j_R/d\Omega^{j-1}_R)$ to be the pullback

$$\begin{tikzcd}
\text{Fil}_{i-1}(\Omega^{j-1}_R/d\Omega^{j-2}_R)[1] \arrow[r] & \Omega^{j-1}_R/d\Omega^{j-2}_R[1] \arrow[r, "{\pi[1]}"] & \Omega^{j-1}_R/\ker d[1] \\
\text{Fil}_i(\Omega^j_R/d\Omega^{j-1}_R) \arrow[u, dashed] \arrow[rr, dashed] & & \Omega^j_R/d\Omega^{j-1}_R \arrow[u, "\delta"']
\end{tikzcd}$$

Here π is the canonical quotient map with kernel $H^{j-1}_{\mathrm{dR}}(R)$, and δ is the connecting map associated to the short exact sequence

$$0 \longrightarrow \Omega^{j-1}_R/\ker d \xrightarrow{d} \Omega^j_R \longrightarrow \Omega^j_R/d\Omega^{j-1}_R \longrightarrow 0. \tag{4.5}$$

Since pulling back a filtration does not change the graded pieces, we see at once that $\mathrm{gr}_i(\Omega^j_R/d\Omega^{j-1}_R)$ is as desired for $i = 1, \dots, 2j-1$.

We now set $\mathrm{Fil}_0(\Omega^j_R/d\Omega^{j-1}_R) := 0$ and $\mathrm{Fil}_{2j+1}(\Omega^j_R/d\Omega^{j-1}_R) = \Omega^j_R/d\Omega^{j-1}_R$; we must show that $\mathrm{gr}_0(\Omega^j_R/d\Omega^{j-1}_R)$ and $\mathrm{gr}_{2j}(\Omega^j_R/d\Omega^{j-1}_R)$ are as desired. Firstly, $\mathrm{gr}_0(\Omega^j_R/d\Omega^{j-1}_R) = \mathrm{Fil}_1(\Omega^j_R/d\Omega^{j-1}_R)$, which was defined to be the pullback of $\mathrm{Fil}_0(\Omega^{j-1}_R/d\Omega^{j-2}_R) = 0 \to \Omega^{j-1}/\ker d[1]$ along δ; that is, it is given by $\mathrm{fib}(\delta)$, which is indeed Ω^j_R thanks to (4.5). Secondly, $\mathrm{gr}_{2j}(\Omega^j_R/d\Omega^{j-1}_R)$ is precisely $\mathrm{cofib}(\pi)[1] = H^{j-1}_{\mathrm{dR}}(R)[2]$, which identifies with $\Omega^{j-1}_R[2]$ via the Cartier isomorphism.

This completes the proof of the existence of the filtration on $\Omega^j_R/d\Omega^{j-1}_R$ when R is smooth. We then obtain the desired filtration on $\mathbb{F}_p(j)^{\mathrm{syn}}(R)$ by recalling that $\mathbb{F}_p(j)^{\mathrm{syn}}(R) = \mathrm{fib}(\Omega^j_R \xrightarrow{C^{-1}-1} \Omega^j_R/d\Omega^{j-1}_R)[-j]$. Finally the desired filtration on $\mathbb{F}_p(j)^{\mathrm{syn}}(A)$, for arbitrary $\mathbb{F}_p$-algebras A, is obtained by left Kan extension from the smooth case. □

As a consequence of the previous lemma and fpqc descent for the cotangent complex, we have that

Corollary 4.17. *The presheaves $\mathbb{Z}_p(j)^{\mathrm{syn}}$ satisfy fpqc descent on the category of $\mathbb{F}_p$-algebras.*

Therefore, by right Kan extension, they extend uniquely to fpqc sheaves

$$\mathbb{Z}_p(j)^{\mathrm{syn}} : \mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}_p} \longrightarrow \mathrm{D}(\mathbb{Z}),$$

thereby defining syntomic cohomology in the non-affine case. Just as derived de Rham cohomology appeared in characteristic zero through the HKR filtration on negative cyclic homology, syntomic cohomology similarly appears through topological cyclic homology:

Theorem 4.18 (BMS filtration [20])**.** *For any qcqs $\mathbb{F}_p$-scheme X, its topological cyclic homology $\mathrm{TC}(X)$ admits a natural, multiplicative, complete, $\mathbb{N}$-indexed filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X)$ with graded pieces*

$$\mathrm{gr}^j_{\mathrm{BMS}}\mathrm{TC}(X) \simeq \mathbb{Z}_p(j)^{\mathrm{syn}}(X)[2j]$$

for $j \geq 0$. Moreover,

1. *The filtration is bounded, i.e., there exists $d \geq 0$ (depending on X) such that, for any $j \geq 0$, the filtered step $\mathrm{Fil}^j_{\mathrm{BMS}}\mathrm{TC}(X)$ is supported in homological degrees $\geq j-d$ (and so the syntomic cohomology $\mathbb{Z}_p(j)^{\mathrm{syn}}(X)$ is supported in cohomological degrees $\leq j+d$).*
2. *The induced filtration on $\mathrm{TC}(X)[\frac{1}{p}]$ is naturally split, so that*

$$\mathrm{TC}(X)[\tfrac{1}{p}] \simeq \bigoplus_{j \geq 0} \mathbb{Q}_p(j)^{\mathrm{syn}}(X)[2j],$$

where $\mathbb{Q}_p(j)^{\mathrm{syn}}(X) := \mathbb{Z}_p(j)^{\mathrm{syn}}(X)[\frac{1}{p}]$.

Proof. The existence of a filtration on TC with graded pieces given by shifts of syntomic cohomology is one of the main theorems of [20], in the case of quasisyntomic $\mathbb{F}_p$-algebras. It was extended, by p-completed left Kan extension, to all $\mathbb{F}_p$-algebras in [5]. It is then obtained for arbitrary qcqs $\mathbb{F}_p$-schemes by right Kan extension. It remains to explain (1) and (2).

The proof of part (1) proceeds via several cases. Firstly, for any quasisyntomic $\mathbb{F}_p$-algebra R, the BMS filtration on $\mathrm{TC}(R)$ is defined by descent from quasiregular semiperfect rings of the two-speed Postnikov filtration; the latter is manifestly complete, which is preserved by the descent. In particular, for smooth $\mathbb{F}_p$-algebras R, the BMS filtration on $\mathrm{TC}(R)$ is complete and each of its graded pieces $\mathrm{gr}^j_{\mathrm{BMS}}\mathrm{TC}(R) \simeq \lim_r R\Gamma_{\mathrm{et}}(R, W_r\Omega^r_{\mathrm{log}})[j]$ is supported in cohomological degrees $[-j, -(j+1)]$; using the completeness it follows in this case that $\mathrm{Fil}^j_{\mathrm{BMS}}\mathrm{TC}(R)$ is supported in homological degrees $\geq j-1$. By left Kan extending we see that $\mathrm{Fil}^j_{\mathrm{BMS}}\mathrm{TC}(A)$ is supported in homological degrees $\geq j-1$ for all $\mathbb{F}_p$-algebras A; so the filtration is bounded on affines. Finally, right Kan extending preserves completeness, so we deduce that the BMS filtration on $\mathrm{TC}(X)$, for any qcqs $\mathbb{F}_p$-scheme X, is at least complete. Therefore it is enough to check boundedness of the filtration on graded pieces, as mentioned in §2.2; we do this next.

Since X is qcqs, it has finite cohomological dimension for quasi-coherent sheaves (even if it does not have finite Krull dimension), and we take d to be one plus this dimension. Then the Zariski sheafification of each graded piece of Lemma 4.16 has global sections supported in cohomological degrees $\leq j+d$, as required.

For part (2) note first that the absolute Frobenius $\phi : X \to X$ induces a natural endomorphism of $\mathrm{TC}(X)$, compatible (by functoriality) with the BMS filtration; its action on the graded piece $\mathbb{Z}_p(j)^{\mathrm{syn}}[2j]$ is as multiplication by p^j (this follows by left Kan extending the same statement for $R\Gamma_{\text{ét}}(-, W_r\Omega^j_{\log})$ of finitely generated polynomial algebras, for all $r \geq 0$). Next observe that the BMS filtration on $\mathrm{TC}(X)[\frac{1}{p}]$ is bounded thanks to part (1), therefore complete. Since $\phi - p^j$ acts invertibly on $\mathbb{Q}_p(i)^{\mathrm{syn}}(X)$ for $i \geq j+1$, we deduce that it acts invertibly on $\mathrm{Fil}^{j+1}_{\mathrm{BMS}}\mathrm{TC}(X)[\frac{1}{p}]$. Similarly it acts invertibly on $\mathrm{TC}(X)[\frac{1}{p}]/\mathrm{Fil}^j\mathrm{TC}(X)[\frac{1}{p}]$. Taking $\phi - p^j$-fixed points, we have shown that the maps

$$\mathbb{Z}_p(j)^{\mathrm{syn}}(X)[2j] = \mathrm{gr}^j_{\mathrm{BMS}}\mathrm{TC}(X) \longleftarrow (\mathrm{Fil}^j_{\mathrm{BMS}}\mathrm{TC}(X))^{\phi=p^j} \longrightarrow \mathrm{TC}(X)^{\phi=p^j}$$

are equivalences after inverting p. This defines a filtered map $\bigoplus_{j\geq 0}\mathbb{Q}_p(j)^{\mathrm{syn}}(X)[2j] \to \mathrm{TC}(X)[\frac{1}{p}]$ which is an equivalence on all graded pieces, therefore an equivalence since the filtrations on both sides are complete (by boundedness). ☐

Remark 4.19 (Variant: cdh-local BMS filtration)**.** Cdh sheafifying the BMS filtration levelwise we see that, for any qcqs $\mathbb{F}_p$-scheme X, there exists a functorial, multiplicative, $\mathbb{N}$-indexed filtration

$$\mathrm{Fil}^\star_{\mathrm{BMS}}L_{\mathrm{cdh}}\mathrm{TC}(X) := L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X),$$

on $L_{\mathrm{cdh}}\mathrm{TC}(X)$ whose graded pieces for $j \geq 0$ are given by

$$\mathrm{gr}^j_{\mathrm{BMS}}L_{\mathrm{cdh}}\mathrm{TC}(X) \simeq L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}(X)[2j].$$

Warning: here $L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}$ is the cdh sheafification of the presheaf $\mathbb{Z}_p(j)^{\mathrm{syn}} : \mathrm{Sch}^{\mathrm{qcqs,op}}_{\mathbb{F}_p} \to \mathrm{D}(\mathbb{Z})$; but since sheafification does not commute with cofiltered limits in general, there is no reason that $L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}$ should land in p-complete complexes.[7] In fact, such an issue already appeared in characteristic zero: $R\Gamma_{\mathrm{cdh}}(-, \widehat{L\Omega}_{-/\mathbb{Q}})$ was not necessarily Hodge complete.

We record the following consequence of the fact that the BMS filtration is split after inverting p; it will be required to control our motivic cohomology after inverting p:

Corollary 4.20. *The presheaf $\mathbb{Q}_p(j)^{\mathrm{syn}} : \mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}_p} \to \mathrm{D}(\mathbb{Z})$ is a cdh sheaf.*

Proof. Since $\mathbb{Q}_p(j)^{\mathrm{syn}}$ is a direct summand of $\mathrm{TC}[\frac{1}{p}]$ by Theorem 4.18(2), it is sufficient to check that the latter is a cdh sheaf. In other words, since cdh sheafification commutes with inverting p, we must show that $\mathrm{TC}(X)[\frac{1}{p}] \to (L_{\mathrm{cdh}}\mathrm{TC}(X))[\frac{1}{p}]$ is an equivalence for any qcqs $\mathbb{F}_p$-scheme X. But this follows from Theorem 3.8 and the result of Weibel that $K(X)[\frac{1}{p}] \xrightarrow{\sim} \mathrm{KH}(X)[\frac{1}{p}]$ [119]. ☐

We next establish an analogue of Proposition 4.6, namely that the trace map in characteristic p is compatible with the motivic and BMS filtrations:

Proposition 4.21. *Let k be a field of characteristic p. Then the trace map $\mathrm{K} \to \mathrm{TC}$, viewed as a map between spectra-valued presheaves on Sm_k, admits a unique, multiplicative extension to a map of filtered presheaves $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K} \to \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}$.*

Proof. We apply the same t-structure argument as Propositions 4.6. Step 1 of that proof was independent of the characteristic, and so shows in the present context that $\mathrm{Fil}^j_{\mathrm{cla}}\mathrm{K}$ is j-connective for each $j \geq 0$. It remains to check that $\mathrm{Fil}^{<j}_{\mathrm{BMS}}\mathrm{TC}(R)$ vanishes in cohomological degrees $\leq -j$ for each smooth k-algebra R. But for each $i = 0, \dots, j-1$ the i^{th} graded piece is $\mathrm{gr}^i_{\mathrm{BMS}}\mathrm{TC}(R) \simeq \mathbb{Z}_p(i)^{\mathrm{syn}}(R)[2i]$, where

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[7] For any fixed qcqs $\mathbb{F}_p$-scheme X, one can show that the following are equivalent:

1. $L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}(X)$ is derived p-complete.
2. Each cohomology group of $\mathrm{fib}(\mathbb{Z}(j)^{\mathrm{mot}}(X) \to \mathbb{Z}(j)^{\mathrm{cdh}}(X))$ is bounded p-power torsion.

If X is quasi-excellent, Noetherian, and of finite Krull dimension, then one can use alterations and the argument of the proof of Proposition 4.22 to check that (1) and (2) are true. However, they definitely do not hold in general. For example, let $A = \mathbb{F}_p[t^{1/p^\infty}]/(t-1)$. Then H^1 of $\mathrm{fib}(\mathbb{Z}(1)^{\mathrm{mot}}(A) \to \mathbb{Z}(1)^{\mathrm{cdh}}(A))$ is the principal units $\ker(A^\times \to \mathbb{F}_p^\times)$, which is not bounded p-power torsion. This also shows that the cdh sheaf $L_{\mathrm{cdh}}\mathbb{Z}_p(1)^{\mathrm{syn}}$ is not invariant for the nil (but not nilpotent) ideal $\ker(A \to \mathbb{F}_p)$.

$\mathbb{Z}_p(i)^{\mathrm{syn}}(R)$ is supported in cohomological degrees $[i, i+1]$; the desired vanishing bound follows by a trivial induction. Multiplicativity follows from the same argument as in characteristic zero using the Postnikov t-structure. □

As in characteristic zero, we need a cdh-local analogue of the previous proposition; unlike characteristic zero,[8] it does not formally follow from the previous proposition:

Proposition 4.22. *The cdh-local trace map* $\mathrm{KH} \to L_{\mathrm{cdh}}\mathrm{TC}$*, viewed as a map between spectra-valued presheaves on* $\mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}_p}$*, admits a unique extension to a multiplicative map of filtered presheaves*

$$\mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH} \longrightarrow \mathrm{Fil}^{\star}_{\mathrm{BMS}} L_{\mathrm{cdh}}\mathrm{TC}$$

(the filtration on the left being the cdh-local motivic filtration of Theorem 3.5(1); the filtration on the right is the cdh-local BMS filtration of Remark 4.19).

Proof. We begin with an argument which is essentially the same as the second half of the proof of Corollary 4.8. Namely, since $L_{\mathrm{cdh}} L_{\mathrm{Sch}^{\mathrm{qcqs,op}}_{\mathbb{F}_p}/\mathrm{Sm}^{\mathrm{op}}_{\mathbb{F}_p}}$ is a left adjoint to restricting from cdh sheaves on $\mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}_p}$ to presheaves on $\mathrm{Sm}_{\mathbb{F}_p}$, and $\mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH} = L_{\mathrm{cdh}} L_{\mathrm{Sch}^{\mathrm{qcqs,op}}_{\mathbb{F}_p}/\mathrm{Sm}^{\mathrm{op}}_{\mathbb{F}_p}} \mathrm{Fil}^{\star}_{\mathrm{cla}}\mathrm{K}$ by definition, the statement of the proposition is equivalent to the following claim: the map of spectra-valued presheaves $\mathrm{K} \to (L_{\mathrm{cdh}}\mathrm{TC})|_{\mathrm{Sm}_{\mathbb{F}_p}}$ on $\mathrm{Sm}_{\mathbb{F}_p}$ admits a unique extension to a multiplicative map of filtered presheaves $\mathrm{Fil}^{\star}_{\mathrm{cla}}\mathrm{K} \to (\mathrm{Fil}^{\star}_{\mathrm{BMS}} L_{\mathrm{cdh}}\mathrm{TC})|_{\mathrm{Sm}_{\mathbb{F}_p}}$.

To prove the claim we apply the same t-structure argument as in Propositions 4.6 and 4.21. We have already noted in the proof of Proposition 4.21 that $\mathrm{Fil}^{j}_{\mathrm{cla}}K$ is j-connective for any $j \geq 0$, so it remains only to show that $L_{\mathrm{cdh}}\mathrm{Fil}^{<j}_{\mathrm{BMS}}\mathrm{TC}(X)$ vanishes in cohomological degrees $\leq -j$ for each smooth $\mathbb{F}_p$-scheme X; by induction it is enough to check that $L_{\mathrm{cdh}}\mathbb{Z}_p(i)^{\mathrm{syn}}(X)$ is supported in cohomological degrees $\geq i$ for each $i \geq 0$. At least modulo p this bound follows from the equivalence

$$R\Gamma_{\mathrm{eh}}(X, \Omega^{i}_{\log})[-i] \xrightarrow{\sim} L_{\mathrm{cdh}}\mathbb{F}_p(i)^{\mathrm{syn}}(X),$$

which holds in fact for any qcqs $\mathbb{F}_p$-scheme and will be explained in the proof of Theorem 4.28 below.

However we must now recall the warning of Remark 4.19: the presheaf $L_{\mathrm{cdh}}\mathbb{Z}_p(i)^{\mathrm{syn}}$ does not take p-complete values on arbitrary qcqs $\mathbb{F}_p$-schemes. To complete the proof we must therefore show that it does take a p-complete value whenever X is a smooth $\mathbb{F}_p$-scheme (as then the coconnectivity bound modulo p yields the same bound for $L_{\mathrm{cdh}}\mathbb{Z}_p(i)^{\mathrm{syn}}(X)$). To prove this p-completeness claim we consider the filtered spectrum $\mathrm{fib}(\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(X) \to \mathrm{Fil}^{\star}_{\mathrm{BMS}} L_{\mathrm{cdh}}\mathrm{TC}(X))$, whose underlying spectrum is zero since $\mathrm{TC}(X) \xrightarrow{\sim} L_{\mathrm{cdh}}\mathrm{TC}(X)$ (using $\mathrm{K}(X) \xrightarrow{\sim} \mathrm{KH}(X)$ and Theorem 3.8) and where the filtration is bounded (see the second paragraph of the proof of Theorem 4.27 for details). Moreover, the associated bounded spectra sequence

$$E_2^{ij} = H^{i-j}(\mathrm{fib}(\mathbb{Z}_p(j)^{\mathrm{syn}}(X) \to L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}(X)) \implies 0$$

degenerates up to bounded denominators thanks to the Frobenius actions, and so each group on the E_2 page is annihilated by a bounded power of p. Since $\mathbb{Z}_p(j)^{\mathrm{syn}}(X)$ is p-complete, we now obtain the same for $L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}(X)$, as required to complete the proof. □

Definition 4.23. For a qcqs $\mathbb{F}_p$-scheme X, let $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X)$ be the filtered spectrum defined as the pullback (in filtered $\mathbb{E}_\infty$-algebras) of the diagram

$$\begin{tikzcd}
\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X) \arrow[r, dotted] \arrow[d, dotted] & \mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(X) \arrow[d] \\
\mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(X) \arrow[r] & \mathrm{Fil}^{\star}_{\mathrm{BMS}} L_{\mathrm{cdh}}\mathrm{TC}(X),
\end{tikzcd} \tag{4.6}$$

where the bottom map is given by Proposition 4.22.

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[8] To be precise it is in fact possible to run the same argument as in Corollary 4.8: we just need to know that the map of filtered spectra $\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(X) \to \mathrm{Fil}^{\star}_{\mathrm{BMS}} L_{\mathrm{cdh}}\mathrm{TC}(X)$ is an equivalence for every smooth $\mathbb{F}_p$-scheme X. Since $\mathrm{TC}(X) \xrightarrow{\sim} L_{\mathrm{cdh}}\mathrm{TC}(X)$, using $\mathrm{K}(X) \xrightarrow{\sim} \mathrm{KH}(X)$ and Theorem 3.8, the problem reduces to checking that $\mathbb{Z}_p(j)(X) \xrightarrow{\sim} L_{\mathrm{cdh}}\mathbb{Z}_p(j)(X)$ for all $j \geq 0$. We will deduce this in Corollary 6.5 using arguments involving motivic cohomology, but the appearance of motivic cohomology is illusory: the core of the proof of Corollary 6.5 is really contained in Theorem 5.14, whose proof does not require the new motivic cohomology. However, to avoid any impression of circular logic, in the body of the text we provide a different proof of the proposition, which only requires the weaker result that $\mathbb{Z}_p(j)(X) \to L_{\mathrm{cdh}}\mathbb{Z}_p(j)(X)$ is an equivalence up to bounded p-power torsion.

For $j \in \mathbb{Z}$, define the *weight-j motivic cohomology* of X to be

$$\mathbb{Z}(j)^{\mathrm{mot}}(X) := (\mathrm{gr}^j_{\mathrm{mot}}\mathrm{K}(X))[-2j],$$

which we will see in Theorem 4.24 lies in $\mathrm{D}(\mathbb{Z})$ and vanishes for $j < 0$. The associated motivic cohomology groups, for $i \in \mathbb{Z}$, are $H^i_{\mathrm{mot}}(X, \mathbb{Z}(j)) := H^i(\mathbb{Z}(j)^{\mathrm{mot}}(X))$.

Here are some fundamental properties of our motivic cohomology in characteristic p:

Theorem 4.24. *Let $j \in \mathbb{Z}$. For any qcqs $\mathbb{F}_p$-scheme X, the weight-j motivic cohomology $\mathbb{Z}(j)^{\mathrm{mot}}(X)$ has the following properties:*

1. *$\mathbb{Z}(j)^{\mathrm{mot}}(X) = 0$ for $j < 0$.*
2. *There is a natural pullback square*

$$\begin{tikzcd}
\mathbb{Z}(j)^{\mathrm{mot}}(X) \arrow[r] \arrow[d] & \mathbb{Z}_p(j)^{\mathrm{syn}}(X) \arrow[d] \\
\mathbb{Z}(j)^{\mathrm{cdh}}(X) \arrow[r] & L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}(X).
\end{tikzcd}$$

3. *The canonical map $\mathbb{Z}(j)^{\mathrm{mot}}(X)[\frac{1}{p}] \to \mathbb{Z}(j)^{\mathrm{cdh}}(X)[\frac{1}{p}]$ is an equivalence. In particular, $\mathbb{Z}(j)^{\mathrm{mot}}[\frac{1}{p}]$ is a cdh sheaf on $\mathrm{Sch}^{\mathrm{qcqs,op}}_{\mathbb{F}_p}$.*
4. *The presheaf $\mathbb{Z}(j)^{\mathrm{mot}} : \mathrm{Sch}^{\mathrm{qcqs,op}}_{\mathbb{F}_p} \to D(\mathbb{Z})$ is a finitary Nisnevich sheaf.*
5. *The endomorphism ϕ^* of $\mathbb{Z}(j)^{\mathrm{mot}}(X)$ induced by the absolute Frobenius $\phi : X \to X$ is multiplication by p^j.*

Proof. (1): The three corners in (4.6) used to define the pullback are $\mathbb{N}$-indexed, whence the same is true of the pullback $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$, i.e., the graded pieces vanish in negative weights. Part (2) is obtained by taking graded pieces in the pullback square (4.6).

(3): Corollary 4.20 states that $\mathrm{fib}(\mathbb{Z}_p(j) \to L_{\mathrm{cdh}}\mathbb{Z}_p(j))[\frac{1}{p}] \simeq 0$, whence the result follows from the cartesian square in (2).

(4): It suffices to prove that $\mathbb{Z}(j)^{\mathrm{mot}}[\frac{1}{p}]$ and $\mathbb{Z}(j)^{\mathrm{mot}}/p$ are finitary. The first follows from part (3) and Theorem 3.5(2). The second reduces, via the pullback square of part (2), to finitariness of $\mathbb{Z}(j)^{\mathrm{cdh}}/p$ (which is indeed finitary by another application of Theorem 3.5(2)), of $\mathbb{F}_p(j)^{\mathrm{syn}}$ (finitary since, on affines, it is left Kan extended from finitely generated polynomial algebras), and of $L_{\mathrm{cdh}}\mathbb{F}_p(j)^{\mathrm{syn}}$ (finitary since it is the cdh sheafification of a finitary presheaf).

(5): As usual it suffices to treat the other corners of the pullback square of part (2). The Frobenius acts on $\mathbb{Z}(j)^{\mathrm{cdh}}$ as multiplication by p^j, by left Kan extending and cdh sheafifying the analogous statement for classical motivic cohomology of smooth $\mathbb{F}_p$-schemes [50]. It acts on $\mathbb{Z}_p(j)^{\mathrm{syn}}$ as multiplication by p^j, as we already noted in the proof of Theorem 4.18(2), and so similarly for $L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}$ by cdh sheafifying. □

Remark 4.25. Neither the presheaf $\mathbb{Z}_p(j)^{\mathrm{syn}}$ nor $L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}$ is finitary. However, the proof of Theorem 4.24(4) shows that their difference, i.e, the fibre

$$X \mapsto \mathrm{fib}\left(\mathbb{Z}_p(j)^{\mathrm{syn}}(X) \to L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}(X)\right),$$

is a finitary presheaf.

Example 4.26 (Weight 0)**.** As in Example 4.11 in characteristic 0, the map

$$\mathbb{Z}(0)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(0)^{\mathrm{cdh}}(X) = R\Gamma_{\mathrm{cdh}}(X, \mathbb{Z})$$

is an equivalence for any qcqs $\mathbb{F}_p$-scheme X. Indeed, from the pullback square Theorem 4.24 it is enough to show that $\mathbb{Z}_p(0)^{\mathrm{syn}}$, or equivalently $\mathbb{F}_p(0)^{\mathrm{syn}}$, satisfies cdh descent. But it is easily checked from the definitions that $\mathbb{F}_p(0)^{\mathrm{syn}} \simeq R\Gamma_{\mathrm{\acute{e}t}}(-, \mathbb{Z}/p\mathbb{Z})$, which even satisfies cdh descent by Deligne (or even arc descent [19]).

Here is the analogue, in characteristic p, of Theorem 4.12 about the existence of the Atiyah–Hirzebruch spectral sequence:

Theorem 4.27. *Let X be a qcqs $\mathbb{F}_p$-scheme. Then the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ is $\mathbb{N}$-indexed, multiplicative, and satisfies $\mathrm{Fil}^0_{\mathrm{mot}}\mathrm{K}(X) = \mathrm{K}(X)$. If X has finite valuative dimension, then:*

1. *the filtration is bounded and so induces a bounded multiplicative Atiyah–Hirzebruch spectral sequence*

$$E_2^{ij} = H^{i-j}_{\mathrm{mot}}(X, \mathbb{Z}(-j)) \implies \mathrm{K}_{-i-j}(X),$$

2. *the filtration is rationally split, i.e., there is a natural equivalence of filtered spectra*

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)_{\mathbb{Q}} \xrightarrow{\sim} \bigoplus_{j\geq 0} \mathbb{Q}(j)^{\mathrm{mot}}(X)[2j],$$

3. *and the spectral sequence degenerates up to bounded denominators.*

Proof. We have already seen in Theorem 4.24(1) that the filtration is $\mathbb{N}$-indexed; it satisfies $\mathrm{Fil}^0_{\mathrm{mot}}\mathrm{K}(X) = \mathrm{K}(X)$ thanks to Theorem 3.8.

Now assume X has finite valuative dimension d (hence also finite Krull dimension $\leq d$). We already know from Theorem 3.5(1) that $\mathrm{Fil}^j_{\mathrm{cdh}}\mathrm{KH}(X)$ is supported in cohomological degrees $\leq d - j$. We also saw in the proof of Theorem 4.18(1) that, on any affine, $\mathbb{F}_p(j)^{\mathrm{syn}}$ is supported in cohomological degrees $\leq j + 1$; by Zariski or cdh sheafifying, it follows that $\mathrm{Fil}^j_{\mathrm{BMS}}\mathrm{TC}(X)$ and $\mathrm{Fil}^j_{\mathrm{BMS}}L_{\mathrm{cdh}}\mathrm{TC}(X)$ are both supported in homological degrees $\geq d + 1 - j$. From the defining pullback square (4.6), we then see that $\mathrm{Fil}^j_{\mathrm{mot}}\mathrm{K}(X)$ is supported in homological degrees $\geq d + 2 - j$, which is good enough to prove the desired boundedness (but not the optimal bound: see §8).

The filtration is split since the Frobenius actions are incompatible in different weights, thanks to Theorem 4.24(5): argue exactly as in Theorem 4.18(2). The Frobenius action also forces the spectral sequence to degenerate up to bounded denominators: each differential $\delta : E_m^{i,j} \to E_m^{i+m,j+1-m}$, where $j \leq 0$, is compatible with the Frobenius, which acts as p^{-j} on the domain and p^{-j-1+m} on the codomain, so that $p^{-j}(p^{m-1} - 1)\delta = 0$. ☐

In the remainder of this section we explicitly describe p-adic motivic cohomology in characteristic p, analogously to Geisser–Levine's identification [50] of classical mod-p^r motivic cohomology $\mathbb{Z}(j)^{\mathrm{cla}}/p^r$ as $R\Gamma_{\mathrm{Zar}}(-, W_r\Omega^j_{\log})[-j]$. More precisely, we show that $\mathbb{Z}(j)^{\mathrm{mot}}/p^r$ can be obtained by glueing syntomic cohomology to cdh and éh cohomologies of $W_r\Omega^j_{\log}$:

Theorem 4.28. *For any qcqs $\mathbb{F}_p$-scheme X and $j, r \geq 0$, there is a natural pullback square in $\mathrm{D}(\mathbb{Z})$:*

$$\begin{tikzcd}
\mathbb{Z}(j)^{\mathrm{mot}}(X)/p^r \arrow[r] \arrow[d] & \mathbb{Z}_p(j)^{\mathrm{syn}}(X)/p^r \arrow[d] \\
R\Gamma_{\mathrm{cdh}}(X, W_r\Omega^j_{\log})[-j] \arrow[r] & R\Gamma_{\mathrm{eh}}(X, W_r\Omega^j_{\log})[-j]
\end{tikzcd}$$

Proof. More precisely, we will obtain the square as the mod-p^r reduction of the square of Theorem 4.24(2). The bottom left corner of the square is indeed $R\Gamma_{\mathrm{cdh}}(X, W_r\Omega^j_{\log})[-j]$ by Theorem 3.5(4), or rather by an analogue for mod-p^r rather than mod-p.

It remains to naturally identify $L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}/p^r$ with $R\Gamma_{\text{éh}}(-, W_r\Omega^j_{\log})[-j]$. Globalising Remark 4.15(2) defines a natural comparison map

$$\mathbb{Z}_p(j)^{\mathrm{syn}}/p^r \longrightarrow R\Gamma_{\text{ét}}(-, W_r\Omega^j_{\log})[-j]$$

which is an isomorphism on any valuation ring since they are Cartier smooth [65, §2] [84, §5.1]; cdh sheafifying therefore defines an equivalence $L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}/p^r \xrightarrow{\sim} L_{\mathrm{cdh}}R\Gamma_{\text{ét}}(-, W_r\Omega^j_{\log})[-j]$ of presheaves on $\mathrm{Sch}^{\mathrm{qcqs,op}}_{\mathbb{F}_p}$.

We finally appeal to Theorem A.3: noting that $X \mapsto R\Gamma_{\text{ét}}(-, W_r\Omega^j_{\log})$ is cohomologically bounded below, we may apply that result to deduce that $L_{\mathrm{cdh}}R\Gamma_{\text{ét}}(-, W_r\Omega^j_{\log})$ is a éh sheaf, and therefore the canonical map $L_{\mathrm{cdh}}R\Gamma_{\text{ét}}(-, W_r\Omega^j_{\log}) \to R\Gamma_{\text{éh}}(-, W_r\Omega^j_{\log})$ is an equivalence. That completes the proof. ☐

In practice we often use Theorem 4.28 in the form of a fibre sequence rather than a pullback square. To formulate the statement we need the following invariant:

Definition 4.29. The *mod-p^r, weight-j Artin–Schreier obstruction* of an $\mathbb{F}_p$-algebra A is the cokernel of the Artin–Schreier map from (4.4)

$$\widetilde{\nu}_r(j)(A) := \operatorname{coker}(C^{-1} - 1 : W_r\Omega^j_A \longrightarrow W_r\Omega^j_A/dV^{r-1}\Omega^j_A).$$

Given a topology τ on qcqs $\mathbb{F}_p$-schemes (notably Zariski, Nisnevich, or cdh), then we write $R\Gamma_\tau(X, \widetilde{\nu}_r(j))$ for the cohomology of the sheafification of $\widetilde{\nu}_r(j)$ in the topology τ, and similarly $H^i_\tau(X, \widetilde{\nu}_r(j))$ for the individual cohomology groups. We warn the reader that $\widetilde{\nu}_r(j)$ is not even a Zariski sheaf on affines, so that in general the map $\widetilde{\nu}_r(j)(A) \to H^0_{\mathrm{Zar}}(A, \widetilde{\nu}_r(j))$ is not an isomorphism.

Remark 4.30. Here are several alternative descriptions of the groups $\widetilde{\nu}_r(j)(A)$:

1. (Cohomological) Since $C^{-1} - 1$ is étale locally surjective and the sheaves $W_r\Omega^j$, $W_r\Omega^j/dV^{r-1}\Omega^{j-1}$ have no higher cohomology on affines, we see that there is a natural isomorphism
$$\widetilde{\nu}_r(j)(A) \cong H^1_{\text{ét}}(A, W_r\Omega^j_{\log}).$$
2. (Syntomic) There is a natural isomorphism
$$\widetilde{\nu}(j)(A) \cong H^{j+1}(\mathbb{Z}_p(j)^{\mathrm{syn}}(A)/p^r),$$
and moreover this is the top degree of $\mathbb{Z}_p(j)^{\mathrm{syn}}(A)/p^r$, by [5, Corol. 5.43]; more precisely, the comparison map of Remark 4.15(2) is an isomorphism in degrees $> j$.
3. (K-theoretic) For A local, there are natural isomorphisms
$$\widetilde{\nu}_r(j)(A) \cong \pi_{j-1}\mathrm{cofib}(\mathrm{K}^{\mathrm{cn}}(A)/p^r \to \mathrm{TC}(A)/p^r)$$
by [32, Thm. 6.11].

At least in the case in which $A = k$ is a field, these invariants have also appeared notably in work of Kato [63], denoted by $H^{j+1}_{p^r}(k)$, and are related to class field theory.

Remark 4.31 (Rigidity)**.** A key property of $\widetilde{\nu}_r(j)$ is its *rigidity*, namely whenever $R \to A$ is a Henselian surjection of $\mathbb{F}_p$-algebras, then the induced map $\widetilde{\nu}_r(j)(R) \to \widetilde{\nu}_r(j)(A)$ is an isomorphism. This can be deduced directly from the definition and Hensel's lemma [32].

For any qcqs $\mathbb{F}_p$-scheme X there is a natural map

$$\mathbb{Z}_p(j)^{\mathrm{syn}}(X)/p^r \longrightarrow R\Gamma_{\mathrm{cdh}}(X, \widetilde{\nu}_r(j))[-j-1], \tag{4.7}$$

defined by Zariski sheafifiying the following composition on affines:

$$\mathbb{Z}_p(j)^{\mathrm{syn}}(A)/p^r \xrightarrow{\text{Rem. 4.30(2)}} \widetilde{\nu}_r(j)(A)[-j-1] \xrightarrow{\text{can. map}} R\Gamma_{\mathrm{cdh}}(A, \widetilde{\nu}_r(j))[-j-1].$$

Our mod-p^r motivic cohomology identifies with the fibre of the map (4.7):

Corollary 4.32 (Fundamental fibre sequence in characteristic p)**.** *For any qcqs $\mathbb{F}_p$-scheme X and $j, r \geq 0$, there is a natural fibre sequence*

$$\mathbb{Z}(j)^{\mathrm{mot}}(X)/p^r \longrightarrow \mathbb{Z}_p(j)^{\mathrm{syn}}(X)/p^r \xrightarrow{(4.7)} R\Gamma_{\mathrm{cdh}}(X, \widetilde{\nu}_r(j))[-j-1].$$

Proof. In terms of the pullback square of Theorem 4.28, the map (4.7) is the dotted composition:

$$\begin{tikzcd}
\mathbb{Z}(j)^{\mathrm{mot}}/p^r \arrow[r] \arrow[d] & \mathbb{Z}_p(j)^{\mathrm{syn}}/p^r \arrow[d] \arrow[rrrd, dashed] & & & \\
R\Gamma_{\mathrm{cdh}}(-, W_r\Omega^j_{\log})[-j] \arrow[r] & R\Gamma_{\mathrm{eh}}(-, W_r\Omega^j_{\log})[-j] \arrow[r, phantom, "="] & L_{\mathrm{cdh}}R\Gamma_{\mathrm{et}}(-, W_r\Omega^j_{\log})[-j] \arrow[r] & (L_{\mathrm{cdh}}\tau^{\geq 1}R\Gamma_{\mathrm{et}}(-, W_r\Omega^j_{\log}))[-j] \arrow[r, phantom, "="] & R\Gamma_{\mathrm{cdh}}(-, \widetilde{\nu}_r(j))[-j-1
\end{tikzcd}$$

(the middle bottom equality having been explained at the end of the proof of Theorem 4.28). The claim to be proved is therefore that the bottom row is a fibre sequence; but this follows from exactness of cdh sheafification and the fibre sequence $W_r\Omega^j_{A,\log} \to R\Gamma_{\mathrm{et}}(A, W_r\Omega^j_{\log}) \to \widetilde{\nu}_r(j)(A)[-1]$ on affines. ☐

4.3 A Beilinson–Lichtenbaum equivalence

The classical Beilinson–Lichtenbaum conjecture states that motivic cohomology with finite coefficients is given by étale cohomology, in the range where cohomological degree is less than or equal to the weight. We refer to [56, §2] for a discussion of the conjecture in the smooth case and exactly how it relates to the other main conjectures, such as Bloch–Kato. Here we record that such a Beilinson–Lichtenbaum equivalence holds for our motivic cohomology, including at the characteristic (where the correct replacement for étale cohomology is syntomic cohomology):

Theorem 4.33. *Let X be a qcqs $\mathbb{F}$-scheme and $j \geq 0$.*

1. *For any integer $\ell > 0$ prime to the characteristic of $\mathbb{F}$, there is a natural map*

$$\mathbb{Z}(j)^{\mathrm{mot}}(X)/\ell \longrightarrow R\Gamma_{\text{ét}}(X, \mu_\ell^{\otimes j}),$$

whose cofibre is supported in degrees $> j$.

2. *If $\mathbb{F} = \mathbb{F}_p$ then for any $r \geq 0$ there is a natural map*

$$\mathbb{Z}(j)^{\mathrm{mot}}(X)/p^r \longrightarrow \mathbb{Z}_p(j)^{\mathrm{syn}}(X)/p^r,$$

whose cofibre is supported in degrees $> j$.

Proof. (1): Recall that $\mathbb{Z}(j)^{\mathrm{mot}}/\ell \to \mathbb{Z}(j)^{\mathrm{cdh}}/\ell$ is an equivalence, by Theorems 4.10(4) and 4.24(3), and the latter is given by $L_{\mathrm{cdh}}\tau^{\leq j}R\Gamma_{\text{ét}}(-, \mu_\ell^{\otimes j})$ by Theorem 3.5(3). It remains only to use that the fiber of the canonical map $L_{\mathrm{cdh}}\tau^{\leq j}R\Gamma_{\text{ét}}(-, \mu_\ell^{\otimes j}) \to R\Gamma_{\acute{e}t}(-, \mu_\ell^{\otimes j})$ is supported in degrees $> j$, since $R\Gamma_{\acute{e}t}(-, \mu_\ell^{\otimes j})$ satisfies cdh descent as recalled in Example 4.26.

(2): This is clear from Corollary 4.32. □

4.4 Comparison maps

In this subsection we explicitly record the canonical comparison maps between the classical motivic cohomology of §3.1, the lisse motivic cohomology of §3.2, the cdh-local motivic cohomology of §3.3 (equivalently, the motivic cohomology of $\mathbb{A}^1$-invariant motivic homotopy theory, as discussed in Remark 3.6), and our new motivic cohomology. These comparisons are induced by various filtered maps between K-theory, KH-theory, and connective K-theory.

Construction 4.34 (Classical vs new motivic cohomology of smooth varieties)**.** We claim, for any smooth $\mathbb{F}$-scheme X, that there is a natural comparison map of filtered spectra

$$\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X) \tag{4.8}$$

given on Fil^0 by $\mathrm{Fil}^0_{\mathrm{cla}}\mathrm{K}(X) = \mathrm{K}(X) = \mathrm{Fil}^0_{\mathrm{mot}}\mathrm{K}(X)$. On shifted graded pieces this induces natural maps

$$\mathbb{Z}(j)^{\mathrm{cla}}(X) \longrightarrow \mathbb{Z}(j)^{\mathrm{mot}}(X) \tag{4.9}$$

for $j \geq 0$.

We define (4.8) as follows. When $\mathbb{F} = \mathbb{Q}$ (resp. $\mathbb{F}_p$), the filtered cdh-local trace map of Corollary 4.8 (resp. Propositoin 4.22) was designed to fit into a commutative diagram

$$\begin{tikzcd}
\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X) \arrow[r] \arrow[d] & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X/\mathbb{Q}) \arrow[d] \\
\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X) \arrow[r] & \mathrm{Fil}^\star_{\mathrm{HKR}}L_{\mathrm{cdh}}\mathrm{HC}^-(X/\mathbb{Q}).
\end{tikzcd}
\quad \text{resp.} \quad
\begin{tikzcd}
\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X) \arrow[r] \arrow[d] & \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X) \arrow[d] \\
\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X) \arrow[r] & \mathrm{Fil}^\star_{\mathrm{BMS}}L_{\mathrm{cdh}}\mathrm{TC}(X),
\end{tikzcd}$$

where the top horizontal arrow is the filtered trace map of Proposition 4.6 (resp. 4.21) for the smooth $\mathbb{F}$-scheme X. From the pullback Definition 4.9 (resp. 4.23) of $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$, there is therefore a natural induced map (4.8) as desired.

In Corollary 6.4 we will prove that (4.8) is an equivalence for every smooth $\mathbb{F}$-scheme X.

Construction 4.35 (LKE vs new motivic cohomology of affines)**.** Restricting (4.8) to smooth $\mathbb{F}$-algebras and then left Kan extending to all $\mathbb{F}$-algebras defines, for any $\mathbb{F}$-algebra A, a natural comparison map of filtered spectra

$$\mathrm{Fil}^\star_{\mathrm{lse}}\mathrm{K}^{\mathrm{cn}}(A) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(A) \tag{4.10}$$

given on Fil^0 by the canonical map $\mathrm{Fil}^0_{\mathrm{lse}}\mathrm{K}^{\mathrm{cn}}(A) = \mathrm{K}^{\mathrm{cn}}(A) \to \mathrm{K}(A) = \mathrm{Fil}^0_{\mathrm{mot}}\mathrm{K}(A)$. On shifted graded pieces this induces natural comparison maps

$$\mathbb{Z}(j)^{\mathrm{lse}}(A) \longrightarrow \mathbb{Z}(j)^{\mathrm{mot}}(A)$$

for $j \geq 0$. We will study these comparison maps in detail in Section 7.

Construction 4.36 (New vs cdh-local motivic cohomology)**.** Tautologically from the pullback definition of $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$, there is a natural comparison map of filtered spectra

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X) \longrightarrow \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)$$

for any qcqs $\mathbb{F}$-scheme X, given on Fil^0 by the canonical map $\mathrm{Fil}^0_{\mathrm{mot}}\mathrm{K}(X) = \mathrm{K}(X) \to \mathrm{KH}(X) = \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)$. On shifted graded pieces this induces the natural comparison maps $\mathbb{Z}(j)^{\mathrm{mot}}(X) \to \mathbb{Z}(j)^{\mathrm{cdh}}(X)$, for $j \geq 0$, which have already appeared in Theorems 4.10(2) and 4.24(2). We will study these maps further in Section 6.

Remark 4.37 ((4.8) is split)**.** While the various comparison maps are displayed, we point out the following: for any smooth $\mathbb{F}$-scheme X and $j \geq 0$, our joint work with Bachmann proves that the composition

$$\mathbb{Z}(j)^{\mathrm{cla}}(X) \xrightarrow{\text{Cons. 4.35}} \mathbb{Z}(j)^{\mathrm{mot}}(X) \xrightarrow{\text{Cons. 4.36}} \mathbb{Z}(j)^{\mathrm{cdh}}(X)$$

is an equivalence. (We will see in Corollary 6.4 that in fact each map is an equivalence, but the result about the composition will be used in the proof.) Indeed, this is the canonical map obtained by left Kan extending $\mathbb{Z}(j)^{\mathrm{cla}}$ to all qcqs schemes, then cdh sheafifying, then restricting back to smooth $\mathbb{F}$-schemes (see (3.5)); it is an equivalence by the special case $k = \mathbb{F}$ of Theorem 3.5(8).

4.5 Extension to derived schemes

We finish this section by briefly explaining that our motivic cohomology extends to derived schemes, though we do not require the theory in such generality in the present article.

We write $\mathrm{CAlg}^{\mathrm{ani}}_{\mathbb{F}}$ for the ∞-category of *animated* $\mathbb{F}$*-algebras*, i.e., the subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\Sigma}_{\mathbb{F}}, \mathcal{S}\mathrm{pc})$ which preserves finite products, where $\mathrm{CAlg}^{\Sigma}_{\mathbb{F}}$ denotes the category of finitely generated polynomial $\mathbb{F}$-algebras. Animated $\mathbb{F}$-algebras are derived affine schemes, out of which we build the ∞-category of *derived* $\mathbb{F}$*-schemes* $\mathrm{dSch}_{\mathrm{k}}$; see [86] for more details.

Construction 4.38. Let $\mathbb{F}$ be a prime field and X a qcqs derived $\mathbb{F}$-scheme. Note that the HKR filtration of Theorem 4.2 and the BMS filtration of Theorem 4.18 extend to the generality of derived $\mathbb{F}$-schemes. Indeed, in the case of the HKR filtration the references [4, 98, 92] work in this degree of generality; for the BMS filtration one p-completely left Kan extends the filtration from discrete algebras, as in [5, Cons. 5.33]

By naturality of these filtrations, there are natural comparison maps of filtered spectra $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X/\mathbb{Q}) \to \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X^{\mathrm{cla}}/\mathbb{Q})$ if $\mathbb{F} = \mathbb{Q}$, and $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X) \to \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X^{\mathrm{cla}})$ and if $\mathbb{F} = \mathbb{F}_p$, where $X^{\mathrm{cla}} \hookrightarrow X$ is the classical locus of X.

The motivic filtration on the K-theory of X is then defined by the following cartesian square

$$\begin{tikzcd}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X) \arrow[r, dotted] \arrow[d, dotted] & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X) \arrow[d] & & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X) \arrow[r, dotted] \arrow[d, dotted] & \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X) \arrow[d] \\
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X^{\mathrm{cla}}) \arrow[r] & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X^{\mathrm{cla}}), & \text{resp.} & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X^{\mathrm{cla}}) \arrow[r] & \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X^{\mathrm{cla}}),
\end{tikzcd}$$

(the first if $\mathbb{F} = \mathbb{Q}$; the second if $\mathbb{F} = \mathbb{F}_p$). In both cases the $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X^{\mathrm{cla}})$ refers to the motivic filtration which we have defined earlier on the K-theory of the classical qcqs $\mathbb{F}$-scheme X^{cla}.

The *weight-j motivic cohomology* of X is then defined to be

$$\mathbb{Z}(j)^{\mathrm{mot}}(X) := \mathrm{gr}^j_{\mathrm{mot}}\mathrm{K}(X)[-2j].$$

It is not our intention to present here an exhaustive account of the motivic filtration on derived schemes; we content ourselves with stating the following summary of the main properties:

Theorem 4.39 (Motivic filtration for derived schemes)**.** *For any qcqs derived $\mathbb{F}$-scheme, there exists a natural $\mathbb{N}$-indexed, multiplicative filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ on $K(X)$ satisfying $\mathrm{Fil}^0_{\mathrm{mot}}\mathrm{K}(X) \simeq \mathrm{K}(X)$. If X is classical then this filtration agrees with the earlier motivic filtration of Definitions 4.9 and 4.23.*

Proof. We just explain the claim that $\mathrm{Fil}^0_{\mathrm{mot}}K(X) = K(X)$, the other statements being clear. By Zariski descent for derived $\mathbb{F}$-schemes, it suffices to prove the result for $X = \mathrm{Spec}(A)$ when A is an animated $\mathbb{F}$-algebra. In this case, the result follows from the same arguments as in the classical case and part of the Dundas–Goodwillie–McCarthy theorem [37], stating that for a simplicial ring A the square of spectra

$$\begin{tikzcd}
\mathrm{K}(A) \arrow[r] \arrow[d] & \mathrm{TC}(A) \arrow[d] \\
\mathrm{K}(\pi_0 A) \arrow[r] & \mathrm{TC}(\pi_0 A)
\end{tikzcd}$$

is cartesian. □

5 THE PROJECTIVE BUNDLE FORMULA AND REGULAR BLOWUP SQUARES

Let k be a field. The following are three important properties concerning classical motivic cohomology of smooth k-schemes:

1. ($\mathbb{A}^1$-invariance) If X is a smooth k-scheme, then the projection map induces an equivalence $\mathbb{Z}(j)^{\mathrm{cla}}(X) \xrightarrow{\simeq} \mathbb{Z}(j)^{\mathrm{cla}}(X \times \mathbb{A}^1)$.
2. ($\mathbb{P}^1$-bundle formula) To each line bundle $\mathcal{L}$ on X, there is natural class $c_1(\mathcal{L}) \in H^2_{\mathrm{cla}}(X, \mathbb{Z}(1))$ which induces, using the multiplicative structure on motivic cohomology, an equivalence for $j \geq 1$:
$$\mathbb{Z}(j)(X)^{\mathrm{cla}} \oplus \mathbb{Z}(j-1)(X)^{\mathrm{cla}}[-2] \xrightarrow{\pi^* \oplus c_1(\mathcal{O}(1))\pi^*} \mathbb{Z}(j)^{\mathrm{cla}}(\mathbb{P}^1_X);$$
3. (Regular blowup formula) Given a closed immersion of smooth k-schemes $Y \to X$ of codimension $c \geq 2$, and letting $\mathrm{Bl_Y X} \to \mathrm{X}$ denote the corresponding blowup, there is a natural equivalence
$$\mathbb{Z}(j)^{\mathrm{cla}}(\mathrm{Bl}_Y X) \simeq \mathbb{Z}(j)^{\mathrm{cla}}(X') \oplus \left(\bigoplus_{1 \leq i \leq c-1} \mathbb{Z}(j-i)(Y)^{\mathrm{cla}}[-2i] \right).$$

Since algebraic K-theory is not in general $\mathbb{A}^1$-invariant the analog of property (1) fails for our motivic cohomology. On the other hand, the analogues of the other two properties do hold for algebraic K-theory, being a direct consequence of additivity. In this section we refine that result by showing that (2) continues to hold for our motivic cohomology, even for non-smooth schemes. Similarly, we also extend property (3) to our motivic cohomology, in the context of blowups along regular immersions of possibly non-smooth schemes; the resulting formula, however, takes the shape of a cartesian square as opposed to a splitting.

Property (2) fits within recent developments in the theory of non-$\mathbb{A}^1$-invariant motives as developed by [1, 3]. In this theory, the projective bundle formula is isolated as the key property of cohomology theories, in lieu of $\mathbb{A}^1$-invariance. In particular, our results imply that the presheaves $\mathbb{Z}(j)^{\mathrm{mot}}$, for $j \geq 0$, assemble into a motivic spectrum in the sense of [3].

5.1 First Chern classes and $\mathbb{P}^1$-bundle formulae

As usual $\mathbb{F}$ denotes a prime field. To formulate the $\mathbb{P}^1$-bundle formula we need the first Chern class:

Lemma 5.1. *There exists a unique map of presheaves on $\mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}$*

$$R\Gamma_{\mathrm{Nis}}(-, \mathbb{G}_m)[-1] \longrightarrow \mathbb{Z}(1)^{\mathrm{mot}}$$

which is given on smooth $\mathbb{F}$-schemes by the $j = 1$ case of (4.9) above (recall that $\mathbb{Z}(1)^{\mathrm{cla}} \simeq R\Gamma_{\mathrm{Nis}}(-, \mathbb{G}_m)[-1]$ on smooth $\mathbb{F}$-schemes).

Proof. As is explained in (4.9), for any smooth $\mathbb{F}$-algebra R, we have a comparison map which takes the form of a natural map $\mathbb{Z}(1)^{\mathrm{cla}}(R) \simeq R\Gamma_{\mathrm{Zar}}(R, \mathbb{G}_m)[-1] \to \mathbb{Z}(1)^{\mathrm{mot}}(R)$; the left Kan extension of the left side to all $\mathbb{F}$-algebras is the functor $(\tau^{\leq 1} R\Gamma_{\mathrm{Zar}}(-, \mathbb{G}_m))[-1]$, so in this way we obtain a functor $(\tau^{\leq 1} R\Gamma_{\mathrm{Zar}}(-, \mathbb{G}_m))[-1] \to \mathbb{Z}(1)^{\mathrm{mot}}$ on affine $\mathbb{F}$-schemes. Nisnevich sheafifying then defines the desired map. Uniqueness follows from the construction. □

Definition 5.2. For any qcqs $\mathbb{F}$-scheme X, we refer to the map of the previous lemma

$$c_1 : R\Gamma_{\mathrm{Nis}}(X, \mathbb{G}_m)[-1] \longrightarrow \mathbb{Z}(1)^{\mathrm{mot}}(X) \tag{5.1}$$

as the *first Chern class*. We will often refer to the induced natural map on H^2, namely

$$c_1 : \mathrm{Pic}(X) \longrightarrow H^2_{\mathrm{mot}}(X, \mathbb{Z}(1)),$$

in the same way.

We now formulate the $\mathbb{P}^1$-bundle formula for any $X \in \mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}$. Thanks to the multiplicative structure of $\bigoplus_{j\geq 0} \mathbb{Z}(j)^{\mathrm{mot}}[2j]$, multiplication by the first Chern class of $\mathcal{O}(1) \in \mathrm{Pic}(\mathbb{P}^1_X)$ defines maps

$$\mathbb{Z}(j)^{\mathrm{mot}}(\mathbb{P}^1_X) \xrightarrow{c_1(\mathcal{O}(1))} \mathbb{Z}(j+1)^{\mathrm{mot}}(\mathbb{P}^1_X)[2]$$

for $j \geq 0$. Denoting by $\pi : \mathbb{P}^1_X \to X$ the canonical projection, we thus have natural maps

$$\mathbb{Z}(j)^{\mathrm{mot}}(X) \oplus \mathbb{Z}(j-1)^{\mathrm{mot}}(X)[-2] \xrightarrow{\pi^* \oplus c_1(\mathcal{O}(1))\pi^*} \mathbb{Z}(j)^{\mathrm{mot}}(\mathbb{P}^1_X). \tag{5.2}$$

for $j \geq 1$. We can also extend (5.2) to $j \leq 0$ by setting $\mathbb{Z}(j)^{\mathrm{mot}} = 0$ for $j < 0$ and so the $c_1(\mathcal{O}(1))\pi^*$ map has domain 0 if $j \leq 0$.

Our goal in this section is to prove the following $\mathbb{P}^1$-bundle formula for motivic cohomology:

Theorem 5.3 ($\mathbb{P}^1$-bundle formula)**.** *For any qcqs $\mathbb{F}$-scheme X and $j \geq 0$, the map* (5.2) *is an equivalence.*

Remark 5.4. Theorem 5.3 establishes a computation of motivic cohomology relative to $\mathbb{P}^1$. We will note, in Theorem 5.24, that this promotes quite automatically to a computation relative to $\mathbb{P}^n$ (or even for $\mathbb{P}(\mathcal{E})$ for any locally free sheaf $\mathcal{E}$) for any $n \geq 1$. The key input is an argument from [3] which allows one to pass from a $\mathbb{P}^1$-bundle formula to a general projective bundle formula using "elementary blowup excision."

In this remainder of this subsection we prove Theorem 5.3 in characteristic zero, as well as a large part in characteristic p. An input which we will use is that the projective bundle formula always holds for additive invariants, as we explain in the next construction and lemma.

Construction 5.5 (Projective bundle formula for additive invariants)**.** Let k be a commutative ring[9] and E be a k-linear additive invariant[10] in the sense of [58, §4], which extends the absolute theory in [27]. Of special interest is the invariant $E = K_{\mathrm{conn}}$. It is corepresented by the unit object in the ∞-category of k-linear additive invariants [58, Thm. 5.24]. Out of this we draw two consequences: first, it acquires a canonical lax monoidal structure and thus its restriction along functor $\mathrm{Perf} : \mathrm{Sch}^{\mathrm{qcqs,op}}_k \to \mathcal{C}\mathrm{at}^k_\infty$ upgrades to a functor into $\mathbb{E}_\infty$-ring spectra:

$$K_{\mathrm{conn}} \circ \mathrm{Perf} : \mathrm{Sch}^{\mathrm{qcqs,op}}_k \to \mathrm{CAlg}(\mathrm{Sp}).$$

Secondly, any localizing invariant E is canonically a K_{conn}-module[11] and thus, in the symmetric monoidal ∞-category of presheaves on $\mathrm{Sch}^{\mathrm{qcqs,op}}_k$, $E \circ \mathrm{Perf}$ is a $K_{\mathrm{conn}} \circ \mathrm{Perf}$-module. From hereon, we suppress the precomposition with Perf whenever the context is clear.

---

[9] For what follows, we can actually work in the generality of k a $\mathbb{E}_\infty$-ring for an appropriate definition of $\mathbb{P}^1$; we refer to [31] for details.

[10] In this paper, we do not assume that E is finitary.

[11] Let us be more precise about this. Connective K-theory defines an additive functor from $\mathcal{C}\mathrm{at}^k_\infty \to \mathrm{Sp}$; the collection of such functors assemble into a symmetric monoidal ∞-category $\mathrm{Fun}_{\mathrm{add}}(\mathcal{C}\mathrm{at}^k_\infty, \mathrm{Sp})$ under Day convolution as explained in [58, pg. 139]. As explained in [58, 5.4], connective K-theory is the unit object of this symmetric monoidal ∞-category and therefore, any other object E acquires a module structure over it. Appealing to Glasman's work identifying lax monoidal functors with commutative monoid objects in functor categories under the Day convolution symmetric monoidal structures [52] lets us translate this into the action of K-theory, as a lax monoidal functor, on E. We note that the absolute version of these monoidal enhancements of the universal property of K-theory is first proved in [28, Thm 5.14].

Now, we have a morphism of presheaves of $\mathbb{E}_\infty$-monoids on $\mathrm{Sch}_k^{\mathrm{qcqs}}$

$$c_1 : \mathcal{P}\mathrm{ic} \to \Omega^\infty K_{\mathrm{conn}}.$$

We call this the *first chern class.* By adjunction $\Sigma^\infty_+ \dashv \Omega^\infty$ there is then an induced map of presheaves of $\mathbb{E}_\infty$-rings, i.e.,

$$c_1 : \Sigma^\infty_+ \mathcal{P}\mathrm{ic} \to K_{\mathrm{conn}}.$$

which we also denote by c_1. This construction (rather, a sheared variant) is discussed more extensively in Appendix B.1. In particular we have morphisms, functorial in $X \in \mathrm{Sch}_k^{\mathrm{qcqs}}$, given by

$$K_{\mathrm{conn}}(X) \oplus K_{\mathrm{conn}}(X) \xrightarrow{\pi^* \oplus c_1(\mathcal{O}(-1))\pi^*} K_{\mathrm{conn}}(\mathbb{P}^1_X), \tag{5.3}$$

More generally, using the K_{conn}-module structure on E, we have functorial morphisms in $X \in \mathrm{Sch}_k^{\mathrm{qcqs}}$, given by

$$E(X) \oplus E(X) \xrightarrow{\pi^* \oplus c_1(\mathcal{O}(-1))\pi^*} E(\mathbb{P}^1_X). \tag{5.4}$$

Lemma 5.6. *For any additive invariant E as in Construction 5.5 and any qcqs $\mathbb{F}$-scheme X, the morphism* (5.4) *is an equivalence.*

Proof. This follows from the semiorthogonal decomposition of $\mathrm{Perf}(\mathbb{P}^1)$; see, for example, [31, Thm. 4.25] for a more general result. ☐

We provide a nontrivial example of E as above by observing that $L_{\mathrm{cdh}}\mathrm{TC}$ extends to an additive invariant.

Lemma 5.7. *Let k be a commutative ring, then $L_{\mathrm{cdh}}\mathrm{TC}$ extends to an additive invariant on schemes. That is, there exists an additive k-linear invariant*

$$L_{\mathrm{cdh}}\mathrm{TC} : \mathcal{C}\mathrm{at}^k_\infty \to \mathit{Sp};$$

such that we have a natural equivalence $(L_{\mathrm{cdh}}\mathrm{TC} \circ \mathrm{Perf})(X) \simeq L_{\mathrm{cdh}}\mathrm{TC}(X)$ for any qcqs k-scheme X.

Proof. First, we remark that KH extends to k-linear categories by the procedure in [76, Def. 3.13] which is stated for $\mathbb{Z}$ but works for more general commutative rings. We define $L_{\mathrm{cdh}}\mathrm{TC}$ by taking the following pushout in ∞-category of functors $\mathcal{C}\mathrm{at}^k_\infty \to \mathrm{Sp}$:

$$\begin{tikzcd} \mathrm{K} \arrow[r] \arrow[d] & \mathrm{TC} \arrow[d, dotted] \\ \mathrm{KH} \arrow[r, dotted] & L_{\mathrm{cdh}}\mathrm{TC}. \end{tikzcd} \tag{5.5}$$

Since the formation of additive invariants is stable under finite colimits, $L_{\mathrm{cdh}}\mathrm{TC}$ is automatically an additive invariant. Now, by Theorem 3.8, and the fact that K, TC and KH of schemes are constructed by precomposing these additive invariants with Perf, we conclude the result. ☐

Combining Lemmas 5.7 and Lemma 5.6, we obtain the following non-obvious projective bundle formulas:

Corollary 5.8. *If $\mathbb{F} = \mathbb{Q}$, the $\mathrm{K}_{\mathrm{conn}}$-module structure on $L_{\mathrm{cdh}}\mathrm{HC}^-(-/\mathbb{Q})$ induces natural equivalences for any $X \in \mathrm{Sch}_{\mathbb{Q}}^{\mathrm{qcqs}}$:*

$$L_{\mathrm{cdh}}\mathrm{HC}^-(X/\mathbb{Q}) \bigoplus L_{\mathrm{cdh}}\mathrm{HC}^-(X/\mathbb{Q}) \xrightarrow{\pi^* \oplus c_1(\mathcal{O}(-1))\pi^*} L_{\mathrm{cdh}}\mathrm{HC}^-(X/\mathbb{Q}).$$

If $\mathbb{F} = \mathbb{F}_p$, the $\mathrm{K}_{\mathrm{conn}}$-module structure on $L_{\mathrm{cdh}}\mathrm{TC}$ induces natural equivalences for any $X \in \mathrm{Sch}_{\mathbb{Q}}^{\mathrm{qcqs}}$:

$$L_{\mathrm{cdh}}\mathrm{TC}(X) \bigoplus L_{\mathrm{cdh}}\mathrm{TC}(X) \xrightarrow{\pi^* \oplus c_1(\mathcal{O}(-1))\pi^*} L_{\mathrm{cdh}}\mathrm{TC}(X).$$

Remark 5.9. The pushout in (5.5) has no reason to promote to a pushout of lax monoidal functors. In particular, from its construction, $L_{\mathrm{cdh}}\mathrm{TC}(X)$, for X a qcqs scheme, need not acquire an $\mathbb{E}_\infty$-ring structure which is compatible with the map K-theory, by purely formal consideration. Nonetheless, such a structure *does exist* because L_{cdh} preserves multiplicative structures. With this remark, we freely use that $L_{\mathrm{cdh}}\mathrm{TC}$ is a presheaf on qcqs schemes,valued in $\mathbb{E}_\infty$-rings, compatible with the maps from K, TC and KH.

A second input into proving Theorem 5.3 is that (5.4) is compatible with filtrations in our cases of interest, at least up to a shearing of the map; this is explained in the next construction and lemma.

Lemma 5.10. *Let $\mathbb{F}$ be a prime field. Then the map of spaces $1 - c_1 : \mathcal{P}\mathrm{ic} \to \Omega^\infty \mathrm{K}$ factors canonically as:*

$$\mathcal{P}\mathrm{ic} \to \Omega^\infty \mathrm{Fil}^{\geq 1}_{\mathrm{lse}}\mathrm{K}_{\mathrm{conn}} \to \Omega^\infty \mathrm{K}_{\mathrm{conn}} \xrightarrow{\simeq} \Omega^\infty \mathrm{K}.$$

Proof. It suffices to prove the claim when restricted to $\mathrm{Sm}_{\mathbb{F}}$, i.e., that the map $\mathcal{P}\mathrm{ic} \to \Omega^\infty \mathrm{K}$ factors through $\Omega^\infty \mathrm{Fil}^{\geq \star}_{\mathrm{cla}}\mathrm{K}$ on $\mathrm{Sm}_{\mathbb{F}}$. In this case, the map $\Omega^\infty \mathrm{K} \to \Omega^\infty \mathrm{gr}^0_{\mathrm{cla}}\mathrm{K} \simeq \mathbb{Z}$ coincides with the rank map, therefore the composite map of presheaves $1 - c_1 : \mathcal{P}\mathrm{ic} \to \Omega^\infty \mathrm{gr}^0_{\mathrm{cla}}\mathrm{K}$ is contractible and thus factors through the fiber, i.e., the presheaf $\Omega^\infty \mathrm{Fil}^{\geq 1}_{\mathrm{lse}}\mathrm{K}_{\mathrm{conn}}$. □

We now construct a filtered refinement of the map (5.4) (up to a shearing), depending only on a multiplicative, filtered refinement of the map $\mathrm{K}_{\mathrm{conn}} \to E$:

Construction 5.11. Let E be a presheaf of spectra on $\mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}$, which comes equipped with a filtration $\mathrm{Fil}^\star E \to E$. Suppose further that E is a K_{conn}-module and that the filtration on E promotes to the structure of a filtered $\mathrm{Fil}^\star_{\mathrm{lse}}\mathrm{K}_{\mathrm{conn}}$-module. Because of Lemma 5.10, we have a morphism

$$\Sigma^\infty_+ \mathcal{P}\mathrm{ic} \otimes \mathrm{Fil}^\star E \xrightarrow{(1-c_1)\otimes \mathrm{id}} \mathrm{Fil}^1_{\mathrm{lse}}\mathrm{K}_{\mathrm{conn}} \otimes \mathrm{Fil}^\star E \to \mathrm{Fil}^{\star+1} E$$

where the second map uses the aforementioned module structure. (To clarify: the previous line is a morphism of presheaves of filtered spectra, where $\Sigma^\infty_+ \mathcal{P}\mathrm{ic}$ and $\mathrm{Fil}^1_{\mathrm{lse}} K^{\mathrm{cn}}$ are given constant filtrations.) Using this structure we have, functorially in X, a morphism "multiplication by $(1 - c_1)(\mathcal{O}(-1))$"

$$\mathrm{Fil}^\star E(\mathbb{P}^1_X) \xrightarrow{(1-c_1)(\mathcal{O}(-1))} \mathrm{Fil}^{\star+1} E(\mathbb{P}^1_X).$$

This shows that the composite:

$$E(X) \oplus E(X) \xrightarrow{\gamma} E(X) \oplus E(X) \xrightarrow{\pi^* \oplus c_1(\mathcal{O}(-1))\pi^*} E(\mathbb{P}^1_X),$$

where γ is the invertible matrix $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, and the second map is the induced by the multiplicative map $K_{\mathrm{conn}} \to E$, is refined by a filtered map

$$\mathrm{Fil}^\star E(X) \oplus \mathrm{Fil}^{\star-1} E(X) \xrightarrow{\pi^* \oplus (1-c_1)(\mathcal{O}(-1))\pi^*} \mathrm{Fil}^\star E(\mathbb{P}^1_X). \tag{5.6}$$

In turn, (5.6) then induces maps on graded pieces

$$\mathrm{gr}^j E(X) \oplus \mathrm{gr}^{j-1} E(X) \xrightarrow{\pi^* \oplus (1-c_1)(\mathcal{O}(-1))\pi^*} \mathrm{gr}^j E(\mathbb{P}^1_X) \tag{5.7}$$

for $j \geq 0$; in our cases of interest these essentially coincide with (5.2):

We next claim that (5.4) promotes to a filtered map in our cases of interest. The starting point is the following observation:

Lemma 5.12. *Supposing in Construction 5.11 that $E = \mathrm{K}$ equipped with its motivic filtration (i.e. Definition 4.9 if $\mathbb{F} = \mathbb{Q}$, resp. Definition 4.23 if $\mathbb{F} = \mathbb{F}_p$). Then, for any $j \geq 1$, the map* (5.7) *is homotopic to the map* (5.2) *up to a shift by 2j.*

Proof. We start with recalling the following observation about classical filtrations on smooth $\mathbb{F}$-schemes. In weight 1, the second summand of (5.7) for $E = K_{\mathrm{conn}}$, i.e.,

$$\mathrm{gr}^0_{\mathrm{cla}}\mathrm{K}(X) = \mathbb{Z}(0)^{\mathrm{mot}}(X) \xrightarrow{(1-c_1)(\mathcal{O}(-1))\pi^*} \mathrm{gr}^1_{\mathrm{cla}}\mathrm{K}(\mathbb{P}^1_X) = \mathbb{Z}(1)^{\mathrm{mot}}(\mathbb{P}^1_X)[2] \tag{5.8}$$

is homotopic to the map $-c_1(\mathcal{O}(-1))\pi^\star \simeq c_1(\mathcal{O}(1))\pi^*$. Indeed, the map classifying the element 1 is nullhomotopic on $\mathrm{gr}^1_{\mathrm{mot}}$ in classical motivic cohomology. Still restricted on smooth schemes, since the new motivic filtration is a filtered $\mathbb{E}_\infty$-algebra under the classical filtration by Construction 4.35, the same fact is true the new motivic filtration. Furthermore, on classical motivic cohomology, the map (5.8) is equivalent to the map induced by the first chern class on classical motivic cohomology, i.e., the one which induces the equivalence $R\Gamma_{\mathrm{Nis}}(-; \mathbb{G}_m)[-1] \xrightarrow{\simeq} \mathbb{Z}(1)^{\mathrm{cla}}$. Therefore, by the construction in Lemma 5.1, the map on the new motivic cohomology for a smooth X:

$$\mathrm{gr}^0_{\mathrm{mot}}\mathrm{K}(X) = \mathbb{Z}(0)^{\mathrm{mot}}(X) \xrightarrow{(1-c_1)(\mathcal{O}(-1))\pi^*} \mathrm{gr}^1_{\mathrm{mot}}\mathrm{K}(\mathbb{P}^1_X) = \mathbb{Z}(1)^{\mathrm{mot}}(\mathbb{P}^1_X)[2],$$

is homotopic to $c_1(\mathcal{O}(1))\pi^*$. By the uniqueness assertion of Lemma 5.1, the claim follows. □

Remark 5.13. For any theory $E(\star)$ admitting a map from motivic cohomology (e.g. $E(\star) = R\Gamma(-, \widehat{L\Omega}^{\geq\star}_{-/\mathbb{Q}})$, $\mathbb{Z}_p(\star)^{\mathrm{syn}}$, and their cdh analogs), precomposing the map from motivic cohomology to $E(\star)$ with the map from Lemma 5.1 gives a first chern class map

$$c_1 : R\Gamma_{\mathrm{Nis}}(-;\mathbb{G}_m)[-1] \to E(1).$$

Since these invariants are modules over the graded ring $\mathbb{Z}(\star)^{\mathrm{mot}}$, we can define the “multiplication by $c_1(\mathcal{O}(1))$" maps:

$$E(j-1)[-2](X) \xrightarrow{c_1(\mathcal{O}(1))\pi^*} E(j)(\mathbb{P}^1_X) \qquad j \in \mathbb{Z}.$$

One can check easily that these coincide with other *a priori* constructions of the first chern classes in the literature for these theories. Since the map from K-theory to theories “downstream" of it (e.g. $\mathrm{TC}, \mathrm{HC}^-(-/\mathbb{Q})$ and their cdh analogs) are multiplicatively compatible on a filtered level, Lemma 5.12 proves that the maps

$$E(j)(X) \oplus E(j-1)[-2] \xrightarrow{\pi^*\oplus c_1(\mathcal{O}(1))\pi^*} E(j)(\mathbb{P}^1_X).$$

are compatible with the ones induced by taking associated graded pieces of the respective filtrations.

We may now prove a large part of Theorem 5.3:

Beginning of proof of Theorem 5.3. We first treat the case $\mathbb{F} = \mathbb{Q}$. By Theorem 4.10(2), it suffices to prove the projective bundle formulae for the theories

$$R\Gamma(-, \widehat{L\Omega}^{\geq j}_{-/\mathbb{Q}}), \qquad R\Gamma_{\mathrm{cdh}}(-, \widehat{L\Omega}^{\geq j}_{-/\mathbb{Q}}), \qquad \mathbb{Z}(j)^{\mathrm{cdh}}.$$

The result for cdh motivic cohomology is in [7] but can be deduced from the smooth case, i.e. the projective bundle formula for $\mathbb{Z}(j)^{\mathrm{cla}}$, by resolution of singularities. The results for derived de Rham and its cdh analog follows from the projective bundle formulae for $\mathrm{HC}^-(-/\mathbb{Q})$ (which is an additive invariant) and $L_{\mathrm{cdh}}\mathrm{HC}^-(-/\mathbb{Q})$ respectively (which follows from an analogous argument as in Lemma 5.7) and the rational degeneration result of Theorem 4.2.

Next, assume that $\mathbb{F} = \mathbb{F}_p$. Then, Theorem 4.24(3) reduces the result for $\mathbb{Z}[\frac{1}{p}]^{\mathrm{mot}}(-)$ to the one for cdh motivic cohomology proved in [7], which does not use resolution of singularities. $\square$

5.2 $\mathbb{P}^1$-bundle formula for cdh sheafified syntomic cohomology

At this juncture, we need only prove Theorem 5.3 in characteristic $p > 0$ after p-adic completion. The key technical input to achieve this is the analogous statement for $L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}$ in characteristic $p > 0$. We note that even though $\mathbb{Z}_p(j)^{\mathrm{syn}}$ has the $\mathbb{P}^1$-bundle formula (see [18, Thm. 9.1.1]) there is no reason, in general, for its cdh sheafification to have the $\mathbb{P}^1$-bundle formula. The technique used to establish this borrows heavily from the ideas in [7].

For this section, we fix $\mathbb{F} = \mathbb{F}_p$. As in Remark 5.13, composing with the natural maps $\mathbb{Z}(j)^{\mathrm{mot}} \to \mathbb{Z}_p(j)^{\mathrm{syn}} \to L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}$, we obtain analogs of the maps (5.2) for $L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}$, multiplicatively compatible with the original c_1 on $\mathbb{Z}(j)^{\mathrm{mot}}$. Our next goal is to prove the following result:

Theorem 5.14. *The maps*

$$L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}(X) \oplus L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}(X)[-2] \xrightarrow{\pi^*\oplus c_1(\mathcal{O}(1))\pi^*} L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}(\mathbb{P}^1_X) \qquad j \geq 0, \tag{5.9}$$

are equivalences for any $X \in \mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}_p}$.

In other words, we want to promote the TC part of Corollary 5.8 to the filtered level. As noted in Remark 5.13, the invertible morphism

$$L_{\mathrm{cdh}}\mathrm{TC}(-) \bigoplus L_{\mathrm{cdh}}\mathrm{TC}(-) \xrightarrow{\pi^*\oplus(1-c_1)(\mathcal{O}(-1))\pi^*} L_{\mathrm{cdh}}\mathrm{TC}(\mathbb{P}^1_{(-)})$$

promotes to a filtered map (as in Construction 5.11) such that the induced map on graded pieces are homotopic to the map of (5.9). We set:

1. $C(j)$ to be the cofiber;
2. and write its p^r-reductions as $C_r(j) := C(j)/p^r$.

By construction and Lemm 5.7 we have spectral sequences, natural in $X \in \mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}_p}$:

$$E_2^{ij} = H^i(C_r(j)(X)) \Rightarrow 0 \qquad E_2^{ij} = H^i(C(j)(X)) \Rightarrow 0. \tag{5.10}$$

We get some automatic vanishing range of this spectral sequence, at least for the $C_r(j)$ variant, because of cohomological dimension reasons. We record this as Proposition 5.16 below which will be used throughout the rest of the section. We will ultimately use these spectral sequences to prove that $C(j) \simeq 0$. Our strategy relies on proving that these spectral sequences degenerate for a large enough class of examples, but we first deduce degeneration up to bounded torsion:

Lemma 5.15 (Theorem 5.14 holds up to p-torsion)**.** *For each $i, j \geq 0$ and for all $X \in \mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}_p}$, we have that the the cohomology groups $H^i(C(j)(X))$ are bounded p-torsion. In particular:*

1. *$C(j)(X)$ is derived p-complete;*
2. *$C(j)(X)[\frac{1}{p}] \simeq 0$.*

As explained in the footnote appearing in Lemma 4.19, $L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}$ need not be derived p-complete in general so it is not immediate that the $C(j)$'s are.

Proof. By construction we get a spectral sequence whose E_2-terms consists of the cohomology of $C(j)(X)$ converging to zero (since the $\mathbb{P}^1$-bundle formula holds for $L_{\mathrm{cdh}}\mathrm{TC}$). By the same argument as in Theorem 4.27 which establishes the degeneration of the motivic spectral sequence up to bounded denominators, we have the same result for this spectral sequence. Therefore, we conclude that each cohomology group $H^i(C(j)(X))$ are bounded p-torsion which, in particular, shows that $C(j)(X)$ is derived p-complete. This also proves that part (2) of the claim. □

In particular, since $C(j)(X)$ is derived p-complete, we get an equivalence

$$C(j)(X) \simeq \lim C_r(j)(X). \tag{5.11}$$

In fact, it suffices to prove that $C(j)(X)/p \simeq 0$. However, our argument proceeds by understanding the integral version of $C(j)$ and so we will make claims about this object. To proceed, we need to understand $C(j)(X)$ more explicitly. By Theorem 4.28, we have that:

$$L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}/p^r \simeq R\Gamma_{\acute{\mathrm{e}}\mathrm{h}}(-, W_r\Omega^j_{\log})[-j].$$

Based on this identification, the next proposition implies a vanishing range of $C(j)(X)$ based on the valuative dimension of X:

Proposition 5.16. *Let $X \in \mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}_p}$ and assume that it is of valuative dimension d, then for all $r \geq 1$,*

$$H^{>d+1}_{\acute{e}h}(X; W_r\Omega^j_{\log}) = 0.$$

Proof. This follows immediately from the fundamental cofiber sequence (which sandwiches $R\Gamma_{\acute{\mathrm{e}}\mathrm{h}}(X; W_r\Omega^j_{\log})$ between two cdh cohomologies of discrete sheaves) in Corollary 4.32 and the fact that the valuative dimension coincides with the cdh cohomological dimension [40]. □

Now, $C_r(j)$ identifies with the cofiber:

$$C_r(j)(-) \simeq \mathrm{cofib}(R\Gamma_{\acute{\mathrm{e}}\mathrm{h}}(-; W_r\Omega^j_{\log}) \oplus R\Gamma_{\acute{\mathrm{e}}\mathrm{h}}(-; W_r\Omega^{j-1}_{\log})[-1] \xrightarrow{\pi^* \oplus c_1(\mathcal{O}(-1))\pi^*} R\Gamma_{\acute{\mathrm{e}}\mathrm{h}}(\mathbb{P}^1_{(-)}; W_r\Omega^j_{\log}))[-j].$$

We also write, for any $X \in \mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}_p}$:

$$R\Gamma_{\acute{\mathrm{e}}\mathrm{h}}(X; W\Omega^j_{\log}) := \lim R\Gamma_{\acute{\mathrm{e}}\mathrm{h}}(X, W_r\Omega^j_{\log}).$$

Note that there is always a comparison map $L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}(X) \to R\Gamma_{\acute{\mathrm{e}}\mathrm{h}}(X; W\Omega^j_{\log})[-j]$, but this is not always an equivalence because of footnote 8 of Remark 4.19; indeed the target witnesses the derived p-completion of the domain. However, we note that thanks to the derived p-completeness assertion and (5.11), we still have an equivalence:

$$C(j)(-) \simeq \mathrm{cofib}(R\Gamma_{\acute{\mathrm{e}}\mathrm{h}}(-; W\Omega^j_{\log}) \oplus R\Gamma_{\acute{\mathrm{e}}\mathrm{h}}(-; W\Omega^{j-1}_{\log})[-1] \xrightarrow{\pi^* \oplus c_1(\mathcal{O}(-1))\pi^*} R\Gamma_{\acute{\mathrm{e}}\mathrm{h}}(\mathbb{P}^1_{(-)}; W\Omega^j_{\log})).$$

One last piece of notation before we embark on the proof: we will reindex $C_r(j)$ and $C(j)$'s up to a shift so that:

$$C_r(j) := C_r(j)[j] \qquad C(j) := C(j)[j].$$

This will ensure that H^0 of the $C(j)$'s and $C_r(j)$'s are related to zeroth éh cohomology of the $W_r\Omega^j_{\log}$ sheaves.

We first establish the main result for fields:

Lemma 5.17 (Theorem 5.14 holds for fields)**.** *For any field F of characteristic $p > 0$, we have that $C(j)(F) \simeq 0$.*

Proof. Setting $H^i(j) := H^i(C(j)(F))$, we claim that the spectral sequence of (5.10) for a field takes the following form:

$$\begin{tikzcd}
0 & H^0(0) \arrow[drr] & H^1(0) & H^2(0) & 0 \\
0 & H^0(1) \arrow[drr] & H^1(1) & H^2(1) & 0 \\
0 & H^0(2) \arrow[drr] & H^1(2) & H^2(2) & 0 \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{5.12}$$

To see this note that the spectral sequence for $C_r(j)$ takes the above form because of the dimension bound in Proposition 5.16 applied to $\mathbb{P}^1_F$ which is valuative dimension ≤ 1. To prove the claim for $C(j)(F)$, it suffices to prove that $\lim^1 H^2(C_r(j)(F)) = 0$. By dimension reasons, for all $r \geq 1$, we have isomorphisms $H^2_{\text{éh}}(\mathbb{P}^1_F; W_r\Omega^j_{\log}) \cong H^2(C_r(j)(F))$. By the Mittag-Leffler condition, it suffices to prove that the maps $H^2_{\text{éh}}(\mathbb{P}^1_F; W_{r+1}\Omega^j_{\log}) \to H^2_{\text{éh}}(\mathbb{P}^1_F; W_r\Omega^j_{\log})$ are surjective. But this follows from the fact that the sheaves $W_{r+1}\Omega^j_{\log} \to W_r\Omega^j_{\log}$ are epimorphisms on strictly henselian valuation rings (hence éh-locally epimorphisms) [65] and the fact that $H^2_{\text{éh}}(\mathbb{P}^1_F, -)$ is right exact since it is the top cohomology group.

From the claimed form of the spectral sequence, we can read off vanishing of $H^1(j)$'s already, along with vanishing of $H^2(0)$. Furthermore, the differentials induce isomorphisms $H^0(j) \xrightarrow{\cong} H^2(j+1)$. But now, $H^0(j)$ is p-torsion-free by Lemma 5.18 and thus $H^2(j+1)$'s are p-torsion-free as well. But we know that the groups are p-torsion by Lemma 5.15 and hence $C(j)(F) \simeq 0$. ☐

The following torsion-freeness assertion was used in the lemma above.

Lemma 5.18. *Let X be an qcqs $\mathbb{F}_p$-scheme. Then $H^0_{\acute{e}h}(X; W\Omega^j_{\log})$ is p-torsion-free.*

Proof. We have an injection

$$H^0_{\text{éh}}(X; W\Omega^j_{\log}) \hookrightarrow \prod_{\operatorname{Spec}(V) \to X} H^0(V; W\Omega^j_{\log}) = W\Omega^j_{\log, V};$$

where the product runs along all spectra of strictly henselian local rings mapping into X. The target is p-torsion-free by [65] and thus we conclude. ☐

Mainly through Gersten injectivity for valuation rings [65], we can boostrap the result for fields to an H^0 result for all schemes.

Lemma 5.19 (Theorem 5.14 holds for H^0)**.** *Let $X \in \mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}_p}$, then*

1. *for all $j \geq 0$, $H^0(C(j)(X)) = 0$ and,*

2. *for all $j \geq 0, r \geq 1$, $H^0(C_r(j)(X)) = 0$.*

Proof. First, we note that (2) proves (1). Indeed, we have an exact sequence for all $r \geq 1$:

$$\begin{tikzcd}
0 \arrow[r] & H^{-1}(C_r(j)(X)) \arrow[r] & \mathrm{H}^0_{\text{éh}}(X; W_r\Omega^j_{\text{log}}) \arrow[r, "\pi^*"] & H^0_{\text{cdh}}(\mathbb{P}^1_X, W_r\Omega^j_{\text{log}}) \arrow[r] & H^0(C_r(j)(V)) \arrow[dll, "\delta"] \\
 & & \mathrm{H}^1_{\text{éh}}(X; W_r\Omega^j_{\text{log}}) \arrow[r, "\pi^*"] & \mathrm{H}^1_{\text{éh}}(\mathbb{P}^1_X; W_r\Omega^j_{\text{log}}) \arrow[r] & H^1(C_r(j)(X)) .
\end{tikzcd}$$

But the maps labeled π^* are split by the the map induced by the inclusion of the ∞-section $\infty : \mathrm{Spec}V \hookrightarrow \mathbb{P}^1_V$ so that $H^{-1}(C_r(j)(V))$ vanishes and we have a direct sum decomposition: $H^0_{\text{éh}}(\mathbb{P}^1_X, W_r\Omega^j) \simeq H^0(C_r(j)(X) \oplus \mathrm{H}^0_{\text{éh}}(X; W_r\Omega^j)$. In particular the $\lim^1$ term involving H^{-1} vanishes so that $H^0(C(j)(X)) \cong \lim H^0(C_r(j)(X))$ and so (2) proves (1).

We now prove (2). We have already proved the result when $X = \mathrm{Spec}(F)$ as a special case of Lemma 5.17. To bootstrap this result, we consider the commutative diagram for all $j \geq 0, r \geq 1$:

$$\begin{tikzcd}
H^0_{\text{éh}}(\mathbb{P}^1_X; W_r\Omega^j_{\text{log}}) \arrow[r] \arrow[d, "\infty^*"] & \prod_{x \in X} H^0_{\text{éh}}(\mathbb{P}^1_x; W_r\Omega^j_{\text{log}}) \arrow[d, "\infty^*"] \\
H^0_{\text{éh}}(X; W_r\Omega^j_{\text{log}}) \arrow[r] & \prod_{x \in X} H^0_{\text{éh}}(x; W_r\Omega^j_{\text{log}}).
\end{tikzcd}$$

It suffices to prove that the left vertical map is injective (since it is already seen to be a split surjection). We have already established that the right vertical map is injective (in fact, an isomorphism), hence it suffices to prove that the top horizontal map is injective.

To do so we note that for all $\mathbb{F}_p$-scheme Y, by the formula for sheafification, the canonical map

$$H^0_{\text{éh}}(Y; W_r\Omega^j_{\text{log}}) \to \prod_{\mathrm{Spec}(V) \to Y} \mathrm{H}^0_{\text{éh}}(V; W_r\Omega^j_{\text{log}}) = W_r\Omega^j_{\text{log},V}$$

is injective where the product runs across all strictly henselian valuation rings (the stalks for the éh topology) mapping to Y. But now, by the validity of the Gersten injectivity for valuation rings in this context [65], we have a further injection

$$H^0_{\text{éh}}(Y; W_r\Omega^j_{\text{log}}) \to \prod_{\mathrm{Spec}(V) \to Y} W_r\Omega^j_{\text{log},V} \to \prod_{\mathrm{Spec}(V) \to Y} W_r\Omega^j_{\text{log},\mathrm{Frac}(V)}.$$

Thus we conclude that the map

$$H^0_{\text{éh}}(Y; W_r\Omega^j_{\text{log}}) \to \prod_{\mathrm{Spec}(F) \to Y} W_r\Omega^j_{\text{log},F},$$

is injective where the product runs across all fields mapping to Y. We use this to prove that for all X, the map $H^0_{\text{éh}}(\mathbb{P}^1_X; W_r\Omega^j_{\text{log}}) \to \prod_{x \in X} H^0_{\text{éh}}(\mathbb{P}^1_x; W_r\Omega^j_{\text{log}})$ is injective. Indeed, let $\alpha \in H^0_{\text{éh}}(\mathbb{P}^1_X; W_r\Omega^j_{\text{log}})$ be an element such that $\alpha|_{x \times \mathbb{P}^1} = 0$ for all $x \in X$. Now, any map $\mathrm{Spec}(F) \to \mathbb{P}^1_X$ factors through $x \times \mathbb{P}^1 \to X \times \mathbb{P}^1$ for some $x \in X$. Therefore, $\alpha|_{\mathrm{Spec}(F)} = 0$, whence we obtain the claim. $\square$

Remark 5.20. The last part of the argument imitates the argument, surely well-known to experts, for reducing $\mathbb{A}^1$-invariance over smooth schemes to $\mathbb{A}^1$-invariance over fields for functors satisfying Gersten injectivity. For example, the first author has used this in joint work with Kulkarni and Wendt [42, Prop. 3.5] to prove $\mathbb{A}^1$-invariance of the Nisnevich sheafification of some cohomology sets. We also use this kind of argument in establishing some properties of cdh-motivic cohomology [7].

To proceed, we further appeal to descent properties of $C(j)$. Indeed, we observe that $C(j)$ is a cdh sheaf since it is the cofiber of two cdh sheaves. The next property we use is *henselian v-excision* (hv-excision for short). Recall that if V is a valuation ring and $\mathfrak{p}$ is a prime ideal, then we can form the following bicartesian square of rings:

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & V_{\mathfrak{p}} \arrow[d] \\
V/\mathfrak{p} \arrow[r] & \kappa(\mathfrak{p}).
\end{tikzcd} \tag{5.13}$$

If V is a henselian valuation ring, then so are the other vertices in the above square by [40, Lem. 3.3.5]. A presheaf of spectra or complexes on $\mathbb{F}$-schemes are said to be *hv-excisive* if it converts (5.13) to a cartesian square.

Lemma 5.21. *For all $j \geq 0$, the presheaves $C(j)$ are*

1. *finitary* cdh *sheaves and, in fact, are éh sheaves;*
2. *hv-excisive.*

Therefore, $C(j) \simeq 0$ if and only if $C(j)(V) \simeq 0$ for any henselian valuation ring of rank ≤ 1.

Proof. As explained above, $C(j)$ is a cdh sheaf; it is also an éh sheaf because it is a cofiber of a map between éh sheaves. The finitary part of $C(j)$ is not quite immediate as $L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}$ is not a finitary sheaf. However, as recorded in the proof of Theorem 4.24, the fiber of the map $\mathbb{Z}_p(j)^{\mathrm{syn}} \to L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}$ which we denote by $W(j)$ is finitary. Since the $\mathbb{P}^1$-bundle formula for $\mathbb{Z}_p(j)^{\mathrm{syn}}$ holds by [18], $C(j)$ is equivalent to the cofibre of the $\mathbb{P}^1$-bundle maps for $W(j)$ (up to a shift) and is thus finitary.

To prove that $C(j)$ is hv-excisive, it suffices to prove the claim for $C(j)[\frac{1}{p}]$ and $C(j)/p$; the former is zero so we are left to prove the claim for the latter. For this claim, it suffices to prove that the functors $L_{\mathrm{cdh}}\mathbb{F}_p(j)^{\mathrm{syn}}$ and $L_{\mathrm{cdh}}\mathbb{F}_p(j)^{\mathrm{syn}}(\mathbb{P}^1 \times -)$ are for all $j \geq 0$. Since L_{cdh} does not change stalks and the terms of the square (5.13) are all henselian valuation rings, the claim for $L_{\mathrm{cdh}}\mathbb{F}_p(j)^{\mathrm{syn}}$ follows from the analogous claim for $\mathbb{F}_p(j)^{\mathrm{syn}}$; this is proved in [7]. We note that the argument reduces to the analogous claim for the cotangent complex by the increasing filtration in Lemma 4.16 which is what we verify in [7]. To prove the claim for $L_{\mathrm{cdh}}\mathbb{F}_p(j)^{\mathrm{syn}}(\mathbb{P}^1 \times -)$ we use [40, Lem. 3.3.7] for $\mathcal{F} = L_{\mathrm{cdh}}\mathbb{F}_p(j)^{\mathrm{syn}}$ and $X = \mathbb{P}^1_V$. Indeed, $\mathcal{F}$ is a finitary cdh-sheaf; it takes values in the derived category of $\mathbb{F}_p$-vector spaces where compact objects are cotruncated. We have already verified hv-excision and thus the cited result applies.

The "therefore" part of the assertion holds because $C(j)$ is a hypercomplete, finitary cdh sheaf (since it is a finitary, cdh sheaf over $\mathbb{F}_p$ we can appeal to [40, Corol. 2.4.16]) and thus its vanishing is detected on henselian valuatioon rings. Using that it is finitary again, we may reduce to the case that V has finite rank and by hv-excision we reduce, by induction, to V being rank ≤ 1. ☐

For the rest of the section, we fix V a henselian valuation ring of rank ≤ 1. This will help us to degenerate enough of the spectral sequences (5.10). The next lemma proves that the spectral sequence $E_2^{ij} = H^i(C_r(j)(V)) \Rightarrow 0$ displays as (after the coconnectedness result of Lemma 5.19):

$$\begin{tikzcd}
0 & H^1(0) \arrow[rrd] & H^2(0) & H^3(0) & 0 \\
0 & H^1(1) \arrow[rrd] & H^2(1) & H^3(1) & 0 \\
0 & H^1(2) \arrow[rrd] & H^2(2) & H^3(2) & 0 \\
\vdots & \vdots & \vdots & \vdots, &
\end{tikzcd} \tag{5.14}$$

and that this patten persists integrally.

Lemma 5.22. *Let V be a henselian valuation ring of rank ≤ 1 over $\mathbb{F}_p$. Then:*

1. *for any $j \geq 0, r \geq 1$, we have that $H^k(C_r(j)(V)) = 0$ whenever $k > 3$;*
2. *for any $j \geq 0, r \geq 1$, $H^2(C_r(j)(V)) = 0$, $H^3(C_r(0)(V)) = 0$, and $H^1(C_r(j)(V)) \xrightarrow{\cong} H^3(C_r(j+1)(V))$.*
3. *for any $j \geq 0$, $H^k(C(j)(V)) = 0$ whenever $k > 3$;*
4. *for any $j \geq 0$, $H^2(C(j)(V)) = 0$, $H^3(C(0)(V)) = 0$, and $H^1(C(j)(V)) \xrightarrow{\cong} H^3(C(j+1)(V))$.*

Proof. First, note that Proposition 5.16 tells us that $H^{\geq 4}_{\text{éh}}(\mathbb{P}^1_V, W_r\Omega^j_{\log}) = 0$ since $\mathbb{P}^1_V$ is valuative dimension ≤ 2. On the other hand, since V is a stalk for the cdh-topology, we have that $H^{\geq 2}_{\text{éh}}(V, W_r\Omega^j_{\log}) = 0$. Therefore, by the definition of $C_r(j)$, we have that $H^{\geq 4}(C_r(j)(V)) = 0$. From the pattern of the spectral sequence (5.14), we conclude (2).

Statement (3) is the integral version of the statement (1). For this, it suffices to prove that $\lim^1 H^3(C_r(j)(V)) = 0$ for all $j \geq 0$. In turn, it suffices to prove that for each $r \geq 1$, the map $H^3(C_{r+1}(V)) \to H^3(C_r(V))$ is surjective. By the definition of $C_r(V)$ and the fact that V is valuative dimension ≤ 1, we have a surjection $H^3_{\text{éh}}(\mathbb{P}^1_V; W_r\Omega^j_{\log}) \to H^3(C_r(j)(V)) \to 0$. Therefore it suffices to prove that $H^3_{\text{éh}}(\mathbb{P}^1_V; W_{r+1}\Omega^j_{\log}) \to H^3_{\text{éh}}(\mathbb{P}^1_V; W_r\Omega^j_{\log})$ is surjective. But this follows from the fact that the sheaves $W_{r+1}\Omega^j_{\log} \to W_r\Omega^j_{\log}$ are epimorphisms on henselian valuation rings (hence éh-locally epimorphisms) and the fact that $H^3_{\text{éh}}(\mathbb{P}^1_V, -)$ is right exact since it is the top cohomology group by Proposition 5.16. Part (4) immediately follows from the pattern of the spectral sequence (5.14), which is a consequence of (2). ☐

To conclude, we need only prove that, in fact, $H^3(C(j)(V)) = 0$ for all $j \geq 1$. Since we know that this is a p-torsion group, we will prove that it is actually p-torsion-free though we need to first prove this under a slightly more restricted hypothesis on V.

Lemma 5.23. *Assume that V is a strictly henselian valuation ring over $\mathbb{F}_p$ of rank ≤ 1. For any $j \geq 1$, the group $H^3(C(j)(V))$ is p-torsion-free. Therefore, it is zero.*

Proof. First, note that $\lim^1 H^2(C_r(j)(V)) = 0$ since the groups vanish by Lemma 5.22(3). Therefore

$$H^3(C(j)(V)) \cong \lim_r H^3(C_r(j)(V)).$$

Therefore, it suffices to prove that the multiplication by p maps

$$H^3(C_{r-1}(j)(V)) \xrightarrow{\times p} H^3(C_r(j)(V)),$$

are all injective. We claim that there is a canonical isomorphism (this uses that V is strictly henselian):

$$H^3_{\text{éh}}(\mathbb{P}^1_V, W_r\Omega^j_{\log}) \cong H^3(C_r(j)(V))$$

To see this, the vanishing of H^2's of Lemma 5.22 produces a short exact sequece:

$$0 \to H^1_{\text{éh}}(V; W_r\Omega^{j-1}_{\log}) \xrightarrow{c_1(\mathcal{O}(1))\pi^*} H^3_{\text{éh}}(\mathbb{P}^1_V; W_r\Omega^j_{\log}) \to H^3(C_r(j)(V)) \to 0.$$

We have an isomorphism $H^1_{\text{éh}}(V; W_r\Omega^{j-1}_{\log}) \cong \widetilde{\nu}_{r-1}(j)(V)$, but the rigidity statement of [32, Proposition 4.31] shows that $\widetilde{\nu}_{r-1}(j)(V) = \widetilde{\nu}_{r-1}(j)(\kappa)$ where κ is the residue field. Since V is assumed to be strictly henselian, this group is zero (it vanishes on any perfect $\mathbb{F}_p$-algebras.

Therefore, it suffices to prove that the maps

$$H^3_{\text{éh}}(\mathbb{P}^1_V, W_{r-1}\Omega^j_{\log}) \xrightarrow{\times p} H^3_{\text{éh}}(\mathbb{P}^1_V, W_r\Omega^j_{\log}),$$

are all injective. We have the short exact sequence of presheaves

$$0 \to W_{r-1}\Omega^j_{\log} \xrightarrow{\times p} W_r\Omega^j_{\log} \to \Omega^j_{\log} \to 0,$$

which induces a long exact sequence

$$\cdots H^2_{\text{éh}}(\mathbb{P}^1_V, W_r\Omega^j_{\log}) \to H^2_{\text{éh}}(\mathbb{P}^1_V, \Omega^j_{\log}) \xrightarrow{\delta} H^3_{\text{éh}}(\mathbb{P}^1_V, W_{r-1}\Omega^j_{\log}) \xrightarrow{\times p} H^3_{\text{éh}}(\mathbb{P}^1_V, W_r\Omega^j_{\log}) \to \cdots.$$

We claim that the map δ is zero for which it suffices to prove that $H^2_{\text{éh}}(\mathbb{P}^1_V, W_r\Omega^j_{\log}) \to H^2_{\text{éh}}(\mathbb{P}^1_V, \Omega^j_{\log})$ is surjective. We have a commutative diagram

$$\begin{tikzcd}
W_{r-1}\Omega^j_{\log,V} \arrow[r] \arrow[d, "\cup c_1(\mathcal{O}(-1))"'] & \Omega^j_{\log,V} \arrow[d, "\cup c_1(\mathcal{O}(-1))"] \\
H^2_{\text{éh}}(\mathbb{P}^1_V, W_{r-1}\Omega^j_{\log}) \arrow[r, "\delta"] & H^2_{\text{éh}}(\mathbb{P}^1_V, \Omega^j_{\log}).
\end{tikzcd}$$

The top map is surjective so it suffices to prove that the right vertical map is surjective. But the cokernel of the right vertical map is exactly $H^2(C_1(j)(V)) = 0$ which vanishes by Lemma 5.22(2). ☐

Proof of Theorem 5.14. By Lemma 5.21, it suffices to prove that $C(j)(V) = 0$ for V a henselian valuation ring of rank ≤ 1. We have already seen that the cohomology groups of $C(j)$, on any $\mathbb{F}_p$-scheme, are all p-torsion by Lemma 5.15. We also know that H^2, H^0 and $H^{\geq 4}$ and $H^3(C(0)(V))$ are all zero for V a henselian valuation ring of rank ≤ 1 by Lemma 5.22. The same lemma also shows that $H^1(C(j)(V)) \cong H^3(C(j+1)(V))$ for a $j \geq 0$ and so it suffices to prove that $H^3(C(j+1)(V))$ is p-torsion free. Now $C(j)$ is an éh sheaf (by the "in fact" part of Lemma 5.21(1)), we may pass to the strict henselization of V. The latter does not change the value group, and hence the rank, of the valuation ring [106, Tag 0ASK] and thus we can use Lemma 5.23 to conclude. □

Conclusion of Proof of Theorem 5.3. It suffices to prove that $\mathbb{F}_p(j)^{\mathrm{mot}}$ has the $\mathbb{P}^1$-bundle formula for $\mathbb{F} = \mathbb{F}_p$. By Theorem 4.24(2), we need to prove the result for the theories:

$$\mathbb{F}_p(j)^{\mathrm{cdh}}, \mathbb{F}_p(j)^{\mathrm{syn}}, L_{\mathrm{cdh}}\mathbb{F}_p(j).$$

The result for $\mathbb{F}_p(j)^{\mathrm{cdh}}$ is verified in [7], while the result for $\mathbb{F}_p(j)^{\mathrm{syn}}$ is verified in [18]. The result for last theory follows from Theorem 5.14. □

We can now prove a general projective bundle formula for motivic cohomology. To formulate this, let $r \geq 0$ and consider for $0 \leq i \leq r$ the map, induced by multiplicativity of motivic cohomology:

$$c_1(\mathcal{O}(1))^i\pi^* : \mathbb{Z}(j-i)^{\mathrm{mot}}[-2i](X) \to \mathbb{Z}(j)(\mathbb{P}^r_X).$$

More generally, we let $\pi : \mathbb{P}_X(\mathcal{E}) \to X$ be the projectivization of $\mathcal{E}$, a rank $r+1$ vector bundle on X. It classifies subbundles of rank 1 and thus comes equipped with a tautological bundle $\mathcal{O}(-1) \subset \pi^*\mathcal{E}$ whose dual is denoted by $\mathcal{O}(1)$. Zariski-locally, $\mathbb{P}_X(\mathcal{E})$ is isomorphic to $\mathbb{P}^r_X$. Then we have an generalization of the previous map:

$$c_1(\mathcal{O}(1))^i\pi^* : \mathbb{Z}(j-i)^{\mathrm{mot}}[-2i](X) \to \mathbb{Z}(j)(\mathbb{P}(\mathcal{E})).$$

Theorem 5.24. *Let X be a qcqs $\mathbb{F}$-scheme and $j \geq 0$.*

1. *Projective bundle formula: for any $r \geq 0$, the map*

$$\sum c_1(\mathcal{O}(1))^i\pi^* \colon \bigoplus_{i=0}^{r} \mathbb{Z}(j-i)^{\mathrm{mot}}[-2i](X) \longrightarrow \mathbb{Z}(j)^{\mathrm{mot}}(\mathbb{P}^r_X) \tag{5.15}$$

is an equivalence.

2. *More generally, if $\mathcal{E}$ is a locally free sheaf on X of rank $r+1$, then the map:*

$$\sum c_1(\mathcal{O}(1))^i\pi^* \colon \bigoplus_{i=0}^{r} \mathbb{Z}(j-i)^{\mathrm{mot}}[-2i](X) \longrightarrow \mathbb{Z}(j)^{\mathrm{mot}}(\mathbb{P}_X(\mathcal{E})),$$

is an equivalence.

3. *Blow-up formula: Let $Y \to X$ be a regular closed immersion (i.e., X admits an open affine cover such that, on each such affine, Y is defined by a regular sequence); then for any $j \geq 0$: we have a cartesian square in $\mathcal{D}(\mathbb{Z})$:*

$$\begin{tikzcd}
\mathbb{Z}(j)^{\mathrm{mot}}(X) \arrow[r] \arrow[d] & \mathbb{Z}(j)^{\mathrm{mot}}(Y) \arrow[d] \\
\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{Bl}_Y(X)) \arrow[r] & \mathbb{Z}(j)^{\mathrm{mot}}(Y \times_X \mathrm{Bl}_Y(X)).
\end{tikzcd}$$

Proof. We first establish the blowup formula (3). The blowup formula of course holds for any cdh sheaf, since cdh sheaves carry arbitrary abstract blowup squares to cartesian squares. So, using the fundamental fibre sequence of Theorem 4.10 in characteristic 0 (resp. the pullback square of Theorem 4.24 in characteristic p) in characteristic 0, it remains to check the blowup formula for $R\Gamma(-, L\Omega^{<j}_{-/\mathbb{Q}})$ (resp. $\mathbb{Z}_p(j)^{\mathrm{syn}}$). In both cases that reduces to the blowup formula for $R\Gamma(-, \mathbb{L}^i_{-/\mathbb{F}})$ for all $i \geq 0$ (here we use Lemma 4.16 in characteristic p, which can be proved directly; see [18, Lem. 9.4.3]).

Knowing the blowup formula, part (1) for arbitrary $j \geq 0$ follows from the special case $r = 1$, i.e., Theorem 5.3. This follows from an argument in [3, Lem. 3.3.5]. More precisely, the argument in the cited lemma shows that as soon as $\mathbb{Z}(j)^{\mathrm{mot}}$ converts the blowup square

$$\begin{tikzcd}
\mathbb{P}^{r-1}_X \arrow[r] \arrow[d] & \mathrm{Bl}_{\{0\}}(\mathbb{A}^r_X) \arrow[d] \\
X \arrow[r, "0"] & \mathbb{A}^r_X
\end{tikzcd} \qquad r \geq 1,$$

to a cartesian square, then the map (5.15) is an equivalence. The above blowup square is a special case of the blowup formula in part (3). Part (2) then follows from part (1) by Zariski descent. □

Remark 5.25. We note that if $Y \to X$ is not a regular closed immersion, then the square of Theorem 5.24(3) need not be cartesian. What is more generally true is the pro cdh descent result of Theorem 8.2. Theorem 5.24(3) does admit two other enhancements which we will leave for the reader to supply details:

1. if $Y \to X$ is the section of a *smooth morphism* $f : Y \to X$ of relative dimension r, then the blowup formula “splits" to give an equivalence

$$\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{Bl}_Y(X)) \simeq \mathbb{Z}(j)^{\mathrm{mot}}(X) \oplus \bigoplus_{0<i<r} \mathbb{Z}(j-i)^{\mathrm{mot}}(Y)[-2i];$$

   this follows by combining parts (2) and (3) of Theorem 5.24.

2. The extension of $\mathbb{Z}(j)^{\mathrm{mot}}$ to derived schemes, as in §4.5, converts a derived blowup square as in [71] to a cartesian square. Indeed, the cdh parts of the theory does not depend on derived structure and thus converts these derived blowup squares to cartesian squares. On the other hand, we reduce the assertions for $\mathbb{Z}_p(j)^{\mathrm{syn}}$ and filtered derived de Rham cohomology to the case of the cotangent complex. Since derived blowup squares are “pulled back" from blowups along regular immersions and therefore, the cotangent complex enjoys the same property.

6 COMPARISON TO $\mathbb{A}^1$-INVARIANT MOTIVIC COHOMOLOGY

We begin by repeating Construction 4.36 for the sake of clarity: on the category of qcqs $\mathbb{F}$-schemes, there are natural comparison maps of $\mathrm{D}(\mathbb{Z})$-valued presheaves

$$\mathbb{Z}(j)^{\mathrm{mot}} \longrightarrow \mathbb{Z}(j)^{\mathrm{cdh}} \tag{6.1}$$

for $j \geq 0$, arising as the shifted graded pieces of a comparison map of filtered presheaves of spectra $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K} \to \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}$. These comparison maps are tautological from the definition of our motivic cohomology, which should be seen a modification of the cdh-local theory. Although it is not strictly necessary for what follows, the reader should also recall from Remark 3.6 that, for X any qcqs $\mathbb{F}$-scheme, the cdh-local motivic cohomology $\mathbb{Z}(j)^{\mathrm{cdh}}(X)$ is in fact the $\mathbb{A}^1$-invariant motivic cohomology of X which arises from motivic homotopy theory, and $\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)$ identifies with the slice filtration.

The goal of this section is to prove the following comparison equivalences related to the maps (6.1):

Theorem 6.1. *Let $\mathbb{F}$ be a prime field and $j \geq 0$.*

1. *The maps (6.1) induce equivalences of $\mathrm{D}(\mathbb{Z})$-valued presheaves on $\mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}$*

$$L_{\mathrm{cdh}}\mathbb{Z}(j)^{\mathrm{mot}} \xrightarrow{\sim} \mathbb{Z}(j)^{\mathrm{cdh}} \qquad \textit{and} \qquad L_{\mathbb{A}^1}\mathbb{Z}(j)^{\mathrm{mot}} \xrightarrow{\sim} \mathbb{Z}(j)^{\mathrm{cdh}}.$$

2. *For any regular Noetherian $\mathbb{F}$-scheme X, the map (6.1) induces an equivalence*

$$\mathbb{Z}(j)^{\mathrm{mot}}(X) \xrightarrow{\sim} \mathbb{Z}(j)^{\mathrm{cdh}}(X).$$

   (Equivalently, using part (1), the maps $\mathbb{Z}(j)^{\mathrm{mot}}(X) \to \mathbb{Z}(j)^{\mathrm{mot}}(\mathbb{A}^m_X)$ are equivalences for all $m \geq 0$.)

The maps in part (1) of the theorem are induced by (6.1), recalling that the presheaf $\mathbb{Z}(j)^{\mathrm{cdh}} : \mathrm{Sch}^{\mathrm{qcqs,op}}_{\mathbb{F}} \to \mathrm{D}(\mathbb{Z})$ is both a cdh sheaf and $\mathbb{A}^1$-invariant by Theorem 3.5(2)&(5). In case of confusion, here $L_{\mathbb{A}^1}$ denotes the endofunctor of presheaves of complexes (or of spectra) on $\mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}$ reflecting onto $\mathbb{A}^1$-invariant presheaves; we will recall the explicit formula for $L_{\mathbb{A}^1}$ in the proof of Lemma 6.6 below.

Remark 6.2. Informally, Theorem 6.1(1) says that our motivic cohomology may be viewed as a "de-cdh-sheafification" or "de-$\mathbb{A}^1$-localisation" of $\mathbb{A}^1$-invariant motivic cohomology. More precisely, it states that on equicharacteristic schemes the comparison equivalences

$$L_{\mathrm{cdh}}\mathrm{K} \xrightarrow{\sim} \mathrm{KH} \qquad \text{and} \qquad L_{\mathbb{A}^1}\mathrm{K} \xrightarrow{\sim} \mathrm{KH}$$

(the first being part of Theorem 3.8, the second being the definition of KH) upgrade to filtered equivalences, where we equip KH with the filtration $\mathrm{Fil}^\star_{\mathrm{cdh}}$ (or equivalently with the slice filtration), and we equip the left sides with L_{cdh}, resp. $L_{\mathbb{A}^1}$, of our motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}$. That is, the slice filtration on KH-theory can be recovered by cdh sheafifying or $\mathbb{A}^1$-localising our motivic filtration on K-theory.

Remark 6.3. Theorem 6.1(2) is a motivic upgrade of the equivalence $\mathrm{K}(X) \xrightarrow{\sim} \mathrm{KH}(X)$ for regular Noetherian $\mathbb{F}$-schemes X. Indeed, combined with this equivalence, it states that the map of filtered spectra $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X) \to \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)$ is an equivalence.

Combined with a comparison isomorphism from the joint project with Bachmann, we obtain the following, starting that our motivic cohomology coincides with the classical theory on smooth $\mathbb{F}$-varieties:[12]

Corollary 6.4. *For any smooth $\mathbb{F}$-scheme X the comparison map (4.8) is an equivalence of filtered spectra, or equivalently the maps $\mathbb{Z}(j)^{\mathrm{cla}}(X) \to \mathbb{Z}(j)^{\mathrm{mot}}(X)$ are equivalences for all $j \geq 0$.*

Proof. As explained in Remark 4.37, we show with Bachmann that the composition $\mathbb{Z}(j)^{\mathrm{cla}}(X) \to \mathbb{Z}(j)^{\mathrm{mot}}(X) \to \mathbb{Z}(j)^{\mathrm{cdh}}(X)$ is an equivalence. Now apply Theorem 6.1(2) to see that the second map is an equivalence, therefore also the first. ☐

Corollary 6.5. *For any regular Noetherian $\mathbb{F}_p$-scheme X and $j \geq 0$, the canonical maps*

$$\mathbb{Z}_p(j)^{\mathrm{syn}}(X) \longrightarrow L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}(X) \qquad \textit{and} \qquad R\Gamma_{\text{ét}}(X, \Omega^j_{\log}) \longrightarrow R\Gamma_{\text{éh}}(X, \Omega^j_{\log})$$

are equivalences.

Proof. The first follows from Theorem 4.27(2) and the cartesian square Theorem 4.24. The second equivalence follows by taking the first equivalence modulo p. ☐

The core of the proof of Theorem 6.1(1) is the fact that derived de Rham and syntomic cohomology are very far from being homotopy invariant [39, 51]:

Lemma 6.6. *In the category of $\mathrm{D}(\mathbb{Z})$-valued presheaves on $\mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}$, the following hold for all $j \geq 0$:*

1. *$L_{\mathbb{A}^1} R\Gamma(-, L^j_{-/\mathbb{F}}) \simeq 0$;*
2. *if $\mathbb{F} = \mathbb{Q}$ then the map $L_{\mathbb{A}^1} R\Gamma(-, \widehat{L\Omega}^{\geq j}_{-/\mathbb{Q}}) \to L_{\mathbb{A}^1} R\Gamma_{\mathrm{cdh}}(-, \widehat{L\Omega}^{\geq j}_{-/\mathbb{Q}})$ is an equivalence.*
3. *if $\mathbb{F} = \mathbb{F}_p$, then $L_{\mathbb{A}^1}\mathbb{Z}_p(j)^{\mathrm{syn}} \simeq 0$ and $L_{\mathbb{A}^1} L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}} \simeq 0$.*

Proof. We will use the following explicit formula for the endofunctor $L_{\mathbb{A}^1}$ of presheaves on $\mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}$: given a presheaf $\mathcal{F}$ then

$$L_{\mathbb{A}^1}\mathcal{F}(X) = \operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathcal{F}(X \times \Delta^\bullet) \tag{6.2}$$

where $\Delta^\bullet$ is the cosimplicial $\mathbb{F}$-scheme built from algebraic m-simplices:

$$\Delta^m = \mathrm{Spec}(\mathbb{F}[T_0, \cdots, T_m]/(\sum_{i=0}^m T_i = 1).$$

Note that $L_{\mathbb{A}^1}$ preserves Nisnevich and cdh sheaves; indeed, this follows from the description of Nisnevich and cdh descent in terms of cd structures and (6.2).

(1): Let A be an $\mathbb{F}$-algebra. By the Künneth formula for the cotangent complex, there is a natural equivalence

$$L^j_{A[\Delta^m]/\mathbb{F}} \simeq \bigoplus_{a+b=j} L^a_{A/\mathbb{F}} \otimes \Omega^b_{\mathbb{F}[\Delta^m]/\mathbb{F}}.$$

---

[12] We stress that our motivic cohomology coincides with the classical theory on smooth varieties over any field. This follows from Theorem 6.1, Remark 3.6, and the fact that motivic homotopy theory recovers classical motivic cohomology in the case of smooth varieties over fields. Here we restrict to the case of varieties over the prime field because that is the important case for our later study of $\mathbb{Z}(j)^{\mathrm{lse}}$, and because formulating the comparison maps over arbitrary base fields is not entirely elementary.

Therefore $(L_{\mathbb{A}^1}L^j_{-/\mathbb{F}})(A) \simeq \bigoplus_{a+b=j} L^a_{A/\mathbb{F}} \otimes (L_{\mathbb{A}^1}\Omega^b_{-/\mathbb{F}})(\mathbb{F})$, which reduces the problem to showing that $(L_{\mathbb{A}^1}\Omega^b_{-/\mathbb{F}})(\mathbb{F}) \simeq 0$ for all $b \geq 0$. The latter vanishing is due to Geller–Weibel [51].

(2): Since $L_{\mathbb{A}^1}$ preserves cdh sheaves, it suffices to prove that $L_{\mathbb{A}^1}R\Gamma(-,\widehat{L\Omega}^{\geq j}_{-/\mathbb{Q}})$ is a cdh sheaf. Since $L_{\mathbb{A}^1}$ preserves fibre sequences, we have a fibre sequence for all $j \geq 0$

$$L_{\mathbb{A}^1}R\Gamma(-,L\Omega^{<j}_{-/\mathbb{Q}})[-1] \to L_{\mathbb{A}^1}R\Gamma(-,\widehat{L\Omega}^{\geq j}_{-/\mathbb{Q}}) \to L_{\mathbb{A}^1}R\Gamma(-,\widehat{L\Omega}_{-/\mathbb{Q}}).$$

The presheaf $R\Gamma(-,L\Omega^{<j}_{-/\mathbb{Q}})$ is killed by $L_{\mathbb{A}^1}$, thanks to part (1) and induction on j. On the other hand, by Lemma 4.5, the last term is a cdh sheaf since $L_{\mathbb{A}^1}$ preserves cdh sheaves. In particular, the middle term is a cdh sheaf, completing the proof.

(3): It suffices to prove the vanishings on any affine $\mathbb{F}_p$-scheme Spec$A$. Firstly, as observed in [39, Lem. 3.0.3], the complex $L_{\mathbb{A}^1}\mathbb{Z}_p(j)^{\mathrm{syn}}(A)$ is p-complete since we have the universal bound that $\mathbb{Z}_p(j)^{\mathrm{syn}}(A[\Delta^m])$ is supported in degrees $\leq j+1$ for any m; so it is enough to prove the vanishing of $L_{\mathbb{A}^1}\mathbb{F}_p(j)^{\mathrm{syn}}(A)$, which in turn follows from part (1) and Lemma 4.16. Next we use the multiplicative morphism of presheaves of $\mathbb{E}_\infty$-rings $\bigoplus_{j\geq 0}\mathbb{Z}_p(j)^{\mathrm{syn}}[2j] \to \bigoplus_{j\geq 0} L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}[2j]$; since $L_{\mathbb{A}^1}$ preserves multiplicative structures, we obtain a morphism of $\mathbb{E}_\infty$-rings $\bigoplus_{j\geq 0}\mathbb{Z}_p(j)^{\mathrm{syn}}(A)[2j] \to \bigoplus_{j\geq 0} L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}(A)[2j]$. But the domain is 0 by the first part, so the target is also 0 as it is receiving a multiplicative map from the zero ring. ☐

Proof of Theorem 6.1(1). Cdh sheafifying the pullback squares of Theorems 4.10(2) or 4.24(2) shows that $L_{\mathrm{cdh}}\mathbb{Z}(j)^{\mathrm{mot}} \xrightarrow{\sim} \mathbb{Z}(j)^{\mathrm{cdh}}$. Similarly, $\mathbb{A}^1$-localising the pullback squares and using the previous lemma yields $L_{\mathbb{A}^1}\mathbb{Z}(j)^{\mathrm{mot}} \xrightarrow{\sim} \mathbb{Z}(j)^{\mathrm{cdh}}$. ☐

The remainder of the section is devoted to the proof of Theorem 6.1(2). The key inputs are the $\mathbb{P}^1$-bundle formula for motivic cohomology (Theorem 5.3), the already proved Theorem 6.1(1), and an argument of Gabber used to prove Gersten injectivity statements [46, 53]. Gabber's argument has been axiomatized by Colliot-Thélène–Hoobler–Kahn [33], and we now review their formalism in a more modern language.

Let k be any field and suppose that we have a presheaf $\mathcal{F} : \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{Sp}$; for $X \in \mathrm{Sm}_k$, we will write $\mathcal{F}^X$ for the presheaf $U \mapsto \mathcal{F}(U \times_k X)$. There are two morphisms of presheaves

$$j^*, \pi^*\infty^* : \mathcal{F}^{\mathbb{P}^1} \to \mathcal{F}^{\mathbb{A}^1},$$

where:

1. π is induced the projection $\mathbb{A}^1 \times_k X \to X$,
2. ∞ is the closed immersion Spec$k \to \mathbb{P}^1$ of the point at ∞,
3. $j : \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ is the open immersion complementary to the point at ∞.

In general, there is no reason for the maps j^* and $\pi^*\infty^*$ to be homotopic. This leads to the next definition:

Definition 6.7. We say that a presheaf $\mathcal{F} : \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{Sp}$ is a *deflatable*[13] if the maps j^* and $\pi^*\infty^*$ are homotopic.

Remark 6.8. More precisely, to ask that two morphisms of presheaves are homotopic means that they are identified in the homotopy category of presheaves. This means that there is an 2-morphism, functorial in smooth k-schemes, between these two morphisms of presheaves. We do not keep track of this 2-morphism, but we note that functoriality in smooth schemes is a substantial amount of extra compatibilities. In fact, calling these 2-morphisms *deflations*, the space of deflations can be parametrized as following: it is the space of sections $s : \mathcal{F}^{\mathbb{P}^1} \to \mathcal{E}$ of the canonical map $\mathcal{E} \to \mathcal{F}^{\mathbb{P}^1}$, were $\mathcal{E}$ is the equaliser of the two maps $j^*, \pi^*\infty^* : \mathcal{F}^{\mathbb{P}^1} \rightrightarrows \mathcal{F}^{\mathbb{A}^1}$.

Remark 6.9. Definition 6.7 implies the validity of axiom "SUB 2" of [33], which is much weaker than deflatability and instead asks only for scheme-wise homotopy commutativity of a relative variant of this axiom.

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[13] We wish to invoke the picture of deflating a balloon: the Riemann sphere is thought of as a balloon and a presheaf is deflatable if "after puncturing at ∞" the sphere deflates onto a point.

Example 6.10. If a presheaf $\mathcal{F} : \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{Sp}$ is $\mathbb{A}^1$-invariant, then it is deflatable. Indeed, the map π^* is an equivalence and there is a natural $\mathbb{A}^1$-homotopy between j^* and ∞^*.

The following lemma is a variant of one of the main results of [33], stated in a convenient language for our use. It proves Gersten injectivity for good cohomology theories satisfying Nisnevich descent. We denote by Reg_k the category of regular Noetherian k-schemes.

Lemma 6.11 ([33])**.** *Let k be a perfect field and $\mathcal{F} : \mathrm{Reg}_k^{\mathrm{op}} \to Sp$ be a finitary, Nisnevich sheaf such that $\mathcal{F}|_{\mathrm{Sm}_k^{\mathrm{op}}}$ is deflatable. Then for any $j \in \mathbb{Z}$ and any regular local k-algebra R with fraction field F, the canonical map*

$$\pi_j(\mathcal{F}(R)) \longrightarrow \pi_j(\mathcal{F}(F))$$

is injective.

Proof. For a regular k-scheme X and closed immersion $Z \hookrightarrow X$ we will write

$$\mathcal{F}_Z(X) := \mathrm{fibre}(\mathcal{F}(X) \to \mathcal{F}(X \setminus Z)),$$

so that we have a long exact sequence functorial in X and Z

$$\cdots \to \pi_j(\mathcal{F}(X)) \to \pi_j(\mathcal{F}(X \setminus Z)) \to \pi_{j-1}(\mathcal{F}_Z(X)) \to \cdots .$$

By Néron–Popescu it suffices to prove the result when $R = \mathcal{O}_{X,x}$ is the local ring of a closed point $x \in X$ where X is a smooth affine k-scheme.

Let $s \in \ker(\pi_j(\mathcal{F}(R)) \to \pi_j(\mathcal{F}(F))$; by possibly shrinking X, we may assume that s is defined on X and vanishes away from some closed subscheme $Z \hookrightarrow X$ of positive codimension, i.e., s lifts to an element $\widetilde{s} \in \pi_j(\mathcal{F}_Z(X))$. To prove the result, it suffices to produce an open neighborhood $U \subseteq X$ of x and a closed subscheme $Z' \hookrightarrow U$ with $Z \cap U \subset Z'$ such that $\widetilde{s}$ vanishes on $\pi_i(\mathcal{F}_{Z'}(U))$.

Gabber's presentation lemma [33, Theorem 3.1.1] (see [57] for the case in which k is a finite field) furnishes an open neighborhood $U \subseteq X$ of x, a smooth affine k-scheme V, a morphism $\phi = (\psi, v) : U \to V \times \mathbb{A}^1$ such that $\psi|_{Z \cap U}$ is finite, and a Nisnevich square

$$\begin{tikzcd}
U \setminus (Z \cap U) \arrow[r] \arrow[d] & U \arrow[d, "\phi"] \\
\mathbb{A}^1_V \setminus (\phi(Z \cap U)) \arrow[r] & \mathbb{A}^1_V .
\end{tikzcd}$$

In particular $\phi(Z \cap U) \hookrightarrow \mathbb{A}^1_V$ is a closed immersion. By Nisnevich excision, we have that $\pi_i(\mathcal{F}_{Z \cap U}(U)) \cong \pi_i(\mathcal{F}_{\mathbb{A}^1_V \cap \phi(Z \cap U)}(\mathbb{A}^1_V))$. Now set $F := \psi(Z \cap U)$ so that $Z \cap U \subset \psi^{-1}(F) =: Z'$. So we have a commutative diagram

$$\begin{tikzcd}
\pi_j(\mathcal{F}_{Z \cap U}(U)) \arrow[r] & \pi_j(\mathcal{F}_{Z'}(U)) \\
\pi_j(\mathcal{F}_{\phi(Z \cap U)}(\mathbb{A}^1_V)) \arrow[r] \arrow[u, "\cong"] & \pi_j(\mathcal{F}_{\mathbb{A}^1_F}(\mathbb{A}^1_V)) \arrow[u]
\end{tikzcd}$$

and, to finish the proof, we need only show that the bottom map is zero. The map of interest is the top horizontal map of the following commutative diagram

$$\begin{tikzcd}
\pi_j(\mathcal{F}_{\phi(Z \cap U)}(\mathbb{A}^1_V)) \arrow[r] & \pi_j(\mathcal{F}_{\mathbb{A}^1_F}(\mathbb{A}^1_V)) & \\
 & & \pi_j(\mathcal{F}_F(V)) \arrow[ul] \\
\pi_j(\mathcal{F}_{\phi(Z \cap U)}(\mathbb{P}^1_V)) \arrow[r] \arrow[uu, "\simeq"] & \pi_j(\mathcal{F}_{\mathbb{P}^1_F}(\mathbb{P}^1_V)) \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

where the triangle commutes exactly because $\mathcal{F}$ is a deflateable. However, the bottom composite is zero, since $\phi(Z \cap U)$ does not meet the ∞-section of $\mathbb{P}^1_V$, and thus the top map is also zero as desired. ☐

The projective bundle formula implies that our motivic cohomology is deflatable:

Lemma 6.12. *For any $j \geq 0$, the presheaf*

$$\mathbb{Z}(j)^{\mathrm{mot}}|_{\mathrm{Sm}_{\mathbb{F}}} : \mathrm{Sm}_{\mathbb{F}}^{\mathrm{op}} \longrightarrow \mathrm{D}(\mathbb{Z})$$

is deflatable.

Proof. This is actually a standard consequence of the $\mathbb{P}^1$-bundle formula. Theorem 5.3 furnishes us with an equivalence

$$\mathbb{Z}(j)^{\mathrm{mot}} \oplus \mathbb{Z}(j-1)^{\mathrm{mot}}[-2] \xrightarrow{\pi^* \oplus c_1(\mathcal{O}(1))\pi^*} (\mathbb{Z}(j)^{\mathrm{mot}})^{\mathbb{P}^1},$$

whence it suffices to explain why the diagram

$$\begin{tikzcd}
\mathbb{Z}(j)^{\mathrm{mot}} \oplus \mathbb{Z}(j-1)^{\mathrm{mot}}[-2] \arrow[rr] \arrow[dr] & & (\mathbb{Z}(j)^{\mathrm{mot}})^{\mathbb{A}^1} \\
& \mathbb{Z}(j)^{\mathrm{mot}} \arrow[ur] &
\end{tikzcd}$$

commutes in the homotopy category of presheaves. On the $\mathbb{Z}(j)^{\mathrm{mot}}$ component, the diagram commutes already at the level of schemes. On the $\mathbb{Z}(j-1)^{\mathrm{mot}}[-2]$ component, the diagram commutes because on $\mathbb{A}^1_{\mathbb{F}}$ there are natural identifications $\pi^*\infty^*\mathcal{O}(1) \cong \mathcal{O} \cong j^*\mathcal{O}(1)$. □

Remark 6.13. More generally, any presheaf of spectra satisfying the $\mathbb{P}^1$-bundle formula as formulated in [3] is deflatable. Hence the conclusion of Lemma 6.11 holds for these theories.

We now have all the necessary ingredients to prove Theorem 6.1(2):

Proof of Theorem 6.1(2). The goal is to prove, for any regular Noetherian $\mathbb{F}$-scheme X, that the map $\mathbb{Z}(j)^{\mathrm{mot}}(X) \to \mathbb{Z}(j)^{\mathrm{cdh}}(X)$ is an equivalence. Since this factors as

$$\mathbb{Z}(j)^{\mathrm{mot}}(X) \to L_{\mathbb{A}^1}\mathbb{Z}(j)^{\mathrm{mot}}(X) \xrightarrow{\sim} \mathbb{Z}(j)^{\mathrm{cdh}}(X), \tag{6.3}$$

where the equivalence is Theorem 6.1(1), it is equivalent to show that $\mathcal{N}(X) \simeq 0$ where $\mathcal{N} := \mathrm{fib}(\mathbb{Z}(j)^{\mathrm{mot}} \to L_{\mathbb{A}^1}\mathbb{Z}(j)^{\mathrm{mot}})$. Furthermore, since $\mathcal{N}$ is finitary and satisfies Zariski descent, it is enough to show that $\mathcal{N}(R) \simeq 0$ for every regular, Noetherian, local $\mathbb{F}$-algebra R. But we will show in the next paragraph that $\mathcal{N}|_{\mathrm{Sm}_{\mathbb{F}}}$ is deflatable, whence Lemma 6.11 implies that $H^n(\mathcal{N}(R)) \to H^n(\mathcal{N}(F))$ is injective for all n, where F is the fraction field of R. This therefore reduces the problem to showing that $\mathcal{N}(F) \simeq 0$ for every field extension F of $\mathbb{F}$; appealing again to (6.3), this time for $X = \mathrm{Spec}F$, it is equivalent to show that $\mathbb{Z}(j)^{\mathrm{mot}}(F) \xrightarrow{\sim} \mathbb{Z}(j)^{\mathrm{cdh}}(F)$. But this follows from the part of Theorem 6.1(1) stating that $L_{\mathrm{cdh}}\mathbb{Z}(j)^{\mathrm{mot}} \xrightarrow{\sim} \mathbb{Z}(j)^{\mathrm{cdh}}$, as fields are local for the cdh topology.

It remains to prove that $\mathcal{N}|_{\mathrm{Sm}_{\mathbb{F}}}$ is deflatable. Firstly, we know from Lemma 6.12 that $\mathbb{Z}(j)^{\mathrm{mot}}|_{\mathrm{Sm}_{\mathbb{F}}}$ is deflatable. Fixing any choice of deflation for it, this deflation induces (using the explicit formula (6.2)) a deflation for $L_{\mathbb{A}^1}\mathbb{Z}(j)^{\mathrm{mot}}|_{\mathrm{Sm}_{\mathbb{F}}}$ which is compatible with the canonical map $\mathbb{Z}(j)^{\mathrm{mot}}|_{\mathrm{Sm}_{\mathbb{F}}} \to L_{\mathbb{A}^1}\mathbb{Z}(j)^{\mathrm{mot}}|_{\mathrm{Sm}_{\mathbb{F}}}$. Passing to the fibre induces a deflation for $\mathcal{N}|_{\mathrm{Sm}_{\mathbb{F}}}$, as desired. □

7 COMPARISON TO LISSE MOTIVIC COHOMOLOGY

We continue to fix a prime field $\mathbb{F}$. The goal of this section is to study the comparison map from lisse motivic cohomology to our new motivic cohomology, as discussed in Construction 4.35. However, we may now adopt a cleaner point of view on this comparison map. Indeed, we now know from Corollary 6.4 that the restriction of $\mathbb{Z}(j)^{\mathrm{mot}}$ to smooth $\mathbb{F}$-algebras coincides with classical motivic cohomology $\mathbb{Z}(j)^{\mathrm{cla}}$. Therefore we will henceforth view $\mathbb{Z}(j)^{\mathrm{lse}}$ as the left Kan extension of $\mathbb{Z}(j)^{\mathrm{mot}}$, restricted to smooth $\mathbb{F}_p$-algebras, back along the inclusion $\mathrm{CAlg}_{\mathbb{F}}^{\mathrm{sm}} \subseteq \mathrm{CAlg}_{\mathbb{F}}$. For any $\mathbb{F}_p$-algebra, this formally induces the same comparison map

$$\mathbb{Z}(j)^{\mathrm{lse}}(A) \longrightarrow \mathbb{Z}(j)^{\mathrm{mot}}(A) \tag{7.1}$$

as Construction 4.35. This map is certainly not an equivalence in general: the left side is supported in cohomological degree $\leq 2j$ (see Proposition 3.3) but this bound cannot always be true for the right side: otherwise the Atiyah–Hirzebruch spectral sequence would then imply that $K(A)$ were always connective.

In general it seems to be a deep question to what extent the right side of (7.1) of is controlled by the left side. In other words, how much of motivic cohomology can be recovered from that of smooth algebras? In this section we provide some partial answers to this question. In particular we will show that, for A local, (7.1) induces an equivalence

$$\mathbb{Z}(j)^{\mathrm{lse}}(A) \xrightarrow{\sim} \tau^{\leq j}\mathbb{Z}(j)^{\mathrm{mot}}(A).$$

Note that, in light of Remark 3.4, this provides a description of $\tau^{\leq j}\mathbb{Z}(j)^{\mathrm{mot}}(A)$ purely in terms of algebraic cycles. We will return to the link between motivic cohomology and algebraic cycles in Section 9.

In this section we also establish some vanishing theorems and prove a Nesterenko–Suslin isomorphism.

7.1 Behaviour of motivic cohomology in degrees $\leq 2j$

We begin with the following rational statement, writing

$$\mathbb{Q}(j)^{\mathrm{lse}}(A) := \mathbb{Z}(j)^{\mathrm{lse}} \otimes_{\mathbb{Z}} \mathbb{Q}, \qquad \mathbb{Q}(j)^{\mathrm{mot}}(A) := \mathbb{Z}(j)^{\mathrm{mot}}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

for the rationalisations of our motivic cohomologies. (Note that $\mathbb{Q}(j)^{\mathrm{lse}}$ is the left Kan extension of the restriction of $\mathbb{Q}(j)^{\mathrm{mot}}$ to smooth $\mathbb{F}$-algebras, since left Kan extension commutes with filtered colimits of functors.)

Lemma 7.1. *For any $\mathbb{F}$-algebra A and $j \geq 0$, the map (7.1) induces an equivalence*

$$\mathbb{Q}(j)^{\mathrm{lse}}(A) \xrightarrow{\sim} \tau^{\leq 2j}\mathbb{Q}(j)^{\mathrm{mot}}(A).$$

In other words, the functor $\tau^{\leq 2j}\mathbb{Q}(j)^{\mathrm{mot}} : \mathrm{CAlg}_{\mathbb{F}} \to D(\mathbb{Z})$ is left Kan extended from smooth $\mathbb{F}$-algebras.

Proof. Rationally, by Theorem 4.12(2) (characteristic zero) and Theorem 4.27(2) (characteristic $p > 0$), there is a natural isomorphism of filtered spectra $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(A)_{\mathbb{Q}} \cong \bigoplus_{j \geq \star} \mathbb{Q}(j)^{\mathrm{mot}}(A)[2j]$ for any $\mathbb{F}$-algebra A of finite valuative dimension. Restricting to smooth $\mathbb{F}$-algebras and left Kan extending back identifies the map

$$\bigoplus_{j \geq 0} \mathbb{Q}(j)^{\mathrm{lse}}(A)[2j] \longrightarrow \bigoplus_{j \geq 0} \mathbb{Q}(j)^{\mathrm{mot}}(A)[2j]$$

(obtained by rationalising the direct sum of (7.1) over all weights), for any $\mathbb{F}$-algebra A of finite valuative dimension, with the canonical map $\mathrm{K}^{\mathrm{cn}}(A)_{\mathbb{Q}} \to \mathrm{K}(A)_{\mathbb{Q}}$. Since the latter map is the connective cover, we deduce the same for the former map, i.e., $\mathbb{Q}(j)^{\mathrm{lse}}(A) \xrightarrow{\sim} \tau^{\leq 2j}\mathbb{Q}(j)^{\mathrm{mot}}(A)$. This proves the result for $\mathbb{F}$-algebras of finite valuative dimension; the general case follows by passing to a filtered colimit. ☐

Corollary 7.2. *For any local $\mathbb{F}$-algebra A and $j \geq 0$, the map (7.1) induces an equivalence*

$$\mathbb{Q}(j)^{\mathrm{lse}}(A) \xrightarrow{\sim} \tau^{\leq j}\mathbb{Q}(j)^{\mathrm{mot}}(A).$$

In other words, the functor $\tau^{\leq j}\mathbb{Q}(j)^{\mathrm{mot}} : \mathrm{CAlg}^{\mathrm{loc}}_{\mathbb{F}} \to \mathrm{D}(\mathbb{Z})$ is left Kan extended from essentially smooth, local $\mathbb{F}$-algebras.

Proof. This follows from the previous lemma since, for A local, the lisse motivic cohomology $\mathbb{Z}(j)^{\mathrm{lse}}(A)$ is supported in cohomological degrees $\leq j$. ☐

The proof of the previous corollary also implies the following rational vanishing result:

Corollary 7.3. *For any local $\mathbb{F}$-algebra A and $0 \leq j < i \leq 2j$, we have $H^i_{\mathrm{mot}}(A, \mathbb{Q}(j)) = 0$.*

Remark 7.4 (Rational Drinfeld vanishing)**.** If A is a Henselian local $\mathbb{F}$-algebra, then we can improve the vanishing bound of the previous corollary by 1; namely we also have $H^{2j+1}_{\mathrm{mot}}(A, \mathbb{Q}(j)) = 0$ for all $j \geq 0$. Indeed, this follows from the theorem of Drinfeld that $\mathrm{K}_{-1}(A) = 0$ and the decomposition $\mathrm{K}(A)_{\mathbb{Q}} \simeq \bigoplus_{j \geq 0} \mathbb{Q}(j)^{\mathrm{mot}}(A)[2j]$ (when A has finite valuative dimension, which we may assume by taking a filtered colimit).

We now prove an integral version of Corollary 7.2. By taking H^1 of the map (5.1), we get a natural map

$$A^{\times} \longrightarrow H^1_{\mathrm{mot}}(A, \mathbb{Z}(1))$$

for any $\mathbb{F}$-algebra A; by multiplicativity this induces *symbol maps*

$$(A^{\times})^{\otimes j} = A^{\times} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} A^{\times} \longrightarrow H^j_{\mathrm{mot}}(A, \mathbb{Z}(j) \tag{7.2}$$

for all $j \geq 1$.

Lemma 7.5. *For any local $\mathbb{F}$-algebra A and $j \geq 1$, the map (7.2) factors through the Milnor K-group* $\mathrm{K}_j^M(A)$

Proof. We must show that the map respects the Steinberg relation, so may assume $j = 2$. Now let $a \in A^\times$ be a unit such that $1 - a$ is also a unit; let $\mathbb{F}[t] \to A$, $t \mapsto a$ be the induced map, and $\mathfrak{p} \subseteq \mathbb{F}[t]$ the pullback of the maximal ideal of A. There is a commutative diagram by naturality

$$\begin{tikzcd}
\mathbb{F}[t]_\mathfrak{p}^\times \otimes_\mathbb{Z} \mathbb{F}[t]_\mathfrak{p}^\times \arrow[r] \arrow[d] & H^2_{\mathrm{mot}}(\mathbb{F}[t]_\mathfrak{p}, \mathbb{Z}(2)) \arrow[d] \\
A^\times \otimes_\mathbb{Z} A^\times \arrow[r] & H^2_{\mathrm{mot}}(A, \mathbb{Z}(2))
\end{tikzcd}$$

in which the left vertical arrow sends $t \otimes 1 - t$ to $a \otimes 1 - a$. The problem therefore reduces to the case of the local ring $\mathbb{F}[t]_\mathfrak{p}$. But, setting $F := \mathrm{Frac}(\mathbb{F}[t])$, we have a second commutative diagram by naturality

$$\begin{tikzcd}
F^\times \otimes_\mathbb{Z} F^\times \arrow[r] & H^2_{\mathrm{mot}}(F, \mathbb{Z}(2)) \\
\mathbb{F}[t]_\mathfrak{p}^\times \otimes_\mathbb{Z} \mathbb{F}[t]_\mathfrak{p}^\times \arrow[r] \arrow[u] & H^2_{\mathrm{mot}}(\mathbb{F}[t]_\mathfrak{p}, \mathbb{Z}(2)) \arrow[u]
\end{tikzcd}$$

in which the right vertical arrow is injective by Gersten injectivity in motivic cohomology (here we are implicitly using that the new motivic cohomology of $\mathbb{F}[t]_\mathfrak{p}]$ coincides with the classical theory, i.e.,). So the problem finally reduces to the case of the field F, in which case it is a theorem of Nesterenko–Suslin and Totaro [94, 109] that the symbol map indeed respects the Steinberg relation. $\square$

We will repeatedly use the following general observation about functors in what follows. Recall that a functor $F : \mathrm{CAlg}_\mathbb{F}^{\mathrm{loc}} \to \mathrm{Sp}$ is said to be *rigid* if for any local $\mathbb{F}$-algebra A and henselian ideal $I \subseteq A$, the canonical map is an equivalence $F(A) \xrightarrow{\simeq} F(A/I)$.

Lemma 7.6. *Let $F : \mathrm{CAlg}_\mathbb{F}^{\mathrm{loc}} \to Sp$ be a rigid functor. Then F is left Kan extended from the subcategory of essentially smooth local $\mathbb{F}$-algebras.*

Proof. As observed in Remark 3.4, we can build for any $B \in \mathrm{CAlg}_\mathbb{F}^{\mathrm{loc}}$ a simplicial resolution $P_\bullet \to B$ where each term P_m is an ind-smooth, local $\mathbb{F}$-algebra and each face map $P_{m+1} \to P_m$ is a henselian surjection. Since F is rigid the simplicial spectrum $m \mapsto F(P_m)$ is equivalent to the constant simplicial diagram at $F(B)$, and so $|F(P_\bullet)| \xrightarrow{\simeq} |F(B)|$. Furthermore, since Δ^{op} is contractible diagram, we have that $|F(B)| \xrightarrow{\simeq} F(B)$. $\square$

The following is the first main theorem of the section, showing that Zariski locally our weight-j motivic cohomology is left Kan extended from smooth algebras in degrees $\leq j$:

Theorem 7.7. *For any local $\mathbb{F}$-algebra A and $j \geq 0$, the map (7.1) induces an equivalence*

$$\mathbb{Z}(j)^{\mathrm{lse}}(A) \xrightarrow{\simeq} \tau^{\leq j}\mathbb{Z}(j)^{\mathrm{mot}}(A).$$

In other words, the functor $\tau^{\leq j}\mathbb{Z}(j)^{\mathrm{mot}} : \mathrm{CAlg}_\mathbb{F}^{\mathrm{loc}} \to \mathrm{D}(\mathbb{Z})$ is left Kan extended from essentially smooth, local $\mathbb{F}$-algebras.

Proof. The result is true rationally by Corollary 7.2, so it suffices to prove the result for $\tau^{\leq j}(\mathbb{Z}(j)^{\mathrm{mot}}(-))/\ell$ for all primes ℓ. We first claim that the result is true for the functor

$$\tau^{\leq j}(\mathbb{Z}(j)^{\mathrm{mot}}(-)/\ell) : \mathrm{CAlg}_\mathbb{F}^{\mathrm{loc}} \to \mathrm{D}(\mathbb{Z}).$$

Indeed, if ℓ is invertible in $\mathbb{F}$ then we have an equivalence $\tau^{\leq j}(\mathbb{Z}(j)^{\mathrm{mot}}/\ell) = \tau^{\leq j} R\Gamma_{\text{ét}}(-, \mu_\ell^{\otimes j})$ by Theorem 4.33(1). Since étale cohomology is even rigid, it is left Kan extended from smooth $\mathbb{F}$-algebras by Lemma 7.6. On the other hand, if $\ell = p = \mathrm{char}(\mathbb{F})$ then $\tau^{\leq j}(\mathbb{Z}(j)^{\mathrm{mot}}/p) = \tau^{\leq j}(\mathbb{F}_p(j)^{\mathrm{syn}})$ by Corollary 4.32, which is also left Kan extended from smooth $\mathbb{F}_p$-algebras: indeed, $\mathbb{F}_p(j)^{\mathrm{syn}}$ is even left Kan extended from finitely generated polynomial $\mathbb{F}_p$-algebras by definition, and $\tau^{>j}\mathbb{F}_p(j)^{\mathrm{syn}}$ identifies with $\widetilde{\nu}(j)[-j-1]$ by Remark 4.30(2), which is rigid by Remark 4.31 and so left Kan extended from smooth algebras by Lemma 7.6.

We now claim, for all local $\mathbb{F}$-algebras A, that the canonical map

$$\tau^{\leq j}(\mathbb{Z}(j)^{\mathrm{mot}}(A)/\ell) \to (\tau^{\leq j}(\mathbb{Z}(j)^{\mathrm{mot}}(A)))/\ell$$

is an equivalence for all primes ℓ; this will complete the proof. For this, it suffices to prove that the map $H^j_{\mathrm{mot}}(A, \mathbb{Z}(j)) \to H^j_{\mathrm{mot}}(A, \mathbb{Z}(j)/\ell)$ is surjective for all local $\mathbb{F}$-algebras A. To see this, we pick a Henselian surjection $P \to A$ where P is an ind-smooth (necessarily local) $\mathbb{F}$-algebra. Since $\tau^{\leq j}(\mathbb{Z}(j)^{\mathrm{mot}}(-)/\ell)$ is left Kan extended from essentially smooth $\mathbb{F}$-algebras, the induced map in top degree $H^j_{\mathrm{mot}}(P, \mathbb{Z}/\ell(j)) \to H^j_{\mathrm{mot}}(A, \mathbb{Z}/\ell(j))$ is surjective. By naturality and Lemma 7.5, this surjective map moreover fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{K}^M_j(P) \arrow[r, two heads] \arrow[d] & H^j_{\mathrm{mot}}(P, \mathbb{Z}(j)) \arrow[r, two heads] \arrow[d] & H^j_{\mathrm{mot}}(P, \mathbb{Z}/\ell(j)) \arrow[d, two heads] \\
\mathrm{K}^M_j(A) \arrow[r] & H^j_{\mathrm{mot}}(A, \mathbb{Z}(j)) \arrow[r] & H^j_{\mathrm{mot}}(A, \mathbb{Z}/\ell(j)).
\end{tikzcd}$$

As indicated, the arrows on the top row are also surjective. Indeed, by taking filtered colimits it is enough to prove such surjectivities for an essentially smooth, local $\mathbb{F}$-algebra in place of P: then for the first top arrow it is a theorem of Kerz, and for the second arrow it follows from the usual bound that $\mathbb{Z}(j)^{\mathrm{mot}}$ is Zariski locally supported in degrees $\leq j$ on smooth $\mathbb{F}$-algebras. From this it follows that the map $H^j_{\mathrm{mot}}(A, \mathbb{Z}(j)) \to H^j_{\mathrm{mot}}(A, \mathbb{Z}/\ell(j))$ is surjective as required. ☐

The proof of the previous result yields the following vanishing theorem:

Corollary 7.8 (Hilbert 90)**.** *For any local $\mathbb{F}$-algebra A and $j \geq 1$, we have $H^{j+1}_{\mathrm{mot}}(A, \mathbb{Z}(j)) = 0$; if A is henselian then also $H^1_{\mathrm{mot}}(A, \mathbb{Z}(0)) = 0$.*

Proof. The surjectivity of the map $H^j_{\mathrm{mot}}(A, \mathbb{Z}(j)) \to H^j_{\mathrm{mot}}(A, \mathbb{Z}/\ell(j))$ from the end of the previous proof means that that $H^{j+1}_{\mathrm{mot}}(A, \mathbb{Z}(j))$ is torsion-free. But it also a torsion group since it vanishes rationally by Corollary 7.3 and Remark 7.4. ☐

To summarise the situation so far, we have shown the following Zariski locally about weight-j motivic cohomology:

1. in degrees $\leq j$ it is left Kan extended from smooth algebras;
2. in degrees $j+1, \ldots, 2j$ it vanishes rationally;
3. in degree $j+1$ it vanishes (unless $j = 0$, in which case it is only true Nisnevich locally).

In general we are not sure what to expect about the behaviour of motivic cohomology in degrees $j+2, \ldots, 2j$, except that it vanishes rationally. In particular we are uncertain whether the following additional Nisnevich local vanishing should be indicative of a more extensive vanishing range:

Proposition 7.9 (Hilbert 90+1)**.** *For any Henselian local $\mathbb{F}$-algebra A and $j \geq 1$, we have $H^{j+2}_{\mathrm{mot}}(A, \mathbb{Z}(j)) = 0$.*

Proof. The vanishing holds rationally by Corollary 7.3, or Remark 7.4 if $j = 1$. So it remains to show that $H^{j+2}_{\mathrm{mot}}(A, \mathbb{Z}(j))$ is torsion-free; we will prove the stronger (actually equivalent, since we already have Corollary 7.8) result that $H^{j+1}_{\mathrm{mot}}(A, \mathbb{Z}/\ell(j)) = 0$ for all prime numbers ℓ.

Let us first suppose that ℓ is invertible in $\mathbb{F}$. Consider the fibre sequence

$$\tau^{\leq j} R\Gamma_{\text{ét}}(-, \mu_\ell^{\otimes j}) \longrightarrow \tau^{\leq j+1} R\Gamma_{\text{ét}}(-, \mu_\ell^{\otimes j}) \longrightarrow H^{j+1}_{\text{ét}}(-, \mu_\ell^{\otimes j})[-j-1] =: \mathcal{F}[-j-1]$$

on qcqs $\mathbb{F}$-schemes. Sheafifying this sequence with respect to the cdh topology and using the identification of Theorem 4.33(1), we get a fibre sequence:

$$\mathbb{Z}^{\mathrm{mot}}(j)/\ell \longrightarrow L_{\mathrm{cdh}} \tau^{\leq j+1} R\Gamma_{\text{ét}}(-, \mu_\ell^{\otimes j}) \longrightarrow (L_{\mathrm{cdh}}\mathcal{F})[-j-1].$$

Since étale cohomology satisfies cdh descent, the middle term agrees with $R\Gamma_{\text{ét}}(-, \mu_\ell^{\otimes j})$ up to degrees $\leq j+1$, whence we deduce that

$$H^{j+1}_{\mathrm{mot}}(A, \mathbb{Z}/\ell(j)) = \ker\left(H^{j+1}_{\text{ét}}(A, \mu_\ell^{\otimes j}) \to H^0(L_{\mathrm{cdh}}\mathcal{F}(A))\right).$$

But now we claim that the map appearing on the right side is injective. Indeed, letting k be the residue field of A, by functoriality it fits into a commutative diagram

$$\begin{tikzcd}
H^{j+1}_{\text{ét}}(A,\mu_\ell^{\otimes j}) \arrow[r] \arrow[d] & H^0(L_{\text{cdh}}\mathcal{F}(A)) \arrow[d] \\
H^{j+1}_{\text{ét}}(k,\mu_\ell^{\otimes j}) \arrow[r] & H^0(L_{\text{cdh}}\mathcal{F}(k))
\end{tikzcd}$$

where the left vertical arrow is an isomorphism (by rigidity of étale cohomology), and the bottom horizontal arrow is also an isomorphism (since fields are cdh points). This completes the proof that $H^{j+1}_{\text{mot}}(A,\mathbb{Z}/\ell(j))=0$.

Next suppose that $\mathbb{F}=\mathbb{F}_p$ and $\ell=p$. Then from the fundamental fibre sequence of Corollary 4.32 we see that there is a natural identification

$$H^{j+1}_{\text{mot}}(A,\mathbb{Z}/p(j))=\ker\big(\widetilde{\nu}(j)(A)\to H^0_{\text{cdh}}(A,\widetilde{\nu}(j))\big).$$

Exactly as in the previous case, this vanishes by comparison with the residue field: namely $\widetilde{\nu}(j)(A)\overset{\simeq}{\to}\widetilde{\nu}(j)(k)\overset{\simeq}{\to}H^0_{\text{cdh}}(k,\widetilde{\nu}(j))$, the first isomorphism being rigidity of $\widetilde{\nu}(j)$ and the second being the fact that k is a point for the cdh topology □

For the next corollary, we let $\mathrm{CAlg}^{\text{h.loc}}_{\mathbb{F}}$ be the category of henselian local $\mathbb{F}$-algebras.

Corollary 7.10. *For $j\geq 1$, the functor $\tau^{\leq j+2}\mathbb{Z}(j)^{\text{mot}}:\mathrm{CAlg}^{\text{h.loc}}_{\mathbb{F}}\to D(\mathbb{Z})$ is left Kan extended from henselisations of essentially smooth, local $\mathbb{F}$-algebras.*

Proof. Corollary 7.8 and Proposition 7.9 imply that $\tau^{\leq j+2}\mathbb{Z}(j)^{\text{mot}}=\tau^{\leq j}\mathbb{Z}(j)^{\text{mot}}$ on Henselian local $\mathbb{F}$-algebras, so the claim reduces to Theorem 7.7. □

An important consequence of the above results is a partial description of weight one motivic cohomology.

Corollary 7.11 (Weight one motivic cohomology)**.** *For any qcqs $\mathbb{F}$-scheme X, the cofibre of the comparison map $R\Gamma_{\text{Nis}}(X,\mathbb{G}_m)[-1]\overset{\simeq}{\to}\mathbb{Z}(1)^{\text{mot}}(X)$ from* (5.1) *is supported in degrees >3. In particular there are natural isomorphisms*

$$H^i_{\text{mot}}(X,\mathbb{Z}(1))\cong\begin{cases}0 & i\leq 0\\ \mathcal{O}(X)^\times & i=1\\ \mathrm{Pic}(X) & i=2\\ H^2_{\text{Nis}}(X,\mathbb{G}_m) & i=3.\end{cases}$$

Proof. It is enough to prove the claim Nisnevich locally, i.e., that the map $A^\times[-1]\to\tau^{\leq 3}\mathbb{Z}(1)^{\text{mot}}(A)$ is an equivalence for any Henselian local $\mathbb{F}$-algebra A. But that is exactly what the Corollary 7.10 states in the case $j=1$. □

7.2 Singular Nesterenko–Suslin isomorphism

In the previous subsection we constructed the symbol map $\mathrm{K}^M_j(A)\to H^j_{\text{mot}}(A,\mathbb{Z}(j))$ and used it in the course of the proof of Theorem 7.7. We now establish an analogue of the theorem of Nesterenko–Suslin and Totaro, namely the symbol map is essentially an isomorphism; we just need to take care to replace Milnor K-theory by the improved variant $\widehat{\mathrm{K}}^M_j(A)$ of Gabber and Kerz [68].

Theorem 7.12 (Singular Nesterenko–Suslin isomorphism)**.** *For any local $\mathbb{F}$-algebra A and $j\geq 0$, the symbol map $\mathrm{K}^M_j(A)\to H^j_{\text{mot}}(A,\mathbb{Z}(j))$ descends to an isomorphism*

$$\widehat{\mathrm{K}}^M_j(R)\overset{\simeq}{\to}H^j_{\text{mot}}(R,\mathbb{Z}(j)).$$

Proof. Let $P_\bullet\to A$ be a simplicial resolution as in Remark 3.4, so that the totalisation of the simplicial complex $m\mapsto\tau^{\leq j}\mathbb{Z}(j)^{\text{mot}}(P_m)$ calculates the evaluation on A of the left Kan extension of $\tau^{\leq j}\mathbb{Z}(j)^{\text{mot}}$ from essentially smooth local $\mathbb{F}$-algebras. In light of Theorem 7.7, the totalisation is equivalent to $\tau^{\leq j}\mathbb{Z}(j)^{\text{mot}}(A)$. Calculating the top degree H^j as a coequaliser, this means that the canonical map

$$\mathrm{coeq}\big(H^j_{\text{mot}}(P_1,\mathbb{Z}(j))\rightrightarrows H^j_{\text{mot}}(P_0,\mathbb{Z}(j))\big)\longrightarrow H^j_{\text{mot}}(A,\mathbb{Z}(j))$$

is an equivalence.

The canonical map

$$\mathrm{coeq}\big(\mathrm{K}_j^M(P_1) \rightrightarrows \mathrm{K}_j^M(P_1)\big) \longrightarrow \mathrm{K}_j^M(A)$$

is also an equivalence; this is the content of [84, Prop. 1.17].

Comparing the two coequaliser diagrams via the natural symbol maps we obtain two immediate conclusions.

1. The symbol map $\mathrm{K}_j^M(A) \to H^j_{\mathrm{mot}}(A, \mathbb{Z}(j))$ is surjective. Indeed, as already used in the proof of Theorem 7.7, the symbol map $\mathrm{K}_j^M(P_0) \to H^j_{\mathrm{mot}}(P_0, \mathbb{Z}(j))$ is surjective by Kerz.

2. Secondly, we may complete the proof in the case that A has big residue field, i.e., its residue field has more than M_j elements in the sense of [68, Prop. 10(5)]. Indeed, in that case the ind-smooth local rings P_i, $i = 0, 1$, also have big residue field and so the symbol maps $\mathrm{K}_j^M(P_i) = \widehat{\mathrm{K}}_j^M(P_i) \to H^j_{\mathrm{mot}}(P_i, \mathbb{Z}(j))$ are isomorphisms by Kerz [68, Prop. 10(11)]. Comparing the two coequaliser diagrams we deduce that the symbol map $\mathrm{K}_j^M(A) = \widehat{\mathrm{K}}_j^M(A) \to H^j_{\mathrm{mot}}(A, \mathbb{Z}(j))$ is also an isomorphism.

It remains to treat the case that A has small (in particular, finite) residue field $\mathbb{F}_q$, which we do by constructing some ad-hoc transfer maps on $H^j_{\mathrm{mot}}(-, \mathbb{Z}(j))$. Let $\ell > 0$ be an integer prime to $|\mathbb{F}_q : \mathbb{F}_p|$, so that $\mathbb{F}_{q^\ell} = \mathbb{F}_q \otimes_{\mathbb{F}_p} \mathbb{F}_{p^\ell}$ (this identity holds because the right side is a tensor product of Galois extensions of coprime degree, therefore a field); this also implies that the semi-local ring $A \otimes_{\mathbb{F}_p} \mathbb{F}_{p^\ell}$ is in fact local, as its quotient by its Jacobson radical is a field. Finally observe that $P_\bullet \otimes_{\mathbb{F}_p} \mathbb{F}_{p^\ell} \to A \otimes_{\mathbb{F}_p} \mathbb{F}_{p^\ell}$ is a simplicial resolution satisfying the conditions of Remark 3.4, and so (replacing A by $A \otimes_{\mathbb{F}_p} \mathbb{F}_{p^\ell}$ above), we have a coequaliser diagram

$$\mathrm{coeq}\big(H^j_{\mathrm{mot}}(P_1 \otimes_{\mathbb{F}_p} \mathbb{F}_{p^\ell}, \mathbb{Z}(j)) \rightrightarrows H^j_{\mathrm{mot}}(P_0 \otimes_{\mathbb{F}_p} \mathbb{F}_{p^\ell}, \mathbb{Z}(j))\big) \xrightarrow{\sim} H^j_{\mathrm{mot}}(A \otimes_{\mathbb{F}_p} \mathbb{F}_{p^\ell}, \mathbb{Z}(j))$$

Since classical motivic cohomology of smooth schemes admits functorial transfer maps along finite morphisms, this diagram induces a transfer map $N : H^j_{\mathrm{mot}}(A \otimes_{\mathbb{F}_p} \mathbb{F}_{p^\ell}, \mathbb{Z}(j)) \to H^j_{\mathrm{mot}}(A, \mathbb{Z}(j))$ such that the pre-composition with the canonical map $H^j_{\mathrm{mot}}(A, \mathbb{Z}(j)) \to H^j_{\mathrm{mot}}(A \otimes_{\mathbb{F}_p} \mathbb{F}_{p^\ell}, \mathbb{Z}(j))$ is multiplication by ℓ. We make no claims that this transfer map is natural, independent of the simplicial resolution, compatible with any transfers on Milnor K-theory, etc.; in fact, we only care about the resulting fact that therefore $\ker\big(H^j_{\mathrm{mot}}(A, \mathbb{Z}(j)) \to H^j_{\mathrm{mot}}(A \otimes_{\mathbb{F}_p} \mathbb{F}_{p^\ell}, \mathbb{Z}(j))\big)$ is annihilated by ℓ.

It now follows formally that the symbol map factors through $\widehat{\mathrm{K}}_j^M(A)$: indeed, given $x \in \ker(\mathrm{K}_j^M(A) \to \widehat{\mathrm{K}}_j^M(A))$ and any ℓ as in the previous paragraph such that $p^\ell > M_j$ then, by functoriality of the symbol map, and the established isomorphism for the local ring $A \otimes_{\mathbb{F}_p} \mathbb{F}_{p^\ell}$, we deduce that ℓx is annihilated by the symbol map. Picking a different value of ℓ, prime to the first value, shows that x is annihilated by the symbol map, i.e., the latter factors through $\widehat{\mathrm{K}}_j^M(A)$. The new symbol map $\widehat{\mathrm{K}}_j^M(A) \to H^j_{\mathrm{mot}}(A, \mathbb{Z}(j))$ is surjective since the original one was.

To prove that the new symbol map $\widehat{\mathrm{K}}_j^M(A) \to H^j_{\mathrm{mot}}(A, \mathbb{Z}(j))$ is injective, we again use a transfer argument; let x be in the kernel. Then the transfer map for improved Milnor K-theory, and the established isomorphism in case of big residue field, shows that $\ell x = 0$ for any ℓ as above. Again picking coprime values of ℓ shows that $x = 0$ and so completes the proof. ☐

8 MOTIVIC SOULÉ–WEIBEL VANISHING AND PRO CDH DESCENT

One of the most influential conjectures concerning the algebraic K-theory of singular schemes has been Weibel's conjecture [118], now a theorem of Kerz–Strunk–Tamme [70]. It states, in particular, that for a Noetherian scheme X of dimension $\leq d$, the negative K-groups $\mathrm{K}_{-n}(X)$ vanish for $n > d$. Kerz–Strunk–Tamme's proof proceeds by first establishing pro cdh descent for K-theory of Noetherian schemes. For earlier work on special cases on Weibel's conjecture and pro cdh descent, see for example [34, 49, 72, 73, 74, 90, 117].

Our goal in this section is to prove the following analogous results about our motivic cohomology; as usual, let $\mathbb{F}$ denote a prime field.

Theorem 8.1 (Motivic Soulé–Weibel vanishing)**.** *Let $j \geq 0$ and let X be a Noetherian $\mathbb{F}$-scheme of finite dimension. Then $H^i_{\mathrm{mot}}(X, \mathbb{Z}(j)) = 0$ for all $i > j + \dim X$.*

Theorem 8.2 (Pro cdh descent)**.** *On the category of Noetherian $\mathbb{F}$-schemes, the presheaf $\mathbb{Z}(j)^{\mathrm{mot}}$ satisfies pro cdh descent for each $j \geq 0$. That is, given any abstract blowup square of Noetherian $\mathbb{F}$-schemes*

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & X \end{tikzcd} , \tag{8.1}$$

the associated square of pro complexes

$$\begin{tikzcd} \mathbb{Z}(j)^{\mathrm{mot}}(X) \arrow[r] \arrow[d] & \mathbb{Z}(j)^{\mathrm{mot}}(X') \arrow[d] \\ \{\mathbb{Z}(j)^{\mathrm{mot}}(rY)\}_r \arrow[r] & \{\mathbb{Z}(j)^{\mathrm{mot}}(rY')\}_r \end{tikzcd}$$

is cartesian.[14]

Remark 8.3 (Relation to Weibel's K-theoretic vanishing conjecture)**.** Let X be a Noetherian $\mathbb{F}$-scheme of dimension $\leq d$. Theorem 8.1 states that the Atiyah–Hirzebruch spectral sequence $E_2^{ij} = H^{i-j}_{\mathrm{mot}}(X, \mathbb{Z}(j)) \Rightarrow \mathrm{K}_{-i-j}(X)$ is supported in the left half plane $x \leq d$. From this one immediately reads off parts of Weibel's package conjectures about lower K-groups: both the vanishing $\mathrm{K}_{-n}(X) = 0$ for $n > d$, and the usual description of $\mathrm{K}_{-d}(X)$ via an edge map isomorphism

$$H^d_{\mathrm{cdh}}(X, \mathbb{Z}) = H^d_{\mathrm{mot}}(X, \mathbb{Z}(0)) \cong \mathrm{K}_{-d}(X).$$

Theorem 8.1 can therefore be seen as a motivic refinement of Weibel's vanishing conjecture. This is moreover reflected in the proof of the theorem, which is based on both the arguments of [34, 35], where Cortiñas–Haesemyer–Schlichting–Weibel proved Weibel's vanishing conjecture and Vorst's conjecture for varieties over a characteristic zero field, and of [70], where Kerz–Strunk–Tamme used pro cdh descent to prove Weibel's conjecture in general.

We stress however that Theorem 8.1 is certainly not a new proof of Weibel's vanishing conjecture in K-theory, since our theory uses the fundamental square Theorem 3.8 which itself relies on the work of Kerz–Strunk–Tamme. We refer the reader to Remark 8.11 for more details on this point.

Remark 8.4 (Applications to Adams eignenspaces)**.** To use the Atiyah–Hirzebruch spectral sequence to deduce the usual K-theoretic Weibel conjecture from a vanishing result in motivic cohomology, it would have been sufficient to establish the following weaker diagonal vanishing line: $H^i_{\mathrm{mot}}(X, \mathbb{Z}(j)) = 0$ for all $i > 2j + \dim X$. The stronger vertical vanishing line of Theorem 8.1 is related to a vanishing theorem of Soulé as follows. By rationalising the Atiyah–Hirzebruch spectral sequence and rewriting in terms of Adams eigenspaces, Theorem 8.1 implies that for any Noetherian $\mathbb{F}$-scheme X of dimension $\leq d$ we have the following vanishing for each $n \in \mathbb{Z}$: the Adams eigenspace $\mathrm{K}_n(X)^{(j)}_{\mathbb{Q}}$ vanishes whenever $j > n + d$. This vanishing is due to SGA6 [12, Exp. VI, Thm. 6.9] in the case of K_0 of Noetherian schemes with an ample line bundle, and when $n > 0$ to Soulé [104, Corol. 1] for the higher algebraic K-groups of Noetherian rings; when $n < 0$ this vanishing of Adams eigenspaces of negative K-groups is new as far as we are aware.

In other words, Theorem 8.1 provides an integral refinement of Soulé's result, as well as an extension beyond the affine case and to negative K-groups.

8.1 Pro cdh descent

Here we prove Theorem 8.2. We begin by noting a similar pro cdh descent property for the Nisnevich cohomology of wedge powers of the cotangent complex $R\Gamma(-, L^i_{-/\mathbb{F}}) : \mathrm{Sch}^{\mathrm{qcqs,op}}_{\mathbb{F}} \to \mathrm{Sp}$. The following is a slight generalization of [90, Thm. 2.10].

---

[14] Here rY denotes the $r-1^{\mathrm{st}}$ infinitesimal thickening of Y inside X, and similarly for rY'. By "cartesian" we simply mean that all pro cohomology groups of the birelative term are zero as pro abelian groups; since $\mathbb{Z}(j)^{\mathrm{mot}}$ of a Noetherian scheme is bounded above depending only on the dimension (this does not require Theorem 8.1, but only the descriptions given in the proof Corollary 8.13), this is equivalent to being cartesian in the ∞-category of pro complexes.

Lemma 8.5. *For any abstract blowup square of Noetherian $\mathbb{F}$-schemes (8.1) and $i \geq 0$, the square of pro complexes*

$$\begin{tikzcd}
R\Gamma(X, L^i_{-/\mathbb{F}}) \arrow[r] \arrow[d] & R\Gamma(X', L^i_{-/\mathbb{F}}) \arrow[d] \\
\{R\Gamma(rY, L^i_{-/\mathbb{F}})\}_r \arrow[r] & \{R\Gamma(rY', L^i_{-/\mathbb{F}})\}_r
\end{tikzcd}$$

is cartesian.

Proof. The proof works in the exact same way as in [90, Thm. 2.10], except that we need to justify why [90, Thm. 2.4] does not require the stated finite dimensionality hypothesis; but this follows from the general formal functions theorem of [86, Lem. 8.5.1.1]. □

Next we establish pro cdh descent for syntomic cohomology; remarkably, the proof uses algebraic K-theory:

Proposition 8.6. *For any abstract blowup square of Noetherian $\mathbb{F}_p$-schemes (8.1) and $j \geq 0$, the square of pro complexes*

$$\begin{tikzcd}
\mathbb{Z}_p(j)^{\mathrm{syn}}(X) \arrow[r] \arrow[d] & \mathbb{Z}_p(j)^{\mathrm{syn}}(X') \arrow[d] \\
\{\mathbb{Z}_p(j)^{\mathrm{syn}}(rY)\}_r \arrow[r] & \{\mathbb{Z}_p(j)^{\mathrm{syn}}(rY')\}_r
\end{tikzcd}$$

is cartesian.

Proof. Since mod-p syntomic cohomology $\mathbb{F}_p(j)^{\mathrm{syn}}$ admits a finite filtration with graded pieces given by shifts of $R\Gamma(-, L^i_{-/\mathbb{F}})$ for various i (by sheafifying Lemma 4.16), it satisfies pro cdh descent thanks to Lemma 8.5. The remaining difficulty is to extend the result from $\mathbb{F}_p(j)^{\mathrm{syn}}$ to $\mathbb{Z}_p(j)^{\mathrm{syn}}$.

Fix $n \in \mathbb{Z}$ and set $A_r := H^n(\mathbb{Z}_p(j)^{\mathrm{syn}}(X, X', rY))$ for each $r \geq 0$; the goal is to show that the pro abelian group $\{A_r\}_r$ vanishes. We claim that each group A_r is bounded p-power torsion. Granting this claim, we may complete the proof as follows. Given $s \geq 1$, pick $c >$ such that $p^c A_s = 0$. By the previous paragraph and an induction, we see that $\{A_r/p^c\}_r = 0$; so there exists $s' > s$ such that the transition map $A_{s'}/p^c \to A_s/p^c$ is zero. But $A_s/p^c = A_s$, so this shows that the transition map $A_{s'} \to A_s$ is zero, as required.

It remains to prove that A_r is bounded p-power torsion. But it is both derived p-complete (since it is H^n of a p-complete complex) and satisfies $A_r[\frac{1}{p}] = 0$ (since $\mathbb{Q}_p(j)^{\mathrm{syn}}$ satisfies cdh descent by Corollary 4.20, which we proved using K-theory), so it is killed by a power of p by [17, Thm. 1.1]. □

Proof of Theorem 8.2. If $\mathbb{F} = \mathbb{F}_p$ then Theorem 4.24(2) shows that $\mathbb{Z}(j)^{\mathrm{mot}}$ differs from $\mathbb{Z}_p(j)^{\mathrm{syn}}$ by a cdh sheaf; since syntomic cohomology satisfies pro cdh descent by Proposition 8.6, the same is true for motivic cohomology.

If instead $\mathbb{F} = \mathbb{Q}$ then the third term in the fundamental fibre sequence Theorem 4.10(3) satisfies pro cdh descent by Lemma 8.5; since the middle term in the fibre sequence is a cdh sheaf, it follows that motivic cohomology also satisfies pro cdh descent. □

8.2 Proof of motivic Soulé–Weibel vanishing

Fix a weight $j \geq 0$. Here we prove Theorem 8.1. In fact, we prove the following stronger statement:

Theorem 8.7. *Let X be a Noetherian $\mathbb{F}$-scheme of dimension $\leq d$. Then the fibre*

$$W(j)(X) := \mathrm{fib}\left(\mathbb{Z}(j)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(j)^{\mathrm{cdh}}(X)\right)$$

vanishes in degrees $> j + d$.

Remark 8.8. The cohomology theory $W(j)(X)$ are the shifts of graded pieces of a filtration on the fibre of $\mathrm{K}(X) \to \mathrm{KH}(X)$. In turn each $W(j)(X)$ admits a filtration whose graded pieces are the "N^r of motivic cohomology," i.e., the fibres of the maps $\mathbb{Z}(j)^{\mathrm{mot}}(X) \to \mathbb{Z}(j)^{\mathrm{mot}}(\mathbb{A}^r \times X)$. These groups refine Bass' N^rK-groups which measure the failure of algebraic K-theory to be $\mathbb{A}^r$-invariant. We intend to explore questions surrounding these groups using motivic methods in the future.

Note that Theorem 8.7 implies Theorem 8.1, as we already know from Theorem 3.5(1) that the cdh-local motivic cohomology $\mathbb{Z}(j)^{\mathrm{cdh}}(X)$ is supported in degrees $\leq j+d$; but the stronger statement also tells us that the map $\mathbb{Z}(j)^{\mathrm{mot}}(X) \to \mathbb{Z}(j)^{\mathrm{cdh}}(X)$ is surjective in degree $j+d$.

First we quote the following result, whose Zariski version has often appeared in previous work on the subject:

Lemma 8.9 (Nisnevich vanishing lemma)**.** *Let $\mathcal{F}$ be a Nisnevich sheaf of abelian groups on a Noetherian scheme X, and $d \geq 0$ such that the stalk $\mathcal{F}_x$ vanishes for all $x \in X$ satisfying $\dim \overline{\{x\}} > d$. Then $H^i_{\mathrm{Nis}}(X, \mathcal{F}) = 0$ for all $i > d$.*

Proof. This is proved by induction using the coniveau spectral sequence, as stated in [64, (1.2.5)] (see also the proof of [96, Thm. 1.32]). □

When proving Weibel and Vorst's conjecture for finite type schemes over characteristic zero fields [34, 35], Cortiñas–Haesemeyer(–Schlichting)–Weibel analysed the relation between of the top degree Nisnevich and cdh cohomologies of sheaves of differential forms; although pro cdh descent did not appear explicitly, it was implicitly encoded in their use of the formal functions theorem. When proving Weibel's conjecture for K-theory [70], Kerz–Strunk–Tamme used pro cdh descent to show that the desired vanishing was of a birational nature. The following proposition may be seen as an axiomatisation of the aforementioned arguments.

Proposition 8.10. *Let k be a base ring and $W : \mathrm{Sch}_k^{\mathrm{qcqs,op}} \to \mathrm{D}(\mathbb{Z})$ a finitary Nisnevish sheaf with the following properties:*

1. *$L_{\mathrm{cdh}} W \simeq 0$.*
2. *W satisfies pro cdh descent on Noetherian k-schemes.*
3. *For any Noetherian, local, henselian k-algebra A and nilpotent ideal $I \subseteq A$, the fibre $W(A, I) = \mathrm{fib}(W(A) \to W(A/I))$ is supported in degrees ≤ 0.*

Then, for any Noetherian k-scheme X of finite dimension, $W(X)$ is supported in degrees $\leq \dim X$.

Proof. We begin by globalising hypothesis (3) by noting the following

> (3'): for any Noetherian k-scheme X of finite dimension and nil immersion $X_0 \to X$, the fibre $W(X, X_0)$ is supported in degrees $\leq \dim X$ (and so $W(X) \to W(X_0)$ is an equivalence in degrees $> \dim X$).

This follows from Nisnevich descent and Nisnevich exactness of the closed embedding $X_0 \hookrightarrow X$, more precisely using that X_{Nis} has cohomological dimension $\leq \dim X$ and the sheaf $W(-, - \times_X X_0)$ on X_{Nis} has stalks supported in degrees ≤ 0 by (3).

Now let X be a Noetherian k-scheme. We must show that $W(X)$ is supported in degree $\leq \dim X$. Using (3') we may assume that X is reduced.

If $\dim X = 0$ then X is a finite disjoint union of the spectra of fields. Since spectra of fields are points for the cdh topology we have $W(X) \xrightarrow{\sim} L_{\mathrm{cdh}} W(X)$, which vanishes by hypothesis (1).

We now proceed by induction on $\dim X$, so assume that $d := \dim X > 0$ and that the desired vanishing has been proved for Noetherian k-schemes of dimension $< d$. We examine the bounded Nisnevich descent spectral sequence

$$E_2^{ab} = H^a_{\mathrm{Nis}}(X, \mathcal{H}^b(W)) \implies H^{a+b}(W(X)),$$

where $\mathcal{H}^b(W)$ is the Nisnevich sheafifcation of the abelian presheaf $Y \mapsto H^b(W(Y))$. The E_2 page of this spectral sequence enjoys various vanishings:

1. $E_2^{ab} = 0$ if $a > d$ (or if $a < 0$), since X has Nisnevich cohomological dimension $\leq d$.
2. $E_2^{ab} = 0$ if $a > 0$ and $b > d$. Indeed, for such b and any $x \in X$ such that $\dim \overline{\{x\}} > 0$, then $\dim \mathcal{O}^h_{X,x} < d$ and so the stalk $\mathcal{H}^b(W)_x = H^b(W(\mathcal{O}^h_{X_x}))$ (the equality is a consequence of W being finitary) vanishes by the inductive hypothesis. Lemma 8.9 now implies that $\mathcal{H}^b(W(j))$ has no higher cohomology.

3. $E_2^{ab} = 0$ if $b \le d$ and $a + b > d$. The proof will be clearest if we start by fixing $b \le d$. Then, for any $x \in X$ such that $\dim \overline{\{x\}} > d - b$, we have that $\dim \mathcal{O}^h_{X,x} < b \le d$, i.e., $b > \dim \mathcal{O}^h_{X,x}$ and $\dim \mathcal{O}^h_{X,x} < d$; so $\mathcal{H}^b(W(j))_x = 0$ by the inductive hypothesis (and again finitariness to compute the stalk in terms of $\mathcal{O}^h_{X,x}$). Lemma 8.9 now implies that $\mathcal{H}^b(W)$ has no cohomology in degrees $> d - b$, or in other words $H^a(\mathcal{H}^b(W)) = 0$ whenever $a + b > d$.

Thanks to vanishings (1)–(4), we can read off from the Nisnevich descent spectral sequence edge map isomorphisms

$$H^n(W(X)) \xrightarrow{\simeq} H^0_{\text{Nis}}(X, \mathcal{H}^n(W))$$

for all $n > d$. For the rest of the proof fix $n > d$. Allowing X to vary, the previous isomorphism may be rephrased as follows:

> (†) On the category of Noetherian k-schemes of dimension $\le d$, the abelian presheaf $H^n(W(-))$ is a Nisnevich sheaf.

In fact, we will only need to know that it is Nisnevich separated.

We now return to our fixed X of dimension $\le d$, and pick a cohomology class $\alpha \in H^n(W(X))$; we must show that $\alpha = 0$. We claim that there exists a modification $f : X' \to X$ (i.e., a proper morphism where X' is also reduced and such there there exists a dense open $U \subseteq X$ satisfying $f^{-1}(U) \xrightarrow{\simeq} U$) such that $f^*\alpha = 0$ in $H^n(W(X'))$. To prove the claim we first use hypothesis (1) to see that, for any $a \in \mathbb{Z}$, the presheaf $H^a(W(-))$ vanishes on valuation rings, therefore vanishes after cdh sheafification. In particular there exists a cdh cover $U \to X$ such that α vanishes in $H^n(W(U))$; we can then refine U to a cdh cover of the form $X_2 \to X_1 \xrightarrow{g} X$ where $X_1 \to X$ is a proper cdh (often called a cdp) cover and $X_2 \to X_1$ is a Nisnevich cover [108, Prop. 5.9]. Next note that there exists a modification $f : X' \to X$ which factors through X_1: for example pick a dense open $U \subseteq X$ such that $f^{-1}(U) \xrightarrow{\simeq} U$, and define X' to be $(-)_{\text{red}}$ of the closure of U in X'. By construction α vanishes when we pull back to X_2, hence also to $X' \times_X X_2$; but $X' \times_X X_2 \to X'$ is a Nisnevich cover of schemes of dimension $\le d$ (since modifications and étale morphisms do not increase dimension), so (†) implies that $H^n(W(X')) \to H^n(W(X' \times_X X_2))$ is injective and therefore α already vanished when pulled back to X'. This completes the proof of the claim.

Our modification $X' \to X$ fits into an abstract blowup square (8.1) in which Y' and Y have dimension $< d$. From hypothesis (2) and the inductive hypothesis applied to the infinitesimal thickenings of Y and Y', we see that $H^n(W(X)) \to H^n(W(X'))$ is an isomorphism. But this map was constructed so as to kill α. Therefore α was already zero in $H^n(W(X))$, completing the proof. □

Example 8.11 (Usual Weibel vanishing)**.** Here we present a revisionist version of Kerz–Strunk–Tamme's proof of Weibel vanishing [70]. First note that Proposition 8.10 applies verbatim to finitary Nisnevich presheaves of spectra $W : \mathrm{Sch}_k^{\mathrm{qcqs}} \to \mathrm{Sp}$ satisfying the same hypotheses; we stated it for preshaves of complexes only for simplicity.

In particular, the proposition applies when $W := \mathrm{fib}(\mathrm{K} \to \mathrm{KH})$ and $k = \mathbb{Z}$. Indeed, hypothesis (1) follows from the fact that $K(V) \xrightarrow{\simeq} \mathrm{KH}(V)$ for any valuation ring V [70, Thm. 6.3] [65, Thm. 3.4]; hypothesis (2) follow fom pro cdh descent of K-theory and cdh descent of KH-theory [70]; hypothesis (3) follows from nil invariance of negative K-theory. We therefore deduce, for any Noetherian scheme X, that $\mathrm{fib}(\mathrm{K}(X) \to \mathrm{KH}(X))$ is supported in homological degrees $\ge -\dim X$. Since there are various ways to show that $\mathrm{KH}(X)$ is supported in homological degrees $\ge -\dim X$ [69] [65, Rmk. 3.5(a)], we deduce that $\mathrm{K}(X)$ is also supported in homological degrees $\ge -\dim X$ as required.

We now verify that the previous proposition may also be applied in our motivic situation of interest, at least up to a harmless shift:

Proposition 8.12. *The presheaf* $W(j) = \mathrm{fib}(\mathbb{Z}(j)^{\mathrm{mot}} \to \mathbb{Z}(j)^{\mathrm{cdh}}) : \mathrm{Sch}_{\mathbb{F}}^{\mathrm{qcqs,op}} \to D(\mathbb{Z})$ *is a finitary Nisnevich sheaf with the following properties:*

1. $L_{\mathrm{cdh}} W(j) = 0$.
2. $W(j)$ *satisfies pro cdh descent on Noetherian* $\mathbb{F}$*-schemes.*
3. *For any* $\mathbb{F}$*-algebra* A *and finitely generated nilpotent ideal* $I \subseteq A$*, the fibre* $W(j)(A, I)$ *is supported in degrees* $\le j$*.*

Proof. The presheaf $W(j)$ is a finitary Nisnevich sheaf since $\mathbb{Z}(j)^{\mathrm{mot}}$ and $\mathbb{Z}(j)^{\mathrm{cdh}}$ are, by Theorems 3.5(2), 4.10(5), and 4.24(4).

(1): $W(j)$ vanishes after cdh sheafification since $L_{\mathrm{cdh}}\mathbb{Z}(j)^{\mathrm{mot}} \xrightarrow{\sim} \mathbb{Z}(j)^{\mathrm{cdh}}$ by Theorem 6.1(1).

(2): $W(j)$ satisfies pro cdh descent on Noetherian schemes, since the same is true of $\mathbb{Z}(j)^{\mathrm{mot}}$ by Theorem 8.2.

(3): We first treat the case that $\mathbb{F} = \mathbb{Q}$. From the fundamental fibre sequence of Theorem 4.10(3), for both A and A/I, we have a description of the relative term $W(j)(A, I)$ as

$$W(j)(A,I) = \mathrm{fib}\left(L\Omega^{<j}_{A/\mathbb{Q}} \to L\Omega^{<j}_{(A/I)/\mathbb{Q}}\right)[-1].$$

This is supported in degrees $< j$ since $\Omega^{j-1}_{A/\mathbb{Q}} \to \Omega^{j-1}_{(A/I)/\mathbb{Q}}$ is surjective.

In the case that $\mathbb{F} = \mathbb{F}_p$, the pullback square of Theorem 4.24(2) shows that $W(j)(A, I) \simeq \mathrm{fib}(\mathbb{Z}_p(j)^{\mathrm{syn}}(A) \to \mathbb{Z}_p(j)^{\mathrm{syn}}(A/I))$, which is derived p-complete; so it is sufficient to prove the claim modulo p, namely that $\mathrm{fib}(\mathbb{F}_p(j)^{\mathrm{syn}}(A) \to \mathbb{F}_p(j)^{\mathrm{syn}}(A/I))$ is supported in degree $< j$. Since nilpotent ideals are Henselian, this is a special case of [5, Thm. 5.2]. □

Proof of Theorem 8.9. Apply Proposition 8.10 to the presheaf $W := W(j)[j]$. The hypotheses of the proposition are satisfied thanks to Proposition 8.12. □

Our arguments implicitly reprove the results of [34, 35] concerning Nisnevich and cdh cohomology of differential forms, as well as a similar style of result in finite characteristic:

Corollary 8.13. *1. For any Noetherian $\mathbb{F}$-scheme of dimension d and $j \geq 0$, the canonical map $H^{j+d}_{\mathrm{mot}}(X, \mathbb{Z}(d)) \to H^{j+d}_{\mathrm{cdh}}(X, \mathbb{Z}(d))$ is surjective.*

2. For $j \geq 1$ and any Noetherian $\mathbb{Q}$-scheme X of dimension $\leq d$, the canonical map

$$H^d_{\mathrm{Nis}}(X, \Omega^{j-1}_{-/\mathbb{Q}}) \to H^d_{\mathrm{cdh}}(X, \Omega^{j-1}_{-/\mathbb{Q}})$$

is surjective.

3. For $j \geq 0$ and any Noetherian $\mathbb{F}_p$-scheme X of dimension $\leq d$, the canonical maps

$$H^d_{\mathrm{Nis}}(X, \widehat{K}^M_j/p) \to H^d_{\mathrm{cdh}}(X, \widehat{K}^M_j/p) \quad \text{and} \quad H^{d-1}_{\mathrm{Nis}}(X, \widetilde{\nu}(j)) \to H^{d-1}_{\mathrm{cdh}}(X, \widetilde{\nu}(j))$$

are surjective, and the canonical map

$$H^d_{\mathrm{Nis}}(X, \widetilde{\nu}(j)) \longrightarrow H^d_{\mathrm{cdh}}(X, \widetilde{\nu}(j))$$

is an isomorphism. Here $\widehat{K}^M_j/p$ denotes improved Milnor K-theory mod p, as an abelian Nisnevich or cdh sheaf.

Proof. The first claim was explained after Remark 8.8. The rest are related to $W(j)(X)$ via the following descriptions, which we state in the generality of qcqs schemes for the sake of possible future reference:

1. For any qcqs $\mathbb{Q}$-scheme X of valuation dimension $\leq d$, then $W(j)(X)$ vanishes in degrees $> j+d+1$ and there is a natural isomorphism $H^{j+d+1}(W(j)(X)) \cong \mathrm{coker}(H^d_{\mathrm{Nis}}(X, \Omega^{j-1}_{-/\mathbb{Q}}) \to H^d_{\mathrm{cdh}}(X, \Omega^{j-1}_{-/\mathbb{Q}}))$.
2. For any qcqs $\mathbb{F}_p$-scheme X of valuative dimension $\leq d$, then $W(j)(X)/p$ vanishes in degrees $>$

$j+d+2$ and there is a natural diagram in which the row and column are exact:

$$\begin{tikzcd}
 & & & H^{d-1}_{\mathrm{Nis}}(X,\widetilde{\nu}(j)) \arrow[d] & \\
 & & & H^{d-1}_{\mathrm{cdh}}(X,\widetilde{\nu}(j)) \arrow[d] & \\
H^d_{\mathrm{Nis}}(X,\widehat{K}^M_j/p) \arrow[r] & H^d_{\mathrm{cdh}}(X,\widehat{K}^M_j/p) \arrow[r, "\delta"] & H^{j+d+1}(W(j)(X)/p) \arrow[r] & \operatorname{coker}\delta \arrow[r] \arrow[d] & 0 \\
 & & & H^d_{\mathrm{Nis}}(X,\widetilde{\nu}(j)) \arrow[d] & \\
 & & & H^d_{\mathrm{cdh}}(X,\widetilde{\nu}(j)) \arrow[d] & \\
 & & & H^{j+d+2}(W(j)(X)/p) \arrow[d] & \\
 & & & 0 &
\end{tikzcd} \tag{8.2}$$

These two claims are clearly sufficient to deduce the corollary, since Theorem 8.7 tells us that $W(j)(X)$ (and so also $W(j)(X)/p$) is supported in degrees $\leq j+d$ whenever X is a Noetherian $\mathbb{F}$-scheme of dimension $\leq d$.

It remains to prove the claims. We first treat the case that $\mathbb{F}=\mathbb{Q}$. From the fundamental fibre sequence Theorem 4.10(3) we have a description of $W(j)(X)$, for any qcqs $\mathbb{Q}$-scheme X, as

$$W(j)(X)=\operatorname{fib}\left(R\Gamma(X,L\Omega^{<j}_{-/\mathbb{Q}})\to R\Gamma_{\mathrm{cdh}}(A/I,L\Omega^{<j}_{-/\mathbb{Q}})\right)[-1].$$

By cohomological vanishing bounds in the Nisnevich and cdh topologies, this fibre is supported in cohomological degrees $\leq j+d+1$ if X has valuative dimension $\leq d$, with its H^{j+d+1} being exactly the desired cokernel.

Next suppose $\mathbb{F}=\mathbb{F}_p$. From the pullback square of Theorem 4.24(2) we see that $W(j)/p=\operatorname{fib}(\mathbb{F}_p(j)^{\mathrm{syn}}\to L_{\mathrm{cdh}}\mathbb{F}_p(j)^{\mathrm{syn}})$, which we compute as follows: on the category of qcqs $\mathbb{F}_p$-schemes, Nisnevich sheafifying Remark 4.30 provides us with a fibre sequence

$$L_{\mathrm{Nis}}\tau^{\leq j}\mathbb{F}_p(j)^{\mathrm{syn}}\longrightarrow\mathbb{F}_p(j)^{\mathrm{syn}}\longrightarrow R\Gamma_{\mathrm{Nis}}(-,\widetilde{\nu}(j))[-j-1],$$

which may be compared to its cdh sheafification to get the following fibre sequence of presheaves on qcqs $\mathbb{F}_p$-schemes:

$$\operatorname{fib}\left(L_{\mathrm{Nis}}\tau^{\leq j}\mathbb{F}_p(j)^{\mathrm{syn}}\to L_{\mathrm{cdh}}\tau^{\leq j}\mathbb{F}_p(j)^{\mathrm{syn}}\right)\longrightarrow W(j)/p\longrightarrow\operatorname{fib}\left(R\Gamma_{\mathrm{Nis}}(-,\widetilde{\nu}(j))\to R\Gamma_{\mathrm{cdh}}(-,\widetilde{\nu}(j))\right)[-j-1]. \tag{8.3}$$

Moreover, $H^j(\mathbb{F}_p(j)^{\mathrm{syn}}(-))$ is Nisnevich locally given by $\widehat{K}^M_j/p$; this follows from the isomorphisms $\widehat{K}^M_j(A)/p\stackrel{\sim}{\to}H^j_{\mathrm{mot}}(A,\mathbb{Z}/p\mathbb{Z}(j))\stackrel{\sim}{\to}H^j_{\mathrm{syn}}(\mathbb{F}_p(A))$ for local $\mathbb{F}_p$-algebras A, the first being the Nesterenko–Suslin isomorphism of Theorem 7.12 (or rather, the mod-p version obtained using Corollary 7.8) and the second isomorphism coming from the fundamental fibre sequence of Corollary 4.32. Since the Nisnevich and cdh sites of X have cohomological dimension $\leq d$ when X has valuative dimension $\leq d$, the claimed vanishing and diagram in (2) can now be read off by by evaluating (8.3) on X. □

9 SOME COMPARISONS TO ALGEBRAIC CYCLES

We present in this section a variety of contexts in which our motivic cohomology admits a description in terms of algebraic cycles. We do not know what to expect in general.

Definition 9.1. For $j\geq 0$ and A a local $\mathbb{F}$-algebra, we say that A has *geometric weight-j motivic cohomology* if $\mathbb{Z}(j)^{\mathrm{mot}}(A)$ is supported in cohomological degrees $\leq j$.

If A has geometric weight-j motivic cohomology in the sense of the definition, then Theorem 7.7 implies that the canonical map $\mathbb{Z}(j)^{\mathrm{lse}}(A) \to \mathbb{Z}(j)^{\mathrm{mot}}(A)$ is an equivalence. But, as explained in Remark 3.4, the complex $\mathbb{Z}(j)^{\mathrm{lse}}(A)$ admits a description purely in terms of algebraic cycles; so in this case we obtain a cycle theoretic description of the whole motivic cohomology $\mathbb{Z}(j)^{\mathrm{mot}}(A)$. This applies to a surprisingly large class of rings:

Theorem 9.2. *Let $j \geq 0$ and let A be a local $\mathbb{F}$-algebra.*

1. *If A is regular Noetherian then it has geometric weight-j motivic cohomology.*
2. *If A is a valuation ring then it has geometric weight-j motivic cohomology.*
3. *If there exists a nil ideal $I \subseteq A$ such that A/I has geometric weight-j motivic cohomology, then so does A.*
4. *If A is Noetherian, henselian of dimension 1, then it has geometric weight-j motivic cohomology.*
5. *If $j \geq 1$ and A is Noetherian, henselian of dimension ≤ 2, then it has geometric weight-j motivic cohomology.*

Proof. (1): Using Néron–Popescu we reduce, by taking a filtered colimit, to the case that A is essentially smooth over $\mathbb{F}$; then we apply the usual Gersten vanishing bound for classical motivic cohomology after knowing the classical comparison result, Corollary 6.4.

(2): If A is a valuation ring then the canonical map $\mathbb{Z}(j)^{\mathrm{mot}}(A) \to \mathbb{Z}(j)^{\mathrm{cdh}}(A)$ is an equivalence since the right vertical maps of the fundamental squares of Theorems 4.10 and 4.24 are equivalences. But now, $\mathbb{Z}(j)^{\mathrm{cdh}}$ is by definition the cdh sheafification of $\mathbb{Z}(j)^{\mathrm{lse}}$ on affines, so also $\mathbb{Z}(j)^{\mathrm{lse}}(A) \xrightarrow{\sim} \mathbb{Z}(j)^{\mathrm{cdh}}(A)$. Finally recall once again from the Gersten bound that $\mathbb{Z}(j)^{\mathrm{lse}}$ is Zariski locally supported in cohomological degrees $\leq j$.

(3): We must show that the relative motivic cohomology $\mathbb{Z}(j)^{\mathrm{mot}}(A, I)$ is supported in degrees $\leq j$; by finitariness we may assume that I is finitary generated, hence nilpotent. First we treat the case that $\mathbb{F} = \mathbb{Q}$. Since $\mathbb{Z}(j)^{\mathrm{cdh}}$ and $R\Gamma_{\mathrm{cdh}}(-, L\Omega^{<j}_{-/\mathbb{Q}})$ are cdh sheaves, they are invariant for the ideal I; by taking the horizontal fibres of the fundamental fibre sequence Theorem 4.10(3) we therefore obtain an equivalence of relative terms

$$\mathbb{Z}(j)^{\mathrm{mot}}(A, I) \simeq L\Omega^{<j}_{A,I/\mathbb{Q}}[-1].$$

The left side is clearly supported in degrees $\leq j$, which completes the proof. Next we assume $\mathbb{F} = \mathbb{F}_p$. Then again $\mathbb{Z}(j)^{\mathrm{cdh}}$ and $L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}$ are invariant for I, and so we obtain an equivalence

$$\mathbb{Z}(j)^{\mathrm{mot}}(A, I) \simeq \mathbb{Z}_p(j)^{\mathrm{syn}}(A, I).$$

In this case it is a non-trivial result that the right side is supported in degrees $\leq j$ [5, Thm. 5.2].

(4): Combine the Soulé-Weibel vanishing bound Theorem 8.1 with Corollary 7.8. Part (5) is proved in the same way, but also using Proposition 7.9. □

Example 9.3 (Dimension 0)**.** Let $j \geq 0$ and let A be a local $\mathbb{F}$-algebra with nil maximal ideal $\mathfrak{m}$. Then, as we saw in the proof of part (3) of the theorem, in characteristic zero there is an equivalence

$$\mathbb{Z}(j)^{\mathrm{mot}}(A, \mathfrak{m}) \simeq L\Omega^{<j}_{A,\mathfrak{m}/\mathbb{Q}}[-1] \tag{9.1}$$

for the relative motivic cohomology, and in characteristic p there is an equivalence

$$\mathbb{Z}(j)^{\mathrm{mot}}(A, \mathfrak{m}) \simeq \mathbb{Z}_p(j)^{\mathrm{syn}}(A, \mathfrak{m}). \tag{9.2}$$

These are remarkable equivalences. Indeed, A has geometric weight-j motivic cohomology (combine cases (1) and (3) of the theorem) so, resolving A as in Remark 3.4, the left sides of (9.1) and (9.2) admit presentations purely in terms of complexes of algebraic cycles. But the right sides are linear invariants ultimately built from differential forms. Isomorphisms between algebraic cycles and differential forms also appear in the theory of Chow groups with modulus [25, 102, 103].

For example, if $A = k[x]/x^e$ where k is a perfect field of characteristic p, then (9.2) states that

$$\mathbb{Z}(j)^{\mathrm{mot}}(A, \mathfrak{m})[1] \simeq \mathbb{W}_{ej}(k)/V^e\mathbb{W}_j(k)$$

(using [107, Thm. 1.1] to describe the syntomic cohomology), thereby offering a cycle theoretic description of the group $\mathbb{W}(k)/V^e\mathbb{W}(k)$.

9.1 Zero cycles on surfaces

In Theorem 9.2 we worked in a local context, but the main ideal globalizes. Suppose that X is a qcqs $\mathbb{F}$-scheme such that, for any $x \in X$, the local ring $\mathcal{O}_{X,x}$ has geometric weight-j motivic cohomology. Then, by checking on stalks, we see that the canonical map

$$(L_{\mathrm{Zar}} L_{\mathrm{Sch}_{\mathbb{F}}^{\mathrm{qcqs,op}}/\mathrm{Sm}_{\mathbb{F}}^{\mathrm{op}}} \mathbb{Z}(j)^{\mathrm{cla}})(X) \longrightarrow \mathbb{Z}(j)^{\mathrm{mot}}(X)$$

is an equivalence; that is, the motivic cohomology $\mathbb{Z}(j)^{\mathrm{mot}}(X)$ is given by Zariski sheafifying the left Kan extension of motivic cohomology from smooth $\mathbb{F}$-schemes, or in other words (using Theorem 7.7) it is given by Zariski sheafifying $\tau^{\leq j}\mathbb{Z}(j)^{\mathrm{mot}}$ on X_{zar}. Similar conclusions holds with Zariski replaced by Nisnevich, if we had instead assumed that each henselian local ring $\mathcal{O}^h_{X,x}$ had geometric weight-j motivic cohomology. These arguments allow us to calculate the motivic cohomology of surfaces in low weights:

Corollary 9.4. *Let X be a Noetherian $\mathbb{F}$-scheme of dimension ≤ 2 (e.g., a curve or surface, with arbitrarily bad singularities, over a field extension of $\mathbb{F}$). Then there are natural equivalences*

$$\mathbb{Z}(j)^{\mathrm{mot}}(X) \simeq \begin{cases} R\Gamma_{\mathrm{cdh}}(X, \mathbb{Z}) & j = 0, \\ R\Gamma_{\mathrm{Nis}}(X, \mathbb{G}_m)[-1] & j = 1, \\ (L_{\mathrm{Nis}}\tau^{\leq j}\mathbb{Z}(j)^{\mathrm{mot}})(X) & j \geq 2, \end{cases}$$

and an isomorphism

$$H^4_{\mathrm{mot}}(X, \mathbb{Z}(2)) \cong H^2_{\mathrm{Nis}}(X, K_2).$$

(The right side denotes Nisnevich cohomology with coefficients in the Nisnevich sheafification of K_2*.)*

Proof. The description of the weight zero motivic cohomology does not depend on the hypotheses on X and may be founded in Examples 4.11 and 4.26.

For weight one, we appeal to Corollary 7.11 and note that $R\Gamma_{\mathrm{Nis}}(X, \mathbb{G}_m)$ and $\mathbb{Z}(1)^{\mathrm{mot}}(X)$ are supported in degrees ≤ 2 and ≤ 3 respectively; for the cohomology of $\mathbb{G}_m$ this is because X has Krull dimension ≤ 2, and for the motivic cohomology we appeal to the Soulé–Weibel vanishing bound, Theorem 8.1.

Now let $j \geq 2$ (in fact, the following argument equally works when $j = 1$). Then the canonical map

$$L_{\mathrm{Nis}}\tau^{\leq j}\mathbb{Z}(j)^{\mathrm{mot}} \to \mathbb{Z}(j)^{\mathrm{mot}} \tag{9.3}$$

of Nisnevich sheaves is an equivalence at all points of X_{Nis} by Theorem 9.2(5) (which, notably, uses the Soulé-Weibel vanishing bound), hence is an equivalence when evaluated on X.

Let us now prove the last statement. We write $\mathcal{H}^j(\mathbb{Z}(j)^{\mathrm{mot}}_X)$ for the sheafification on X_{Nis} of $X_{\mathrm{Nis}} \ni U \mapsto H^j_{\mathrm{mot}}(U, \mathbb{Z}(j)^{\mathrm{mot}})$. By the equivalence of (9.3) and the fact that X has Nisnevich cohomological dimension ≤ 2, there is a natural edge map isomorphism

$$H^{j+2}_{\mathrm{mot}}(X, \mathbb{Z}(j)) \cong H^2_{\mathrm{Nis}}(X, \mathcal{H}^j(\mathbb{Z}(j)^{\mathrm{mot}}_X)).$$

But the singular Nesterenko–Suslin Theorem 7.12 defines a symbol isomorphism $\widehat{K}^M_j \xrightarrow{\simeq} \mathcal{H}^j(\mathbb{Z}(j)^{\mathrm{mot}}_X))$ where $\widehat{K}^M_j$ is the Nesnevich sheaf of improved Milnor K-groups on X; in case $j = 2$ we moreover have $\widehat{K}^M_2 \xrightarrow{\simeq} K_2$ [68, Prop. 10(3)], completing the proof. □

We are very grateful to F. Binda for help with the following proof:

Theorem 9.5. *Let X be a reduced, equi-dimensional, quasi-projective surface over a field k; then there is a natural isomorphism*

$$H^4_{\mathrm{mot}}(X, \mathbb{Z}(2)) \cong \mathrm{CH}_0(X),$$

where $\mathrm{CH}_0(X)$ *denotes the lci Chow group of zero cycles*[15] *of [21].*

Proof. In light of the final isomorphism of Corollary 9.4, we must produce a natural isomorphism $\mathrm{CH}_0(X) \cong H^2_{\mathrm{Nis}}(X, \mathcal{K}_2)$. Such Bloch–Quillen formulae for singular surfaces are are due to Levine [77] and Binda–Krishna–Saito [23]; since the precise form we need does not quite explicitly appear in the papers, we provide the extra details.

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[15] If k is infinite then this is isomorphic to the older Levine–Weibel Chow group of zero cycles $\mathrm{CH}_0^{\mathrm{LW}}(X)$.

Firstly, citing from [23, Lem. 3.4] and the reference there to [54, Lem. 3.2], there is a commutative diagram

$$\begin{tikzcd}
H^2_{\mathrm{Zar}}(X, K_2) \arrow[r, "\lambda'_X"] & H^2_{\mathrm{Nis}}(X, K_2) \arrow[dd, bend left=60, "\text{edge}"] \\
H^2_{\mathrm{Zar}}(X, K_2^M) \arrow[u] \arrow[r, "\lambda_X"] & H^2_{\mathrm{Nis}}(X, K_2^M) \arrow[u] \arrow[d, "\gamma_X"] \\
\mathrm{CH}_0(X) \arrow[u, "\rho_X"] \arrow[r, "\mathrm{cyc}_X"'] & \mathrm{K}_0(X)
\end{tikzcd}$$

where

- cyc_X and ρ_X are cycle class maps from the lci Chow group;
- λ_X and λ'_X are change of topology maps;
- the two vertical maps at the top are induced by the canonical map $\mathrm{K}_2^M \to \mathrm{K}_2$;
- the edge map is the edge map in the Nisnevich descent spectral sequence; and γ_X is defined to make the curvy triangle commute.

We will explain that the cycle class map $\xi_X : \mathrm{CH}_0(X) \to H^2_{\mathrm{Nis}}(X, \mathcal{K}_2)$, defined to be the composite from the bottom left to the top right of the diagram, is an isomorphism.

We first treat the case that k is finite (or more generally perfect). According to [23, Thm 8.1], the cycle class maps cyc_X is injective (this does not require the hypothesis on k); so ξ_X is also injective. Furthermore, according to [64, Thm. 2.5], the Nisnevich cycle class map $\lambda_X \rho_X$ is surjective (this does require the hypothesis on k, as it means that the regular locus X_{reg} is "nice" in the terminology of [op. cit.]). Finally note that $H^2_{\mathrm{Nis}}(X, K_2^M) \to H^2_{\mathrm{Nis}}(X, K_2)$ is surjective, because X_{Nis} has cohomological dimension 2 and the map of Nisnevich sheaves $K_2^M \to K_2$ is surjective. The last two sentences show ξ_X is surjective, completing the proof in this case.

Next we treat the case that k is infinite. Then the cycle class map $\mathrm{CH}_0(X) \to H^2_{\mathrm{Nis}}(X, K_2)$ (i.e., bottom left to top left of the diagram) is an isomorphism by [23, Corol. 7.8]. It remains to show that λ'_X is an isomorphism (which does not require the hypothesis on k); this is well-known to experts but we could not find a reference. This isomorphism is proved by comparing the Zariski descent spectral sequence $E_2^{i,j} = H^i_{\mathrm{Zar}}(X, K_{-j}) \Rightarrow \mathrm{K}_{-i-j}(X)$ to the analogous Nisnevich descent spectral sequence, as follows. Both spectral sequences are supported in columns $i = 0, 1, 2$ since X_{Zar} and X_{Nis} have cohomological dimension 2; therefore the only non-zero differentials ∂ are on the first page, from the 0^{th} column to the 2^{ed} column, and so the abutement filtrations on $\mathrm{K}_0(X)$ are described via a commutative diagram

$$\begin{tikzcd}
 & & & H^0_{\mathrm{Zar}}(X, \mathbb{Z}) & \\
H^0_{\mathrm{Zar}}(X, \mathbb{G}_m) \arrow[r, "\partial"] & H^2_{\mathrm{Zar}}(X, K_2) \arrow[r, "\text{edge}"] & \mathrm{K}_0(X) \arrow[r] & \mathrm{K}_0(X)/\mathrm{edge}(H^2_{\mathrm{Zar}}(X, K_2)) \arrow[u] \arrow[r] & 0 \\
 & & & H^1_{\mathrm{Zar}}(X, \mathbb{G}_m) \arrow[u] & \\
 & & & 0 \arrow[u] &
\end{tikzcd}$$

and similarly replacing Zar by Nis everywhere. The Zariski diagram maps to the Nisnevich one, involving in particular the map λ'_X, and one sees from a diagram chase that λ'_X being an isomorphism follows from the following standard facts:

1. $H^0_{\mathrm{Zar}}(X, \mathbb{Z}) \to H^0_{\mathrm{Nis}}(X, \mathbb{Z})$ is injective (it is even an isomorphism);
2. $H^1_{\mathrm{Zar}}(X, \mathbb{G}_m) \to H^1_{\mathrm{Nis}}(X, \mathbb{G}_m)$ is injective (it is even an isomorphism);

3. the boundary maps ∂ in both the Zariski and Nisnevich diagrams is zero because $\mathbb{G}_m$ is representable by a one-dimensional scheme: more precisely, given $f \in H^0_{\mathrm{Zar}}(X, \mathbb{G}_m)$ (resp. Nisnevich), let $X \to \mathrm{Spec}(k[t^{\pm 1}])$ be the induced map; then the analogous boundary map in the Zariski (resp. Nisnevich) descent spectral sequence for $\mathrm{Spec}(k[t^{\pm 1}])$ is zero, simply because $H^2_{\mathrm{zar}}(\mathrm{Spec} k[t^{\pm 1}], \mathcal{K}_2) = 0$ (resp. Nisnevich) for dimensional reasons; so by functoriality we deduce $\partial(f) = 0$, as desired.

This completes the proof. □

Remark 9.6 (Zero cycles)**.** The argument at the end of the proof of Corollary 9.4 shows, for any $j \geq 0$ and any qcqs $\mathbb{F}$-scheme X of Krull dimension $\leq d$, that there is a natural map

$$H^d_{\mathrm{Nis}}(X, \widehat{K}^M_j) \cong H^{j+d}(L_{\mathrm{Nis}} \tau^{\leq j} \mathbb{Z}(j)^{\mathrm{mot}}(X)) \longrightarrow H^{j+d}_{\mathrm{mot}}(X, \mathbb{Z}(j)).$$

Taking $j = d = \dim(X)$, we hope that $H^{2d}_{\mathrm{mot}}(X, \mathbb{Z}(d))$ provides a "good" group of zero cycles on X. This point of view will be explored further elsewhere.

A THE cdh-SHEAFIFICATION OF AN ÉTALE SHEAF

In this technical appendix we prove a result about éh sheafification, stating that it is equivalent to étale sheafification followed by cdh sheafification. While this might appear to be merely a curiosity, it is a crucial input into controlling our mod-p motivic cohomology in characteristic p.

We begin by recalling the definition:

Definition A.1. Letting R be a commutative ring, the *éh topology* on $\mathrm{Sch}^{\mathrm{qcqs}}_R$ is the Grothdendieck topology generated by abstract blowup squares[16] and étale covers.

It is thus relatively formal that a presheaf $\mathcal{F} : \mathrm{Sch}^{\mathrm{qcqs}}_R \to \mathrm{Sp}$ is an éh sheaf if and only if it is both a cdh and étale sheaf; in particular, assuming $\mathcal{F}$ is an étale sheaf, then it is an éh sheaf if and only if it sends abstract blowup squares to cartesian squares.

Remark A.2. The éh topology is finer than the cdh topology but coarser than the h topology; the latter can be defined as the topology generated by abstract blowup squares and fppf covers. It is an insight of Geisser, when defining his *arithmetic cohomology* of separated, finite type schemes over finite fields, that the éh topology is better suited to capturing mod-p information than the h topology [48, Page 30, Remark]. Furthermore, the points of the éh topology are given by strictly henselian valuation rings.

To prove the main result of the appendix, let us recall the following notion introduced in [40]. We say that an ∞-category $\mathcal{C}$ is *compactly generated by cotruncated objects* if it is compactly generated and each compact object is contruncated. The main point of $\mathcal{C}$ is that filtered colimits commute with cosimplicial limits as noted in [40, Lem. 3.1.7(1)].

Theorem A.3. *Let R be a commutative ring. Let $\mathcal{C}$ be a stable ∞-category which is compactly generated by cotruncated objects. Then for any finitary, étale sheaf of spectra*

$$F : \mathrm{Sch}^{\mathrm{qcqs,op}}_R \to \mathcal{C},$$

we have that $L_{\mathrm{cdh}} F$ is an éh-sheaf.

Proof. It suffices to prove that $L_{\mathrm{cdh}} F$ satisfies étale descent. Since cdh-sheaves are, in particular, Nisnevich sheaves, it suffices to prove that $L_{\mathrm{cdh}} F$ satifies finite étale descent by the structure result of [86, Thm. B.6.4.1]. Let $X \to Y$ be a finite étale cover and let $C^\bullet_Y(X)$ be the Čech nerve; our goal is to prove that the map $L_{\mathrm{cdh}} F(X) \to \lim_\Delta L_{\mathrm{cdh}} F(C^\bullet_Y(X))$ is an equivalence. Set

$$G : \mathrm{Sch}^{\mathrm{qcqs,op}}_X \to \mathrm{Sp} \qquad U \mapsto \mathrm{Fib}(L_{\mathrm{cdh}} F(U) \to \lim_{\Delta} L_{\mathrm{cdh}} F(C^\bullet_Y(X) \times_X U)).$$

We claim that $G \simeq 0$. Since a cdh and Nisnevich squares are stable under taking products of schemes, G is a cdh sheaf on Sch_X. Since totalisations commute with filtered colimits in $\mathcal{C}$ [40, Lem. 3.1.7(1)], G is finitary. Furthermore, by part (2) of the same lemma, G is hypercomplete. Hence to prove that $G \simeq 0$, it suffices to argue that for any henselian valuation ring V with a map $X \to \mathrm{Spec}(V)$, we have that $G(V) \simeq 0$.

---

[16] As in §3.3, our convention for abstract blowup squares means that p and i are assumed to be finitely presented.

Now, since finite étale morphisms are stable under base change, the map $C_Y^\bullet(X) \times_X \mathrm{Spec}(V) \to \mathrm{Spec}(V)$ is a cosimplicial diagram such that the face maps are all finite étale. But now, by a theorem of Nagata which we review below, each term of the Čech nerve is a coproduct of henselian valuation rings. Therefore, since $L_{\mathrm{cdh}}F$ does preserve coproduct decompositions and henselian valuation rings are cdh points, we have an equivalence of cosimplicial objects.

$$L_{\mathrm{cdh}}F(C_Y^\bullet(X) \times_X V) \simeq F(C_Y^\bullet(X) \times_X V).$$

The limit of the latter is $F(V)$ since F was assumed to be an étale sheaf and we conclude by the fact that $L_{\mathrm{cdh}}F(V) \simeq F(V)$. □

Corollary A.4. *Let R be a commutative ring. With the same hypotheses as in Theorem A.3 on $\mathcal{C}$, we have a canonical equivalence of endofucntors on $\mathcal{C}$-valued presheaves on $\mathrm{Sch}_R^{\mathrm{qcqs}}$:*

$$L_{\mathrm{cdh}}L_{\acute{e}t} \simeq L_{\acute{e}h}.$$

Lemma A.5. *[Nagata's Hensel lemma] Let R be a henselian valuation ring. Then, for any finite étale morphism $R \to S$, S is a product of henselian valuation rings.*

Proof. We give a proof using more modern references. Since S is finite over a henselian local ring, by [106, Tag 04GH], S is a finite product of henselian local rings, each of which is finite over S. It then suffices to prove to assume furthermore that S is a henselian local ring and prove that S is, in fact, a valuation ring. Since étale morphisms are flat and the diagonal is an open immersion (hence flat), [59, Corol. 2.15] asserts that localizations of S at any prime ideal is a valuation ring. But we are done because S is, in fact, a local ring. □

Remark A.6. Let V be a valuation ring and let $\mathrm{\acute{E}t}_V$ be its étale site; the underlying category are étale V-schemes. The above proof shows a little more we have an equivalence:

$$L_{\mathrm{Zar}}\left(F|_{\mathrm{\acute{E}t}_V}\right) \simeq L_{\mathrm{cdh}}F|_{\mathrm{\acute{E}t}_V}.$$

Remark A.7. The order of sheafifications in Theorem A.3 is important: the canonical map $L_{\mathrm{\acute{e}t}}L_{\mathrm{cdh}}\mathcal{F} \to L_{\mathrm{\acute{e}h}}\mathcal{F}$ need not be an equivalence, because the étale sheafification of a cdh sheaf need not be a cdh sheaf.

B A SPECTRUM LEVEL CORTIÑAS–HAESEMEYER–WEIBEL THEOREM

The goal of this appendix is to prove the following technical enhancement of the main theorem of [36].

Theorem B.1. *Let $n \geq 2$ and suppose that k is a $\mathbb{Q}$-algebra. Then for any qcqs k-scheme X we have a commutative diagram of $\mathbb{E}_\infty$-rings, functorial in X:*

$$\begin{tikzcd} \mathrm{K}(X)[\frac{1}{n}] \arrow[r, "\psi^n"] \arrow[d, "ch"] & \mathrm{K}(X)[\frac{1}{n}] \arrow[d, "ch"] \\ \mathrm{HC}^-(X/k) \arrow[r, "\psi^n"] & \mathrm{HC}^-(X/k). \end{tikzcd} \tag{B.1}$$

Upon taking homotopy groups, we get back the main compatibility result asserted in [36]. As mentioned in the introduction of [36], the classical Adams operations on K-theory and negative cyclic homology are defined in "very different ways." The key point of our proof of Theorem B.1 is that, given the new context of [3], the two constructions are not so different after all. In particular, our proof is quite different from the one [36] where the key point is to use Goodwillie's theorem about $\mathrm{K}_{\mathbb{Q}}^{\mathrm{inf}}$ and Cathelineu's result on compatibility of Adams operations in the infinitesimal context (whose proof was repaired in [36, App. B]). In fact, what we will need in the main body of the paper is a filtered enhancement of Theorem B.10, in the context of smooth k-schemes when the classical motivic filtration (in the sense of §3.1) is defined.

B.1 Construction of the Adams operations via the Annala-Iwasa theorem

To begin, we offer a construction of the Adams operations on n-periodic K-theory using the technology of [3]. Fix a commutative ring k and consider the category of smooth k-schemes; as in [3] we denote by

$$\mathrm{St}_k := \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k; \mathcal{S}\mathrm{pc}),$$

the ∞-category of Zariski sheaves of spaces on smooth k-schemes; we call the latter the ∞-category of *Zariski k-stacks.* We have the *Picard stack*[17] $\mathcal{P}\mathrm{ic} \in \mathrm{St}_k$[3, 2.1.4] which comes equipped with a canonical map of Zariski k-stacks

$$\mathcal{P}\mathrm{ic} \to \Omega^\infty \mathrm{K}.$$

In fact, $\mathcal{P}\mathrm{ic}$ is a grouplike $\mathbb{E}_\infty$-monoid in k-stacks (under tensor products of line bundles/induced by the group structure on $\mathbb{G}_m$); this is equivalent to saying that the functor $\mathcal{P}\mathrm{ic} : \mathrm{Sm}_k^{\mathrm{op}} \to \mathcal{S}\mathrm{pc}$ in fact promotes to a functor $\mathcal{P}\mathrm{ic} : \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{CMon}$ where CMon denote the ∞-category of $\mathbb{E}_\infty$-monoids in spaces. The (pointed) suspension spectrum functor $\Sigma^\infty_+ : \mathcal{S}\mathrm{pc} \to \mathrm{Sp}$ is lax monoidal whence it promotes to a functor $\Sigma^\infty_+ : \mathrm{CMon} \to \mathrm{CAlg}(\mathrm{Sp})$. As in [3, 5.3.1] we set

$$\mathbb{S}[\mathcal{P}\mathrm{ic}] := \Sigma^\infty_+ \mathcal{P}\mathrm{ic} \in \mathrm{CAlg}(\mathrm{St}_k).$$

Since the canonical map of Zariski stacks $\mathcal{P}\mathrm{ic} \to \Omega^\infty K$ is a morphism of commutative monoids (where the right hand side is given the multiplicative structure), the composite

$$\mathbb{S}[\mathcal{P}\mathrm{ic}] \to \Sigma^\infty_+ \Omega^\infty K \to K \tag{B.2}$$

defines a morphism in $\mathrm{CAlg}(\mathrm{St}_k)$. The result [3, Thm. 5.3.3] concerns this map, where it is proved that it becomes an equivalence under a certain localization. To make use of this result, we introduce several notation from [3]:

1. we have the pointed version of S-stacks St_{k*} which is a symmetric monoidal ∞-category; it contains the Yoneda image of $\mathbb{P}^1$ pointed at ∞.
2. In St_{k*} we have the $\mathbb{E}_\infty$-ring $Q(\mathcal{P}\mathrm{ic}) := \Omega^\infty \mathbb{S}[\mathcal{P}\mathrm{ic}]$; whence we may speak of $Q(\mathcal{P}\mathrm{ic})$-modules in St_{k*}.
3. We have the *Bott element* which is a map in St_{k*}

$$\beta := 1 - [\mathcal{O}(-1)] : \mathbb{P}^1 \to \Omega^\infty \mathbb{S}[\mathcal{P}\mathrm{ic}];$$

   classifying the above named bundle.
4. By the previous construction, $Q(\mathcal{P}\mathrm{ic})$-module comes equipped with a canonical map

$$\beta : E \to E^{\mathbb{P}^1};$$

   and we say that E is $\mathbb{P}^1$-periodic if this map is an equivalence.
5. we have the stabilized version of St_k denoted by $\mathrm{Sp}(\mathrm{St}_k)$, modeled by Zariski sheaves of spectra om Sm_k; it comes equipped with the functor $\Sigma^\infty_+ : \mathrm{St}_k \to \mathrm{Sp}(\mathrm{St}_k)$ with a right adjoint Ω^∞. Since Σ^∞_+ is lax monoidal, we obtain an induced adjunction:

$$Q(\mathcal{P}\mathrm{ic})\text{-}\mathcal{M}\mathrm{od}(\mathrm{St}_k) \leftrightarrows \mathbb{S}[\mathcal{P}\mathrm{ic}]\text{-}\mathcal{M}\mathrm{od}(\mathrm{Sp}(\mathrm{St}_k)).$$

6. the morphism $\mathbb{S}[\mathcal{P}\mathrm{ic}] \to K$ is automatically a morphism of $\mathbb{E}_\infty$-algebra objects in $\mathbb{S}[\mathcal{P}\mathrm{ic}]\text{-}\mathcal{M}\mathrm{od}(\mathrm{Sp}(\mathrm{St}_k))$.

Here is the main theorem of [3]:

Theorem B.2 (Annala–Iwasa). *We have an equivalence of $\mathbb{E}_\infty$-algebras in $\mathbb{S}[\mathcal{P}\mathrm{ic}]\text{-}\mathcal{M}\mathrm{od}(\mathrm{Sp}(\mathrm{St}_k))$:*

$$\mathbb{S}[\mathcal{P}\mathrm{ic}][\beta^{-1}] \to \mathrm{K}.$$

---

[17] For concreteness, we can regard the Picard stack as the functor on Sm_k which assigns X to the groupoid of line bundles on X (regarded as an object of $\mathcal{S}\mathrm{pc}$. Alternatively, we may take the connective cover $\tau_{\geq 0}\,(R\Gamma_{\mathrm{Zar}}(-; \mathbb{G}_m)[1]))$ and apply the Dold-Kan correspondence to get a presheaf of spaces.

With this we can define the Adams operations on the n-periodization of non-connective K-theory as follows. First, consider the map of group schemes:

$$\mathbb{G}_m \to \mathbb{G}_m \qquad z \mapsto z^n.$$

This induces a map of commutative monoids:

$$(-)^n : \mathcal{P}\text{ic} \to \mathcal{P}\text{ic};$$

whence a map in $\mathrm{CAlg}(\mathrm{St}_k)$:

$$\psi^n : Q(\mathcal{P}\text{ic}) \to Q(\mathcal{P}\text{ic}).$$

Composing the Bott element with the map $Q(\mathcal{P}\text{ic}) \to \Omega^\infty K$ we get a map in St_{k*}:

$$\beta_K : \mathbb{P}^1 \to \Omega^\infty K.$$

By the exact same argument as in [8, Lem. 3.11] we have:

Lemma B.3. *The map*

$$\mathbb{P}^1 \to Q(\mathcal{P}\text{ic}) \xrightarrow{\psi^n} Q(\mathcal{P}\text{ic}) \to \Omega^\infty K$$

is homotopic to $n\beta_K$.

The next construction follows [8, §3.3.1].

Construction B.4. Since $\psi^n : Q(\mathcal{P}\text{ic}) \to Q(\mathcal{P}\text{ic})$ is a morphism of $\mathbb{E}_\infty$-rings, we can restrict the $Q(\mathcal{P}\text{ic})$ $\Omega^\infty K$ along ψ^n; which we denoted by $\Omega^\infty K^{[n]}$. We then have a morphism of $Q(\mathcal{P}\text{ic})$-modules

$$Q(\mathcal{P}\text{ic}) \xrightarrow{\psi^n} Q(\mathcal{P}\text{ic}) \to \Omega^\infty K^{[n]},$$

whence a morphism of $\mathbb{S}[\mathcal{P}\text{ic}]$-modules:

$$\mathbb{S}[\mathcal{P}\text{ic}] \xrightarrow{\psi^n} \mathbb{S}[\mathcal{P}\text{ic}] \to K^{[n]};$$

here we have abusively denoted the suspension of ψ^n by the same name and $K^{[n]}$ is the $\mathbb{S}[\mathcal{P}\text{ic}]$-module obtained by the same restriction.

Inverting β along the above morphism and applying Lemma B.3 and Theorem B.2 yields a morphism of $\mathbb{E}_\infty$-algebras in $\mathbb{S}[\mathcal{P}\text{ic}]$-modules:

$$K \to K^{[n]}[n\beta_K^{-1}] \simeq K[\tfrac{1}{n}],$$

which factors through the n-periodization of K and thus gives a map of $\mathbb{E}_\infty$-algebras in $\mathbb{S}[\mathcal{P}\text{ic}]$-modules

$$\psi^n : K[\tfrac{1}{n}] \to K[\tfrac{1}{n}].$$

We regard ψ^n as a $\mathbb{E}_\infty$-algebra maps in $\mathrm{Sp}(\mathrm{St}_k)$.

B.2 Adams operations and the slice filtration

We will now study the interaction of the Adams operations, as defined above, with the classical motivic filtration $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)$ on K-theory, at least after rationalization. As reviewied in §3.1 the classical motivic filtration is incarnated by Voevodsky's the slice filtration in the sense of stable motivic homotopy theory. This filtration is *a priori* defined on KH as it relies on $\mathbb{A}^1$-invariance properties. However, in this section, we assume that k is a field so for X a smooth k-scheme, we have that $\mathrm{KH}(X) \simeq \mathrm{K}(X)$ and thus $\mathrm{KH}(X)_\mathbb{Q} \simeq \mathrm{K}(X)_\mathbb{Q}$.

To begin, note that the Construction B.4 agrees with the construction from motivic homotopy theory. Indeed, our construction manifestly coincides[18] with the construction from [8, §3.3] which has been compared to prior constructions due to Riou by [8, Prop. 3.12]. Therefore, in the context of the motivic stable ∞-category over k, $\mathcal{SH}(k)$, Construction B.4 furnishes multiplicative maps in $\mathcal{SH}(k)$:

$$\psi^n : \mathrm{KGL}_\mathbb{Q} \to \mathrm{KGL}_\mathbb{Q} \qquad n \geq 2.$$

---

[18] In more details: the construction above naturally takes place in the ∞-category $\mathrm{Sp}_{\mathbb{P}^1}(\mathrm{St}_k)$. By [3, Rem. 2.2.11] the ∞-category $\mathcal{SH}(k)$ embeds fully faithfully inside $\mathrm{Sp}_{\mathbb{P}^1}(\mathrm{St}_k)$.

Here KGL is the motivic spectrum representing (homotopy invariant) algebraic K-theory and $\mathrm{KGL}_{\mathbb{Q}}$ denotes its rationalization. For the purposes of this paper we need to check that $\mathrm{KGL}_{\mathbb{Q}}$ decomposes in a multiplicatively compatible with the slice filtration, which is formal once one gathers some results already available in the literature.

The main result of [99] furnishes a decomposition in $\mathcal{SH}(S)$[19]:

$$\mathrm{KGL}_{\mathbb{Q}} \simeq \bigoplus_{j\in\mathbb{Z}} \mathrm{KGL}_{\mathbb{Q}}^{\psi^n - n^j} \tag{B.3}$$

One then defines the *Beilinson motivic cohomology spectrum* as the piece

$$\mathrm{H}\mathbb{Q} := \mathrm{KGL}_{\mathbb{Q}}^{\psi^n - \mathrm{id}},$$

this construction is independent of n. This is a motivic spectrum which represents rationalized motivic cohomology whenever the latter is defined.

Remark B.5. In Spitweck's theory of $\mathbb{A}^1$-invariant motivic cohomology [105] $\mathrm{KGL}_{\mathbb{Q}}^{\psi^n - \mathrm{id}}$ is, by construction, the rational part of his motivic cohomology.

Furthermore, the Bott element $\beta = 1 - [\mathcal{O}(-1)] \in K_0(\mathbb{P}^1_k)$ induces an invertible map $\beta : \Sigma^{2,1}\mathrm{KGL} \to \mathrm{KGL}$ in $\mathcal{SH}(k)$ such that the composite

$$\Sigma^{2,1}\mathrm{KGL}_{\mathbb{Q}}^{\psi^n - n^j} \to \Sigma^{2,1}\mathrm{KGL}_{\mathbb{Q}} \xrightarrow{\beta} \mathrm{KGL}_{\mathbb{Q}} \to \mathrm{KGL}_{\mathbb{Q}}^{\psi^n - n^{j+1}}$$

is an equivalence [30, Lem. 14.1.4]. Therefore, we have equivalences, for all $j \in \mathbb{Z}$:

$$\Sigma^{2j,j}\mathrm{H}\mathbb{Q} \simeq \Sigma^{2,1}\mathrm{KGL}_{\mathbb{Q}}^{\psi^n - n^j}.$$

The interaction between the multiplicative structures of $\mathrm{H}\mathbb{Q}$ and $\mathrm{KGL}_{\mathbb{Q}}$ was examined in [30, §14.2]. Here are the key points: the inclusion of the summand (as in the decomposition B.3), $\mathrm{H}\mathbb{Q} \to \mathrm{KGL}_{\mathbb{Q}}$ is an $\mathbb{E}_\infty$-map [30, Corol. 14.2.17]. Furthermore we consider the motivic spectrum $\mathrm{H}\mathbb{Q}[t, t^{-1}]$, the free $\mathbb{E}_\infty$-$\mathrm{H}\mathbb{Q}$-algebra generated by a single invertible generator in degree $(2, 1)$ (this construction is also studied more explicitly in [105, App. C]). The underlying graded spectrum of this object is $\bigoplus_{j\in\mathbb{Z}} \Sigma^{2j,j}\mathrm{H}\mathbb{Q}$. We have a multiplicative filtration[20] $\mathrm{Fil}^\star\mathrm{H}\mathbb{Q}[t, t^{-1}]$ on $\mathrm{H}\mathbb{Q}[t, t^{-1}]$ where, on underlying object, we have

$$\mathrm{Fil}^i\mathrm{H}\mathbb{Q}[t, t^{-1}] = \bigoplus_{j\geq i} \Sigma^{2j,j}\mathrm{H}\mathbb{Q}.$$

The canonical map

$$\mathrm{H}\mathbb{Q}[t, t^{-1}] \to \mathrm{KGL}_{\mathbb{Q}}, \tag{B.4}$$

induced by the inclusion of the summand $\Sigma^{2,1}\mathrm{H}\mathbb{Q} \simeq \mathrm{KGL}_{\mathbb{Q}}^{\psi^n - n} \to \mathrm{KGL}_{\mathbb{Q}}$ is an equivalence. We record the following result about the compatibility of this equivalence with the filtration. We use the notation from Remark 3.6: if E is a motivic spectrum then $f^\star E$ denotes the slice filtration.

Theorem B.6. *Let k be a field. The equivalence* (B.4) *enhances to a filtered $\mathbb{E}_\infty$-equivalence in $\mathcal{SH}(k)$:*

$$\mathrm{Fil}^\star\mathrm{H}\mathbb{Q}[t, t^{-1}] \xrightarrow{\simeq} f^\star\mathrm{KGL}_{\mathbb{Q}}.$$

Proof. Over any base, by the functoriality and the lax monoidality of the slice filtration [9], the map $\mathrm{H}\mathbb{Q}[t, t^{-1}] \to \mathrm{KGL}_{\mathbb{Q}}$ from (B.4) induces an $\mathbb{E}_\infty$-map of filtered objects:

$$f^\star\mathrm{H}\mathbb{Q}^{\mathrm{Spi}}[t, t^{-1}] \to f^\star\mathrm{KGL}_{\mathbb{Q}}.$$

To prove the claim, restricting ourselves to the hypothesis on k, it suffices to prove that for all $i \in \mathbb{Z}$, we have $f^{\geq i}(\bigoplus_{j<i} \mathrm{H}\mathbb{Q}(i)[2i]) \simeq 0$. Since $f^\star$ commutes with colimits [43, Prop. 3.5] and twists [101, Lem. 2.1], it suffices to know that $f^{\geq 1}\mathrm{H}\mathbb{Q}^{\mathrm{Spi}} \simeq 0$ which follows from the fact that negative weight motivic cohomology vanishes. □

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[19] There is a standing assumption in [99] that S is a regular scheme; this is only used to ensure that KGL indeed represents algebraic K-theory as opposed to homotopy K-theory. The arguments work over any qcqs scheme given this caveat. In any event, we only need the relevant compatibility for smooth schemes over a Dedekind domain or a field.

[20] In more details. If G^* is a graded object, then applying the functor $\iota_! : \mathcal{C}^{\mathbb{Z}^\delta} \to \mathcal{C}^{\mathbb{Z}^{\mathrm{op}}}$ produces a filtered object $\iota_! G^*$ where

$$(\iota_! G^*)_n := \bigoplus_{m\geq n} G_m,$$

and the transition maps are given by projections. This is what we refer to as the natural filtration on a graded object G^*.

Translating the result above to the classical motivic filtration reviewed in §3.1 and Remark 3.6, we have proved:

Corollary B.7. *Let $n \geq 2$ and k a field. For any essentially smooth k-scheme X, we have that:*

1. *the Adams operations $\psi^n : \mathrm{K}(X)_{\mathbb{Q}} \to \mathrm{K}(X)_{\mathbb{Q}}$ promotes to a multiplicative map of filtered objects*
$$\psi^n : \mathrm{Fil}^{\star}_{\mathrm{cla}}\mathrm{K}(X)_{\mathbb{Q}} \to \mathrm{Fil}^{\star}_{\mathrm{cla}}\mathrm{K}(X)_{\mathbb{Q}};$$
2. *on the graded pieces $\mathrm{gr}^j_{\mathrm{cla}}$, ψ^n acts by $\cdot n^j$;*
3. *there is a natural equivalence of filtered $\mathbb{E}_\infty$-rings:*
$$\bigoplus_{j \in \mathbb{Z}} \mathbb{Q}(j)^{\mathrm{cla}}(X)[2j] \xrightarrow{\simeq} \mathrm{K}(X)_{\mathbb{Q}}$$
which preserves the natural filtration on the graded object on the domain and the slice filtration on the target.
4. *The spectral sequence degenerates rationally.*

B.3 Adams operations and the Chern character

Let us now recall the Adams operations in negative cyclic homology. While these operations are classically defined for the negative cyclic homology groups and even complexes [83, 116], we use Raksit's thesis [98] as our main reference. To the best of our knowledge, it is the first instance where Adams operations on negative cyclic homology was proved to be a functorially a filtered map, on the level of spectra.

As in the case of K-theory, we write $\mathrm{HH}(X/k)^{[n]}$ be the S^1-equivariant $\mathbb{E}_\infty$-ring spectrum[21], obtained by restricting the S^1-action along the n-power map $[n] : S^1 \to S^1$. As in [98, Cons. 6.4.3], we obtain an $\mathbb{E}_\infty$-ring map $\mathrm{HH}(X/k) \to \mathrm{HH}(X/k)^{[n]}$. As proved in [98, Prop. 6.4.4], the Adams operations promote to a multiplicative, S^1-equvariant, filtered map

$$\psi^n : \mathrm{Fil}_{\mathrm{HKR}}\mathrm{HH}(X/k) \to \mathrm{Fil}_{\mathrm{HKR}}\mathrm{HH}(X/k)^{[n]}.$$

Passing to fixed points and applying [98, Lems. 6.4.5-6.4.6] we get a the *Adams operation* on HC^-:

$$\psi^n : \mathrm{HC}^-(X/k) \to \mathrm{HC}^-(X/k); \tag{B.5}$$

as in [98, Cons. 6.4.7]. This refines to a filtered map as verified in [98, Cons. 6.4.8]:

$$\psi^n : \mathrm{Fil}_{\mathrm{HKR}}\mathrm{HC}^-(X/k) \to \mathrm{Fil}_{\mathrm{HKR}}\mathrm{HC}^-(X/k). \tag{B.6}$$

Lemma B.8. *We have the following commutative diagram in* $\mathrm{CAlg}(\mathrm{St}_k)$*:*

$$\begin{tikzcd} Q(\mathcal{P}\mathrm{ic}) \arrow[r] \arrow[d] & \Omega^\infty \mathrm{HC}^-(-/k) \arrow[d] \\ Q(\mathcal{P}\mathrm{ic}) \arrow[r] & \Omega^\infty \mathrm{HC}^-(-/k). \end{tikzcd} \tag{B.7}$$

Proof. The endofunctor $Q : \mathrm{St}_k \to \mathrm{St}_k$ is a monad for grouplike $\mathbb{E}_\infty$-monoids in St_k and $Q(\mathcal{P}\mathrm{ic})$ is the free monad on the object $\mathcal{P}\mathrm{ic}$. Since $\Omega^\infty\mathrm{HC}^-(-/k)$ is a grouplike $\mathbb{E}_\infty$-monoid, to prove commutation of (B.7) we need only check commutation of the diagram in St_k:

$$\begin{tikzcd} \mathcal{P}\mathrm{ic} \arrow[r] \arrow[d, "(-)^n"'] & \Omega^\infty \mathrm{HC}^-(-/k) \arrow[d, "\psi^n"] \\ \mathcal{P}\mathrm{ic} \arrow[r] & \Omega^\infty \mathrm{HC}^-(-/k). \end{tikzcd} \tag{B.8}$$

---

[21] The Hochschild complex of discrete commutative rings or, more genereally, animated rings, have a natural structure of a *derived commutative algebra* as in [98, Def. 4.2.22] which we can forget to a $\mathbb{E}_\infty$-ring. We will not make use of this richer structure explicitly, but it is used systematically in [98] to formulate a universal property of Hochschild homology with the HKR filtration.

There are several ways to verify this: for example, using (the proof of) [3, Corol. 4.3.3] we see that both composites coincide with the element

$$nc_1 \in \mathrm{HC}_0^-(k)[[c_1]] = k[[c_1]].$$

□

Proof of Theorem B.1. It follows immediately from the commutativity of (B.7) from Lemma B.8 that we have the following commutative diagram of $\mathbb{E}_\infty$-algebras in $\mathbb{S}[\mathcal{P}\mathrm{ic}]$-$\mathcal{M}\mathrm{od}$:

$$\begin{tikzcd} \mathbb{S}[\mathcal{P}\mathrm{ic}] \arrow[r] \arrow[d] & \mathrm{HC}^-(-/k) \arrow[d] \\ \mathbb{S}[\mathcal{P}\mathrm{ic}] \arrow[r] & \mathrm{HC}^-(-/k). \end{tikzcd} \tag{B.9}$$

Since $\mathrm{HC}^-(-/k)$ has the projective bundle formula it is already $\mathbb{P}^1$-periodic, after applying Theorem B.2 and Construction B.4 we get a commutative diagram of $\mathbb{E}_\infty$-algebras in $\mathbb{S}[\mathcal{P}\mathrm{ic}]$-$\mathcal{M}\mathrm{od}(\mathrm{Sp}(\mathrm{St}_k))$ given by

$$\begin{tikzcd} \mathrm{K}[\frac{1}{n}] \arrow[r] \arrow[d] & \mathrm{HC}^-(-/k) \arrow[d] \\ \mathrm{K}[\frac{1}{n}] \arrow[r] & \mathrm{HC}^-(-/k), \end{tikzcd} \tag{B.10}$$

as desired. □

Remark B.9. In the main text, we only need Adams operations for X a smooth scheme over a field. Hence the Adams operations on K-theory could have been defined using motivic stable homotopy theory as explained in §B.2. However since HC^- is not $\mathbb{A}^1$-invariant, Theorem B.1 could not have been proved within the environment of motivic stable homotopy theory. The main theorem of [3] furnishes a universal property of K-theory outside of the $\mathbb{A}^1$-invariant setting which is key in our proof of Theorem B.1, even in the smooth setting.

Corollary B.10. *Let $n \geq 2$ and let k be a field of characteristic zero. Then for any essentially smooth k-scheme X, the commutative diagram B.1 canonically enhances to a commutative diagram of filtered $\mathbb{E}_\infty$-algebras:*

$$\begin{tikzcd} \mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}_\mathbb{Q} \arrow[r, "\psi^n"] \arrow[d, "ch"] & \mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}_\mathbb{Q} \arrow[d, "ch"] \\ \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-/k) \arrow[r, "\psi^n"] & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-/k). \end{tikzcd} \tag{B.11}$$

Proof. The two circuits of (B.1) promote to naturally equivalent, multiplicative filtered morphisms $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}_\mathbb{Q} \to \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-$ by Remark 4.7. The commutativity of (B.11) then follows from the fact that the Adams operations on both $\mathrm{K}_\mathbb{Q}$ and HC^- admits unique, multiplicative filtered refinement by Corollary B.7 and [98, Cons. 6.4.8] respectively.

□

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